Open Problems in Discrete Differential Geometry

Collected by Günter Rote

Problem 1 (Günter M. Ziegler)**.** What is the smallest possible maximum vertex degree $f(d)$ for a centrally symmetric triangulation of the d-sphere? (Or a simplicial $(d+1)$-polytope.) The known bounds are $f(1) = 2$, $f(2) = 4$, and $d+1 < f(d) \le 2d$. The upper bound comes from the cross-polytope. If one could show $f(d) < 3d/2$ for some large d that would have interesting consequences.

Problem 2 (Richard Kenyon)**.** Let M be a closed polyhedral surface homeomorphic to S^2 which is entirely composed of equal regular pentagons. If M is immersed in 3-space, is it necessarily the boundary of a union of solid dodecahedra that are glued together at common facets? The pentagonal faces may intersect each other (and the "union of solid dodecahedra" must be defined appropriately) but two different faces are not allowed to coincide.

(The corresponding question for equal *squares* has an affirmative answer.)

Note (Ulrich Brehm): The great dodecahedron (Kepler-Poinsot polyhedron) is an interesting example, but it has genus 4. Moreover, the vertex-figures are self-crossing pentagrams, and therefore the surface is not immersed. The question would be interesting even if the self-crossing near vertices is ignored and the local embedding property is required only away from the vertices, and for surfaces of low genus.

Problem 3 (Wolfgang Kühnel)**.** Let M^d be a triangulated d-manifold (simply connected, closed) with a discrete metric such that the discrete curvature (angle defect) along any $(d-2)$-simplex is positive. Give a *discrete* proof that M^d is homeomorphic to S^d. (Or give a counterexample.) Cf. Matveev/Shevshichin for $d = 3$, using Ricci flow, and Cheeger for general dimensions. (Note: Nonnegative curvature is not sufficient. $\mathbb{C}P^2$ has a metric of positive sectional curvature. But the polyhedral condition should correspond to the stronger condition of positive curvature operator.)

Problem 4 (Joseph O'Rourke)**.** Can a finite number of disjoint (closed) line segments in the plane —acting as 2-sided mirrors— and a point light source be arranged so that no light ray escapes to infinity? It seems most natural to treat the mirrors as open segments, but they should be disjoint when closed. The conjecture is that this is impossible.

Problem 5 (Günter Rote)**.** Take the complete graph graph K_4 embedded in the plane in general position, with vertices p_1, p_2, p_3, p_4. Pick two arbitrary points a and b and define two functions ω_{ij} on f_{ij} on the six edges of this graph:

$$\omega_{ij} := \frac{1}{[p_i p_j p_k][p_i p_j p_l]}, \qquad f_{ij} := [a p_i p_j][b p_j p_j], \tag{1}$$

where k and l are the two vertices different from i and j, and $[q_1 q_2 \dots q_n]$ denotes signed area of the polygon $q_1 q_2 \dots q_n$. (The function ω_{ij} is a *self-stress*: the equilibrium condition $\sum_j \omega_{ij}(p_j - p_i) = 0$ holds for every vertex i, where the summation is over all edges ij incident to i.) Then we have the following identity:

$$\sum_{1 \le i < j \le 4} \omega_{ij} f_{ij} = 1 \tag{2}$$

This generalizes to any wheel (graph of a pyramid) with a vertex p_0 that is connected to vertices $p_1, \dots, p_n$ forming a cycle, with the self-stress

$$\omega_{i,i+1} := \frac{1}{[p_i p_{i+1} p_0][p_1 p_2 \dots p_n]}, \qquad \omega_{0,i} := \frac{[p_{i-1} p_i p_{i+1}]}{[p_{i-1} p_i p_0][p_i p_{i+1} p_0][p_1 p_2 \dots p_n]}.$$

(The summation in (2) extends over all edges of the graph.) A different formula for f_{ij} that fulfills (2) is given by a line integral over the segment $p_i p_j$:

$$f'_{ij} := \tfrac{3}{2} \cdot \|p_i - p_j\| \cdot \int_{x \in p_i p_j} \|x\|^2 \, ds = \tfrac{1}{2} \cdot \|p_i - p_j\|^2 \cdot \big(\|p_i\|^2 + \|p_j\|^2 + \langle p_i, p_j \rangle \big)$$

Question 1. *Are there other graphs with n vertices and $2n - 2$ edges, for which a self-stress ω satisfying* (2) *can be defined? The next candidates with 6 vertices are the graph of a triangular prism with an additional edge, and the complete bipartite graph $K_{3,3}$ with an additional edge.*

Question 2. *Are these formulas an instance of a more general phenomenon? What are the connections to homology and cohomology?*

Question 3. *By positive scaling of the function f given by* (1) *or by adding a function that is orthogonal to the space of self-stresses* (*i. e., it lies in the range of the rigidity map*)*, one obtains different functions f for which the expression in* (2) *is positive, see* [1, Lemma 3.10]. *Are all functions f that assign a number to each segment in the plane and that make* (2) *positive for all embeddings of K_4 obtained in this way?*

[1] Günter Rote, Francisco Santos, and Ileana Streinu, *Expansive motions and the polytope of pointed pseudo-triangulations*, in: Discrete and Computational Geometry—The Goodman-Pollack Festschrift, Springer, 2003, pp. 699–736, arXiv:math/0206027 [math.CO].

Problem 6 (Jürgen Richter-Gebert)**.** Take $2 \cdot 5 = 10$ vectors A, B, C, D, E and A', B', C', D', E' in $\mathbb{C}^2$. Consider 2×2 determinants $[AB]$ etc. Take the quotient

$$\alpha(A, B, C, D, E | A', B', C', D', E') := \frac{[AB][BC][CD][DE][EA]}{[A'B'][B'C'][C'D'][D'E'][E'A']}$$

The alternating sum of α over all 4! simultaneous permutations of $A \dots E$ and $A' \dots E'$ that start with A/A' is 0. This should also be true for a general number of $2n$ points. Thus we conjecture in [1]:
Let $A_1, \dots, A_n, A'_1, \dots, A'_n \in \mathbb{K}^2$ be $2n$ points in a 2-dimensional vector space over a commutative field $\mathbb{K}$. Then the following formula holds

$$\sum_{\pi = (1, \pi_2, \dots, \pi_n) \in S_n} \sigma(\pi) \alpha(\pi(A_1, \dots, A_n) | \pi(A'_1, \dots, A'_n)) = 0.$$

where $\alpha(A_1, , \dots, A_n | A'_1, , \dots, A'_n)$ is a cyclic quotient analogous to the above one.

(This follows trivially from symmetry arguments for $n = 3, 4, 7, 8, 11, 12, \ldots$, has been proved for 5 and 6, and seems true for 9 and 10.)

[1] Alexander Below, Jürgen Richter-Gebert, Vanessa Krummeck, *Complex matroids, phirotopes and their realizations in rank 2*, in: Discrete and Computational Geometry—The Goodman-Pollack Festschrift, Springer, 2003, pp. 203–233.

Problem 7 (Ivan Izmestiev)**.** An embedded graph in S^3 is called *linked* if it contains two disjoint linked cycles. Is there a convex 4-polytope P such that its 1-skeleton $P^{(1)}$ is linked as a graph in $\partial P \approx S^3$, but has a different embedding that is not linked?

Background: Let G be a graph with Colin de Verdière invariant $\mu(G) = 4$, and let M be a Colin de Verdière matrix for G. According to a conjecture of [1] and [2], the null-space of M yields then a non-linked embedding of G. The positive solution of the problem would disprove this conjecture: Since $G = P^{(1)}$ has a non-linked embedding, $\mu(G) = 4$. By [3], there is a Colin de Verdière matrix M whose null-space represents G as the skeleton of P, and thus in a linked way.

[1] László Lovász, Alexander Schrijver, *On the null space of a Colin de Verdière matrix*, Symposium à la Mémoire de François Jaeger (Grenoble, 1998). Ann. Inst. Fourier (Grenoble) **49** (1999), no. 3, 1017–1026.

[2] László Lovász, *Steinitz representations of polyhedra and the Colin de Verdière number*, J. Combin. Theory Ser. B **82** (2001), no. 2, 223–236.

[3] Ivan Izmestiev, *Colin de Verdière number and graphs of polytopes*, to appear in Israel J. Math., arXiv:0704.0349 [math.CO].

Problem 8 (Serge Tabachnikov)**.** The standard origami model of a hyperbolic paraboloid is made from a square paper, folded zig-zag along many concentric squares and along the two diagonals. What is really going on in this model? Can it be realized with straight creases and with (developable) faces that are isometric to plane faces, or does it necessarily involve some stretching of the paper?

Problem 9 (Serge Tabachnikov)**.** We are given a partition of the unit square into T triangles (not necessarily a triangulation). A vertex that lies on an edge of some triangle or of the bounding square has 1 degree of freedom, all other interior vertices have 2. In total, there are $T-2$ degrees of freedom for moving the vertices while maintaining the combinatorial structure. If we consider the map sending any configuration to the T-tuple of signed areas, the image must satisfy two relations. One is that the sum of the areas is 1. What is the other?

Problem 10 (Ken Stephenson)**.** Uniqueness of inversive distance packings

Let K be a doubly periodic hexagonal lattice in the plane, that is, some affine image of a regular hexagonal lattice. Define a circle packing P for K by centering a circle of radius r at every lattice point, where r is sufficiently small that no two circles intersect. For each pair of neighboring lattice points record the inversive distance between their circles. (The Möbius invariant *inversive distance* between separated circles $C_1 = C(z, s)$ and $C_2 = C(w, t)$ is given by $(C_1, C_2) = |s^2 + t^2 - |z-w|^2|/2st$. See [1, Appendix E] for details on inversive distance circle packings.)

By lattice periodicity, just three inversive distances occur, $a, b, c \geq 1$, each associated with an axis direction for K: every pair of circles that are neighbors in the parallel direction share that inversive distance.

Question. *Is the packing P rigid? That is, suppose Q is a second circle packing for K whose circles realize the corresponding inversive distances. Is it the case that all circles of Q share a common radius?*

Dennis Sullivan [2] proved that the answer is yes when $a = b = c = 1$, that is, when every circle is tangent to its six neighbors. Zheng-Xu He [3] extended this to the case $a, b, c \in [0, 1]$, in which the circles overlap with specified overlap angles up to $\pi/2$.

[1] Ken Stephenson, *Introduction to Circle Packing: the Theory of Discrete Analytic Functions*, Cambridge Univ. Press (2005).

[2] Dennis Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*, in: Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference, Ann. of Math. Stud. **97** (1981), 465–496.

[3] Zheng-Xu He, *Rigidity of infinite disk patterns*, Annals of Mathematics **32** (1999), 1–33.

Problem 11 (John Sullivan)**.** Given a triangulated torus, it can be realized in $\mathbb{R}^3$. In what isotopy classes? (Which homology classes can be meridian etc.?)

Problem 12 (Feng Luo and Igor Pak)**.** Given a simplicial convex 3-polytope, is it infinitesimally rigid when at each edge either length *or* dihedral angle is fixed and at least one edge length is given?

Dehn's infinitesimal rigidity says the answer is yes if each edge length is given. Also the recent work of A. Pogorelov [1] shows infinitesimal rigidity up to scaling when dihedral angles at all edges are fixed, assuming all face angles are acute. Most recently, R. Mazzeo announced a complete solution of the Stoker conjecture, thus in particular removing the acute angle condition. This question of mixed type is motivated by the variational principle point of view. For instance, it is now known that a circle packing metric on a triangulated surface is determined when at each vertex either the curvature or the radius is given. On the other hand, *global* rigidity is not always true.

[1] Pogorelov, A. V.: *On a problem of Stoker* (Russian), Dokl. Akad. Nauk, Ross. Akad. Nauk **385** (2002), no. 1, 25–27; English translation in Dokl. Math. **66** (2002), no. 1, 19–21.

Problem 13 (Joseph O'Rourke)**.** Given a simple piecewise-geodesic curve on S^2, develop it onto the plane (same lengths and angles). Find conditions which guarantee that the developed image does not intersect itself. This is true if we start with a closed convex polygon on S^2. What about star-shaped polygons?

Problem 14 (Alexander Bobenko)**.** Are there (discrete) integrable systems in dimension four and higher? An n-dimensional discrete system is an equation for a function (field) defined at the vertices of an n-dimensional cube. Given values of the function at all the vertices of the cube but one, the discrete system determines the value at the last vertex. Following [1], a discrete system is called integrable if it is consistent, i. e., can be consistently set at all n-dimensional faces of an $(n+1)$-dimensional cube. The same problem can be also formulated for the systems with the fields on edges or faces. Many incidence relations in discrete differential geometry are discrete integrable systems.

There are many 2-dimensional and a few 3-dimensional discrete integrable systems (see the talk of Suris in this workshop). No examples in dimension four and higher are known.

[1] Alexander I. Bobenko, Yuri B. Suris, *Discrete Differential Geometry: Integrable Structure*, Graduate Studies in Mathematics, Vol. 98, AMS, 2008.