Majority Colorings of Sparse Digraphs

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Abstract

A *majority coloring* of a directed graph is a vertex-coloring in which every vertex has the same color as at most half of its out-neighbors. Kreutzer, Oum, Seymour, van der Zypen and Wood [KOS+17] proved that every digraph has a majority 4-coloring and conjectured that every digraph admits a majority 3-coloring. They observed that the Local Lemma implies the conjecture for digraphs of large enough minimum out-degree if, crucially, the maximum in-degree is bounded by a(n exponential) function of the minimum out-degree.

Our goal in this paper is to develop alternative methods that allow the verification of the conjecture for natural, broad digraph classes, *without* any restriction on the in-degrees. Among others, we prove the conjecture 1) for digraphs with chromatic number at most 6 or dichromatic number at most 3, and thus for all planar digraphs; and 2) for digraphs with maximum out-degree at most 4. The benchmark case of r-regular digraphs remains open for $r \in [5, 143]$. Our inductive proofs depend on loaded inductive statements about precoloring extensions of list-colorings. This approach also gives rise to stronger conclusions, involving the choosability version of majority coloring.

We also give further evidence towards the existence of majority-3-colorings by showing that *every* digraph has a fractional majority 3.9602-coloring. Moreover we show that every digraph with large enough minimum out-degree has a fractional majority $(2 + \varepsilon)$-coloring.

1 Introduction

Preliminiaries. Digraphs considered in this paper are loopless, have no parallel edges, but are allowed to have anti-parallel pairs of edges (*digons*). A directed edge with tail u and head v is denoted by (u, v). For a digraph D and a vertex $v \in V(D)$, we let $N^+(v)$ and $N^-(v)$ denote the out- and in-neighborhood of v in D and $d^+(v)$, $d^-(v)$ the respective sizes. We denote by $\delta^+(D)$, $\delta^-(D)$, $\Delta^+(D)$, $\Delta^-(D)$ the minimum or maximum out- or in-degree of D, respectively, and let $\Delta(D) = \max\{d^+(v) + d^-(v) | v \in V(D)\}$ denote the maximum degree in D. The *underlying graph* of a digraph D, denoted by $U(D)$, is the simple undirected graph with vertex set $V(D)$ in which two vertices $x \neq y$ are adjacent iff $(x, y) \in E(D)$ or $(y, x) \in E(D)$. We say that D is r-regular for an integer $r \geq 1$ if $d^+(x) = d^-(x) = r$ for every $x \in V(D)$.

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A *majority coloring* of a digraph D with k colors is an assignment $c : V(D) \to \{1, \dots, k\}$ such that for every $v \in V(D)$, we have $c(w) = c(v)$ for at most half of all out-neighbors $w \in N^+(v)$. This notion of coloring was first introduced and investigated by Kreutzer, Oum, Seymour, van der Zypen, and Wood [KOS+17]. Related questions concerning splittings of digraphs with degree restrictions have been well studied, see for instance [Alo96, Alo06, NAB20]. The main result obtained by Kreutzer et al. shows that every digraph has a majority 4-coloring. Their elegant argument is based on the observation that every acyclic digraph can be majority 2-colored. The relevant property of an acyclic digraph is that there is an ordering of its vertices, in which every vertex is preceded by its complete out-neighborhood. Then coloring vertices along this ordering with two colors such that each vertex is assigned the color that appears least frequently in the (already colored) out-neighborhood will produce a majority 2-coloring.

It is easy to construct digraphs which require three colors for a majority coloring. The canonical examples are the odd directed cycles $\vec{C}_{2k+1}, k \geq 1$, which are not majority 2-colorable since for digraphs with maximum out-degree 1 majority-coloring and proper graph coloring of the underlying graph are equivalent. However, no example of a digraph is known that requires the use of four colors. Kreutzer et al. conjectured that there are none.

Conjecture 1 ([KOS+17])**.** *Every digraph is majority* 3*-colorable.*

Kreutzer et al. [KOS+17] also provide ample evidence for their conjecture by establishing that it holds for "most" digraphs. They show, using the Lovász Local Lemma, that the uniform random 3-coloring is a majority 3-coloring with non-zero probability if certain local density conditions hold, namely if

- $\delta^+(D) > 72 \ln(3|V(D)|)$, or
- $\delta^+(D) \geq 1200$ and $\Delta^-(D) \leq \frac{\exp(\delta^+(D)/72)}{12\delta^+(D)}$.

In [KOS+17] it is also mentioned at the end that a more careful analysis of the Local Lemma approach works for r-regular digraphs provided $r \geq 144$. Subsequently Girão, Kittipassorn, and Popielarz [GaKP17] studied tournaments in particular, and showed, also using the probabilistic method, that every tournament with minimum out-degree at least 55 is majority 3-colorable.

These are all the results we are aware of about Conjecture 1. All the proofs use the Local Lemma for a random coloring and hence require some upper bound on the maximum in-degree in terms of the minimum out-degree (in order to control the number of "bad" events that are adjacent to any fixed bad event in some dependency graph of the events). As it is the case in many related open problems on splitting/coloring digraphs with large minimum out-degree [Alo06, YBWW18, AL89, BHL+17], large maximum in-degrees seem to be outside the realm of any such probabilistic approach and it looks like it constitutes the main difficulty of the problem. This is also illustrated by the fact that it was not even known whether planar digraphs are majority 3-colorable.

In this paper our main motivation is to complement the existing results on digraphs with balanced in- and out-degrees, and provide approaches for natural, broad families of digraphs, without any restriction on the maximum in-degree.

1.1 Our results

1.1.1 Majority 3-Colorability

Since a proper coloring is also a majority coloring, Conjecture 1 is immediately true for digraphs with chromatic number at most three. For 4-chromatic digraphs this is already not obvious.

Our first result resolves the conjecture for digraphs with low chromatic number, including planar digraphs (the chromatic number of a digraph here simply denotes the chromatic number of its underlying undirected graph).

Theorem 1. *Let D be a digraph such that $\chi(D) \leq 6$. Then D is majority 3-colorable.*

The most commonly used digraph coloring concept which captures also the orientation of edges, is the dichromatic number. For a digraph D, its *dichromatic number* $\vec{\chi}(D)$ is defined as the smallest integer k such that there exists a k-coloring $f : V(D) \to [k]$ such that there is no monochromatic cycle, that is, each color class $f^{-1}(i)$ is acyclic.[1] This parameter was first introduced in 1982 by Victor Neumann-Lara [NL82][2] and grew in importance ever since.

In the introduction above we mentioned how to give a majority 2-coloring of acyclic digraphs, i.e. digraphs with dichromatic number 1. In our second main result we prove Conjecture 1 for digraphs with dichromatic number at most three.

Theorem 2. *Let D be a digraph such that $\vec{\chi}(D) \leq 3$. Then D is majority 3-colorable.*

1.1.2 The Proofs and Majority List Coloring

For our proofs it will be crucial to work in a more general framework, involving the list coloring version of majority coloring. This allows us to formulate appropriately loaded inductive statements from which our theorems follow.

The notion of majority choosability of digraphs was first proposed in [KOS+17]. For an assignment $L : V(D) \to 2^{\mathbb{N}}$ of subsets $L(v) \subseteq \mathbb{N}$ of colors to each vertex $v \in V(D)$, we call a coloring $f : V(D) \to \mathbb{N}$ an L-coloring if $f(v) \in L(v)$ for every $v \in V(D)$. When $L(v) = [k]$, then L-coloring and k-coloring coincide. We call a digraph *majority k-choosable* if for every k-list assignment (i.e., assignment L with $|L(v)| = k$ for every $v \in V(D)$) there is a majority L-coloring.

It was noted in [KOS+17] that all the results about dense digraphs using the Local Lemma remain valid for majority 3-choosability (instead of majority 3-colorability). Moreover, Anholcer, Bosek, and Grytczuk [ABG17] gave a beautiful proof to show that every digraph is majority 4-choosable (not only majority 4-colorable).

The following theorem is at the heart of all of our proofs and is interesting in its own right.

Theorem 3. *Let D be a digraph and for each $v \in V(D)$ let $L(v)$ be a list of two colors. Suppose that there exists no odd directed cycle in D all whose vertices are assigned the same list. Then there is a majority-coloring c of D such that $c(v) \in L(v)$ for all $v \in V(D)$.*

This statement has several nice consequences, some immediate, some less so. We collect these in the next subsection.

1.1.3 Consequences for majority 3- and 2-colorings

We start by stating choosability analogues of our first two theorems. The analogue of Theorem 1 connects the choosability of the underlying graph to majority choosability.

Theorem 4. *Let D be a digraph whose underlying undirected graph is 6-choosable. Then D is majority 3-choosable. In particular any digraph with a 5-degenerate underlying graph is majority 3-choosable.*

[1] A coloring with no monochromatic cycle is called *acyclic coloring.*

[2] Some authors simply refer to $\vec{\chi}(D)$ as the chromatic number of the digraph D.

The *list dichromatic number* $\vec{\chi}_\ell(D)$ of a digraph D was introduced by Bensmail, Harutyunyan, and Le [BHL18]. It is defined as the minimum integer $k \geq 1$ such that for any k-list assignment, we can choose colors from the respective lists without producing monochromatic directed cycles. We have the following analogue of Theorem 2 involving this parameter.

Theorem 5. *Let D be a digraph with $\vec{\chi}_\ell(D) \leq 3$. Then D is majority 3-choosable.*

The results of [KOS$^+$17] and [GaKP17] cited in the introduction indicate that the case of r-regular digraphs for constant r constitute an important benchmark in the study of Conjecture 1. Recall in particular that the Local Lemma approach works for r-regular digraphs provided $r \geq 144$. Next we obtain conditions at the other end of the local density spectrum, which imply that r-regular digraphs are majority 3-colorable for $r \leq 4$.

Note first the crucial non-monotonicity in the problem: even though we do not know whether Conjecture 1 is true for $r = 143$, it does hold (quite easily) for $r = 1$ and 2. Indeed, a 1-regular digraph is the disjoint union of directed cycles, and hence we can 3-color it properly to obtain a majority-coloring. Then Conjecture 1 also follows for 2-regular digraphs. Even more generally, the validity of the conjecture for any odd regularity $r-1$ implies it for the next even regularity r. This is the consequence of the fact[3] that for even r any r-regular digraph D contains a 1-regular spanning subgraph F and any 3-majority coloring of the $(r-1)$-regular digraph $D - F$ is also a majority coloring of D. Most generally, if a digraph D' is obtained from a digraph D by adding an edge (u, v) whose tail has odd out-degree $d^+_D(u)$ then a majority coloring of D is also a majority coloring of D'.

From our next result it follows that 3- and 4-regular digraphs are majority 3-choosable and hence Conjecture 1 holds for them as well.

Theorem 6. *If $\Delta^+(D) \leq 4$ or $\Delta(U(D)) \leq 6$ or $\Delta(D) \leq 7$, then D is majority 3-choosable.*

An open question posed in [KOS$^+$17] asked whether there is a characterisation of digraphs that have a majority 2-coloring (or a polynomial time algorithm to recognise such digraphs). This was answered (most likely) in the negative by Bang-Jensen, Bessy, Havet, and Yeo [BJBHY18] who showed that deciding whether a 3-out-regular digraph is majority 2-colorable is NP-complete. With no hope for an efficient characterization of majority 2-colorability, any simple sufficient condition comes in handy.

For a condition, it is natural to exclude odd directed cycles, as they are canonical examples of graphs with no majority 2-coloring. It turns out that excluding them already implies 2-choosability.

Theorem 7. *If D is a digraph without odd directed cycles, then D is majority 2-choosable.*

1.1.4 Fractional Majority Colorings

The concept of fractional majority coloring emerges as the natural LP-relaxation of the problem of majority coloring, much in the same way as the usual fractional colorings of graphs. This notion was first introduced in [KOS$^+$17]. The definition is somewhat technical and we postpone it to Section 4. To appreciate our results here, it is sufficient to keep in mind that the minimum total weight of a fractional majority coloring is at most the majority chromatic number.

Kreutzer et al [KOS$^+$17] ask what is the smallest constant K such that every digraph admits a fractional majority coloring with total weight at most K. This is yet another direction

[3] To see this, consider the undirected bipartite graph obtained from D by splitting every vertex v into two vertices v^+, v^- and adding an edge u^+v^- for every arc $(u, v) \in A(D)$. Then this bipartite graph is r-regular and hence has a perfect matching, which yields a 1-regular spanning subdigraph of D.

to approach Conjecture 1 from. Proving that there is a fractional majority coloring with total weight 3 for every digraph would certainly be an easier task. Here we take the first step in this direction and show that the upper bound of 4, which follows from the fact that every digraph is majority 4-colorable, can be slightly improved.

Theorem 8. *Every digraph D admits a fractional majority coloring with total weight at most* 3.9602*.*

Our proof is the combination of an intricate probabilistic coloring with some deterministic alteration.

In the second theorem of the section we show that digraphs with sufficiently large minimum out-degree have fractional majority colorings with total weight arbitrarily close to 2. This improves the corresponding result in [KOS$^+$17] obtained using the Local Lemma, as the upper bound on the maximum in-degree is not necessary here.

Theorem 9. *There exists a constant $C > 0$ such that for every $\varepsilon > 0$ and every digraph D with $\delta^+(D) \geq C(1/\varepsilon)^2 \ln(2/\varepsilon)$, there exists a fractional majority coloring of D with total weight at most $2 + \varepsilon$.*

Organization of the paper. In Section 2 we obtain Theorem 7 as a consequence of a more general result (Theorem 3). This result is crucial for the proofs of Theorems 1, 2, 6, 4, 5, which are presented in Section 3. In Section 4 we treat fractional majority colorings and prove Theorems 8 and 9. We conclude with final remarks and some open problems in Section 5.

2 Digraphs without Odd Directed Cycles

We have seen that acyclic digraphs as well as bipartite digraphs are majority 2-colorable. We have also seen that odd directed cycles are canonical examples of digraphs having no majority 2-coloring. It is therefore natural to try unifying these results and ask whether every digraph without an odd directed cycle is majority 2-colorable. In this section, we answer this question positively. We start with a simple observation:

Lemma 10. *A digraph D contains no odd directed cycles if and only if all its strong components are bipartite.*

Proof Sketch. The sufficiency of this condition is obvious, as a directed cycle is always contained in a single strong component. For the reverse direction, it suffices to observe that if D is strongly connected and all directed cycles have even length, then D is bipartite. However, this statement can be easily verified by considering an ear decomposition of D. □

Proposition 1. *Let D be a digraph which contains no odd directed cycles. Then D is majority 2-colorable. Moreover, any given pre-coloring of the sinks of D can be extended to a majority 2-coloring of D.*

Proof. We prove the statement by induction on the number $s \geq 1$ of strong components of D. Suppose first that $s = 1$, i.e. D is strongly connected. Then by Lemma 10 D is bipartite and therefore majority 2-colorable. Since D is either a single vertex or contains no sinks, the claim follows.

Now let $s \geq 2$ and suppose that the statement holds true for all digraphs with at most $s-1$ strong components. We now distinguish two cases: Either, D is an independent set of s vertices, and therefore, the claim holds trivially true. If there exists at least one arc in D, there has to be a strong component of D containing no sinks such that there are no arcs entering the component. Let X be the vertex set of this component.

Now let a pre-coloring of the sinks of D with $1, 2$ be given. By the choice of X, $D - X$ has the same set of sinks as D and $s-1$ strong components. By the inductive assumption, there exists a majority 2-coloring $c : V(D) \setminus X \to \{1, 2\}$ of $D - X$ which extends the pre-coloring of the sinks. By Lemma 10, there exists a bipartition $\{A, B\}$ of $D[X]$.

For any subset $W \subseteq X$ equipped with a vertex-coloring $c_W : V(D) \setminus W \to \{1, 2\}$ of $D - W$, any vertex $x \in W$, and any $i \in \{1, 2\}$, denote by $d(c_W, i, x)$ the number of out-neighbors of x which lie in $V(D) \setminus W$ and have color i under c_W.

We now claim that there exists a subset $U \subseteq X$ and a $1, 2$-coloring c_U of $D - U$ which extends c, such that

- Every vertex $x \in V(D) \setminus U$ has at least $\frac{d^+(x)}{2}$ out-neighbors in $V(D) \setminus U$ with a color different from $c_U(x)$.
- Every vertex $x \in U$ fulfills $\max\{d(c_U, 1, x), d(c_U, 2, x)\} < \frac{1}{2}d^+(x)$.

In order to find such a set, we apply the following procedure:

We keep track of a pair (W, c_W), consisting of a subset $W \subseteq X$ and a vertex-coloring $c_W : V(D) \setminus W \to \{1, 2\}$ extending c. As an invariant we will keep the first of the two above properties, i.e. we assert that every vertex $x \in V(D) \setminus W$ has at least $\frac{d^+(x)}{2}$ out-neighbors with a different color according to c_W.

We initialize $W := X$, $c_W := c$. It is clear that this assignment satifies the invariant (remember that c is a majority coloring of $D - X$, and that there are no edges entering X).

As long as a vertex $x_0 \in W$ with $\max\{d(c_U, 1, x_0), d(c_U, 2, x_0)\} \geq \frac{1}{2}d^+(x_0)$ exists, we choose such a vertex. We put $W' := W \setminus \{x_0\}$, and define a coloring $c_{W'}$ of $D - W'$ according to

$$c_{W'}(x) := \begin{cases} c_W(x), & \text{if } x \neq x_0 \\ 1, & \text{if } x = x_0, d(c_W, 1, x_0) < d(c_W, 2, x_0) \\ 2, & \text{if } x = x_0, d(c_W, 1, x_0) \geq d(c_W, 2, x_0) \end{cases}.$$

It is easily verified that the coloring $c_{W'}$ also fulfills the invariant, since by definition x_0 has at least $\max\{d(c_U, 1, x_0), d(c_U, 2, x_0)\} \geq \frac{1}{2}d^+(x_0)$ out-neighbors in $D - W'$ of different color.

Finally we update according to $(W, c_W) := (W', c_{W'})$.

In the moment the procedure terminates, we have found a subset $U := W \subseteq X$ and a $1, 2$-coloring c_U of $D - U$ extending c with the property that every vertex $x \in V(D) \setminus U$ has at least $\frac{d^+(x)}{2}$ out-neighbors with different color according to c_U. Since the procedure terminated, we furthermore have $\max\{d(c_U, 1, x), d(c_U, 2, x)\} < \frac{1}{2}d^+(x)$ for every vertex $x \in U$. This shows that U satisfies both of the conditions stated above.

We now finally extend the coloring c_U of $V(D) \setminus U$ to a $1, 2$-coloring of D by giving color 1 to each vertex in $A \cap U$ and color 2 to every vertex in $B \cap U$. This coloring extends c and therefore the initial pre-coloring of the sinks, and is a majority coloring: By the first of the two conditions, every vertex $x \in V(D) \setminus U$ has at least $\frac{d^+(x)}{2}$ out-neighbors with a different color. For each vertex $x \in U$, since $\{A, B\}$ is a bipartition of $D[X]$, all out-neighbors in U have a different color, and among the out-neighbors in $D - U$, at most $\max\{d(c_U, 1, x), d(c_U, 2, x)\} < \frac{1}{2}d^+(x)$ can share its color. Therefore every vertex satifies the condition for a majority-coloring, and this concludes the proof of the claim. $\square$

We are now ready for the proof of Theorem 3.

Proof of Theorem 3. We may assume w.l.o.g. that color lists of adjacent vertices always intersect: Otherwise, we remove all edges between vertices with disjoint color lists to obtain a digraph D'. Any majority-coloring of D' with colors chosen from the lists will also be a majority-coloring of D.

Now consider an arbitrary pair $\{a,b\}$ of colors and let $X_{\{a,b\}} := \{x \in V(D) | L(x) = \{a,b\}\}$. By assumption $D[X_{\{a,b\}}]$ contains no odd directed cycles. Let $D'_{\{a,b\}}$ be the digraph obtained from $D[X_{\{a,b\}}]$ by adding all arcs $(x,y) \in E(D)$ with $x \in X_{\{a,b\}}$ and $y \notin X_{\{a,b\}}$ and their endpoints. Since we only add sinks to $D[X_{\{a,b\}}]$, also $D'_{\{a,b\}}$ contains no odd directed cycles. For each vertex $y \in N^+(X_{\{a,b\}}) \setminus X_{\{a,b\}}$, there is a unique color $p_{\{a,b\}}(y)$ in $L(y) \cap \{a,b\}$. Pre-color the sinks of $D'_{\{a,b\}}$ in such a way that every vertex $y \in N^+(X_{\{a,b\}}) \setminus X_{\{a,b\}}$ receives color $p_{\{a,b\}}(y)$. By Proposition 1 we can now find a majority-coloring $c_{\{a,b\}}$ of $D'_{\{a,b\}}$ extending this pre-coloring with colors a and b.

Now define a coloring c of all vertices in D by setting $c(x) := c_{\{a,b\}}(x)$ if $L(x) = \{a,b\}$. Clearly, we have $c(x) \in L(x)$ for all $x \in V(D)$. We claim that c is a majority-coloring of D. Indeed, for any vertex $x \in V(D)$, if $L(x) = \{a,b\}$, then we have $N^+(x) = N^+_{D'_{\{a,b\}}}(x)$, and $\{y \in N^+(x) \mid c(y) = c(x)\} \subseteq \{y \in N^+_{D'_{\{a,b\}}}(x) \mid c_{\{a,b\}}(x) = c_{\{a,b\}}(y)\}$. Hence, at most half of the out-neighbors of x share its color, and the claim follows. $\square$

Theorem 7 is now obtained from Theorem 3 as a direct consequence.

3 Majority 3-Colorings of Sparse Digraphs

As a consequence of Theorem 3, we obtain our main result:

Theorem 11. *Let D be a digraph. Suppose there is a partition $\{X_1, X_2, X_3\}$ of the vertex set such that for every $i \in \{1,2,3\}$, $D[X_i]$ contains no odd directed cycles. Then D is majority 3-colorable.*

Proof. We assign lists of size two to the vertices of D, namely, we assign the list $\{2,3\}$ to all vertices in X_1, the list $\{1,3\}$ to all vertices in X_2, and the list $\{1,2\}$ to all vertices in X_3. Because $D[X_i], i = 1,2,3$ contains no odd directed cycle, we can apply Theorem 3 to conclude that there exists a majority-coloring of D which uses only colors $1,2$ and 3. This proves the claim. $\square$

From this we now directly derive Theorems 1 and 2.

Proof of Theorem 1. If $\chi(D) \leq 6$, then D admits a partition $Y_1, \ldots, Y_6$ into independent sets. Using the partition $\{Y_1 \cup Y_2, Y_3 \cup Y_4, Y_5 \cup Y_6\}$ of the vertex set to apply Theorem 11 now shows that D is indeed majority 3-colorable. $\square$

Proof of Theorem 2. If $\vec{\chi}(D) \leq 3$, then there exists a partition $\{X_1, X_2, X_3\}$ of the vertex set such that $D[X_i]$ contains no directed cycles, for $i = 1,2,3$. The claim now follows by Theorem 11. $\square$

The fact that Theorem 3 deals with an assignment of lists can be further exploited to show analogues of Theorem 11, Theorems 1 and 2 for list colorings.

For this purpose we need the following notion: Call a digraph D *OD-3-choosable* if for any assignment of color lists $L(x)$, $x \in V(D)$ of size 3 to the vertices, there exists a choice function c (i.e. $c(x) \in L(x)$ for all $x \in V(D)$) such that no odd directed cycle in (D,c) is monochromatic.

Theorem 12. *Let D be a digraph. If D is OD-3-choosable, then D is majority 3-choosable.*

Proof. Let $L(v)$ for all $v \in V(D)$ be a given color list of size three. We have to show that there is a majority-coloring c of D such that $c(v) \in L(v)$ for all $v \in V(D)$. For every $v \in V(D)$, let $L^*(v) := \{\{C_1, C_2\} | C_1 \neq C_2 \in L(v)\}$ contain all three unordered color-pairs in $L(v)$. Since D is OD-3-choosable, there exists a choice function c^* on $V(D)$ such that $c^*(v) \in L^*(v)$ for each vertex $v \in V(D)$ is a subset of $L(v)$ of size two and such that there exists no odd directed cycle in D which is monochromatic with respect to c^*. If we now consider $c^*(v), v \in V(D)$ as a 2-list assignment of D, we can apply Theorem 3 to conclude that there is a majority-coloring c of D such that $c(v) \in c^*(v) \subseteq L(v)$ for every vertex $v \in V(D)$. As $L(\cdot)$ was arbitrary, we conclude that D is majority 3-choosable. $\square$

We are now ready to prove Theorems 4 and 5.

Proof of Theorem 4. We show that D is OD-3-choosable, the claim then follows by Theorem 12. Let $L(v)$ for each vertex $v \in V(D)$ be an assigned list of three colors. For each color C used in one of the lists, let C' be a distinct copy of this color. We now consider the assignment $L_6(\cdot)$ of lists of size 6 to the vertices of D, where for each vertex $v \in V(D)$, $L_6(v) := \{C_1, C_1', C_2, C_2', C_3, C_3'\}$ if C_1, C_2, C_3 denote the colors contained in $L(v)$. Because the underlying graph of D is 6-choosable, there is a proper coloring c_6 of D such that $c_6(v) \in L_6(v)$ for all $v \in V(D)$. Now consider the coloring c of D obtained from c_6 by identifying each copy C' of an original color C with C again. We then have $c(v) \in L(v)$ for every $v \in V(D)$. Since c_6 was a proper coloring of the undirected underlying graph of D, each color class with respect to c induces a bipartite subdigraph of D, and hence there are no monochromatic odd directed cycles in (D, c). Hence, D is OD-3-choosable. $\square$

Proof of Theorem 5. This follows directly since any digraph with $\vec{\chi}_\ell(D) \leq 3$ is clearly OD-3-choosable. $\square$

The rest of this section is devoted to proving Theorem 6. The proof uses the following Lemma, which in turn uses Theorems 4 and 5.

Lemma 13. *Let D be a digraph such that $\min\{d^+(x), d^-(x)+1\} \leq 3$ for every $x \in V(D)$. Then D is OD-3-choosable.*

Proof. Suppose the claim was false and consider a counterexample D minimizing $|V(D)|+|E(D)|$. We have $|V(D)| \geq 4$, D is connected and every proper subdigraph of D must be OD-3-choosable.

We first consider the case that there is a vertex v with $d^-(v) \leq 2$. Since $D - v$ is OD-3-choosable, given any assignment $L(v), v \in V(D)$ of lists of size at least 3 to the vertices, we can choose colors $c(w) \in L(w)$ from the lists for every $w \in V(D) \setminus \{v\}$ such that in $D - v$, there exists no monochromatic odd directed cycle. Now assign to v a color $c(v) \in L(v) \setminus \{c(w) \mid w \in N^-(v)\}$. We claim that c is a coloring of D without monochromatic odd directed cycles. In fact, such a cycle would have to pass v, however no edge entering v is monochromatic. Therefore D is OD-3-choosable, a contradiction.

Hence we know for every $x \in V(D)$ that $d^-(x) \geq 3$. Since $\min\{d^+(x), d^-(x) + 1\} \leq 3$, we also must have $d^+(x) \leq 3$. We conclude

$$3|V(D)| \leq \sum_{v \in V(D)} d^-(x) = \sum_{v \in V(D)} d^+(x) \leq 3|V(D)|$$

and thus we have $d^+(x) = d^-(x) = 3$ for all $x \in V(D)$. Consequently, the underlying simple graph $U(D)$ has maximum degree $\Delta(U(D)) \leq 6$. If $U(D)$ is 6-choosable, then it follows as in the proof of Theorem 4 that D is OD-3-choosable, a contradiction.

Therefore, by the list coloring version of Brooks' Theorem [Viz76], we must have $U(D) = K_7$. Since D is 3-out- and 3-in-regular, it follows that D is a tournament on 7 vertices. However, every tournament on 7 vertices has list dichromatic number at most 3 and is therefore OD-3-choosable according to Theorem 5. This can be seen using two results from [BHL18]. Clearly, we have $\vec{\chi}(D) \leq 3$. Now if $\vec{\chi}(D) = 3$, then we have $|V(D)| = 7 \leq 2\vec{\chi}(D) + 1$ and by Theorem 2.2 in [BHL18], we conclude that $\vec{\chi}_\ell(D) = \vec{\chi}(D) = 3$. Otherwise, we have $\vec{\chi}(D) \leq 2$. In this case, we can apply Theorem 3.3 in [BHL18] to conclude $\vec{\chi}_\ell(D) \leq 2\ln(7) < 4$. Therefore we have $\vec{\chi}_\ell(D) \leq 3$ in each case.

Finally, since we obtained that D is OD-3-choosable in each case, the initial assumption was wrong, which concludes the proof by contradiction. □

Corollary 14. *Let D be a digraph with $\min\{d^+(x), d^-(x) + 2\} \leq 4$ for every $x \in V(D)$. Then D is majority 3-choosable.*

Proof. For a proof by contradiction, suppose the claim was false and consider a counterexample D minimizing the number of edges.

Consider first the case that there is a $v \in V(D)$ with $d^+(v) = 4$. Let e be an edge leaving v and put $D' := D - e$. By the minimality of D, D' is majority 3-choosable. We now claim that any majority-coloring of D' also defines a majority-coloring of D. Clearly, such a coloring satisfies the condition for a majority-coloring at any vertex distinct from v. Since v has out-degree 3 in D', it has at most one out-neighbor in D' of the same color. Thus there are at most two out-neighbors of v in D which share its color, and so the majority condition is fulfilled at v. We conclude that also D must be majority 3-choosable, which gives the desired contradiction.

Now for the second case, assume that no vertex has out-degree 4. This means that for every $x \in V(D)$, we either have $d^+(x) \leq 3$ or $d^+(x) \geq 5$ and therefore $d^-(x) \leq 2$. We can therefore apply Lemma 13 to D, which shows that D is OD-3-choosable. From Theorem 12 we get that D is majority 3-choosable. This again is a contradiction to D being a counterexample to the claim.

Therefore the initial assumption was wrong, and this concludes the proof. □

Proof of Theorem 6. If $\Delta^+(D) \leq 4$ or $\Delta(D) \leq 7$, then the claim follows by applying Corollary 14. If $\Delta(U(D)) \leq 6$, then by the list coloring version of Brook's Theorem either $U(D)$ is 6-choosable, and then the claim follows from Theorem 4, or $U(D) = K_7$.

Now let $L(v_1), \ldots, L(v_7)$ be lists of size three assigned to the vertices $\{v_1, \ldots, v_7\}$ of D. We first consider the case that all lists are equal, i.e., show that D is majority 3-colorable.

If there exists a vertex $v \in V(D)$ which is contained in at most 3 digons, then there are vertices $u_1 \neq u_2 \in V(D) \setminus \{v\}$ such that u_1, u_2, v do not form a directed triangle. Therefore, any partition $\{X_1, X_2, X_3\}$ of $V(D)$ where $X_1 = \{v, u_1, u_2\}$ and $|X_2| = |X_3|$ shows, by Theorem 11, that D is majority 3-colorable. Otherwise, every vertex in D is contained in at least 4 digons and thus has out-degree at least 4. Now any 3-coloring of D with color classes of sizes $2, 2, 3$ defines a majority-coloring of D.

Now suppose that not all lists are equal. In this case we can choose for each vertex v_i a sublist $L_2(v_i) \subseteq L(v_i)$ of size two such that no three vertices are assigned the same sublist (minimize the number of edges whose ends are assigned the same sublist). By Theorem 3 we obtain a majority-coloring c of D where $c(v_i)$ is contained in $L_2(v_i) \subseteq L(v_i)$. Hence, D is majority 3-choosable in each case, which concludes the proof. □

4 Fractional Majority Colorings

Another concept introduced in [KOS+17] is that of a fractional majority coloring. Given a subset $S \subseteq V(D)$, a vertex v is *popular in S* if $v \in S$ and more than half of its out-neighbors are in S. A subset $S \subseteq V(D)$ is *stable* if it contains no popular vertices. Let $S(D)$ be the set of all stable sets of D, and $S(D, v)$ the set of all stable sets containing v. A *fractional majority coloring* is a function that assigns a weight $w_T \geq 0$ to every set $T \in S(D)$, satisfying $\sum_{T \in S(D,v)} w_T \geq 1$ for every $v \in V(D)$. The total weight of a fractional majority coloring is simply $\sum_{T \in S(D)} w_T$. Kreutzer et al. asked for the minimum constant K such that every digraph admits a fractional majority coloring with total weight at most K.

We will show two results related to this question, namely Theorem 8 and Theorem 9. The proof of these two theorems will be based on the dual of the linear program defined by the restrictions on a fractional majority coloring:

Observation 1. *For a digraph D, the minimum possible total weight of a fractional majority coloring equals the maximum total weight $\sum_{v \in V(D)} w_v$ in a non-negative weight assignment of $V(D)$ in which every stable set T satisfies $\sum_{v \in T} w_v \leq 1$.*

The main idea of the proof of both theorems is that, given any choice of weights on $V(D)$, we can construct a stable set in which the weight is at least a given fraction of the total weight, using the probabilistic method.

Lemma 15. *Let D be a digraph and let $0 < p < 1$. Suppose that one can take a random subset $X \subseteq V(D)$ with the property that, for every $v \in V(D)$, the probability that v is in X but not popular in X is at least p. Then D admits a fractional majority coloring with total weight at most $\frac{1}{p}$.*

Proof. Suppose that D is a counterexample to our statement, and we will reach a contradiction. By Observation 1, we can assign weights to $V(D)$ so that the total weight is $w > \frac{1}{p}$, and every stable set in D has a sum of weights at most one. Let Y be the set of popular vertices in X. By linearity of expectation, the expected total weight of $X \setminus Y$ is at least $pw > 1$.

Take an instance of $X \setminus Y$ with weight greater than 1. Every vertex in $X \setminus Y$ has at least half of its out-neighbors outside of X, which implies that it is not popular in $X \setminus Y$. Hence $X \setminus Y$ is stable in D and has total weight greater than 1, producing a contradiction. $\square$

The proof of Theorem 9 is a straightforward application of this lemma:

Proof of Theorem 9. Let $\varepsilon > 0$, let $C > 0$ be a sufficiently large absolute constant and let $N \geq C(1/\varepsilon)^2 \ln(2/\varepsilon)$ be an integer. Let D be a digraph with $\delta^+(D) \geq N$. Our goal is to show that D admits a fractional coloring with total weight at most $2 + \varepsilon$. We may w.l.o.g. assume that $\varepsilon < 1.9602$, as otherwise the claim follows by applying Theorem 8, which is given below. This ensures that $\ln(2/\varepsilon)$ is lower-bounded by a positive constant. Set $p = \frac{1}{2} - \sqrt{\frac{\ln N}{N}}$. Let X be a random subset of $V(D)$ in which every element is included independently with probability p. Hoeffding's inequality states that for a binomial random variable $B(n,p)$ with n trials and success probability p, and for any $\varepsilon > 0$, it holds that $\Pr(B(n,p) \geq (p+\varepsilon)n) \leq \exp(-2\varepsilon^2 n)$. For every fixed vertex $v \in V(D)$, the random variable $|X \cap N^+(v)|$ counting the number of out-neighbors of v contained in X is distributed binomially with $n = d^+(v)$ and probability p. Hence, putting $\varepsilon = \frac{1}{2} - p = \sqrt{\frac{\ln N}{N}}$ we find that for any vertex v the probability that at least half of its out-neighbors are in X is at most

$$\Pr\left(|X \cap N^+(v)| \geq \frac{1}{2} d^+(v)\right) \leq e^{-2\left(\frac{1}{2}-p\right)^2 d^+(v)} \leq e^{-2\ln N} = N^{-2}$$

Setting $q = N^{-2}$, from Lemma 15 we find a fractional majority coloring of total weight at most $\frac{1}{p-q} \leq 2 + 2\sqrt{\frac{\ln N}{N}}$. For C chosen sufficiently large, we now obtain:

$$2\sqrt{\frac{\ln N}{N}} \leq 2\sqrt{\frac{\ln(C(1/\varepsilon)^2 \ln(2/\varepsilon))}{C(1/\varepsilon)^2 \ln(2/\varepsilon)}} = \varepsilon \frac{2}{\sqrt{C}} \sqrt{\frac{2\ln(2/\varepsilon) + \ln(\frac{C}{4}\ln(2/\varepsilon))}{\ln(2/\varepsilon)}}$$

$$= \varepsilon \cdot \underbrace{\frac{2}{\sqrt{C}}\sqrt{2 + \frac{\ln(\frac{C}{4}\ln(2/\varepsilon))}{\ln(2/\varepsilon)}}}_{\ll 1} < \varepsilon.$$

This shows that D admits a fractional majority coloring with weight at most $2+\varepsilon$, as required. $\square$

For Theorem 8, we need to be more careful. Consider again the set X containing each vertex independently with probability p, where p is slightly lower than $\frac{1}{2}$. If the out-degree of v is not 1, one can show that the probability that v is popular in X is upper-bounded by a constant, strictly smaller than $p - \frac{1}{4}$. However, if v has out-degree 1, the probability that v is popular in X is $p^2 > p - \frac{1}{4}$. For this reason, the vertices with out-degree 1 deserve extra consideration.

Observe that, in the graph induced by the vertices of out-degree 1, all cycles are directed, pairwise disjoint and act as sinks. Consequently, removing one vertex from each directed cycle produces an acyclic graph, where the vertices can be given an ordering in which every edge goes from a larger vertex to a smaller one.

Proof of Theorem 8. Set $p_1 = 0.4594$ and $p_2 = 0.4503$. Assign independently to each vertex v a random indicating variable X_v, which takes the value 1 with probability p_1 if $d^+(v) = 1$ and with probability p_2 otherwise. Now construct the random subset X as follows:

- Add to X all vertices v with $d^+(v) \neq 1$ and $X_v = 1$.
- For every cycle C formed by vertices with $d^+(v) = 1$ and $X_v = 1$, select a vertex $v \in C$ uniformly at random and set $X_v = 0$.
- Take an ordering of the vertices v with $d^+(v) = 1$ and $X_v = 1$, in which if we have an edge (v, w) then v comes after w (this is possible because these vertices form an acyclic digraph). Following this order, add v to X if its out-neighbor is not in X.

We will show that, for every vertex v, the probability that v is in X but not popular in X is at least $\frac{1}{4} + \varepsilon$, for a fixed value of $\varepsilon > 0$. Suppose first that $d^+(v) = 1$. If we draw the vertices with out-degree 1 in red and those with other out-degrees in blue, then the successive out-neighborhoods of v must have one of these forms:

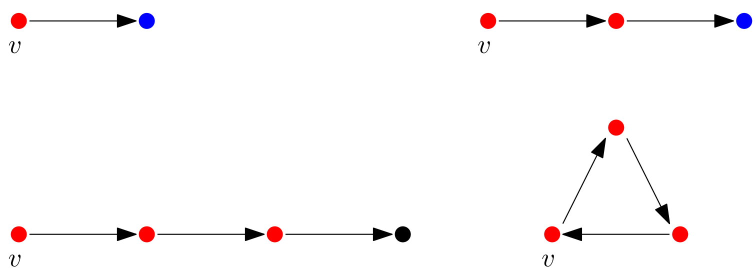

Figure 1: The four possible out-neighborhoods of a red vertex. The black vertex here can be either red or blue.

We label the cases as Case 1 through Case 4, left to right and top to bottom in Figure 1. We denote $v = v_0$, and v_{i+1} as the out-neighbor of v_i, if it is unique. We go through each case:

- If v is in Case 1, then whenever $X_v = 1$ and $X_{v_1} = 0$ we have $v \in X$. This happens with probability $p_1(1 - p_2)$.

- If v is in Case 2, then whenever $X_v = 1$ and $X_{v_1} = 0$, or whenever X_v, X_{v_1} and X_{v_2} all equal 1, we have $v \in X$. This happens with probability $p_1(1-p_1) + p_1^2 p_2$.
- If v is in Case 3, then whenever $X_v = 1$ and $X_{v_1} = 0$, or whenever $X_v = 1$, $X_{v_1} = 1$, $X_{v_2} = 1$ and $X_{v_3} = 0$ we have $v \in X$. This happens with probability at least $p_1(1-p_1)+p_1^3(1-p_1)$.
- If v is in Case 4, if $X_v = 1$ and $X_{v_1} = 0$, or if X_v, X_{v_1} and X_{v_2} all initially equal 1 and X_{v_1} is selected to be modified, then we have $v \in X$. This happens with probability $p_1(1-p_1) + \frac{1}{3}p_1^3$.

Suppose now that $d^+(v) \neq 1$. The probability that $v \in X$ is p_2. If v is popular in X, then over half of its out-neighbors w have $X_w = 1$ (this is necessary for $w \in X$). Since the X_w are independent, and each of them takes the value 1 with probability at most p_1, the probability that v is popular on X, conditioned on $v \in X$, is at most $\Pr\left(B(d^+(v), p_1) > \frac{d^+(v)}{2}\right)$. For $d^+(v) = 3$, this probability is $3p_1^2 - 2p_1^3$. We claim that this is the worst case:

Proposition 2. *For every $k \neq 1$,*

$$\Pr\left(B(k, p_1) > \frac{k}{2}\right) \leq \Pr(B(3, p_1) \geq 2).$$

Proof. Consider an infinite sequence $X_1, X_2, \ldots$ of indicating random variables, each taking value 1 independently with probability p_1. Let I_i be the event "among the first i variables more than half take value 1". Then $\Pr(I_k) = \Pr\left(B(k, p_1) > \frac{k}{2}\right)$. Clearly $\Pr(I_0) = 0$. Moreover, if k is even then I_k implies I_{k+1}, so we can restrict ourselves to odd k.

We will prove our statement by induction, by showing that $\Pr(I_{2k+1}) < \Pr(I_{2k-1})$ for $k \geq 2$. Indeed, the event $I_{2k-1} \setminus I_{2k+1}$ is precisely the case in which exactly k of the first $2k-1$ variables take value 1, and $X_{2k} = X_{2k+1} = 0$. Thus $\Pr(I_{2k-1} \setminus I_{2k+1}) = \binom{2k-1}{k} p_1^k (1-p_1)^{k+1}$. Similarly, the event $I_{2k+1} \setminus I_{2k-1}$ is precisely the case in which exactly $k-1$ of the first $2k-1$ variables take value 1, and $X_{2k} = X_{2k+1} = 1$. Thus $\Pr(I_{2k+1} \setminus I_{2k-1}) = \binom{2k-1}{k-1} p_1^{k+1} (1-p_1)^k$. Now $P(I_{2k-1}) - P(I_{2k+1}) = \Pr(I_{2k-1} \setminus I_{2k+1}) - \Pr(I_{2k+1} \setminus I_{2k-1}) = \binom{2k-1}{k} p_1^k (1-p_1)^k (1-2p_1) > 0$. $\square$

With this, we know that for every vertex v the probability that v is in X and not popular in X is at least

$$\begin{aligned}
&\min\{p_1(1-p_2), p_1(1-p_1) + p_1^2 p_2, p_1(1-p_1) + p_1^3(1-p_1), p_1(1-p_1) + \frac{1}{3}p_1^3, p_2(1 - 3p_1^2 + 2p_1^3)\} \\
&= p_2(1 - 3p_1^2 + 2p_1^3) = 0.252513 =: p.
\end{aligned}$$

Applying Lemma 15, there is a fractional majority coloring of D with total weight at most $\frac{1}{p} < 3.9602$. $\square$

5 Conclusive Remarks and Discussion

Girão et al. [GaKP17] and independently Knox and Šámal [KS18] investigated a natural generalization of majority colorings: For any $\alpha \in [0, 1]$, define an *α-majority coloring* of a digraph D to be a vertex-coloring in which for every vertex v, at most $\alpha \cdot d^+(v)$ vertices in $N^+(v)$ have the same color as v. If such a coloring can be found for any ℓ-list-assignment, we call the digraph *α-majority ℓ-choosable.*

Generalizing the result by Anholcer et al. it was proved both in [GaKP17] and [KS18] that for every integer $k \geq 1$, every digraph is $\frac{1}{k}$-majority $2k$-choosable. Girão et al. proposed the following generalization of Conjecture 1:

Conjecture 2. *For every integer $k \geq 1$, every digraph D has a $\frac{1}{k}$-majority $(2k-1)$-coloring. In fact, every digraph is $\frac{1}{k}$-majority $(2k-1)$-choosable.*

It is natural to try and generalize the results presented in this paper for majority colorings with $\alpha = \frac{1}{2}$ to arbitrary values $\alpha \in [0, 1]$. Among our results, we can only generalize a special case of Theorem 2, namely for digraphs of dichromatic number 2, we verify the first part of Conjecture 2 for all $k \geq 1$.

Proposition 3. *Let D be a digraph with $\vec{\chi}(D) \leq 2$. Then for every $k \in \mathbb{N}$, $k \geq 2$, D admits a $\frac{1}{k}$-majority coloring using $2k-1$ colors.*

Proof. Consider first an acyclic digraph F with a pre-coloring of its sinks using colors from $\{1, \dots, k\}$. We claim that such a coloring can always be extended to a $\frac{1}{k}$-majority coloring of F also using colors from $\{1, \dots, k\}$. To find such a coloring, we take a topological ordering $x_1, \dots, x_n$ of the vertices (i.e. $(x_i, x_j) \notin E(D)$ for all $i \leq j$) such that $\{x_1, \dots, x_t\}$ are the pre-colored sinks. Now we color the vertices one by one, starting with x_{t+1}, then x_{t+2} etc. When coloring the vertex x_i with $i > t$, we assign to it a color from $\{1, \dots, k\}$ appearing least frequently among its (already colored) out-neighbors. This procedure eventually yields a k-coloring of F where any vertex has at most a $\frac{1}{k}$-fraction of its out-neighbors with the same color.

Now let $\{X_1, X_2\}$ be a partition of $V(D)$ such that $D[X_1]$, $D[X_2]$ are acyclic. For $i = 1, 2$ let D_i' be the digraph obtained from $D[X_i]$ by adding all arcs in D leaving X_i together with their endpoints. Clearly, also D_1' and D_2' are acyclic. By the above observation, D_i' for $i = 1, 2$ has a majority $\frac{1}{k}$-coloring c_i with k colors in which all sinks receive color 1. After renaming we may suppose that c_1 uses colors from $\{1, 2, \dots, k\}$, while c_2 uses colors from $\{1, k+1, k+2, \dots, 2k-1\}$. We now define a $(2k-1)$-coloring of all vertices in D by putting $c(x) := c_i(x)$ for $x \in X_i$. For any vertex $x \in X_i$, we have that $N^+(x) = N^+_{D_i'}(x)$, and, since all vertices in $V(D_i') \setminus X_i$ received color 1 under c_i, it follows that $\{y \in N^+(x) | c(y) = c(x)\} \subseteq \{y \in N^+_{D_i'}(x) | c_i(y) = c_i(x)\}$. Therefore, and since c_i is a majority $\frac{1}{k}$-coloring of D_i', at most a $\frac{1}{k}$-fraction of vertices in $N^+(x)$ have the same color as x. This shows that c is a coloring as requested and concludes the proof. $\square$

It is worth noting that the above bound is tight. Consider for example the circulant digraph $\vec{C}(2k-1, k)$ which has as vertex set $\mathbb{Z}_{2k-1}$, and where we have an edge (i, j) if and only if $j - i \in \{1, 2, 3 \dots, k-1\}$. It is easy to see that in any majority $\frac{1}{k}$-coloring of D, the $2k-1$ vertices must receive pairwise distinct colors, however, the partition $X_1 = \{0, 1, \dots, k-1\}$, $X_2 = \{k, k+1, \dots, 2k-2\}$ of the vertex set shows that $\vec{\chi}(\vec{C}(2k-1, k)) = 2$.

The methods used in this paper are unlikely to resolve Conjecture 1 for the open cases of 5- and 6-regular digraphs. One possible approach could be via an extension to hypergraphs: Given a 5-regular digraph D, consider the hypergraph $\mathcal{H}(D)$ with vertex set $V(D)$ and whose edges are $\{v\} \cup N^+(v)$, $v \in V(D)$. This hypergraph is 6-regular and 6-uniform. If we could now find a vertex-3-coloring of $\mathcal{H}(D)$ such that no hyperedge contains four vertices of the same color, this coloring would certainly be a majority coloring of D. We are therefore interested in deciding the following question.

Problem 1. *Let H be a 6-regular 6-uniform hypergraph. Is there a 3-coloring of $V(H)$ such that no hyperedge contains four vertices of the same color?*

The setting of k-regular k-uniform hypergraphs could be fruitful, as it is known that these hypergraphs have property B for all $k \geq 4$ (as noted in [Vis03]). We conclude with a small selection of open questions.

- Is every 5-regular digraph $\frac{1}{3}$-majority 5-colorable? We can show that it is possible to color with 5 colors such that in each connected component, at most one vertex violates the majority condition.
- Does every digraph with $\chi(D) \leq 6$ have a $\frac{1}{3}$-majority 5-coloring?
- Does every digraph D with $\vec{\chi}(D) \leq 3$ have a $\frac{1}{k}$-majority $(2k-1)$-coloring for every $k \geq 1$?

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