A short introduction to Sheaf Theory

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Contents

1 Presheaves and Sheaves

1.0 Categorical preliminaries

Definition 1. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors from the category $\mathcal{C}$ to the category $\mathcal{D}$. A **natural transformation** $\eta : F \to G$ from F to G is a family of morphisms that satisfies the following requirements

1. For each object X in $\mathcal{C}$ there is a morphism $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$.
2. For each morphism $f : X \to Y$ in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

   commutes.

η is a **natural isomorphism** if for each object X in $\mathcal{C}$ the morphism η_X is an isomorphism in $\mathcal{D}$.

1.1 Definitions of (pre)sheaves and examples

For us a category $\mathcal{C}$ will be usually one of the following: **Sets**, **Ab**, **Rings**, $\mathbf{R-Mod}$. In any case the underlying structure of the objects of $\mathcal{C}$ will be sets so that we can talk about elements of objects of $\mathcal{C}$.

Definition 2. Suppose $\mathcal{G}$ and $\mathcal{C}$ are categories. A **$\mathcal{C}$-valued presheaf** on $\mathcal{G}$ is a functor

$$F : \mathcal{G}^{op} \to \mathcal{C}.$$

If $\mathcal{G}$ is small, the collection of all $\mathcal{C}$-valued presheaves forms a category, which we denote by $\mathrm{PSh}(\mathcal{G}, \mathcal{C})$. Objects are presheaves, morphisms are natural transformations.

Definition 3. If X is a topological space and $\mathcal{C}$ is a category, we denote by $\mathrm{PSh}(X, \mathcal{C})$ the category of $\mathcal{C}$-valued presheaves on the category

$$\mathrm{Open}(X) = \{U \subset X \mid U \text{ open}\}, \quad \mathrm{Hom}(U, V) = \begin{cases} \emptyset, & U \not\subset V \\ \{i : U \hookrightarrow V\}, & U \subset V. \end{cases}$$

Let $F \in \mathrm{PSh}(X, \mathcal{C})$. Any inclusion $U \subset V$ of open sets of X gets mapped to a $\rho_{VU} : F(V) \to F(U)$, a so called **restriction morphism**. Moreover if we have inclusions $U \subset V \subset W$ of opens of X, then $\rho_{WU} = \rho_{VU} \circ \rho_{WV}$:

$$\begin{tikzcd} U \arrow[rr, hook] \arrow[dr, hook] & & W \\ & V \arrow[ur, hook] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F(U) & & F(W) \arrow[ll, "\rho_{WU}"'] \arrow[dl, "\rho_{WV}"] \\ & F(V) \arrow[ul, "\rho_{VU}"] & \end{tikzcd}.$$

We refer to $F(V)$ as the **sections** of F over the open set V. If $s \in F(V)$ we sometimes write $s|_U$ instead of $\rho_{VU}(s)$.

Sheaves are now just presheaves that satisfy certain conditions.

Definition 4. A presheaf $F \in \mathrm{PSh}(X, \mathcal{C})$ is a **sheaf** if it satisfies the following two conditions for any open set $U \subset X$ and any open covering $\{U_i\}_{i \in I}$ of U:

(S1) If $s_1, s_2 \in F(U)$ are elements such that $s_1|_{U_i} = s_2|_{U_i}$ for all $i \in I$, then $s_1 = s_2$.

(S2) If we have elements $s_i \in F(U_i)$ for each $i \in I$ with the property that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for each $i, j \in I$, then there is an element $s \in F(U)$ such that $s|_{U_i} = s_i$ for each $i \in I$.

We denote by $\mathrm{Sh}(X, \mathcal{C})$ the category of $\mathcal{C}$-valued sheaves on X.

Remark 5. (S1) implies that the element s in (S2) is unique.

Remark 6. Note that if $\mathcal{C}$ is **Ab**, **Rings** or **R-Mod** we always have that for a sheaf $F(\emptyset) = \{0\}$: For the open $\emptyset \subset X$ choose the empty covering with $I = \emptyset$: $\{U_i\}_{i \in \emptyset}$. Then for two sections $s_1, s_2 \in F(\emptyset)$ the condition $s_1|_{U_i} = s_2|_{U_i}$ is vacuously fulfilled and we conclude $s_1 = s_2$ by (S1).

So much for now with definitions. Let us train our intuition with some examples.

Example 7. Let X be a space and for each open $U \subset X$ let

$$F(U) := \{f : U \to \mathbb{R}\},$$

the set of continuous real-valued functions on U. For $U \subset V$ let further

$$\rho_{VU} : F(V) \to F(U),\ f \mapsto f|_U$$

be the usual restriction map. Then F is a sheaf of rings on X.

Example 8. Consider the real line $\mathbb{R}$ and let for each open $U \subset \mathbb{R}$

$$F(U) := \{f : U \to \mathbb{R} \mid f \text{ bounded}\}.$$

For $U \subset V$ let further

$$\rho_{VU} : F(V) \to F(U),\ f \mapsto f|_U$$

Then F is a presheaf of rings that is *not* a sheaf. It does satisfy (S1) but not (S2), because gluing together bounded functions may not yield bounded functions.

Example 9. Suppose $X = \{x, y\}$ and X has the discrete topology. Set

$$F(\{x\}) = \mathbb{R}, \quad F(\{y\}) = \mathbb{R}, \quad F(\{x, y\}) = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}.$$

Let the two (non-trivial) restrictions maps be projection two the first and second factor. $\{x\}$ and $\{y\}$ form an open covering of $\{x, y\}$ and the two restrictions to them of the distinct sections $(0, 1, 1), (0, 1, 2) \in F(\{x, y\})$ are the same. Hence (S1) is not satisfied and F is a presheaf that is *not* a sheaf. Note that (S2) holds true, though.

Example 10. Let X be a space and let G be an abelian group. The **constant sheaf F_G on X determined by** G is defined as follows. Give G the discrete topology and for an open $U \subset X$ let

$$F_G(U) := \{f : U \to G\}$$

be the group of all continuous maps from U to G. Then with the usual restriction maps

$$\rho_{VU} : F_G(V) \to F_G(U),\ f \mapsto f|_U$$

F_G is a sheaf of abelian groups. Note that on every connected open U any $f \in F_G(U)$ must be constant and conversely any choice of a constant map $U \to G$ is, of course, continuous. Hence $F_G(U) \approx G$ and the name "constant sheaf". If U is an open set whose connected components are also open (this is always the case in locally connected spaces), then $F_G(U)$ is a direct product of copies of G, one for each connected component of U.

2 Stalks and morphisms of (pre)sheaves

2.0 Categorical preliminaries

Definition 11. A **directed set** $(I, \leq)$ is a nonempty set I together with a *reflexive* and *transitive* relation $\leq$ with the property that for each $i, j \in I$ there is a $k \in I$ such that

$$i \leq k \quad \text{and} \quad j \leq k.$$

Definition 12. Let $(I, \leq)$ and be a directed set and $\{G_i\}_{i\in I}$ be a family of abelian groups indexed by I. For all $i \leq j$ let further $f_{ij} : G_i \to G_j$ be a homomorphism such that

1. f_{ii} is the identity on G_i,
2. $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$.

Then $(G_i, f_i)_{i\in I}$ is called a **direct system of abelian groups** over I.

Given a direct system $(G_i, f_i)_{i\in I}$ of abelian groups, define the following equivalence relation on the disjoint union $\bigsqcup_{i\in I} G_i$ of the groups G_i: For $x_i \in G_i$ and $x_j \in G_j$ we let $x_i \sim x_j$ if and only if there is a k with $i, j \leq k$ and $f_{ik}(x_i) = f_{jk}(x_j)$.

Definition 13. Let $(G_i, f_i)_{i\in I}$ be a direct system of abelian groups, then the **direct limit** of this system is defined by

$$\varinjlim G_i := \bigsqcup_{i\in I} G_i \Big/ \sim,$$

where the equivalence relation on the right hand side is the one described above.

The direct limit naturally carries an abelian group structure: Given equivalence classes $[x_i]$ and $[x_j]$ with $x_i \in G_i$ and $x_j \in G_j$ there is a $k \in I$ with $i, j \leq k$ and we define

$$[x_i] + [x_j] := [f_{ik}(x_i) + f_{jk}(x_j)].$$

One only has to check that this is well defined, which is a nice exercise.

Proposition 14. *The direct limit of a direct system of abelian groups is an abelian group.* □

2.1 Definition of stalks and morphisms of (pre)sheaves and examples

Definition 15. Let X be a topological space, $F \in \mathrm{PSh}(X, \mathcal{C})$ a presheaf and assume that the category $\mathcal{C}$ is cocomplete. Then for $x \in X$ we define

$$F_x := \varinjlim_{U \ni x} F(U),$$

where the direct limit is taken over all open neighborhoods U of x. F_x is called the **stalk of F at x**.

An element of the stalk F_x is represented by a pair (U, s), where $s \in F(U)$ is a section and U is an open neighborhood of x. Two such pairs (U, s) and (V, t) are equivalent if there is an open neighborhood W of x with $W \subset U \cap V$ and $s|_W = t|_W$. This is why we may speak of elements of the stalk F_x as **germs** of sections of F at the point x. Sometimes we will denote a germ in F_x, represented by a pair (U, s), by s_x.

Example 16. Let G be an abelian group, X be a space and $x \in X$. For open $U \subset X$ we set

$$x_*G(U) = \begin{cases} G, & \text{if } x \in U \\ \{0\}, & \text{otherwise.} \end{cases}$$

For two open sets $U \subset V$ we set

$$\rho_{VU} = \begin{cases} \mathrm{id}_G, & \text{if } x \in U \\ 0, & \text{otherwise.} \end{cases}$$

Then we can check that it is a presheaf. But it even is a sheaf (with values in abelian groups): Suppose $\{U_i\}$ is an open covering of the open $U \subset X$ and $s_1, s_2 \in x_*G(U)$ satisfy $s_1|_{U_i} = s_2|_{U_i}$ for all i. If $x \notin U$, then $s_1 = s_2 = 0$. If $x \in U$, there is a U_i containing x. Because ρ_{UU_i} is just the identity, we conclude $s_1 = s_1|_{U_i} = s_2|_{U_i} = s_2$. If we are given sections $s_i \in x_*G(U_i)$ with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ and $x \notin U$, then the zero section $s = 0$ clearly restricts to the s_i on U_i. If $x \in U$, then $x \in U_i$ for some i and we set $s = s_i$.
If X is a T_1-space, we observe that all stalks of this sheaf are $\{0\}$ except $(x_*G(U))_x$, which is G. For this reason it is called a **skyscraper sheaf**.

Definition 17. Let F, G be (pre)sheaves on a space X. A **morphism of (pre)sheaves** $\varphi : F \to G$ is a natural transformation from F to G. Hence it consists of a morphism $\varphi_U : F(U) \to G(U)$ for each open set $U \subset X$ such that for all inclusions $U \subset V$ of open sets the following diagram commutes

$$\begin{array}{ccc} F(V) & \xrightarrow{\rho_{VU}} & F(U) \\ \downarrow{\scriptstyle \varphi_V} & & \downarrow{\scriptstyle \varphi_U} \\ G(V) & \xrightarrow{\rho'_{VU}} & G(U). \end{array}$$

Here ρ and ρ' are the restriction maps of F and G. φ is an **isomorphism of (pre)sheaves** if φ is a natural isomorphism.

Example 18. Let F be the sheaf from Example 7. Then $\varphi_U(f) := 2f$, which sends any real-valued function to its double, defines a morphism of sheaves. It is even an isomorphism.

Proposition 19. *Let $\varphi : F \to G$ be a morphism of presheaves on a space X. Then φ induces a map on the stalk $\varphi_x : F_x \to G_x$ for every $x \in X$.*

Proof. Let $x \in X$, then $\varphi_x[U, s] = [U, \varphi_U(s)]$, where $[U, s]$ stands for the equivalence class of the pair (U, s) in F_x. We have to show that this is well defined. So assume (V, t) is another pair equivalent to (U, s). Then there is an open neighborhood W of x with $W \subset U \cap V$ such that $s|_W = t|_W$. It follows that $\varphi_U(s)|_W = \varphi_W(s|_W) = \varphi_W(t|_W) = \varphi_V(t)|_W$. This can be seen in the following diagram

$$\begin{tikzcd}
s \arrow[r, mapsto] \arrow[dddd, mapsto, bend right] & s|_W = t|_W & t \arrow[l, mapsto] \arrow[dddd, mapsto, bend left] \\
{} \arrow[u, phantom, sloped, "\in"] & {} \arrow[u, phantom, sloped, "\in"] & {} \arrow[u, phantom, sloped, "\in"] \\
F(U) \arrow[r, "\rho_{UW}"] \arrow[d, "\varphi_U"] & F(W) \arrow[d, "\varphi_W"] & F(V) \arrow[l, "\rho_{VW}"'] \arrow[d, "\varphi_V"] \\
G(U) \arrow[r, "\rho'_{UW}"] & G(W) & G(V) \arrow[l, "\rho'_{VW}"'] \\
\varphi_U(s) \arrow[r, mapsto] \arrow[u, phantom, sloped, "\in"] & \varphi_W(s|_W) \arrow[u, phantom, sloped, "\in"] & \varphi_V(t). \arrow[l, mapsto] \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

□

This means that for each $x \in X$ we get a functor

$$\cdot_x : \mathrm{PSh}(X, \mathbf{Ab}) \to \mathbf{Ab}$$

which sends a presheaf F to its stalk at x

$$F \mapsto F_x$$

and a presheaf morphism to its induced map on the stalk at x

$$(\varphi : F \to G) \mapsto (\varphi_x : F_x \to G_x).$$

One easily verifies the rest of the properties that make this a functor.

Proposition 20. *Let $\varphi : F \to G$ be a morphism of sheaves on a topological space X. Then φ is an isomorphism if and only if the induced map on the stalk $\varphi_x : F_x \to G_x$ is an isomorphism for every $x \in X$.*

Proof. Assume first that $\varphi : F \to G$ is an isomorphism of sheaves. Taking stalks is a functor as mentioned above, which, of course, sends isomorphism to isomorphisms. Hence φ_x is an isomorphism for each $x \in X$.
Assume now for all $x \in X$ the map $\varphi_x : F_x \to G_x$ is an isomorphism. We have to show that for any open $U \subset X$ the map $\varphi_U : F(U) \to G(U)$ is an isomorphism. So let $s, t \in F(U)$ be sections satisfying $\varphi_U(s) = \varphi_U(t)$. Then the pairs $(U, \varphi_U(s))$ and $(U, \varphi_U(t))$ are equivalent in any stalk G_x with $x \in U$. Since φ_x is injective, the pairs (U, s) and (U, t) are equivalent. So there is an open neighborhood $U_x \subset U$ of x such that

$$s|_{U_x} = t|_{U_x}.$$

$\{U_x\}_{x \in U}$ forms an open covering of U and by (S1) $s = t$ and injectivity of φ_U follows. Let now $t \in G(U)$ be any section. For any $x \in U$ the element represented by (U, t) in G_x is equivalent to $(U_x, \varphi_{U_x}(s^x))$ for some open neighborhood $U_x \subset U$ of x and some section $s^x \in F(U_x)$, since φ_x is surjective. We may assume that $t|_{U_x} = \varphi_{U_x}(s^x)$ by shrinking U_x a little if necessary. Let now s^x and s^y two such sections for $x, y \in U$. Then $s^x|_{U_x \cap U_y} = s^y|_{U_x \cap U_y}$ because

$$\varphi_{U_x \cap U_y}(s^x|_{U_x \cap U_y}) = t|_{U_x \cap U_y} = \varphi_{U_x \cap U_y}(s^y|_{U_x \cap U_y})$$

and $\varphi_{U_x \cap U_y}$ is injective as proofed above. Again $\{U_x\}_{x \in U}$ forms an open covering of U and by (S2) there is an $s \in F(U)$ with $s|_{U_x} = s^x$ for all $x \in U$. Since $\varphi_U(s)|_{U_x} = t|_{U_x}$ for all points $x \in U$ we can appeal (S1) to deduce $\varphi_U(s) = t$. □

The proposition illustrates the local nature of sheaves and would be false for presheaves.

2.2 Kernels, cokernels and images of morphisms of presheaves

For this section let the category $\mathcal{C}$ be **Ab**, the category of abelian groups.

Definition 21. Let $\varphi : F \to G$ be a morphism of presheaves on a space X. We define the

1. **presheaf kernel** of φ to be the presheaf given by $U \mapsto \ker \varphi_U$,
2. **presheaf image** of φ to be the presheaf given by $U \mapsto \operatorname{im} \varphi_U$,
3. **presheaf cokernel** of φ to be the presheaf given by $U \mapsto \operatorname{coker} \varphi_U$.

Let us elaborate a little bit on this. If $U \subset V$ are open sets. Then the first two definitions above can be illustrated in the following diagram:

$$\begin{array}{ccc}
\ker \varphi_V & \xrightarrow{\rho_{VU}} & \ker \varphi_U \\
\cap & & \cap \\
F(V) & \xrightarrow{\rho_{VU}} & F(U) \\
\Big\downarrow{\scriptstyle \varphi_V} & & \Big\downarrow{\scriptstyle \varphi_U} \\
G(V) & \xrightarrow{\rho'_{VU}} & G(U) \\
\cup & & \cup \\
\operatorname{im} \varphi_V & \xrightarrow{\rho'_{VU}} & \operatorname{im} \varphi_U .
\end{array}$$

Using commutativity it is easy to verify that for an $s \in \ker \varphi_V$ we have that $s|_U \in \ker \varphi_U$ and for a $t \in \operatorname{im} \varphi_V$ we have $t|_U \in \operatorname{im} \varphi_U$. For the third definition we only have to show that the maps induced by the restriction maps ρ'_{VU} are well defined. So let $t + \operatorname{im} \varphi_V \in \operatorname{coker} \varphi_V$ and let $t' \in G(V)$ such that $t - t' \in \operatorname{im} \varphi_V$. Then $t|_U - t'|_U = (t - t')|_U \in \operatorname{im} \varphi_U$ which is exactly what we wanted.

We have the following proposition regarding the presheaf kernel.

Proposition 22. *Let $\varphi : F \to G$ be a morphism of sheaves on a space X. Then the presheaf kernel is a sheaf.*

Proof. We have to check that (S1) and (S2) in the definition of sheaves are satisfied. So let $U \subset X$ be an open set and $\{U_i\}$ be an open covering of U. (S1) is obviously true since F is a sheaf. Let now $s_i \in \ker \varphi_{U_i}$ be sections with the property that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for each i, j. Because F is a sheaf there is an element $s \in F(U)$ with $s|_{U_i} = s_i$ for all i. Now $\varphi_U(s)$ is locally zero, so it is the zero section: $\varphi_U(s)|_{U_i} = \varphi_{U_i}(s|_{U_i}) = \varphi_{U_i}(s_i) = 0$ and since G is a sheaf we conclude with (S2) that $\varphi_U(s) = 0$. □

Note that the above proposition is not true for presheaf images and presheaf cokernels. So taking presheaf image and cokernel of a morphism of sheaves might only yield presheaves. So it is natural to ask how to make a sheaf out of a presheaf - or how to *sheafify* a presheaf. Remember that sheaves are presheaves where the sections have two additional properties. (S1) roughly says, that if sections agree locally they do so globally. So if a presheaf does not satisfy this, it violates uniqueness and one could say that there are "too many" sections. (S2) roughly says, that if we are given sections locally such that they agree on the overlaps, we can glue them together to get a global section. This is an existence statement and if a presheaf does not satisfy this, we could on the other hand say that there are "too few" sections. Thus we could try to "throw away" the sections that are superfluous while "adding the missing" sections and it turns out that there is an optimal way to do it.

Proposition 23. *Given a presheaf F on a space X, then there is a sheaf F^+ on X and a (presheaf) morphism $\theta : F \to F^+$ such that for any sheaf G on X and any morphism $\varphi : F \to G$ there is a unique (sheaf) morphism $\psi : F^+ \to G$ such that $\varphi = \psi \circ \theta$:*

$$\begin{tikzcd} F \arrow[r, "\theta"] \arrow[d, "\varphi"] & F^+ \arrow[dl, dashed, "\exists!\psi"] \\ G. & \end{tikzcd}$$

Proof. **Construction of F^+:**
We construct F^+ as follows. For any open $U \subset X$ let $F^+(U)$ be the set of functions $s : U \to \bigcup_{x \in U} F_x$, from U to the disjoint union of stalks of F over points in U, such that

1. for all $x \in U$ we have $s(x) \in F_x$,
2. for all $x \in U$ there is an open neighborhood $V \subset U$ of x and an element $t \in F(V)$ such that for all $y \in V$ the germ t_y is equal to $s(y)$.

This means that locally around any point $x \in U$ a section $s \in F^+(U)$ looks like a section t of $F(U)$. Let the restriction maps be the usual restrictions of functions. We now have to verify that F^+ is indeed a sheaf. Let $U \subset X$ be open. First of all note that $F^+(U)$ is nonempty because it always contains the zero section $s(x) = 0$. It is also a presheaf and $F^+(U)$ is an abelian group with pointwise addition (note that for $s_1, s_2 \in F^+(U)$ the section $s_1 + s_2$ satisfies both conditions above). Let now $\{U_i\}$ be an open covering of U and suppose we are given $s_1, s_2 \in F^+(U)$ such that $s_1|_{U_i} = s_2|_{U_i}$ for all i. Since s_1 and s_2 really are functions, it follows that $s_1 = s_2$. If we are given $s_i \in F^+(U_i)$ such that for all i, j we have $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, we define $s(x) := s_i(x)$, where $x \in U_i$. This is independent of the choice of i. We have to show that $s \in F^+(U)$. For any $x \in U$ there is a U_i containing x such that $s|_{U_i} = s_i$. Hence by condition 2 above there is a neighborhood $V_i \subset U_i$ of x and an element $t \in F(V_i)$ such that for all $y \in V_i$ the germ t_y is equal to $s_i(y) = s(y)$ and we conclude that F^+ is a sheaf.

Construction of $\theta : F \to F^+$:

Now we construct the presheaf morphism $\theta : F \to F^+$. For open $U \subset X$ we set

$$\theta_U : F(U) \to F^+(U), \quad t \mapsto (x \mapsto t_x).$$

Then the two conditions above are immediate and θ_U indeed maps to $F^+(U)$. To show that θ is a presheaf morphism we have to establish commutativity of

$$\begin{tikzcd} F(V) \arrow[r, "\rho_{VU}"] \arrow[d, "\theta_V"] & F(U) \arrow[d, "\theta_U"] \\ F^+(V) \arrow[r, "\rho'_{VU}"] & F^+(U) \end{tikzcd} \qquad \begin{tikzcd} t \arrow[r, mapsto, "\rho_{VU}"] \arrow[d, mapsto, "\theta_V"] & t|_U \arrow[d, mapsto, "\theta_U"] \\ x \mapsto t_x \arrow[r, mapsto, "\rho'_{VU}"] & \cdot \end{tikzcd}$$

for another open $V \supset U$ and that θ_U is a homomorphism. Going from the top left corner first down and then right, we end up with $U \ni x \mapsto t_x$. Going first right and then down we obtain $U \ni x \mapsto (t|_U)_x = t_x$. For another section $t' \in F(U)$ we have $\theta_U(t + t') = x \mapsto (t + t')_x = x \mapsto (t_x + t'_x) = \theta_U(t) + \theta_U(t')$, so θ_U is a homomorphism.

Construction of a $\psi : F^+ \to G$:

Let now G be any sheaf on X and let $\varphi : F \to G$ be a morphism. By the second condition there is for any section $s \in F^+(U)$ and any $x \in U$ an open neighborhood $V_x \subset U$ of x and a section $t^x \in F(V_x)$ with $s(z) = t^x_z$ for all $z \in V_x$. Then $\{V_x\}_{x \in U}$ is an open covering of U. We set

$$\psi_U(s) := \text{"The section } g \in G(U) \text{ with } g|_{V_x} = \varphi_{V_x}(t^x)\text{"}.$$

We have to verify that $\varphi_{V_x}(t^x)|_{V_x \cap V_y} = \varphi_{V_y}(t^y)|_{V_x \cap V_y}$ for all $x, y \in U$ and then by (S2) this will give us a section $g \in G(U)$. For all $z \in V_x \cap V_y$ it is true that $t^x_z = s(z) = t^y_z$ and so there is an open neighborhood $W_z \subset V_x \cap V_y$ of z such that $t^x|_{W_z} = t^y|_{W_z}$. $\{W_z\}_{z \in V_x \cap V_y}$ forms an open covering of $V_x \cap V_y$ and

$$\left(\varphi_{V_x}(t^x)|_{V_x \cap V_y}\right)\Big|_{W_z} = \varphi_{W_z}(t^x|_{W_z}) = \varphi_{W_z}(t^y|_{W_z}) = \left(\varphi_{V_y}(t^y)|_{V_x \cap V_y}\right)\Big|_{W_z}$$

so by (S1) we obtain $\varphi_{V_x}(t^x)|_{V_x \cap V_y} = \varphi_{V_y}(t^y)|_{V_x \cap V_y}$.

We also have to show that the obtained section $g \in G(U)$ does not depend on the choices of the V_x and t^x. So let U_x and u^x another such choice and let $h \in G(U)$ be the section obtained from them. Then $t^x_x = s(x) = u^x_x$ and we obtain an open neighborhood $W_x \subset V_x \cap U_x$ such that $t^x|_{W_x} = u^x|_{W_x}$. So $\{W_x\}$ forms an open covering of U,

$$g|_{W_x} = \varphi_{V_x}(t^x)|_{W_x} = \varphi_{W_x}(t^x|_{W_x}) = \varphi_{W_x}(u^x|_{W_x}) = \varphi_{U_x}(u^x)|_{W_x} = h|_{W_x}$$

and by (S1) we conclude $g = h$. Also one verifies that ψ_U is a homomorphism using (S1) again. Lastly for open $U \subset V$ we need to establish the commutativity of

$$\begin{tikzcd}
F^+(V) \arrow[r] \arrow[d, "\psi_V"] & F^+(U) \arrow[d, "\psi_U"] & & s \arrow[r, mapsto] \arrow[d, mapsto, "\psi_V"] & s|_U \arrow[d, mapsto, "\psi_U"] \\
G(V) \arrow[r] & G(U) & & \psi_V(s) \arrow[r, mapsto] & \cdot
\end{tikzcd}$$

For $s \in F^+(V)$ we choose the V_x and t^x such that for $x \in U$ we have $V_x \subset U$. Then we can use the same V_x and t_x for $s|_U$. Now going first right then down we get $\psi_U(s|_U)$ and first down then right yields $\psi_V(s)|_U$. Observe that

$$\Big(\psi_V(s)|_U\Big)\Big|_{V_x} = \psi_V(s)|_{V_x} = \varphi_{V_x}(t^x) = \psi_U(s|_U)|_{V_x},$$

and $\{V_x\}$ is an open covering of U, hence by (S1) commutativity is established. The assertion $\varphi = \psi \circ \theta$ is immediately verified.
Uniqueness of $\psi : F^+ \to G$:
Assume we have another sheaf morphism ψ' as above. Then for open $U \subset X$ consider the diagram

$$\begin{tikzcd}
F(U) \arrow[r, "\theta_U"] \arrow[d, "\varphi_U"] & F^+(U) \arrow[dl, "\psi_U"] \arrow[dl, bend left, "\psi'_U"] \\
G(U) &
\end{tikzcd}$$

Let $s \in F^+(U)$ be any section and suppose we use V_x, t^x for the definition of $\psi_U(s)$. Once again $\{V_x\}$ is an open covering of U and

$$\psi_U(s)|_{V_x} = \varphi_{V_x}(t^x) = \psi'_{V_x}(\theta_{V_x}(t^x)) = \psi'_{V_x}(s|_{V_x}) = \psi'_U(s)|_{V_x},$$

so by (S1) $\psi_U(s) = \psi'_U(s)$ and we are done (finally!). □

Remark 24. As usual with universal properties, the sheaf F^+ is unique up to isomorphism. It follows, that if F is already a sheaf, then F^+ is isomorphic to F via θ.

Proposition 23 reinforces the perspective that the sections $s \in F(U)$ of a sheaf F are functions defined on U. Since if you look at sections of the sheafification F^+ (being isomorphic to F) they are exactly that.

Definition 25. For a presheaf F on a topological space X we call the sheaf F^+ together with the morphism $\theta : F \to F^+$ constructed above the **sheafification** of F or the **sheaf associated** to F.

Remark 26. For a morphism of presheaves $\varphi : F \to G$ there is a unique morphism $\varphi^+ : F^+ \to G^+$ making the following diagram commute

$$\begin{tikzcd}
F \arrow[r, "\varphi"] \arrow[d, "\theta_F"] & G \arrow[d, "\theta_G"] \\
F^+ \arrow[r, "\varphi^+"] & G^+.
\end{tikzcd}$$

Verifying the functorial properties we conclude that sheafifying is a functor

$$\cdot^+ : \mathrm{PSh}(X, \mathbf{Ab}) \to \mathrm{Sh}(X, \mathbf{Ab}).$$

Proposition 27. *For a presheaf F the morphism $\theta : F \to F^+$ induces an isomorphism on the stalks, so $F_x \approx F_x^+$ for all $x \in X$.*

Proof. Let $x \in X$ be a any point, V an open neighborhood of x and $t \in F(V)$ a section such that $\theta_x(t_x) = 0$. Then there is an open $U \ni x$ with $0 = \theta_V(t)|_U = (U \ni y \mapsto t_y)$. Hence $t_x = 0$.
Take now any $s_x \in F_x^+$. By the construction of F^+ there is an open $U \ni x$ and a $t \in F(U)$ with $s(y) = t_y$ for all $y \in U$. So $\theta_U(t) = s|_U$ and we obtain $\theta_x(t_x) = s_x$. $\square$

Example 28. Consider the presheaf from Example 9. It satisfies (S2) but not (S1). The stalks are $F_x \approx \mathbb{R} \approx F_y$. We obtain

$$F^+(\{x, y\}) = \{s : \{x, y\} \to F_x \cup F_y\} \approx \{s : \{x, y\} \to \mathbb{R}\} \approx \mathbb{R} \oplus \mathbb{R},$$

so we got rid of one factor $\mathbb{R}$ and have therefore "thrown away" a lot of sections. Note also that the two sections $(0, 1, 1), (0, 1, 2) \in F(\{x, y\})$ both get mapped to $(x \mapsto 0, y \mapsto 1) \in F^+(\{x, y\})$ by θ.

Example 29. Now consider the presheaf F from Example 8. The presheaf of bounded continuous functions on $X = \mathbb{R}$ does satisfy (S1) but not (S2). We claim that its sheafification F^+ is isomorphic to the sheaf of continuous functions (here with values in abelian groups), which we denote by C^0. Let $U \subset \mathbb{R}$ be open, G a sheaf on $\mathbb{R}$ and $\varphi : F \to G$ a morphism. We have to show that there is a unique morphism $\psi : C^0 \to G$ making the diagram

$$\begin{tikzcd} F(U) \arrow[r, hook, "\mathrm{incl}"] \arrow[d, "\varphi"] & C^0(U) \arrow[dl, dashed, "\exists!\psi"] \\ G(U) & \end{tikzcd}$$

commutative. For a continuous function $f : U \to \mathbb{R}$ we can find an open covering $\{U_i\}$ of U such that $f|_{U_i}$ is bounded. We then set

$$\psi_U(f) = \text{"The section } g \in G(U) \text{ with } g|_{U_i} = \varphi_{U_i}(f|_{U_i})\text{"},$$

which makes the diagram commute. Note that this does not depend on the open cover $\{U_i\}$ and that ψ thus is a well defined morphism. If ψ' is another such morphism we use the cover $\{U_i\}$ again to see that $\psi_U(f)|_{U_i} = \varphi_{U_i}(f|_{U_i}) = \psi'_U(f)|_{U_i}$. Hence by (S1) $\psi = \psi'$.

Definition 30. A **subsheaf** of a sheaf F is a sheaf F' such that $F'(U)$ is a subgroup of $F(U)$ for every open $U \subset X$ and the restriction maps of F' are induced by those of F.

Remark 31. Observe that for any point $x \in X$ the stalk F'_x can be naturally identified with a subgroup of F_x.

Definition 32. If $\varphi : F \to G$ is a morphism of sheaves, we define the **kernel** of φ, denoted by $\ker \varphi$, to be the presheaf kernel of φ which is a sheaf by Proposition 22.

Remark 33. $\ker \varphi$ is a subsheaf of F.

Definition 34. A morphism of sheaves $\varphi : F \to G$ is said to be **injective** if $\ker \varphi = 0$.

Remark 35. φ is injective if and only if $\varphi_U : F(U) \to G(U)$ is injective for every open $U \subset X$.

For a morphism of sheaves $\varphi : F \to G$, we let for the moment the sheafification of the presheaf image of φ be denoted by $\widetilde{\mathrm{im}\varphi}$. By the universal property of the sheafification there is a natural morphism $\widetilde{\mathrm{im}\varphi} \to G$. We will show that this is injective and thus $\widetilde{\mathrm{im}\varphi}$ can be identified with a subsheaf of G.

Lemma 36. *Let $\varphi : F \to G$ be a morphism of presheaves such that $\varphi_U : F(U) \to G(U)$ is injective for each open $U \subset X$. Then the induced morphism $\varphi^+ : F^+ \to G^+$ is injective.*

Proof. Let $s \in F^+(U)$ be any section such that $\varphi^+(s) = 0$. There is an open neighborhood $U_x \subset U$ of each x and a section $t \in F(U_x)$ such that $s(y) = t_y$ for each $y \in U_x$. Consider the diagram

$$\begin{array}{ccc} F(U_x) & \xrightarrow{\varphi_{U_x}} & G(U_x) \\ \downarrow & & \downarrow \\ F^+(U_x) & \xrightarrow{\varphi^+_{U_x}} & G^+(U_x). \end{array}$$

Mapping t first right, then down we obtain the map

$$U_x \ni y \mapsto (\varphi_{U_x}(t))_y$$

which is zero due to commutativity. Thus we find an open neighborhood $V_x \subset U_x$ of x such that $\varphi_{U_x}(t)|_{V_x} = \varphi_{V_x}(t|_{V_x}) = 0$. Using injectivity of φ_{V_x} yields $t|_{V_x} = 0$ and so $s(x) = t_x = 0$, so $s = 0$. $\square$

Proposition 37. *If $\varphi : F \to G$ is a morphism of sheaves, then the natural morphism $\widetilde{\mathrm{im}\varphi} \to G$ is injective. Hence we can identify $\widetilde{\mathrm{im}\varphi}$ with a subsheaf of G.*

Proof. The inclusion $\mathrm{im}\, \varphi_U \hookrightarrow G(U)$ is clearly injective for each open $U \subset X$, so the statement follows from the lemma above. $\square$

Definition 38. For a morphism of sheaves $\varphi : F \to G$ we define the **image** of φ, denoted by $\mathrm{im}\, \varphi$, to be the subsheaf of G identified with $\widetilde{\mathrm{im}\varphi}$.

Definition 39. A morphism of sheaves $\varphi : F \to G$ is said to be **surjective** if $\mathrm{im}\, \varphi = G$.

We saw that a morphism of sheaves $\varphi : F \to G$ is injective if and only if $\varphi_U : F(U) \to G(U)$ is injective for each open $U \subset X$. The corresponding statement for surjective morphisms is *not* true, as we will see now.

Example 40. Consider the circle $X = S^1$ and the sheaf C^∞ of smooth functions on it. Further let $\gamma(t) := (\cos t, \sin t)$. Define the sheaf morphism $d : C^\infty \to C^\infty$ by

$$d_U : C^\infty(U) \to C^\infty(U), \quad f \mapsto ((\cos t, \sin t) \mapsto (f \circ \gamma)'(t)).$$

Let $P := (0, 1) \in S^1$ be the north pole and $Q = (0, -1) \in S^1$ be the south pole. For the two sections $f \in C^\infty(S^1 - P), f(\cos t, \sin t) = t$ and $g \in C^\infty(S^1 - Q), g(\cos t, \sin t) = t$ we have $d(f) = 1 \in C^\infty(S^1 - P)$ and $d(g) = 1 \in C^\infty(S^1 - Q)$. Hence they agree on the overlaps of their domains. Their domains also cover the whole S^1 but the global section obtained by gluing them together is the constant function $1 \in C^\infty(S^1)$ which is not in the image of d_{S^1}. So d_{S^1} is *not* surjective.
d as a morphism of sheaves is still surjective. Roughly because all smooth functions are locally in the image of d.

Definition 41. A sequence

$$\ldots \to F^{i-1} \xrightarrow{\varphi^{i-1}} F^i \xrightarrow{\varphi^i} F^{i+1} \to \ldots$$

of sheaves and morphisms is said to be **exact** if $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$.

Remark 42. A sequence $0 \to F \xrightarrow{\varphi} G$ is exact if and only if φ is injective. A sequence $F \xrightarrow{\varphi} G \to 0$ is exact if and only if φ is surjective.

Definition 43. Let F' be a subsheaf of F. We define the **quotient sheaf** F/F' to be the sheafification of the presheaf $U \mapsto F(U)/F'(U)$.

Remark 44. Observe that for the stalks we have $(F/F')_x \approx F_x/F'_x$.

References

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