# Odd independent transversals are odd

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(joint works with Penny Haxell and Gábor Tardos)

## The problem

A graph $G$ is $r$-partite if $V(G) = V_1 \cup \cdots \cup V_r$, with $V_i$ independent.

$\mathcal{G}_r(n)$ = set of $r$-partite graphs with parts of size $n$.

$T \subseteq V(G)$ is called independent transversal of $G \in \mathcal{G}_r(n)$ if

- $T$ spans no edge
- $|T \cap V_i| = 1$ for every $i = 1, \ldots r$.

**Definition** $\Delta(r, n)$ is the smallest integer such that there is a $G \in \mathcal{G}_r(n)$ with maximum degree $\Delta(r, n)$ which has no independent transversal.

**Obvious facts:** $n = \Delta(2, n) \geq \Delta(3, n) \geq \ldots$

**Problem** (Bollobás, Erdős, Szemerédi — 1975)

Determine $\Delta(r, n)$.

Or at least determine $\Delta_r = \lim_{n\to\infty} \frac{\Delta(r,n)}{n}$.

Or at least determine $\mu = \lim_{r\to\infty} \Delta_r$.

## Appearences of independent transversals

- The linear arboricity conjecture (Akiyama, Exoo, Harary) is asymptotically correct (Alon)

- Strong chromatic number (Alon)

- List coloring (Reed's Conjecture, Haxell)

- Vertex partitions with bounded components (Alon, Ding, Oporowski and Vertigan)

## 3-, 4-, and 5-partite graphs

**Theorem.** (Graver, 1975)

$$\Delta(3,n) = n. \quad \Delta_3 = 1.$$

*Proof.*

Consider a vertex $v \in V(G)$ which maximizes $d_{V_i}(x)$ over all $x \in V(G)$ and $i = 1, 2, 3$.

**Construction.** (Alon, 1988)

$$\Delta_4 \leq \frac{2}{3}.$$

**Theorem.** (Jin, 1992; Yuster, 1997)

$$\Delta_4 = \frac{2}{3}.$$

**Theorem.** (Jin, 1992)

$$\Delta_5 = \frac{2}{3}.$$

## Upper bounds

**Construction** (Bollobás, Erdős, Szemerédi,1975)

$$\Delta_r \leq \frac{1}{2} + \frac{1}{r-2}.$$

**Conjecture** (BESz)

$$\Delta_r \to \frac{1}{2}.$$

**A product construction.** (Jin, 1992; Yuster, 1997)
If $r$ is a power of 2, then

$$\Delta_r \leq \frac{r}{2(r-1)} = \frac{1}{2} + \frac{1}{2r-2}.$$

## Lower bounds

**Theorem.** (Bollobás, Erdős, Szemerédi, 1975)

$$\frac{2}{r} \leq \Delta_r$$

**Theorem.** (Alon, 1992)

$$\frac{1}{2e} \leq \Delta_r.$$

*Proof.* Select a vertex uniformly at random from each part and apply the Lovász Local Lemma.

**Theorem.** (Haxell, 2001)

$$\frac{1}{2} \leq \Delta_r.$$

In particular $\Delta_r \to \frac{1}{2}$.

## Lower bound through triangulations

**Theorem.** (Alon, 2002 based on theorem of Aharoni and Haxell, 2001)

$$\frac{r}{2(r-1)} \leq \Delta_r.$$

In particular if $r$ is a power of 2,

$$\Delta_r = \frac{r}{2(r-1)}.$$

**TOOL # 1**

**Sperner's Lemma.** Let $\mathcal{T}$ be a triangulation of the $n$-dimensional simplex $\sigma$. Suppose that the vertices of $\mathcal{T}$ are colored with $n+1$ colors, such that

- The vertices of $\sigma$ receive distinct colors
- The vertices of $\mathcal{T}$ on any face $\tau$ of $\sigma$ are colored with the colors of the vertices of $\tau$.

Then there exists a multicolored $n$-dimensional simplex in $\mathcal{T}$.

**TOOL # 2**

$2k$**-degenerate filling of the simplex.**
(Aharoni-Chudnovsky-Kotlov)
Any triangulation of the boundary of the $k$-dimensional simplex could be filled up (vertex by vertex) into a triangulation of the full simplex, such that each new vertex, when introduced, is connected to at most $2k$ previous ones.

## Demonstration of the method

*Proof of Thm.* Consider the $(r-1)$-dimensional simplex $\sigma$ with vertex set $\{v_1, \ldots, v_r\}$. We construct a labeled triangulation $(\mathcal{T}, l)$ of $\sigma$, such that

- for every vertex $x$ of $\mathcal{T}$ and face $\tau$ of $\sigma$,
  $l(x) \in \cup_{i:v_i \in \tau} V_i$
- for any edge $\{x, y\}$ of $\mathcal{T}$,
  $l(x)$ and $l(y)$ are *not* adjacent in $G$.

Such a labeled triangulation is possible to construct if $\Delta(G) < \frac{rn}{2(r-1)}$. (More precisely if $2(s-1)$ vertices cannot dominate the union $V_{i_1} \cup \cdots \cup V_{i_s}$ of any $s$ parts.)

Apply Sperner's Lemma to this triangulation, where the color of a vertex is defined to be the index of the part where its label lies. $\square$

## Values

$$
\begin{aligned}
\Delta_2 &= 1 \\
\Delta_3 &= 1 \\
\Delta_4 &= \frac{2}{3} \\
\Delta_5 &= \frac{2}{3} \\
\frac{6}{2 \cdot 5} = \frac{3}{5} \leq \Delta_6 &\leq \frac{2}{3} \\
\frac{7}{2 \cdot 6} = \frac{7}{12} \leq \Delta_7 &\leq \frac{2}{3} \\
\Delta_8 &= \frac{4}{7} = \frac{8}{2 \cdot 7} \\
&\vdots \\
\frac{14}{2 \cdot 13} \leq \Delta_{14} &\leq \frac{12}{2 \cdot 11} < \Delta_8 = \frac{4}{7} \\
&\cdot \\
\Delta_{16} &= \frac{8}{15} = \frac{16}{2 \cdot 15}
\end{aligned}
$$

## Construction for even $r$

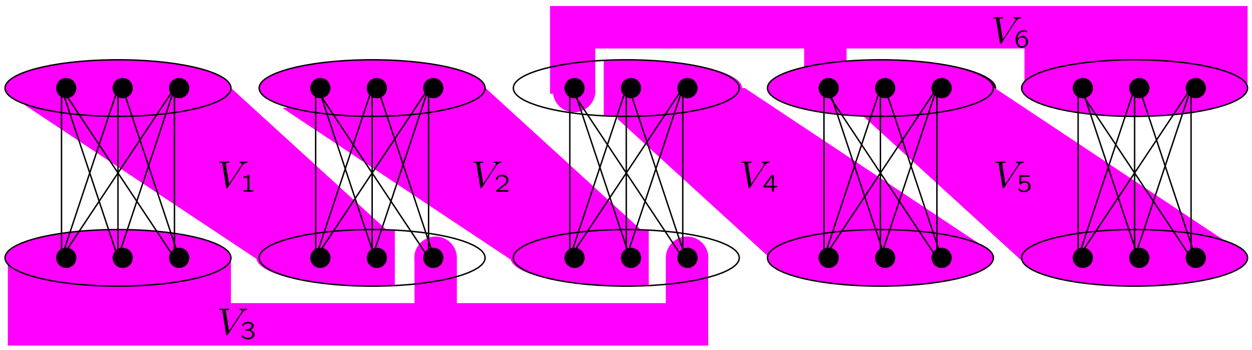
**Theorem.** (Sz, Tardos, 2002) For every $n \geq 1$ and even $r \geq 2$

$$\Delta(r,n) = \left\lceil \frac{rn}{2(r-1)} \right\rceil .$$

In particular

$$\Delta_r = \frac{r}{2(r-1)}.$$

*Example*: $r = 6$, $n = 5k$ and $\Delta = \frac{6 \cdot 5k}{2(6-1)} = 3k$.



# Odd transversals are odd

**Theorem.** (Haxell, Sz 2003) For every $n \geq 1$ and odd $r \geq 7$

$$\Delta(r, n) = \Delta(r-1, n) = \left\lceil \frac{(r-1)n}{2(r-2)} \right\rceil.$$

In particular

$$\Delta_r = \frac{r-1}{2(r-2)}.$$

## “Proof”

**Theorem 1** Let $V(G) = V_1 \cup \ldots \cup V_r$, $|V_i| = n$, and $r \geq 7$ arbitrary integer. Let

$$\Delta(G) < \frac{(r-1)n}{2(r-2)}$$

and suppose that $G$ has NO independent transversal, but $G - e$ has for every $e \in E(G)$.

THEN

$G$ is the disjoint union of $r - 1$ complete bipartite graphs.

**Theorem 2** $V(G) = V_1 \cup \ldots \cup V_r$, $|V_i| = n$, $r$ odd.

$$\Delta(G) < \frac{(r-1)n}{2(r-2)}.$$

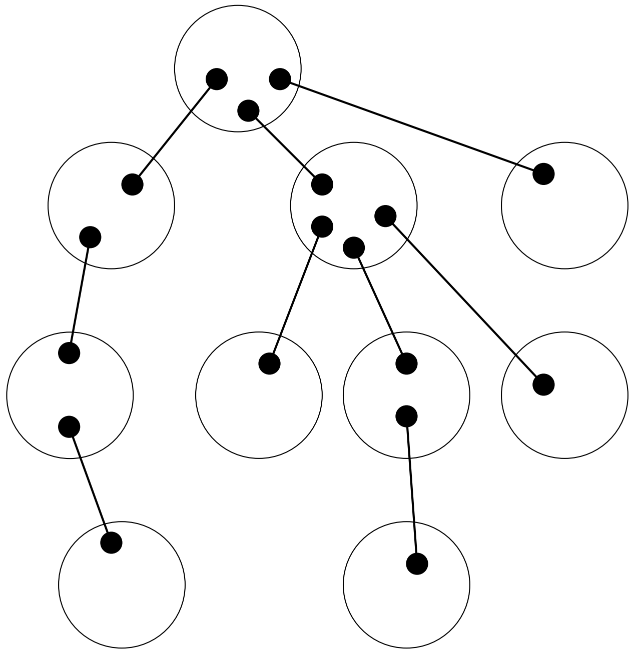
$G$ is the disjoint union of $r - 1$ complete bipartite graphs.

THEN

$G$ has an independent transversal.

## What is an IMC?

THIS:

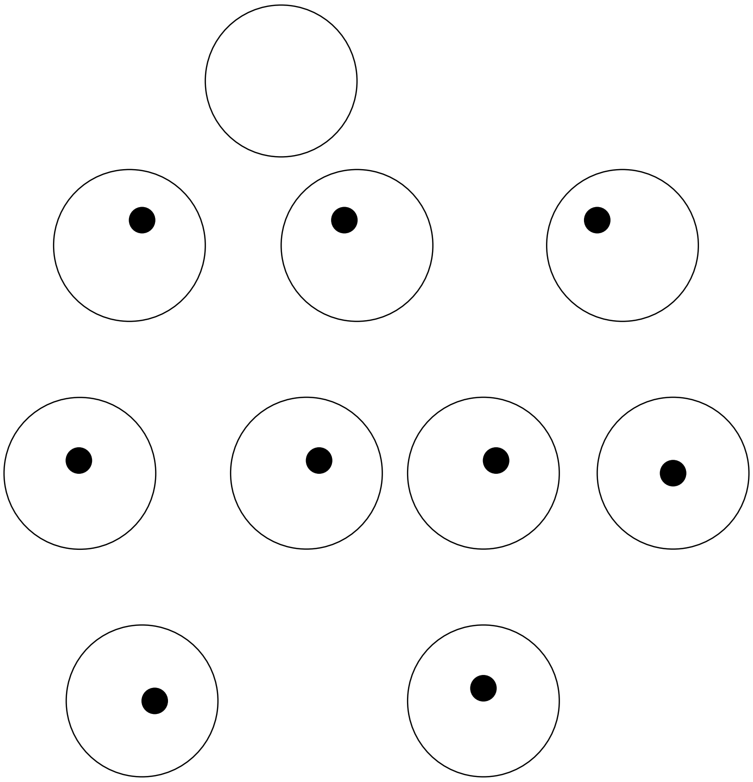


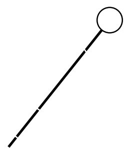
$I$ is an *induced matching configuration* if
$(i)$ $G[I]$ is a perfect matching and
$(ii)$ the class graph $\mathcal{G}_I$ is a tree on $r$ vertices.

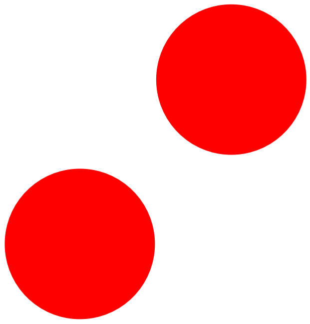
**Lemma** For any index $i$ there is a partial independent transversal $T_i^I \subseteq I$, such that $T_i^I \cap V_j = \emptyset$ iff $j = i$. Moreover, if $v$ is not dominated by $I$, then there is an independent transversal $T_v^I$ containing $v$.

## How to build an IMC?

Suppose $G$ contains no independent transversal.







those parts which contain a vertex of $I$ called active

Choose a vertex $v$ from active classes and a max independent transversal $T$ (agreeing with the current one in active classes) which

minimizes $deg_T(v)$

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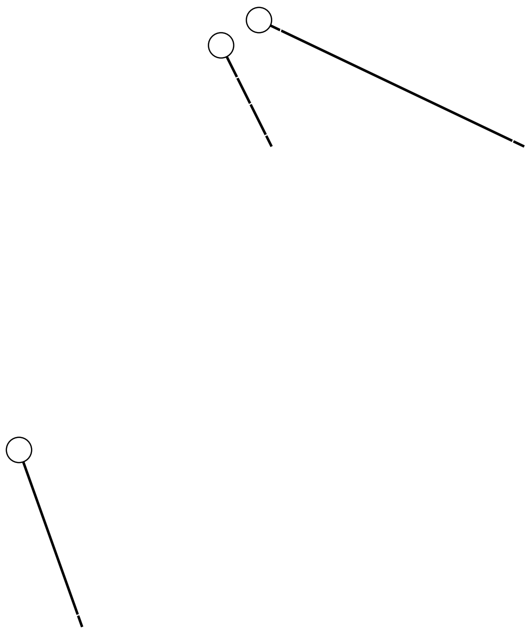

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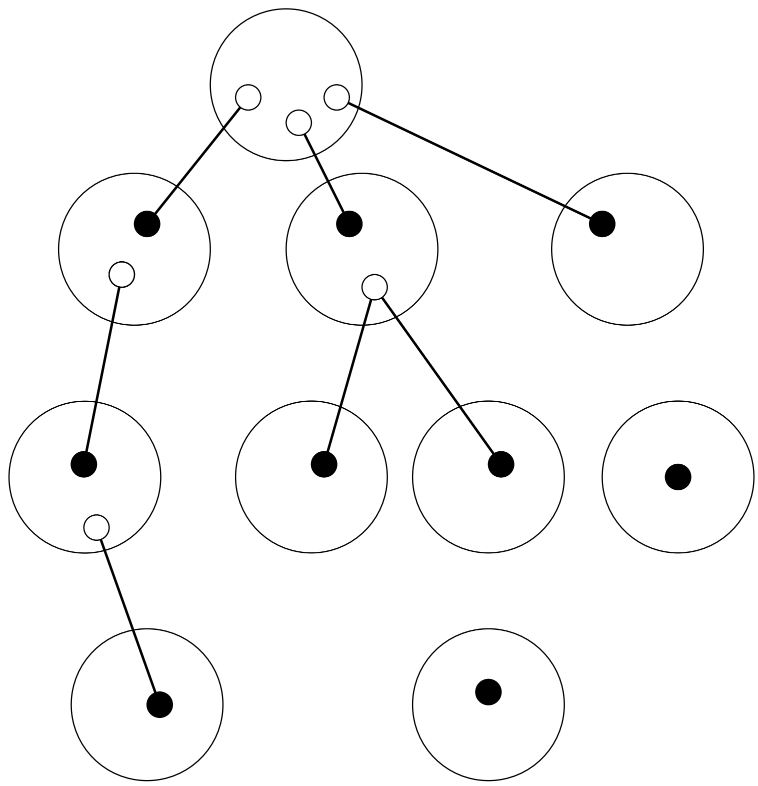
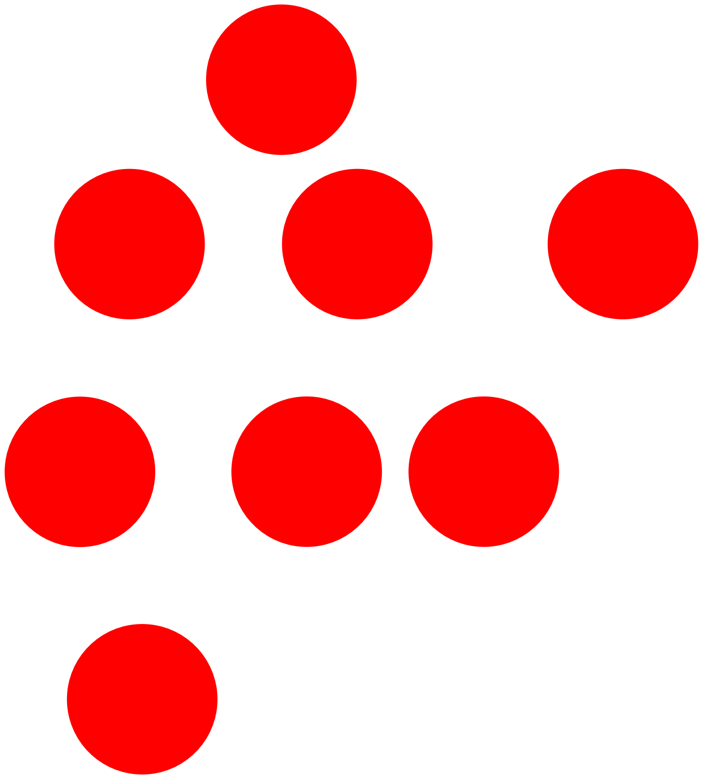
## Why is it an IMC?

Suppose $G$ contains no independent transversal.



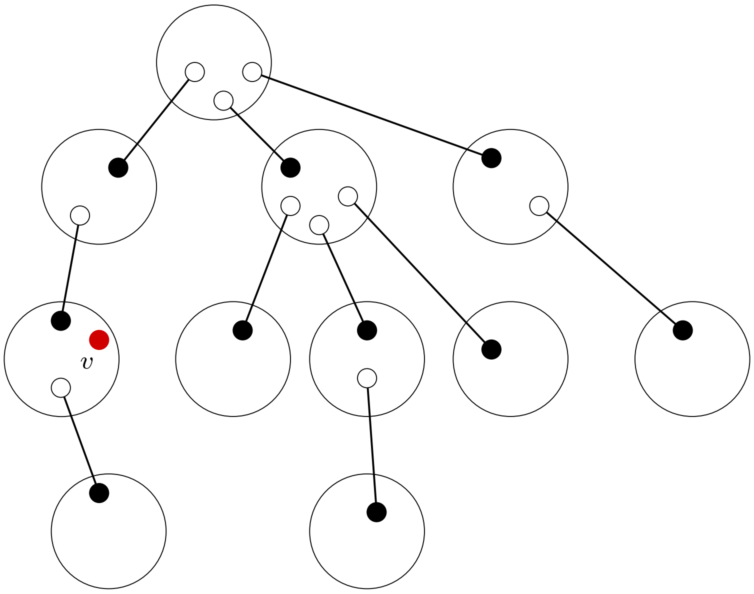


Eventually: $s$ active parts are dominated by the at most $2(s-1)$ vertices of $I$

If $\Delta < \frac{r-1}{2(r-2)}n$, then at the end $I$ is an IMC which dominates $V(G)$.

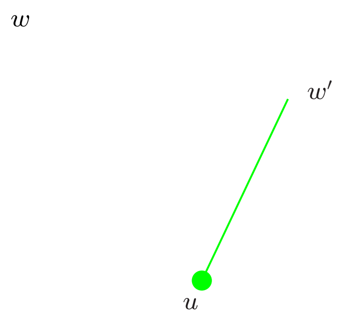
## Sketch of Proof of Thm 2

- Select an IMC $I$
- Select a root where all subtrees not containing the root have order at most $t$. Remember, $r = 2t+1$.



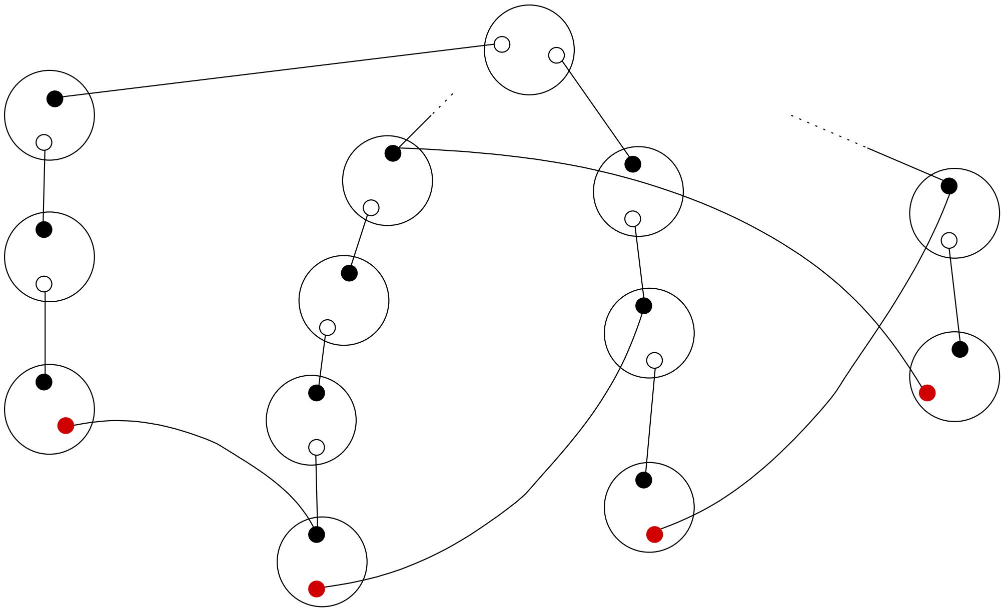

**Fact 1** The unique neighbor of $v$ in $I$ is either a white vertex on the path from the class of $v$ towards the root or a black vertex not on this path.

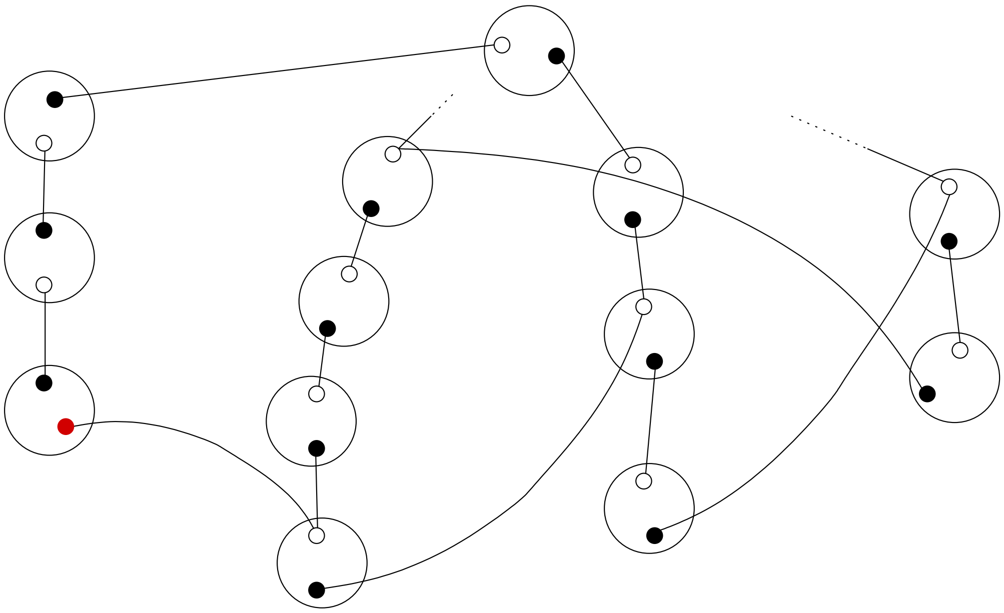
**Fact 2** For any white vertex $w$ there is a vertex $v \notin I$ in its subtree $\mathcal{G}_w$, which has its $I$-neighbor $w' \neq w$ outside $\mathcal{G}_w$.



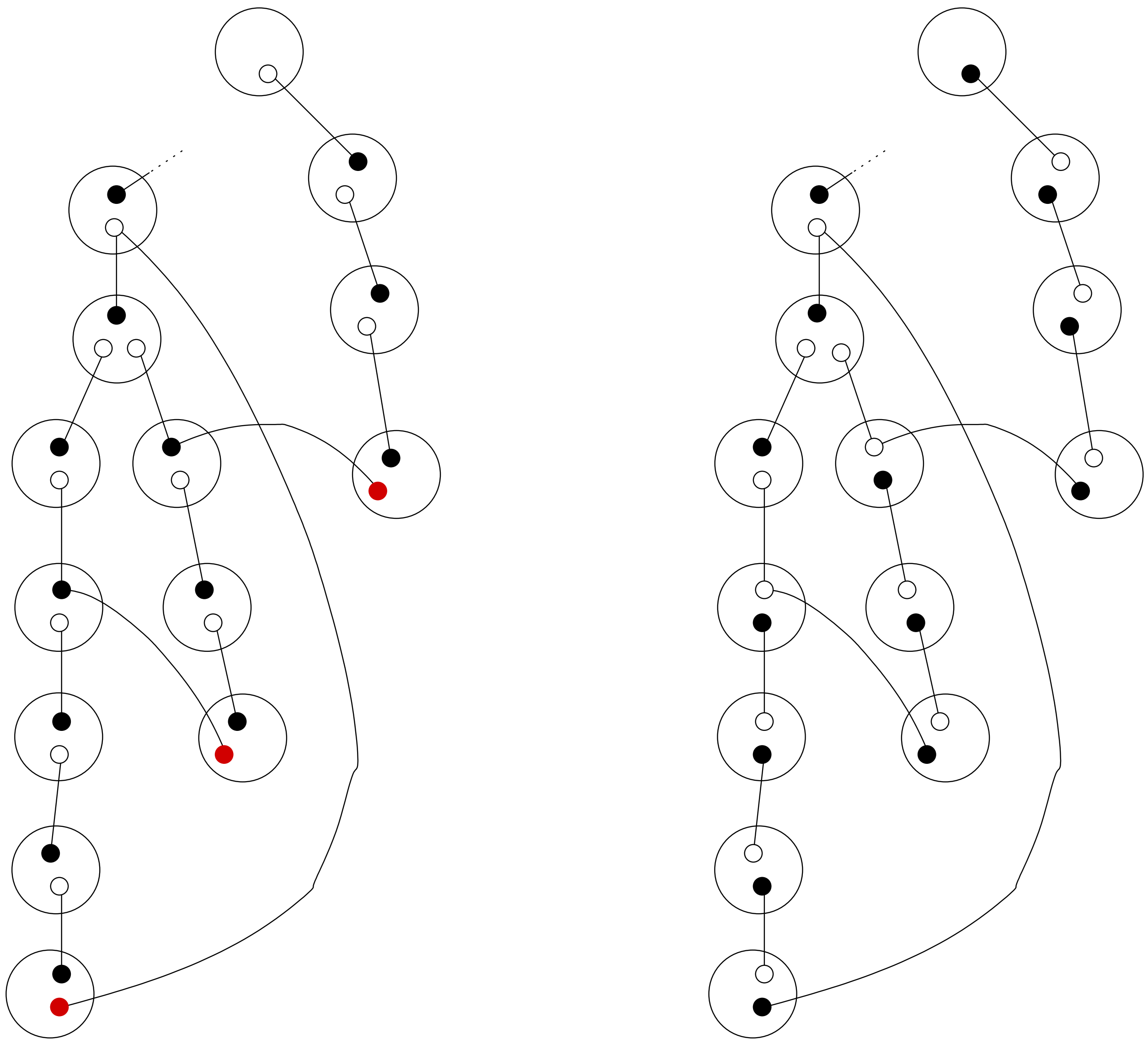
$w$
$w'$
$u$

# Switching to a complete IT, Case 1





## Switching to a complete IT, Case 2



## Related open problems

- **# of independent transversals**

  $f_r(n)$ := largest $f$, such that every $G \in \mathcal{G}_r(n)$ with $\Delta(G) < \Delta(r,n)$ has AT LEAST $f$ independent transversal.

  $f_2(n) = n$

  $f_3(n) = 4$ (BESz, 1975)

  $f_4(n) = \Theta(n^3)$ (Jin, 1998)

  $f_5(n) =???$

  $f_r(n) \stackrel{???}{=} \Theta(n^{r-1})$ for $r$ even

  $f_r(n) \stackrel{???}{=} o(n^{r-1})$ or maybe $\Theta(1)$ for $r$ odd

- **Topological reason for the different behaviour of odd and even**

## Matching transversals — a problem

$T \subseteq V(G) = V_1 \cup \cdots \cup V_r$ is called matching transversal of $G \in \mathcal{G}_r(n)$ if

- $T$ induces a matching
- $|T \cap V_i| = 1$ for every $i = 1, \dots r$.

**Definition** $\Psi(r,n)$ is the smallest integer such that there is a $G \in \mathcal{G}_r(n)$ with maximum degree $\Psi(r,n)$ which has no matching transversal.

**Problem**

Determine $\Psi(r,n)$.
Or at least determine $\Psi_r = \lim_{n\to\infty} \frac{\Psi(r,n)}{n}$.
Or at least determine $\lambda = \lim_{r\to\infty} \Psi_r$.

**Known facts:** (Haxell, Sz., Tardos — 2003)
$\frac{3}{2}n = \Psi(3,n) \geq \Psi(4,n) \geq \dots$
For every $r \geq 3$, $\quad \Psi(r,n) \geq \frac{2}{3}n$
For every $r \geq n^2 + n + 1$, $\quad \Psi(r,n) \leq n + 1$
That is $\frac{2}{3} \leq \lambda \leq 1$.