Turán's Theorem in sparse (pseudo)random graphs

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(based on joint works with
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Triangle-free subgraphs

Theorem. (Turán, 1941) Among n vertex K_t-free graphs the $(t-1)$-partite Turán graph has *the* most number of edges.

Corollary. (Turán, 1941) The maximum number of edges in an n-vertex K_t-free graph is $\left(\frac{t-2}{t-1} + O\left(\frac{1}{n}\right)\right)\binom{n}{2}$.

Asymptotic extension

ex(G, H) is the largest integer e such that G has an H-free subgraph with e edges.

Claim. For every graph G, $ex(G, K_t) \geq \frac{t-2}{t-1}|E(G)|$.

Definition We call a graph G t-Turán, if $ex(G, K_t) \leq \left(\frac{t-2}{t-1} + o(1)\right)|E(G)|$.

Question. For what edge probability p is the random graph $G(n, p)$ t-Turán (with probability tending to 1)?

Asymptotic extension – the conjecture

Claim.

$$ex(G(n,p),K_t) = |E| \; a.s. \qquad \text{if } p \ll n^{-\frac{2}{t-1}},$$
$$ex(G(n,p),K_t) = |E|(1+o(1)) \; a.s. \quad \text{if } p \ll n^{-\frac{2}{t+1}}.$$

In particular there is a constant $c > 0$ such that if $p \leq cn^{-\frac{2}{t+1}}$, then the random graph $G(n,p)$ IS NOT t-Turán.

Proof.

(a) $\quad \binom{n}{t} p^{\binom{t}{2}} \ll 1;$ $\quad$ there isn't a single K_t.

(b) $\quad \binom{n}{t} p^{\binom{t}{2}} \ll \binom{n}{2} p;$ $\quad$ delete an edge from each K_t.

$\square$

Conjecture. (Kohayakawa-Łuczak-Rödl, 1997) There is a constant $C > 0$ such that if $p \geq Cn^{-\frac{2}{t+1}}$, then the random graph $G(n,p)$ IS t-Turán.

Asymptotic extension – results

Confirmed for

K_3 (Frankl - Rödl, 1986), 10pp.
K_4 (Kohayakawa - Łuczak - Rödl, 1997) 28pp.
K_5 (Gerke - Schickinger - Steger, 200?) 34pp.
K_6 (Gerke - Kohayakawa - Rödl - Steger, 200?) ??pp.

Theorem. (Sz.-Vu, 2003) For every $\epsilon > 0$ and integer $t \geq 4$ there exists a constant $C = C(\epsilon, t)$, such that

$$\mathbf{P}\left(|\mathrm{ex}(G(n,p), K_t) - \frac{t-2}{t-1}\binom{n}{2}p| \geq \epsilon\binom{n}{2}p\right) \leq$$

$$\leq 2\exp\left(-\frac{\epsilon^2 p}{16}\binom{n}{2}\right),$$

provided $p \geq Cn^{-\frac{1}{t-1.5}}$. □

Independently, Kohayakawa-Rödl-Schacht (2003) obtained

$$\mathbf{P}\left(\mathrm{ex}(G(n,p), t) = \left(\frac{t-2}{t-1} \pm \eta\right)\binom{n}{2}p\right) \to 1,$$

provided $p \gg (\log n)^4 n^{-\frac{1}{t-1}}$.

The statement — Notations

Theorem. (Turán, 1941) To make K_n K_t-free one must delete at least $\frac{n^2}{2(t-1)} - \frac{n}{2}$ edges.

Proof. Induction on t. $t = 2$ trivial.

Let R be a minimum set of edges, such that $K_n - R$ is K_{t+1}-free. The elements of R called "red" edges, the rest is called "blue". We set to prove $|R| \geq \frac{n^2}{2t} - \frac{n}{2}$.

r_v is the red degree of v.
b_v is the blue degree of v.

$r = \frac{1}{n}\sum_{v\in V} r_v$ is the average red degree.
$b = \frac{1}{n}\sum_{v\in V} b_v$ is the average blue degree.

For every $v \in V$, $b_v + r_v = n - 1 = b + r$.

N_i is the number of triangles with i red edge.

Double counting

$$
\begin{aligned}
N_1 &\geq \sum_{v\in V} \frac{b_v^2}{2(t-1)} - \frac{b_v}{2} \geq n\left(\frac{b^2}{2(t-1)} - \frac{b}{2}\right) \\
N_1 &\leq N_1 + N_2 = \frac{1}{2}\sum_{v\in V} b_v r_v \leq \frac{n}{2}br
\end{aligned}
$$

So

$$
\begin{aligned}
n\left(\frac{b^2}{2(t-1)} - \frac{b}{2}\right) &\leq \frac{n}{2}br \\
\frac{b}{t-1} - 1 &\leq r \\
n - 1 - r - (t-1) &\leq r(t-1) \\
n - t &\leq rt \\
\frac{n}{t} - 1 &\leq r
\end{aligned}
$$

$$
\Longrightarrow |R| = \frac{rn}{2} \geq \frac{n^2}{2t} - \frac{n}{2}
$$

A RECIPE...

...to adapt deterministic proofs for random graphs

- Identify (pseudorandom) properties which make the proof work.
- Prove the statement for any graph having these properties.
- Prove that a random graph has these properties.

G is (k, δ, p)-normal if the following three conditions hold.

(i) For every two subsets $X, Y \subseteq V(G)$ of size at least δnp, $e(X, Y) = (1 \pm \delta)pxy$.

(ii) For any vertex u, $d(u) = (1 \pm \delta)np$.

(iii) For every set X of size $x \geq \delta np$, one needs to delete at least $(\frac{1}{k-1} - \delta)\binom{x}{2}p$ edges from the subgraph $G[X]$ induced by X in order to destroy all k-cliques in this subgraph.

Proof by induction

Start at $k = 4$ (or 5 or 6??).

Y_k: r.v. of minimum number of edges one must delete from $G(n,p)$ to make it K_k-free.

μ_k: expectation of Y_k.

Assume that for some $k \geq 4$ there exists a constant $K(\eta,k)$ for any $\eta > 0$ such that

$$\mathbf{P}\left(Y_k \leq \left(\frac{1}{k-1} - \eta\right)\binom{n}{2}p\right) \to 0,$$

provided $np^{k-\frac{3}{2}} \geq K(\eta,k)$.
Then for any $\epsilon > 0$ there exists a constant $C(\epsilon,k)$

$$\mu_k \geq \left(\frac{1}{k-1} - \frac{\epsilon}{2}\right)\binom{n}{2}p,$$

provided $np^{k-\frac{3}{2}} \geq C(\epsilon,k)$. Then

$$\begin{aligned}\mathbf{P}\left(Y_k \leq \left(\frac{1}{k-1} - \epsilon\right)\binom{n}{2}p\right) &\leq \mathbf{P}\left(Y_k - \mu_k \leq -\frac{\epsilon}{2}\binom{n}{2}p\right) \\ &\leq \exp\left(-\frac{\epsilon^2 p}{16}\binom{n}{2}\right).\end{aligned}$$

Strong concentration

An Azuma-Hoeffding type martingale lemma

A function $Y = Y(t_1, \ldots, t_m)$ has Lipschitz coefficient L if $|Y(t) - Y(t')| \leq L$ for any two vectors t and t' which differ at exactly one coordinate.

Lemma 1 *(Kahn, Kim) Let $t_1, \ldots, t_m$ be independent identically distributed (i.i.d.) binary random variables with mean p. For any function $Y = Y(t_1, \ldots, t_m)$ with Lipschitz coefficient 1,*

$$\mathbf{P}(Y \leq \mu - \lambda) \leq \exp\left(-\rho\lambda + \rho^2 pm\right),$$

where μ denotes the expectation of Y and $0 < \rho \leq \log 2$.

Applying Lemma 1 for Y_k with $m = \binom{n}{2}$, $L = 1$, $\rho = \frac{\lambda}{2mp}$ and $\lambda = \epsilon mp$:

Corollary 2 *For any integer $k \geq 2$ and $2 \log 2 \geq \epsilon > 0$*

$$\mathbf{P}\left(Y_k \leq \mu_k - \epsilon\binom{n}{2}p\right) < \exp\left(-\frac{\epsilon^2 p}{4}\binom{n}{2}\right).$$

A probabilistic and a deterministic lemma

Lemma 3 *For every $\delta > 0$ and integer $k \geq 2$ there is a constant $C' = C'(\delta, k+1)$, such that if $np^{k-\frac{1}{2}} \geq C'$ then $\mathrm{P}(G(n,p)$ is (k,δ,p)-normal$) > 1-\delta$.*

Proof of (iii) needs strong concentration!

Lemma 4 *If G is a (k,δ,p)-normal graph on n vertices and $pn \geq 4/\delta$, then*

$$\mathrm{ext}(G, K_{k+1}) \geq (\frac{1}{k} - 11\delta)\binom{n}{2}p.$$

Proof: Adaptation of deterministic argument.

From the lemmas, with $\delta = \epsilon/24$, we have

$$\mu_{k+1} \geq (1-\delta)\left(\frac{1}{k} - 11\delta\right)\binom{n}{2}p \geq \left(\frac{1}{k} - \frac{\epsilon}{2}\right)\binom{n}{2}p,$$

Open questions

- KŁR-conjecture

  Conjecture 5 *For every integer $k \geq 3$ and for every $\eta > 0$, there exists a constant $K = K(\eta, k)$ such that*

  $$\mathbf{P}\left(\text{ext}(G(n,p),k) \leq \left(\frac{1}{k-1} - \eta\right)\binom{n}{2}p\right) \to 0,$$

  provided $p = p(n) \geq Kn^{-\frac{2}{k+1}}$.

- Refine the error term η in the conjecture. Maybe even $\eta = n^{-\alpha}$?

- Prove results for $k = k(n)$. (even for $p = 1/2$) Possibly up to $k = c \log n$?

Extension for sparse pseudorandom graphs

A d-regular graph G on n vertices with spectrum $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ is called an (n, d, λ)-graph if $\lambda = \max\{|\lambda_i| : i = 2, \ldots n\}$.

Theorem (Sudakov-Sz.-Vu, 2003) Let $t \geq 3$ an integer and G be an (n, d, λ)-graph with $d^{t-1}/n^{t-2} \gg \lambda$. Then G is t-Turán. $\square$

Remarks.

- K_n is an $(n, n-1, 1)$-graph; (Weak) Turán's theorem is a special case.
- If $\lambda = \Theta(\sqrt{d})$, then the theorem is valid from $d = \Omega(n^{\frac{2t-4}{2t-3}})$.
- Best for K_3

Core of the proof

G has the (t, δ, p)-property if the following two conditions hold.

(i) For every two subsets $X, Y \subseteq V(G)$ of cardinality at least $(\delta p)^{t-2}n$, $e(X,Y) = (1 \pm \delta)p|X||Y|$.

(ii) For every subset $X \subseteq V(G)$ of cardinality at least $(\delta p)^{t-3}n$ there are at most $(\delta p)^{t-1}n$ vertices v of G with

$$d_X(v) = (1 \pm \delta)p|X|.$$

Lemma 6 *Let $t \geq 2$ be an integer and G be an (n, d, λ)-graph, such that $d^{t-1}/n^{t-2} = \omega(n)\lambda$. Then for n large enough G has the $(n, d/n, \delta)$-property with $\delta = \omega(n)^{-1/(t-1)}$.*

Lemma 7 *Let G be a graph of order n, which has the (t, δ, p)-property with some $t \geq 2$, $0 < p = p(n) \leq 1$ and $\delta = \delta(n) \to 0$. Then*

$$\text{ext}(G, K_t) \geq (\frac{1}{t-1} + o(1))|E(G)|.$$

Constructions and questions

Alon (1994): Explicit triangle-free (n, d, λ)-graphs with $d = \Theta(n^{2/3})$ and $\lambda = \Theta(n^{1/3})$

Blow up: for any $\Omega(n^{2/3}) \leq d \leq n$ there is a triangle-free $d_1 = \Theta(d)$-regular graph with $n_1 = O(n)$ vertices and second eigenvalue $\lambda(G_1) = O(d_1^2/n_1)$. So $ex(G_1, K_3) = |E(G_1)|$.

Alon, Krivelevich (1997): (n, d, λ)-graphs with $d = n^{\frac{t-3}{t-2}}(1 + o(1))$, $\lambda = n^{\frac{t-3}{2(t-2)}}(1 + o(1))$, which can be made K_t-free by deleting at most $n^{\frac{t-3}{t-2}}$ vertices.

Our theorem is not true with $\lambda < Cd \cdot (d/n)^{\frac{t-3}{2}}$.

Where is the true "threshold" for being t-Turán?

We know from $\lambda \gg d \cdot (d/n)^{t-2}$.

Extension for arbitrary H instead of a clique

Theorem. (Erdős-Simonovits-Stone, 1966)

$$ex(K_n, H) = \left(\frac{\chi(H) - 2}{\chi(H) - 1} + o(1)\right)\binom{n}{2}.$$

Claim. For every graph G and H,

$$ex(G, H) \geq \frac{\chi(H) - 2}{\chi(H) - 1}|E(G)|.$$

Question. For what edge probability p is the Erdős-Simonovits-Stone Theorem true in $G(n, p)$?

Conjecture. (Kohayakawa - Łuczak - Rödl, 1997) There is a constant $C > 0$ such that if $p \geq Cn^{-\frac{1}{d_2(H)}}$, then the ESS Theorem is true for H in $G(n, p)$.

Here $d_2(H) = \max\left\{\frac{e(J)-1}{|J|-2} : J \subseteq H, |J| \geq 3\right\}$.

$K_3, \dots, K_6$ (FR, KLR, GSS, GKRS, 1986–200?)
C_l (Haxell-Kohayakawa-Łuczak, 1995, 1996)

Theorem (Sz.-Vu, 2003) Suppose for every $\eta > 0$,

$$\mathbf{P}(\mathrm{ex}(G(n,p),H) > \left(\frac{\chi(H)-2}{\chi(H)-1} + \eta\right)\binom{n}{2}p) \to 0$$

provided $p \gg n^{-1/\alpha}$. Let H' be a supergraph of H, such that $V(H') = V(H) \cup \{u\}$, $E(H'|_{V(H)}) = E(H)$ and $\chi(H') = \chi(H) + 1$.
Then for every $\eta > 0$,

$$\mathbf{P}(\mathrm{ex}(G(n,p),H') > \left(\frac{\chi(H')-2}{\chi(H')-1} + \eta\right)\binom{n}{2}p) \to 0$$

provided $p \gg n^{-1/(\alpha+1)}$. $\square$

Theorem (Kohayakawa-Rödl-Schacht) (2003)

$$\mathbf{P}\left(\mathrm{ex}(G(n,p),H) > \left(\frac{\chi(H)-2}{\chi(H)-1} + \eta\right)\binom{n}{2}p\right) \to 0,$$

provided $p \gg (\log n)^4 n^{-\frac{1}{d(H)}}$.

Suggested approach of KŁR

Definition 8 *A bipartite graph $B = (U \cup W, E)$ with $|U| = |W| = n$ and $|E| = m$ is called (ϵ, n, m)-regular if for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geq \epsilon n$ and $|W'| \geq \epsilon n$,*

$$\left| |E(U', W')| - m \cdot \frac{|U'||W'|}{n^2} \right| \leq \epsilon m \cdot \frac{|U'||W'|}{n^2}.$$

Definition 9 *Let H be a graph on l vertices. An l-partite graph is called $(H, n, m; \epsilon)$-regular if $G[V_i, V_j]$ is (ϵ, n, m)-regular for all $\{i, j\} \in E(H)$, and there are no edges between V_i and V_j otherwise.*

$\mathcal{S}(H, n, m; \epsilon)$:= set of $(H, n, m; \epsilon)$-regular graphs.
$\mathcal{F}(H, n, m; \epsilon) := \{G \in \mathcal{S}(H, n, m; \epsilon) | H \not\subseteq G\}$

Conjecture 10 *Let H be an arbitrary graph. For any $\beta > 0$ there exist constants $\epsilon_0 > 0, C > 0, n_0 > 0$ such that*

$$|\mathcal{F}(H, n, m; \epsilon)| \leq \beta^m \binom{n^2}{m}^{e(H)},$$

for all $m \geq Cn^{2-1/d_2(H)}$, $n \geq n_0$, and $0 < \epsilon \leq \epsilon_0$.

Confirmed for K_3 (KŁR),
K_4 (Gerke, Prömel, Schickinger, Steger, Taraz)
K_5 (GSS)
K_6 (GKRS)