

# Discrete Geometry I

## Homework # 4— due November 12th

Please do **two** of the exercises. The problem marked with a  $\square$  is **mandatory**.

### Exercise 1. $\square$

- i) Let  $Q = \{t \in \mathbb{R} : a_i t \leq b_i, i = 1, \dots, m\}$  be a polyhedron on the real line with  $a_i, b_i \in \mathbb{R}$  for  $i = 1, \dots, m$ . Give a constructive way to check if  $Q = \emptyset$ .
- ii) Let  $\pi_d : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$  be the coordinate projection  $\pi_d(x_1, \dots, x_{d-1}, x_d) = (x_1, \dots, x_{d-1})$ . Let  $Q = \{x \in \mathbb{R}^d : a_i^t x \leq b_i, i = 1, \dots, n\}$  be a polyhedron and for  $y \in \mathbb{R}^{d-1}$  define  $Q_y := \{x \in \mathbb{R}^d : (y, x) \in Q\}$ . Show that for all  $y \in \mathbb{R}^{d-1}$ , the set  $Q_y$  is a polyhedron and give an explicit inequality description in terms of the inequalities describing  $Q$ .
- iii) Argue (using (i)) that the image

$$\pi_d(Q) = \{y \in \mathbb{R}^{d-1} : Q_y \neq \emptyset\}$$

is a polyhedron.

[Hint: use (i) and (ii) to write the condition  $Q_y \neq \emptyset$  as an affine inequality.]

**(10 points)**

- Exercise 2.** i) Iterating Exercise 1iii), show that this yields an algorithm for checking if  $Q \neq \emptyset$ .  
 [This uses the trivial fact that  $Q \neq \emptyset \iff \pi_d(Q) \neq \emptyset$ .]
- ii)  $\square$  Implement this algorithm in SAGE and test if

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ -2 & 2 & -1 \\ 2 & -4 & -4 \\ -3 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \leq \begin{pmatrix} 2 \\ -1 \\ 3 \\ 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ -2 & 2 & -1 \\ 2 & -4 & -4 \\ -3 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \leq \begin{pmatrix} 1 \\ 2 \\ -3 \\ 4 \\ -5 \end{pmatrix}$$

are empty or not.

Can the number of inequalities in each step vary if you change the order of coordinate projections?

**(10 points)**

- Exercise 3.** Let  $C \subset \mathbb{R}^d$  be a full dimensional finitely generated convex polyhedral cone and let  $\mathbb{R}_{\geq 0}^m = \{x \in \mathbb{R}^m : x_i \geq 0 \text{ for all } i = 1, \dots, m\}$ .
- i) Show that there is an  $n$  and a linear map  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^d$  such that  $C = \pi(\mathbb{R}_{\geq 0}^n)$ .

- ii) Let  $\pi : \mathbb{R}^d \rightarrow \mathbb{R}^e$  be a surjective linear map (a projection) and let  $L = \ker \pi$ . Show that

$$\pi(C)^\circ \cong C^\circ \cap L^\perp,$$

where  $\cong$  denotes linear isomorphism.

- iii) Show that there is some  $m \geq 0$  and a linear subspace  $L \subseteq \mathbb{R}^m$  such that  $C$  is linearly isomorphic to  $\mathbb{R}_{\geq 0}^m \cap L$ .

**(10 points)**

**Bonus Exercise.** Let  $P = \{x \in \mathbb{R}^d : Ax \leq b\}$  be a polyhedron. Prove that its recession cone is  $\text{rec}(P) = \{x \in \mathbb{R}^d : Ax \leq 0\}$ .

**(+ 5 points)**