Karl-Franzens-Universität Graz & Technische Universität Graz

SPEZIALFORSCHUNGSBEREICH F 003

OPTIMIERUNG
und
KONTROLLE

Projektbereich
DISKRETE OPTIMIERUNG

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Bericht Nr. 134 – August 1998

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Abstract

We consider all planar oriented curves that have the following property depending on a fixed angle φ. For each point B on the curve, the rest of the curve lies inside a wedge of angle φ with apex in B. This property restrains the curve's meandering, and for $\varphi \leq \frac{\pi}{2}$ this means that a point running along the curve always gets closer to all points on the remaining part. For all $\varphi < \pi$, we provide an upper bound $c(\varphi)$ for the length of such a curve, divided by the distance between its endpoints, and prove this bound to be tight. A main step is in proving that the curve's length cannot exceed the perimeter of its convex hull, divided by $1 + \cos\varphi$.

Keywords. Self-approaching curves, convex hull, detour, arc length.

1 Introduction

Let f be an oriented curve in the plane running from A to Z, and let φ be an angle in $[0, \pi)$. Suppose that, for every point B on f, the curve segment from B to Z is contained in a wedge of angle φ with apex in B. Then the curve f is called *φ-self-approaching*, generalizing the self-approaching curves introduced by Icking and Klein [4].

At the 1995 Dagstuhl Seminar on Computational Geometry, Seidel [1] posed the following open problems. Is there a constant, $c(\varphi)$, so that the length of every φ-self-approaching curve is at most $c(\varphi)$ times the distance between its endpoints? If so, how small can one prove $c(\varphi)$ to be?

Both questions are answered in this paper. We provide, for each φ in $[0, \pi)$, a constant $c(\varphi)$ with the above property, and prove it minimal by constructing a φ-self-approaching curve such that this factor $c(\varphi)$ is achieved.

Self-approaching curves are interesting for different reasons. If a *mobile robot* wants to get to the kernel of an unknown star-shaped polygon and continuously follows the angular bisector of the innermost left and right reflex vertices that are visible, the resulting path is self-approaching for $\varphi = \pi/2$; see [4]. Since the value of $c(\pi/2)$ is known to be ≈ 5.3331, as already shown in Icking et al. [5], one immediately obtains

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This work was partially supported by the Deutsche Forschungsgemeinschaft, grant Kl 655/8-2, and by the Spezialforschungsbereich *Optimierung und Kontrolle*.

an upper bound for the competitive factor of the robot's strategy. Improving on this, Lee and Chwa [6] give a tight upper bound of $\pi + 1$ for this factor, and Lee et al. [7] present a different strategy that achieves a factor of 3.829, while a lower bound of 1.48 is shown by López-Ortiz and Schuierer [8].

In the construction of *spanners* for euclidean graphs, one can proceed by recursively adding to the spanning structure a point from a cone of angle φ, resulting in a sequence $p_1, p_2, \ldots, p_n$ such that for each index i, $p_{i+1}, \ldots, p_n$ are contained in a cone of angle φ with apex p_i; see Ruppert and Seidel [10] or Arya et al. [2]. (Note, however, that such a sequence of points does not necessarily define a φ-self-approaching polygonal chain because the property may not hold for every point in the interior of an edge.)

Finally, such properties of curves are interesting in their own right. For example, in the book by Croft et al. [3] on open problems in geometry, *curves with increasing chords* are defined by the property that for every four consecutive points A, B, C, D on the curve, B is closer to C than A to D. The open problem of how to bound the length of such curves divided by the distance between its endpoints has been solved by Rote [9]; he showed that the tight bound equals $\frac{2}{3}\pi$. There is a nice connection to the curves studied in this paper. Namely, a curve has increasing chords if and only if it is $\pi/2$-self-approaching in both directions.

In this paper we generalize the results of [5] to arbitrary angles $\varphi < \pi$. In Section 2 we prove, besides some elementary properties, the following fact. Let B, C and D denote three consecutive points on a φ-self-approaching curve f. Then

$$CD \leq BD - \cos\varphi \cdot \mathrm{length}(f[B, C]),$$

where CD denotes the euclidean distance between points C and D and $\mathrm{length}(f[B, C])$ denotes the length of f between B and C. This property accounts for the term "self-approaching"; in fact, for $\varphi \leq \pi/2$ the factor $\cos\varphi$ is not negative, so that $CD \leq BD$ holds: The curve always gets closer to each point on its remaining part. Although this property does not hold for $\varphi > \frac{\pi}{2}$, we will nevertheless see that our analysis of the tight upper bound $c(\varphi)$ directly applies to this case, too.

In Section 3 we show that the length of a φ-self-approaching curve cannot exceed the perimeter of its convex hull, divided by $1 + \cos\varphi$. This fact is the main tool in our analysis. It allows us to derive an upper bound for the curve's length by circumscribing it with a simple, closed convex curve whose length can be easily computed, see Section 4. Finally, in Section 5, we demonstrate that the resulting bound is tight, by constructing φ-self-approaching curves for which the upper bounds are achieved.

2 Definitions and properties

For two points B and C let $\vec{r}(B, C)$ denote the ray starting at B and passing through C. We simply write BC for the euclidean distance between B and C.

We consider oriented curves f in the plane, i. e., each curve f has a specified direction from beginning to end. We do not make any assumptions about smoothness or rectifiability of the curve, although it will turn out that φ-self-approaching curves are rectifiable.

For two consecutive points B and C on f, we write $f[B, C]$ for the part of f between B and C. By $f^{\geq B}$ ($f^{>B}$) we denote the part of f from B to the end (without B) and length(f), length($f[B, C]$), etc., means the length of the curve or of an arc. For a curve $f[B, C]$ and a point $D \notin f[B, C]$ let $\gamma(D, f[B, C])$ be the positive angle of rotation around D the curve goes through from B to C. So if we consider $f[B, C]$ as a continous function in polar coordinates centered at D running from (ψ_B, BD) to (ψ_C, CD) then $\gamma(D, f[B, C]) = |\psi_B - \psi_C|$.

Definition 1 A curve f is called *φ-self-approaching* for $0 \leq \varphi < \pi$ if for any point B on f there is a wedge of angle φ at point B which contains $f^{\geq B}$. In other words, for any three consecutive points B, C, D on f, the angle $\gamma(B, f[C, D])$ is at most φ.

Let f be an oriented curve from A to Z. Then the quantity

$$\frac{\text{length}(f[A, Z])}{AZ}$$

is called the *detour* of f.

The detour of a curve is the reciprocal of the *minimum growth rate* used in [9].

We wish to bound the detour of φ-self-approaching curves. The first definition of φ-self-approaching curves also makes sense for $\varphi \geq \pi$, but then any circular arc connecting two points A and Z is φ-self-approaching, which means that the detour of such curves is not bounded. Therefore, we restrict our attention to the case $\varphi < \pi$. Each 0-self-approaching curve is a line segment and its detour equals 1.

Lemma 2 *A φ-self-approaching curve does not go through any point twice.*

Proof. Suppose the curve visits point B twice. Shortly after the first visit of B there is a point on the curve for which the φ-self-approaching property is violated. $\square$

So a φ-self-approaching curve $f[B, C]$ cannot visit B again, now we want to show a stronger restriction, that is, a φ-self-approaching curve $f[B, C]$ cannot loop around B.

Lemma 3 *Let B, C, D be three consecutive points on a φ-self-approaching curve. If C lies on $\vec{r}(D, B)$ then $CD \leq BD$.*

Proof. The assumption $CD > BD$ would lead to an angle $\gamma(B, f[C, D]) = \pi$, violating the φ-self-approaching property. $\square$

The following lemma shows, roughly speaking, that a φ-self-approaching curve $f[A, Z]$ is enclosed by two oppositely winding φ-logarithmic spirals through A with pole Z, see Figure 1. For $\varphi > \pi/2$, this is true only locally, as long as the curve does not leave the vicinity of A. In polar coordinates (r, ψ), a φ-logarithmic spiral is the set of all points with $r = e^{\psi \cot \varphi}$ or $r = e^{-\psi \cot \varphi}$. Each ray through the origin intersects the spiral in the same angle φ. In the appendix we give a short summary of known facts of φ-logarithmic spirals. For $\varphi \neq \pi/2$, the length of an arc $S[A, B]$ of a φ-logarithmic spiral S around pole Z is given by $\frac{1}{\cos \varphi}(AZ - BZ)$. Note that for $\varphi = \pi/2$, the spiral degenerates to a circle.

The property of Lemma 4a is used in Section 4 for the circumscription of a φ-self-approaching curve by a convex area whose perimeter is easy to determine. With the help of Lemma 4b we are able to prove a close connection between the length of a φ-self-approaching curve and the perimeter of its convex hull, see Section 3.

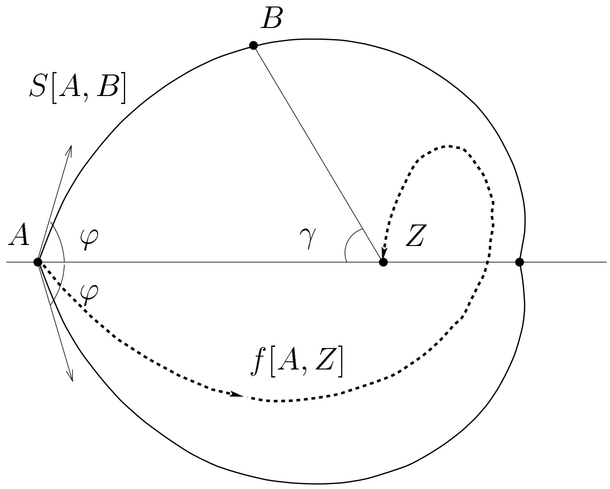

Figure 1: For $0 < \varphi < \pi/2$ the curve is enclosed by two congruent arcs of a φ-logarithmic spiral S.

Lemma 4 *Let B, C, D be three consecutive points on a φ-self-approaching curve f.*

(a)
$$CD \leq BD \cdot e^{-\gamma(D, f[B,C]) \cot \varphi}$$

(b)
$$CD \leq BD - \cos \varphi \cdot \mathrm{length}(f[B, C])$$

The inequalities of Lemma 4 are fulfilled by equality for the arc $S[A, B]$ of a φ-logarithmic spiral S around a pole Z, see Figure 1, i. e., we have $BZ = AZ \cdot e^{-\gamma \cot \varphi}$ and $AZ = BZ - \cos \varphi \cdot \mathrm{length}(S[A, B])$.

In order to proof Lemma 4 we will first establish a somewhat weaker bound for the bound of the lemma, from which the stronger bounds will follow by a limiting argument.

Lemma 5 *Let B, C, D be three consecutive points on a φ-self-approaching curve f. Let α denote the angle $\gamma(D, f[B, C]) \geq 0$ and let $\bar{\varphi}$ denote the angle $\gamma(B, f[C, D]) \leq \varphi$. Assume that $\alpha < \pi - \varphi$.*

(a) *Then*
$$CD = BD \cdot \frac{\sin \bar{\varphi}}{\sin(\bar{\varphi} + \alpha)} \leq BD \cdot \frac{\sin \varphi}{\sin(\varphi + \alpha)} \tag{1}$$

For $\alpha \cdot \cot \varphi \leq 1$, $\alpha \leq \frac{1}{2}$, it follows that
$$CD \leq BD \cdot e^{-\alpha \cdot \cot \varphi} \cdot e^{K\alpha^2}, \tag{2}$$

where the constant $K = 2(1 + \cot^2 \varphi)$ depends only on φ.

(b) *We have*
$$CD - BD + BC \cdot \cos \bar{\varphi} = BC \cdot \sin \bar{\varphi} \cdot \frac{1 - \cos \alpha}{\sin \alpha}. \tag{3}$$

For $\alpha \leq 1$, it follows that
$$CD - BD + BC \cdot \cos \varphi \leq BC \cdot \alpha. \tag{4}$$

Proof. Within this proof we make use of some elementary inequalities shown in Lemma 12 in the appendix. Equations (1) and (3) follow from the sine law. From (3), we obtain (4) by using Lemma 12e, $\cos\bar{\varphi} \geq \cos\varphi$ and $|\sin\bar{\varphi}| \leq 1$.

Now, to prove (2), we have to show the inequality

$$\frac{\sin(\varphi+\alpha)}{\sin\varphi} \cdot e^{2(1+\cot^2\varphi)\alpha^2} \geq e^{\alpha\cdot\cot\varphi}\,.$$

We have

$$\begin{aligned}\frac{\sin(\varphi+\alpha)}{\sin\varphi} &= \frac{\sin\varphi\cos\alpha+\cos\varphi\sin\alpha}{\sin\varphi}\\ &= \cos\alpha+\sin\alpha\cot\varphi\\ &\geq 1-\alpha^2+\sin\alpha\cot\varphi\,,\end{aligned}$$

using Lemma 12a,

$$e^{\alpha\cdot\cot\varphi} \leq 1+\alpha\cdot\cot\varphi+\alpha^2\cdot\cot^2\varphi\,,$$

using Lemma 12d, and finally

$$e^{2(1+\cot^2\varphi)\alpha^2} \geq 1+2(1+\cot^2\varphi)\alpha^2\,,$$

using Lemma 12c with $x = 2(1+\cot^2\varphi)\alpha^2$. So we only have to show

$$(1-\alpha^2+\sin\alpha\cot\varphi)(1+2(1+\cot^2\varphi)\alpha^2) \geq 1+\alpha\cdot\cot\varphi+\alpha^2\cdot\cot^2\varphi\,.$$

We show separately

$$\sin\alpha\cot\varphi\,(1+2(1+\cot^2\varphi)\alpha^2) \geq \alpha\cdot\cot\varphi \tag{5}$$

and

$$(1-\alpha^2)\,(1+2(1+\cot^2\varphi)\alpha^2) \geq 1+\alpha^2\cdot\cot^2\varphi\,, \tag{6}$$

from which the previous inequality follows by summation. Inequality (5) follows from Lemma 12b and $\cot^2\varphi \geq 0$. Multiplying out and simplifying (6) leads to the equivalent inequality

$$\alpha^2(1-2\alpha^2)(1+\cot^2\varphi) \geq 0\,,$$

which is obviously true. □

Now we show how Lemma 4 follows from Lemma 5.

Proof of Lemma 4. First we consider part (a) of Lemma 4 and Lemma 5. We divide the angular range of $\gamma(D, f[B,C])$ into n equal sectors of angle $\alpha := \gamma(D, f[B,C])/n$. By choosing n large enough, we can ensure that α fulfills the conditions for (2) in Lemma 5.

By continuity, the curve $f[B,C]$ must pass through $n+1$ consecutive points $B = A_0, A_1, A_2, \ldots, A_n = C$ with $\gamma(D, f[A_i, A_{i+1}]) = \alpha$ for $i = 0, \ldots, n-1$. Then we can apply (2) to the successive distances $A_0D = BD$, A_1D, $\ldots$, $A_nD = CD$ to obtain

$$A_{i+1}D \leq A_iD \cdot e^{-\alpha\cdot\cot\varphi} \cdot e^{K\alpha^2}\,,$$

and hence

$$CD \le BD \cdot e^{-\gamma(D,f[B,C])\cdot\cot\varphi} \cdot e^{K\frac{\gamma(D,f[B,C])^2}{n}}.$$

Since we can choose n arbitrarily large, we get $\frac{\gamma(D,f[B,C])^2}{n} \to 0$ as $n \to \infty$, and Lemma 4a follows.

Now we consider part (b) of Lemma 4 and Lemma 5. The proof is similar as for part (a), cutting the curve into pieces which are small enough that the "error term" $BC \cdot \alpha$ in (4) can be neglected. However, in the way in which the subdivision of the curve is defined, we have to make a case distinction.

Case 1. $\varphi = \pi/2$. Then $\cos\varphi = 0$ and Lemma 4b follows from Lemma 4a.

Case 2. $\varphi < \pi/2$. In this case, $\cos\varphi > 0$, and Lemma 4b gives a lower bound for the decrease $BD - CD$ of the distance to D from B to C in terms of length$(f[B,C])$. From Lemma 4a we conclude that the distance to D is strictly decreasing from B to C, and hence we have $C'D \ge CD$ for all points C' on the curve $f[B,C]$. Let us choose an arbitrary $\varepsilon > 0$. We will show that

$$BD - CD \ge \text{length}(f[B,C]) \cdot (\cos\varphi - \varepsilon).$$

Let us set $\alpha := \min\{\varepsilon, 1\} \cdot |\cos\varphi|$; so α fulfills the conditions for (4) in Lemma 5.

The length of a curve $f[B,C]$ is, by definition, the supremum of the lengths of all polygonal chains Q which are obtained as polygonal subdivisions of f with consecutive vertices on f. Let us take such a subdivision $Q = (A_0, A_1, \ldots, A_n)$ with $n+1$ consecutive points A_i on $f[B,C]$ and $A_0 = B$, $A_n = C$. We would like to apply (4) to the segments A_iA_{i+1}. Therefore, whenever $\gamma(D, f[A_i, A_{i+1}]) > \alpha$, we have to refine the subdivision Q by inserting at least one additional point between A_i and A_{i+1}. Denoting the newly inserted point again by A_{i+1} (and renumbering the points behind it), we select A_{i+1} in such a way that we get $\gamma(D, f[A_i, A_{i+1}]) = \alpha$. Then we have

$$\text{length}(f[A_i, A_{i+1}]) \ge A_iA_{i+1} \ge 2 \cdot CD \cdot \sin\frac{\alpha}{2},$$

which is a fixed positive constant. It follows that each newly inserted point consumes a certain part of length$(f[B,C])$, and therefore we have to insert only finitely many points. We end up with a refined subdivision Q' with length$(Q') \ge$ length(Q) and with the desired property.

Now we apply (4) to the segments A_iA_{i+1}, obtaining

$$A_iD - A_{i+1}D \ge A_iA_{i+1} \cdot (\cos\varphi - \alpha) \ge A_iA_{i+1} \cdot \cos\varphi \cdot (1 - \varepsilon).$$

Summation gives

$$\begin{aligned} BD - CD &\ge \text{length}(Q') \cdot \cos\varphi \cdot (1-\varepsilon) \\ &\ge \text{length}(Q) \cdot \cos\varphi \cdot (1-\varepsilon). \end{aligned}$$

Since this holds for any $\varepsilon > 0$ and any subdivision Q, the lemma is proved for this case.

Case 3. $\pi/2 < \varphi < \pi$. In this case, $\cos\varphi < 0$, Lemma 4b gives an upper bound for the increase $BD - CD$ of the distance to D from B to C in terms of length$(f[B,C])$. We will proceed similarly as in the proof of Lemma 4a. Let us choose α as in case 2. On the curve $f[B,C]$, we find points $n+1$ consecutive points

$A_0 = A, A_1, A_2, \ldots, A_n = C$ with $\gamma(D, f[A_i, A_{i+1}]) = \gamma(D, f[B, C])/n$, choosing n large enough so that $\gamma(D, f[B, C])/n \leq \alpha$. We apply (4) to the successive pieces and obtain

$$\begin{aligned} A_{i+1}D - A_iD &\leq A_iA_{i+1} \cdot (-\cos\varphi + \alpha) \\ &\leq \text{length}(f[A_i, A_{i+1}]) \cdot (-\cos\varphi) \cdot (1 + \varepsilon)\,. \end{aligned}$$

Summation gives

$$CD - BD \leq \text{length}(f[B, C]) \cdot (-\cos\varphi) \cdot (1 + \varepsilon)\,,$$

for any $\varepsilon > 0$, and the proof of part (b) of Lemma 4 is complete. □

3 φ-self-approaching curves and the perimeter of their convex hull

In this section we prove that the length of a φ-self-approaching curve is bounded by the perimeter of its convex hull, divided by $1 + \cos\varphi$.

Let $\text{conv}(P)$ denote the convex hull of a point set P and $\text{peri}(P)$ the length of the perimeter of $\text{conv}(P)$.

Theorem 6 *For a φ-self-approaching curve f with $0 \leq \varphi < \pi$,*

$$(1 + \cos\varphi)\,\text{length}(f) \leq \text{peri}(f)\,.$$

Proof. The length of a curve f is, by definition, the supremum of the lengths of all polygonal chains Q which are obtained as polygonal subdivisions of f with consecutive vertices on f. Therefore, an upper bound for the length of all such chains is also an upper bound for the length of f. Let $Q = (A_0, A_1, \ldots, A_n)$ be a polygonal subdivision of $f[A_0, A_n]$ with $n+1$ consecutive points on $f[A_0, A_n]$. Before we prove that $(1+\cos\varphi)\,\text{length}(Q)$ is bounded by $\text{peri}(f)$, we will introduce additional subdivision points of the curve into Q. This may only increase $\text{length}(Q)$, but it will make the proof simpler. We go through the vertices of Q, starting at the end. When considering A_i, we have already added all additional subdivision points after A_{i+1}, and we now consider which subdivision points we may add between A_i and A_{i+1}. Let P denote the convex hull of all vertices of Q which come after A_{i+1}, inclusive of A_{i+1} and inclusive of the additional points which were added in previous steps. By the φ-self-approaching property, A_{i+1} lies on the boundary of P. There are two cases, depending on whether the point A_{i+1} lies on the boundary of $\text{conv}(P \cup \{A_i\})$ or not.

If A_{i+1} lies in the interior of $\text{conv}(P \cup \{A_i\})$, we do nothing, and we proceed by looking at A_{i-1}. Suppose now that the point A_{i+1} lies on the boundary of $\text{conv}(P \cup \{A_i\})$. Let $B_1, B_2, \ldots$ be the sequence of vertices which lie clockwise from A_{i+1} on P, let $B_{-1}, B_{-2}, \ldots$ be the sequence of vertices anti-clockwise from A_{i+1} on P, and let $B_0 := A_{i+1}$, see Figure 2.

Suppose the left tangent from A_i to P touches P in B_j, and the right tangent from A_i to P touches P in B_{-k}. Depending on whether the curve $f[A_i, B_0]$ "winds" counter-clockwise or clockwise around P, it will either intersect the extended sides $\vec{r}(B_j, B_{j-1})$,

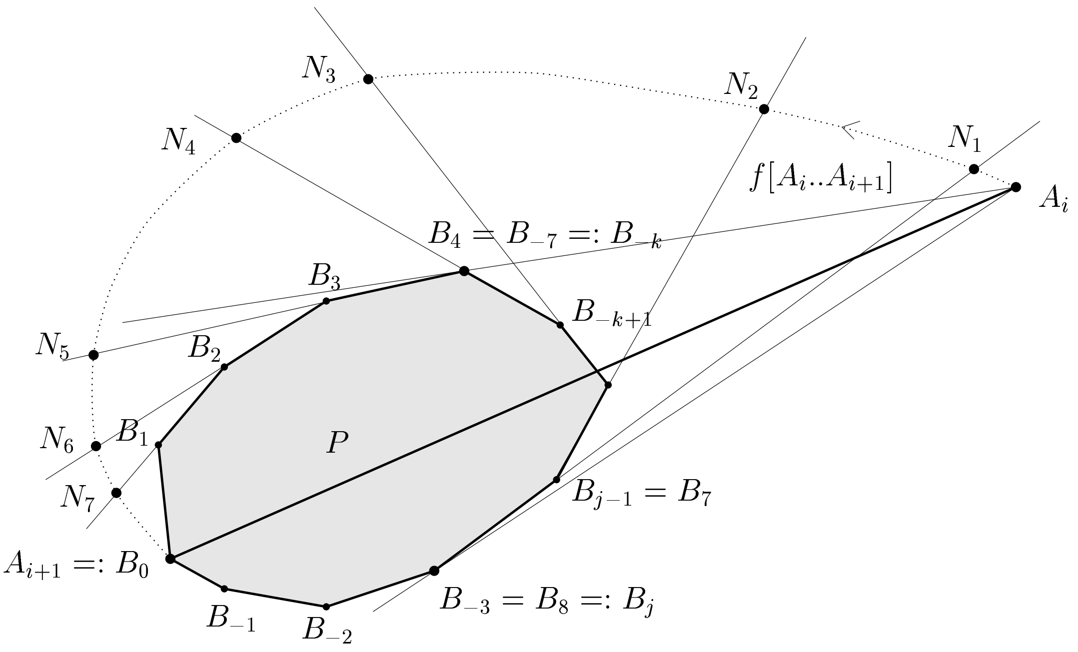

Figure 2: The additional subdivision points between A_i and A_{i+1} are $N_1, N_2, \ldots, N_7$.

$\vec{r}(B_{j-1}, B_{j-2})$, ..., $\vec{r}(B_2, B_1)$, or the extended sides $\vec{r}(B_{-k}, B_{-k+1})$, $\vec{r}(B_{-k+1}, B_{-k+2})$, ..., $\vec{r}(B_{-2}, B_{-1})$, in the given order. This follows from the fact that the curve f is disjoint from P, by the φ-self-approaching property. We add these finitely many subdivision points between A_i and A_{i+1} and proceed by looking at A_{i-1}.

In this way, after processing the whole chain Q, we obtain a possibly refined subdivision, which we again denote by $Q = (A_0, A_1, \ldots, A_n)$. This subdivision has the property that, if A_{i+1} lies on the boundary of $\mathrm{conv}(\{A_i, \ldots, A_n\})$, each vertex of $P := \mathrm{conv}(\{A_{i+1}, \ldots, A_n\})$ is also on the boundary of $\mathrm{conv}(\{A_i, \ldots, A_n\}) = \mathrm{conv}(P \cup \{A_i\})$ (although not necessarily as a vertex).

Now we have to distinguish between the cases $\varphi \le \pi/2$ and $\varphi \ge \pi/2$. If $0 \le \varphi \le \pi/2$, we show that, for a subdivision Q of f with the additional property mentioned above,

$$\mathrm{peri}(Q) \ge (1 + \cos\varphi) \cdot \mathrm{length}(Q)\,. \tag{7}$$

If $\pi/2 \le \varphi < \pi$, then $\cos\varphi \le 0$, and we show the weaker statement

$$\mathrm{peri}(Q) \ge \mathrm{length}(Q) + \cos\varphi \cdot \mathrm{length}(f)\,. \tag{7$'$}$$

The left side is bounded by $\mathrm{peri}(f)$, whereas the right side of each inequality can be made arbitrarily close to $(1 + \cos\varphi) \cdot \mathrm{length}(f)$, thus proving the theorem.

We will use induction on the number of vertices of Q. The assertion is true for Q being a line segment, so let us assume that Q has at least three vertices, the first two are called A and B. Let Q' denote the chain Q without the initial segment AB. The induction hypothesis is that (7) or (7′) is fulfilled for Q' and $f^{\ge B}$. For the inductive step, it is then sufficient to prove

$$\mathrm{peri}(Q) - \mathrm{peri}(Q') \ge (1 + \cos\varphi) \cdot AB\,, \tag{8}$$

if $0 \le \varphi \le \pi/2$, or

$$\mathrm{peri}(Q) - \mathrm{peri}(Q') \ge AB + \cos\varphi \cdot \mathrm{length}(f[A, B])\,, \tag{8$'$}$$

if $\pi/2 \leq \varphi < \pi$. Note that for φ being in either domain, the inequality of (8) or (8′) that we need to prove is weaker than the other inequality. Hence, independently of φ, it is sufficient to prove any of (8) or (8′).

We distinguish two cases, depending on whether B lies on the boundary of conv(Q) or not.

Case 1. The point B is on the boundary of conv(Q). In this case we prove (8′). We have a situation as depicted in Figure 3. When passing from conv(Q') to conv(Q),

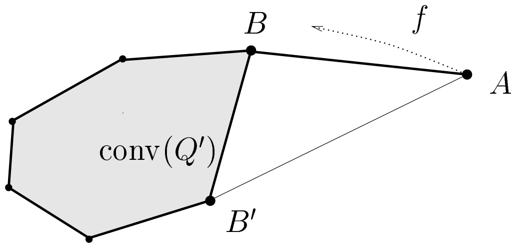

Figure 3: $(AB + AB') - BB' \geq AB + \cos\varphi \cdot \mathrm{length}(f[A, B])$.

the segment between B and one of its neighboring vertices B' is replaced by the chain BAB'. So we need to show

$$(AB + AB') - BB' \geq AB + \cos\varphi \cdot \mathrm{length}(f[A, B])\,,$$

which follows from Lemma 4b, by considering the three consecutive points A, B, B' on f.

Case 2. The point B is not on the boundary of conv(Q). In this case we prove (8). We use the notation $\ldots, B_{-2}, B_{-1}, B_0 = B, B_1, B_2, \ldots$ for the vertices of Q' that was introduced above. The two vertices of conv(Q') which are adjacent to B are B_{-1} and B_1. Then A must lie in the wedge included between $\vec{r}(B_{-1}, B)$ and $\vec{r}(B_1, B)$, see Figure 4. W.l.o.g. we may assume that B_{-1} appears before B_1 on f so $\gamma(B, f[B_{-1}, B_1])$ is at most φ since f is φ-self-approaching.

Suppose the left tangent from A to P touches P in B_j, and the right tangent from A to P touches P in B_{-k}. Then the points $B_{-k+1}, B_{-k+2}, \ldots, B_{j-2}, B_{j-1}$. are not vertices of conv(Q). As we move continuously along Q from B to A, the points where these points disappear from the boundary of the convex hull are point A and the intersections of the segment AB with the extended sides $\vec{r}(B_j, B_{j-1})$, …, $\vec{r}(B_2, B_1)$, and $\vec{r}(B_{-k}, B_{-k+1})$, …, $\vec{r}(B_{-2}, B_{-1})$, see Figure 4. We prove (8) by showing that it is true for each transition from one intersection point D' on AB to the next intersection point D''. We have to show

$$(D''B_{-k'} + D''B_{j'}) - (D'B_{-k'} + D'B_{j'}) \geq (1 + \cos\varphi) \cdot D''D'\,.$$

We may use the fact that D' is contained in the triangle $D''B_{-k'}B_{j'}$, and the angle at D' in the triangle $B_{-k'}D'B_{j'}$ is less than the angle $\gamma(B, f[B_{-1}, B_1])$, which is at most φ. This elementary geometric inequality is proved in Lemma 11 in the appendix. □

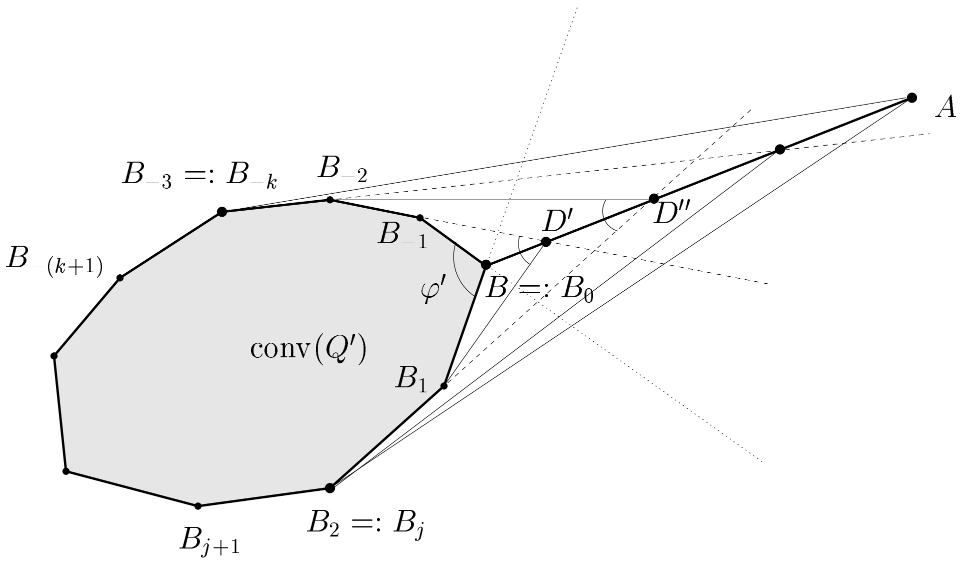

Figure 4: $(D''B_{-2} + D''B_1) - (D'B_{-2} + D'B_1) \geq (1 + \cos\varphi) \cdot D''D'$.

4 An upper bound for the detour

Theorem 7 *The length of a φ-self-approaching curve is not greater than $c(\varphi)$ times the distance of its endpoints, where*

$$c(\varphi) := \begin{cases} 1 & \textit{for } \varphi = 0, \\ \max\limits_{\beta\in[0..\frac{\pi}{2}]} \dfrac{2\beta+\pi+2}{\sqrt{5-4\cos\beta}} & \textit{for } \varphi = \pi/2, \\ \max\limits_{\beta\in[0..\varphi]} \dfrac{(1+e^{\pi\cot\varphi})e^{\beta\cot\varphi} - \frac{2}{1+\cos\varphi}}{\cos\varphi\sqrt{((1+e^{\pi\cot\varphi})e^{\beta\cot\varphi}-\cos\beta)^2+\sin^2\beta}} \quad (*) & \textit{otherwise.} \end{cases}$$

Proof. For $\varphi = 0$ we have a straight line segment, and the theorem is obvious. So let us assume $0 < \varphi < \pi$ from now on.

Let f be a φ-self-approaching curve from A to Z. W.l.o.g., we may assume that f does not cross the segment AZ. Otherwise we apply the bound $c(\varphi)$ successively to each subcurve between to successive curve points on AZ and add up the results. Due to the self-approaching property the curve points on AZ appear in the same order as on f, so the overall bound is less than or equal to $c(\varphi)$.

Assume w.l.o.g. that the curve starts by leaving A on the left side of the directed line AZ. Let AC be the right tangent from A to the curve, touching the curve in the point C. (The curve is on the left side of AC.) We select C as far as possible from A, and we denote the angle at point C in the triangle ACZ by β, see Figure 5. (For $C = Z$ we set $\beta = 0$.) Note that $\beta \leq \varphi$ holds, otherwise there is a point C_0 before C on f such that the angle $\gamma(\bar{C}_0, f[C,Z])$ is also greater than φ which contradicts the self-approaching property at C_0. Let B denote the first point of intersection of the curve with $\vec{r}(C,Z)$. (For $C = Z$ we take $B = A$.) Since the curve neither crosses $\vec{r}(A,C)$ nor the segment AZ, Z must lie between B and C. We apply Lemma 4a to the arc $f[B,C]$, considering the three consecutive points B, C, Z on f, and we get

$$CZ \leq BZ \cdot e^{-\pi\cot\varphi}.$$

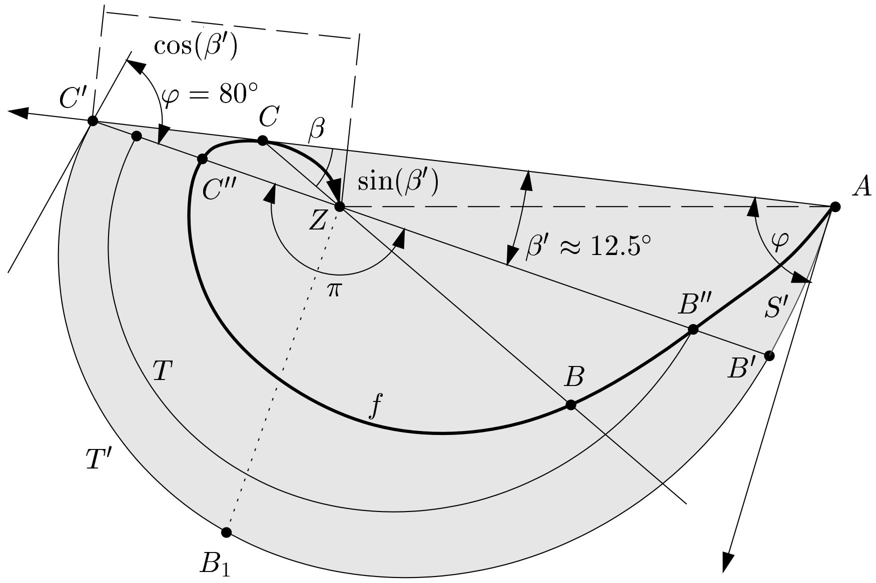

Figure 5: A φ-self-approaching curve must lie in this area. The angle β' shown here maximizes $(*)$ in the definition of $c(\varphi)$.

Applying the lemma to $f[A, B]$, considering the three consecutive points A, B, C on f, we get

$$CB \leq CA \cdot e^{-\beta \cot \varphi}.$$

Note that the last inequalities trivially hold for $C = Z$, $\beta = 0$ and $A = B$. All in all, since $CB = CZ + BZ$, we obtain

$$CA \geq CZ \cdot (1 + e^{\pi \cot \varphi}) e^{\beta \cot \varphi}. \tag{9}$$

Now we select a point C' on $\vec{r}(A, C)$ with $C'A \geq CA$, let β' be the angle at C' in the triangle $AC'Z$. We choose C' such that the equation

$$C'A = C'Z \cdot (1 + e^{\pi \cot \varphi}) e^{\beta' \cot \varphi} \tag{10}$$

is fulfilled. Such a point C' exists, since, as we move C' further away from A, the ratio $C'A : C'Z$ converges to 1, the angle β' decreases towards 0 and hence $e^{\beta' \cot \varphi}$ also converges to 1, whereas $1 + e^{\pi \cot \varphi}$ is a constant bigger than 1. Therefore the inequality (9) changes direction as $AC' \to \infty$. We have $\beta' \leq \beta \leq \varphi$, and also $C' \neq Z$.

In the following let B' be the point on $\vec{r}(C', Z)$ with $B'Z = C'Z \cdot e^{\pi \cot \varphi}$ and $B' \notin C'Z$, see Figure 5.

Lemma 8 *A φ-self-approaching curve from A to Z is contained in the convex region bounded by the following three curves, see Figure 5.*

(1) *a φ-logarithmic spiral from A to B' of polar angle β' with pole C';*

(2) *a φ-logarithmic spiral from B' to C' of polar angle π with pole Z;*

(3) *the segment AC'.*

Proof. Let B'' be the first intersection point of f with $\vec{r}(C', Z)$. By Lemma 4a applied to the pole C, the arc $f[A, B'']$ lies inside the φ-logarithmic spiral S with pole C with polar angle β starting at A. The φ-logarithmic spiral S' with pole C' through A is obtained from S by stretching it about the center A, and hence $f[A, B'']$ is also contained inside S', and in part (1) of the region boundary. In particular, $B''Z \le B'Z$. It follows with the help of Lemma 3 that, between the rays ZA and ZB', no point of the whole curve $f[A, Z]$ can lie outside S'.

Now let C'' be the first intersection point of f with $\vec{r}(Z, C')$. By Lemma 4a applied to B'', C'' and Z, the arc $f[B'', C'']$ lies inside the φ-logarithmic spiral T around pole Z with polar angle π starting at B''. Since $B''Z \le B'Z$, the arc lies inside the logarithmic spiral T' which forms part (2) of the region boundary. Again, it follows with the help of Lemma 3 that, below the line $C'B'$, no point of the whole curve f can lie outside T'.

Finally, in the region between the rays ZC' and ZA, the curve cannot escape across the segment AC', and the proof is complete. $\square$

Now we can prove Theorem 7. We treat only the case $\varphi \neq \pi/2$, where we have proper spirals. The case $\varphi = \pi/2$, where we have circular arcs, has already been treated in [5].

We choose a scale such that $C'Z$ equals 1. Now $AC' = (1 + e^{\pi \cot\varphi})e^{\beta' \cot\varphi}$ and $B'Z = e^{\pi\cot\varphi}$ holds by construction. Therefore the lengths of the three curves in Lemma 8 are given by

$$\begin{aligned}
L_1 &:= \frac{1}{\cos\varphi}(AC' - B'C') = \frac{1}{\cos\varphi}\left((1 + e^{\pi\cot\varphi})e^{\beta'\cot\varphi} - (1 + e^{\pi\cot\varphi})\right)\\
L_2 &:= \frac{1}{\cos\varphi}(B'Z - C'Z) = \frac{1}{\cos\varphi}\left(e^{\pi\cot\varphi} - 1\right)\\
L_3 &:= AC' = (1 + e^{\pi\cot\varphi})e^{\beta'\cot\varphi}
\end{aligned}$$

and for the distance of the endpoints of f we have

$$AZ = \sqrt{((1 + e^{\pi\cot\varphi})e^{\beta'\cot\varphi} - \cos\beta')^2 + \sin^2\beta'}\,.$$

Altogether we conclude from Theorem 6

$$\frac{\operatorname{length}(f)}{AZ} \le \frac{L_1 + L_2 + L_3}{AZ} \cdot \frac{1}{1 + \cos\varphi}\,.$$

The right term can be transformed to

$$\frac{(1 + e^{\pi\cot\varphi})e^{\beta'\cot\varphi} - \frac{2}{1+\cos\varphi}}{\cos\varphi\sqrt{((1 + e^{\pi\cot\varphi})e^{\beta'\cot\varphi} - \cos\beta')^2 + \sin^2\beta'}}\,,$$

and it is easy to compute the maximum of the last expression for $\beta' \in [0, \varphi]$. $\square$

The function $c(\varphi)$ is strictly monotone and continuous for $\varphi \in [0, \pi)$. It tends to infinity if φ tends to π. The graph of the function for $0 \leq \varphi \leq 1.8$ is shown in Figure 6.

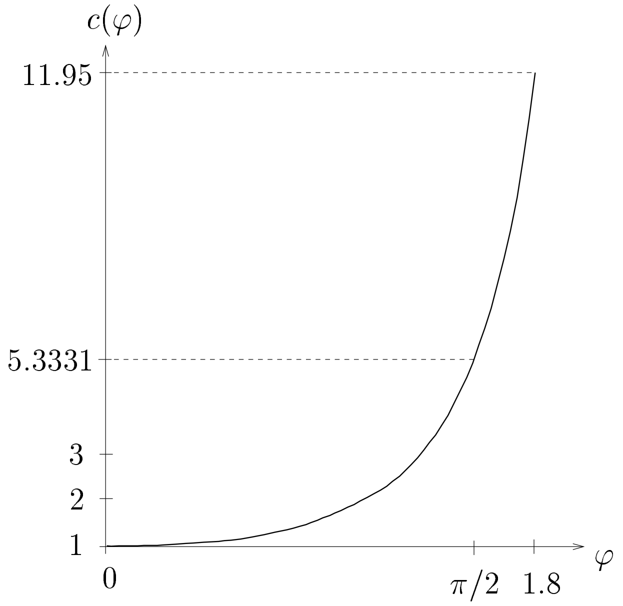

Figure 6: The function $c(\varphi)$ is strictly monotone and continuous for $\varphi \in [0, \pi)$.

5 Tightness of the upper bound

Theorem 9 *The upper bound $c(\varphi)$ given in Theorem 7 for the detour of φ-self-approaching curves is tight.*

Proof. We construct a φ-self-approaching curve f from A to Z as in the proof of Lemma 8, the construction is shown in Figure 7. The curve starts with two logarithmic spirals similar to part (1) and part (2) in the proof of Lemma 8, except that part (2) is split into two parts at point B_1. The segment $C'Z$ of length 1 is replaced by a φ-self-approaching zigzag curve of length $L_3' = \frac{2}{1+\cos\varphi}$ from C' to Z, which moves inside a thin rectangle along the segment $C'Z$. This last part of the curve, see Figure 8, is obtained by "stacking" small cycloids $(x, y) = (r(\alpha - \sin\alpha), r(1 - \cos\alpha))$, for $0 \leq \alpha \leq 2\varphi$ as described in the appendix. One piece of cycloid has "height" $H = r(1 - \cos 2\varphi) = 8r\sin^2\varphi\cos^2\varphi$ and length $L = 8r\sin^2\frac{\varphi}{2}$. We must choose the size parameter r in such a way that $1/H$ is an even integer n; then the curve of n pieces of cycloids with "height" H will precisely connect the points C' and Z. The length of the curve is then $n \cdot L = L/H = \frac{2}{1+\cos\varphi}$, which is what we need. The width of the construction is $W = r(2\varphi - \sin 2\varphi)$. We arrange the cycloid pieces on the left side of the segment $C'Z$, as indicated in Figure 8; then the curve is contained in a rectangle $\bar{C}'Z\bar{Z}\bar{C}$ of width $W = Z\bar{Z}$. The long side $Z\bar{C}'$ slightly extends the segment ZC' by the amount $2r - H$.

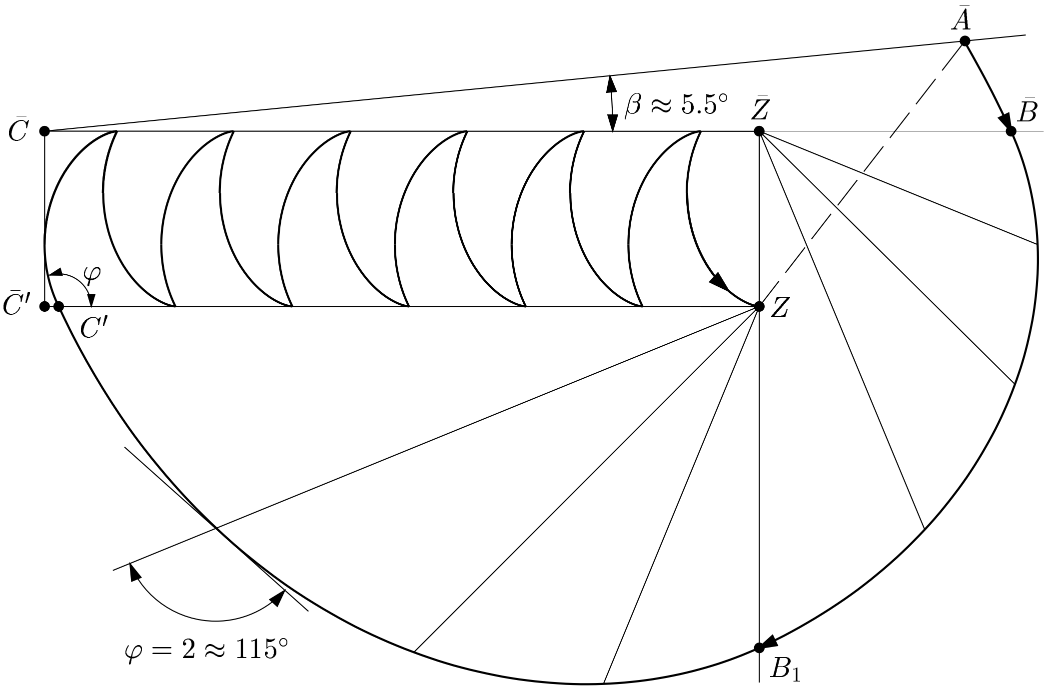

Figure 7: The construction of the φ-self-approaching curve with maximal detour for $\varphi = 2$.

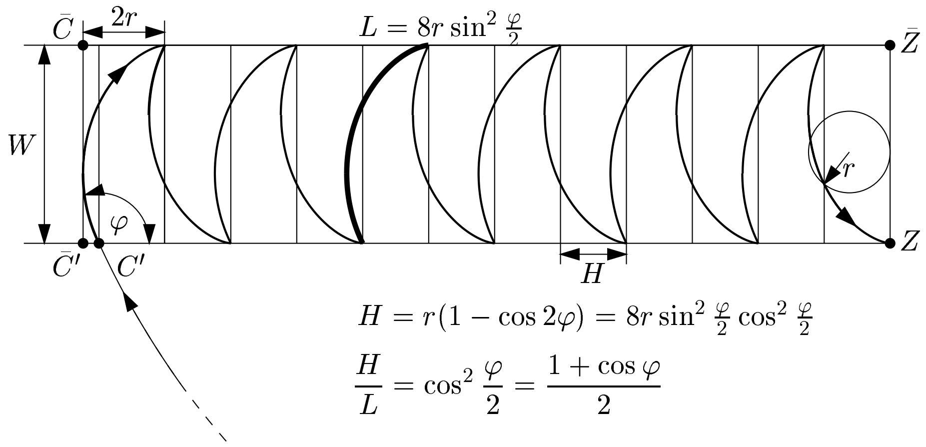

Figure 8: A sequence of cycloid parts forming a φ-self-approaching curve inside a thin rectangle.

Now we can describe the whole construction of the curve, in reverse direction. The curve consists of the following parts, numbered in accordance with Lemma 8, using the notation from there:

(3′) The curve ends with the φ-self-approaching curve from C' to Z described above, whose length is $\frac{2}{1+\cos\varphi}$;

(2) before, there is a φ-logarithmic spiral of polar angle $\pi/2$ with pole Z connecting C' to some point B_1 on $\vec{r}(\bar{Z}, Z)$.

(2′) then, a φ-logarithmic spiral of polar angle $\pi/2$ with pole $\bar{Z}$ connecting B_1 to $\bar{B}$;

(1′) the curve starts with a φ-logarithmic spiral of polar angle β with pole $\bar{C}$ between a point $\bar{A}$ and $\bar{B}$, where $\beta \leq \varphi$ is the value for which the maximum in the definition of $c(\varphi)$ is attained.

It can be checked that the parts which are logarithmic spirals are always φ-self-approaching, as the line from any point X to the current pole contains the whole curve on one side. So obviously the whole curve is φ-self-approaching. Since r can be made as small as we like, we have $\bar{C} \to C'$, $\bar{Z} \to Z$, $\bar{A} \to A$, $\bar{B} \to B'$ as $r \to 0$, and the logarithmic spirals that we use will approach the "ideal" logarithmic spirals that appear in Lemma 8, see Figure 5. This means that

$$\begin{aligned} \frac{\text{length}(f)}{\bar{A}Z} \quad \to \quad & \frac{L_1 + L_2 + L_3'}{AZ} = \\ & \frac{(1+e^{\pi\cot\varphi})e^{\beta\cot\varphi} - 2 + \frac{2}{1+\cos\varphi}\cdot\cos\varphi}{\cos\varphi\sqrt{((1+e^{\pi\cot\varphi})e^{\beta\cot\varphi} - \cos\beta)^2 + \sin^2\beta}} \end{aligned}$$

which equals $c(\varphi)$. □

6 Conclusions

Here we analyze the maximum length of curves with an upper bound on the angular wedge at A. This condition is not symmetric since it distinguishes the source and the target of the curve. One might also consider a symmetric situation, where curves are φ-self-approaching in both directions.

Generalizations to three dimensions are also completely open.

A Appendix

Lemma 10 *Let ABC be a triangle with angle $\pi - \varphi$ at B ($0 \leq \varphi \leq \pi$), as in Figure 9. Then $BC \leq AC - AB \cdot \cos\varphi$.*

Proof. From the cosine law and from $|\cos\varphi| \leq 1$ we conclude

$$\begin{aligned} AC^2 &= AB^2 + BC^2 - 2\cdot AB\cdot BC\cdot\cos(\pi-\varphi) \\ &= BC^2 + 2\cdot AB\cdot BC\cdot\cos\varphi + (AB\cos\varphi)^2 + \\ &\quad AB^2(1-\cos^2\varphi) \\ &\geq (BC + AB\cos\varphi)^2 . \end{aligned}$$

□

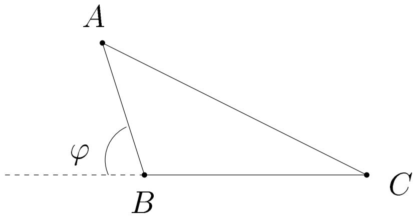

Figure 9: $BC \leq AC - AB \cdot \cos\varphi$

Lemma 11 *Let V be a point inside a triangle ABC. We connect each vertex to v using segments $l_1 = BV$, $r_1 = CV$, and $z = AV$. Let $l_2 = AB$ and $r_2 = AC$ be two edges of the triangle; see Figure 10. Let $0 \leq \varphi \leq \pi$ be the angle between l_1 and r_1 then for the lengths of the segments $l_1 + r_1 + (1 + \cos\varphi)z \leq l_2 + r_2$ holds.*

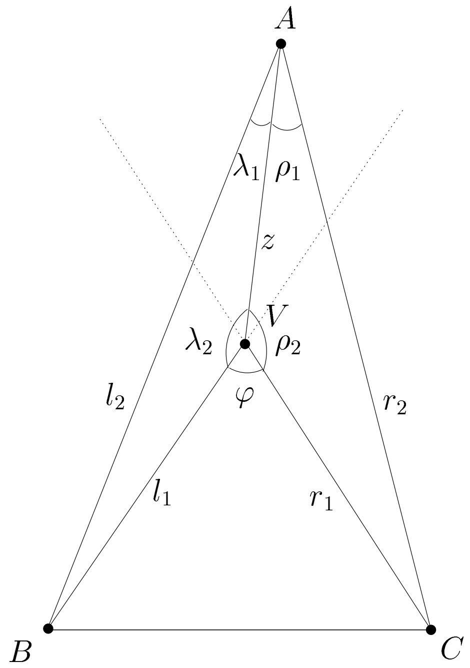

Figure 10: $l_1 + r_1 + (1 + \cos\varphi)z \leq l_2 + r_2$ holds for $0 \leq \varphi \leq \pi$.

Proof. The claim is obviously true for $z = 0$. Let $z > 0$. We have to prove the inequality $l_2 - l_1 + r_2 - r_1 \geq (1 + \cos\varphi)z$. Using Lemma 10 we conclude

$$\begin{aligned} l_1 &\leq l_2 - z\cos(\pi - \lambda_2) \\ r_1 &\leq r_2 - z\cos(\pi - \rho_2)\,. \end{aligned}$$

So $l_2 - l_1 + r_2 - r_1 \geq -(\cos\lambda_2 + \cos\rho_2)z$ holds and it is sufficient to show that $-(\cos\lambda_2 + \cos\rho_2) \geq 1 + \cos\varphi$ is fulfilled. We know that $-(\cos\lambda_2 + \cos\rho_2)$ is equivalent to $-2\cos\left(\frac{\lambda_2+\rho_2}{2}\right)\cos\left(\frac{\lambda_2-\rho_2}{2}\right)$. Also the following equations are obviously true:

$$\begin{aligned} \frac{\lambda_2 + \rho_2}{2} &= \frac{2\pi - \varphi}{2} = \pi - \frac{\varphi}{2} \\ \frac{\lambda_2 - \rho_2}{2} &= (\pi - \rho_2) - \frac{\varphi}{2}\,. \end{aligned}$$

From the position of A we conclude $0 \le (\pi - \rho_2) \le \varphi$ and so $\left|(\pi - \rho_2) - \frac{\varphi}{2}\right| \le \frac{\varphi}{2}$ holds. Altogether we conclude

$$
\begin{aligned}
-(\cos\lambda_2 + \cos\rho_2) &= -2\cos\left(\frac{\lambda_2 + \rho_2}{2}\right)\cos\left(\frac{\lambda_2 - \rho_2}{2}\right) \\
&= 2\cos\left(\frac{\varphi}{2}\right)\cos\left(\left|(\pi - \rho_2) - \frac{\varphi}{2}\right|\right) \\
&\ge 2\cos^2\left(\frac{\varphi}{2}\right) \\
&= 2\left(\frac{1+\cos\varphi}{2}\right) = 1 + \cos\varphi\,.
\end{aligned}
$$

This completes the proof. □

Lemma 12

(a) $$\cos x \ge 1 - x^2\,, \quad \text{for all } x \in \mathbb{R}.$$

(b) $$x \le \sin x \cdot (1 + 2x^2)\,, \quad \text{for } 0 \le x \le 1.$$

(c) $$e^x \ge 1 + x\,, \quad \text{for all } x \in \mathbb{R}.$$

(d) $$e^x \le 1 + x + x^2\,, \quad \text{for } x \le 1.$$

(e) $$\frac{1 - \cos x}{\sin x} \le x\,, \quad \text{for } 0 \le x \le 1.$$

Proof. In parts (a)–(d), the difference between the two sides of the inequality is in each case a convex function achieving its minimum value of 0 for $x = 0$. (For part (d), this is only true for $x \le \ln 2$.) For $\ln 2 \le x \le 1$ in part (d), and for part (e), where the expression on the left-hand side equals $\tan\frac{x}{2}$, the difference between the two sides is a concave function, taking nonnegative values at the boundaries of the definition interval. □

Logarithmic spirals. Logarithmic spirals, directed to the center, are used to construct interesting examples of φ-self-approaching curves. In polar coordinates (r, ψ), a logarithmic spiral with pole at the origin Z is the set of all points $r = r_0 q^{\psi}$, $\psi \in (-\infty, \infty)$, for some $q > 0$. We have right spirals and left spirals, depending on whether $q > 1$ or $q < 1$. For $q = 1$ we get a circle. The pole Z may be regarded as a limit point of the spiral. Each ray through the origin intersects the spiral in the same angle α with $\cot\alpha = \pm\ln q$. We call such a spiral an α-logarithmic spiral, see Figure 11 as an example with $\alpha = 1.3$. α-logarithmic spirals with α small enough, directed to the center, are simple examples of φ-self-approaching curves. Parts of φ-logarithmic spirals are used to construct φ-self-approaching curves with maximum detour.

For $\varphi \ne \pi/2$, the length of an arc $S[A, B]$ of a φ-logarithmic spiral S is given by

$$\text{length}(S[A, B]) = \frac{1}{\cos\varphi}\,(AZ - BZ)\,.$$

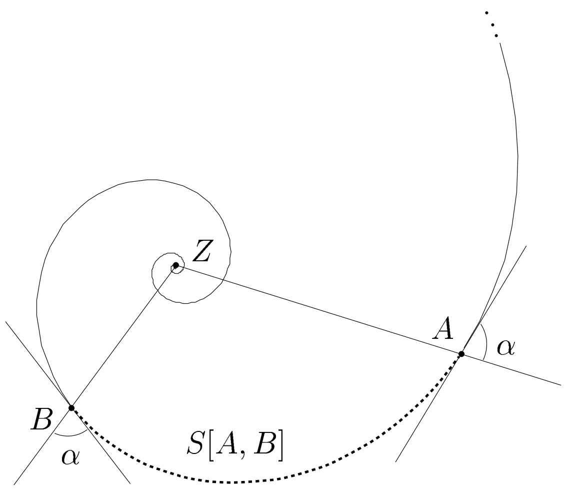

Figure 11: A 1.3-logarithmic spiral.

Cf. also Lemma 4b, where the same ratio between change of radius and arc length appears as an upper bound for φ-self-approaching curves.

Cycloids and their φ-evolutes. A cycloid is the curve traced by a point M on the circumference of a circle rolling on a line without slipping, see Figure 12. In coordinates, a cycloid generated by a circle of radius r rolling on the x-axis is given by $x = (\alpha - \sin\alpha)r$, $y = (1 - \cos\alpha)r$ for $\alpha \in \mathbb{R}$. Cycloids play a role in our construction of extreme self-approaching curves. It is well-known that the evolute of a cycloid C', i. e., the envelope of all normals, or the locus of the centers of curvature, is another congruent cycloid C: Each tangent t of the cycloid C (the evolute) intersects the other cycloid C' (the involute) in a point where it has a tangent t' that is perpendicular to t. One may generalize this relation between evolute and involute to an angle φ different from the right angle: Each tangent t of the *φ-evolute* intersects the *φ-involute* in an angle φ. It turns out that, even for this more general situation, the φ-evolute of a cycloid C', i. e., the envelope of all lines which are obtained from the tangents of C' by a rotation of φ about the point of tangency, is another congruent cycloid C, see Figure 12.

It is known that the tangent MT of a cycloid always goes through the highest point T of the current position of the circle. It is convenient to measure directions by the clockwise oriented angle α with the vertical downward direction. The tangent direction TM (taken always in the direction pointing leftward) turns precisely half as fast as the radius OM of the rolling circle. When OM has direction α, TM has direction $\alpha/2$. Now we trace out the arc of a cycloid C from the lowest point Z until the radius OM has direction 2φ. At this point Z', we start another arc of a congruent cycloid C' for which Z' is the lowest point, this time rolling counterclockwise, stopping again when the circle has been rotated by an angle of 2φ. The length of each of the two arcs of the cycloid is $L = 8r\sin^2\frac{\varphi}{2}$. We can think of C and C' being generated by two circles rolling clockwise simultaneously at the same speed. The circle generating C' rolls on a line which is higher by the distance $H = (1-\cos 2\varphi)r$, and it is always in a position where the radius $O'M'$ is by an angle $2\pi - 2\varphi$ clockwise from the radius OM. The centers O and O' of the two circles always have the same relative position; they are translated horizontally. The highest point T of the lower circle lies on the other

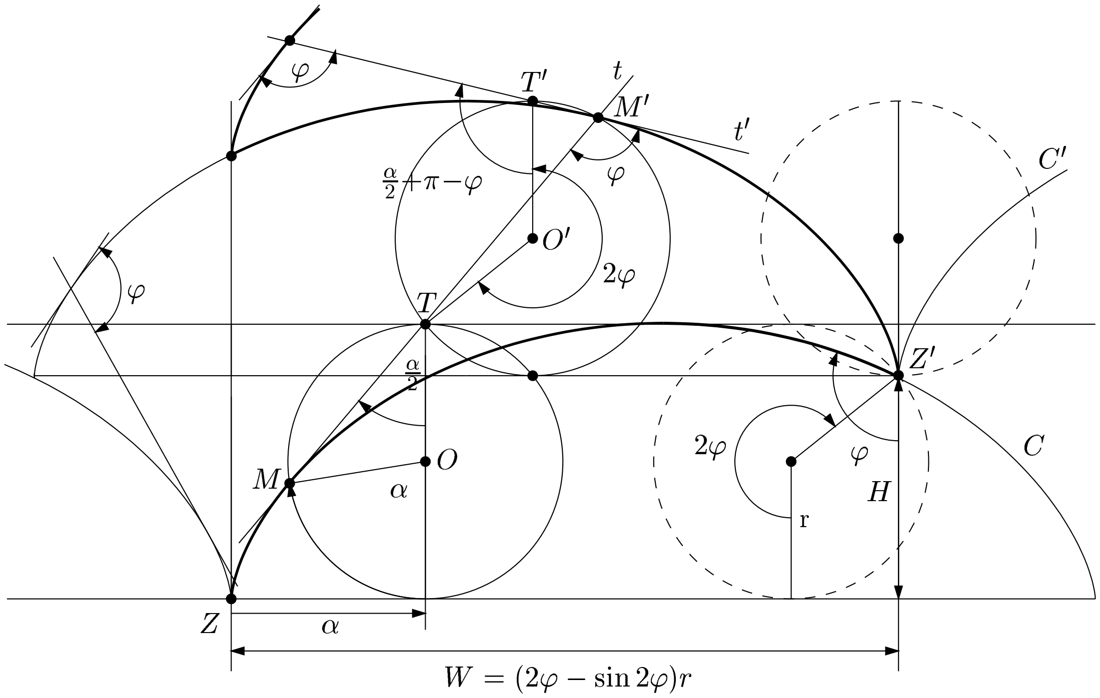

Figure 12: The envelope of all lines which intersect a cycloid in a given angle φ is another congruent cycloid.

circle, and similarly for the lowest point on the higher circle. The directed clockwise angle between TO and TO' is 2φ.

To see that C is the φ-evolute of C', consider a tangent $t = MT$ of C, having direction $\alpha/2$. At the same time the direction of the tangent $t' = M'T'$ (taken always in the direction showing leftwards) is $\alpha/2 + \pi - \varphi$. The peripheral angle between the tangent $T'M'$ and $M'T$ is equal to φ, since it corresponds to a central angle $T'OT = 2\varphi$. It follows that the direction of $M'T$ is $\alpha/2$, and hence $M'T$ coincides with the tangent $t = MT$, and, as we have already seen, the angle between TM and $T'M'$ is φ.

Thus, if we go through the curve in the reverse direction in which we discussed its generation, starting at C', the curve will always be enclosed in the wedge of angle φ between the tangents to C' and C. Thus we have a φ-self-approaching curve.

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