(Improved) Optimal Triangulation of Saddle Surfaces

*Computational Geometric Learning* (CGL) supported by EU FET-Open grant Transregio-SFB *Discretization in Geometry and Dynamics* (DGD)

D. Atariah  G. Rote  M. Wintraecken

Freie Universität Berlin,  
Rijksuniversiteit Groningen

SFB DGD Workshop,  
Schloss Schley, November 2013
Motivation

- Smooth surface is locally approximated by a quadratic patch.
- Euclidean motion transforms the quadratic patch to graph of a bi-variate polynomial.
- \( \rightarrow \) approximate graphs of quadratic polynomials!

\[
\left\{ (x, y, z) : z = F(x, y) \right\}
\]

Outline

Introduction

Interpolating Approximation

Non-interpolating Approximation
We are interested in a neighborhood of some point.

Make the surface normal vertical.

The direction in which Hausdorff distance is measured becomes almost vertical.

**Definition (Vertical Distance, $L_\infty$ Distance)**

Given two domains $D_1, D_2 \subset \mathbb{R}^2$ and two graphs $f: D_1 \to \mathbb{R}$ and $g: D_2 \to \mathbb{R}$ then the *vertical distance* is

$$\text{dist}_V(f, g) = \max_{(x,y) \in D_1 \cap D_2} |f(x, y) - g(x, y)|$$
Lemma

Let $A, B \subset \mathbb{R}^3$ be two sets with equal projection to the plane. Then

$$\text{dist}_H(A, B) \leq \text{dist}_V(A, B)$$
Lemma (Every two points are the same)

Let $S$ be the graph of a quadratic function. For every point $p \in S$, there is an affine transformation $\mathcal{T}_p : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies the following:

- $\mathcal{T}_p(p) = \vec{0}$
- $\mathcal{T}_p(S)$ is a quadratic graph $\tilde{S}$ with a homogeneous polynomial of the form $\tilde{F}(x, y) = ax^2 + bxy + cy^2$ (\(*\))

- For all $q, r \in \mathbb{R}^3$ on a vertical line,

$$|q - r| = |\mathcal{T}_p(q) - \mathcal{T}_p(r)|.$$ 

- $\mathcal{T}_p(p)$ on the first two coordinates is a translation in $\mathbb{R}^2$. 
If $S$ is negatively curved, the maximum distance to a triangle never occurs in the interior.

**Lemma**

For a line segment $pq$ between two points $p = (p_x, p_y, p_z)$ and $q = (q_x, q_y, q_z)$ on a quadratic graph $S$, $\text{dist}_V(pq, S) = \frac{1}{4} |\tilde{F}(q_x - p_x, q_y - p_y)|$

- $\tilde{F}(x, y)$ is the homogeneous polynomial ($\ast$).
- The max. vertical distance is attained at the midpoint.
Setup

From now on,

\[ S = \{ (x, y, z) : z = xy \} \]

(by a linear transformation of the x-y-plane)

Goal

Given \( \varepsilon > 0 \), find a triangle \( T \) with vertices \( p_0, p_1, p_2 \in S \) of largest area such that

\[ \text{dist}_V (T, S) \leq \varepsilon \]

Translated and reflected copies of \( T \) have the same error and tile the plane:

\[ \text{max. AREA} \Leftrightarrow \text{min. NUMBER of triangles} \]
Maximize the Area of Planar Triangles

\[ \left| xy \right| = \frac{4}{\pi} p_0 \]
Maximize the Area of Planar Triangles

\[ |xy| = \frac{4}{\pi} \]

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Maximize the Area of Planar Triangles

\[ |xy| = \frac{4}{\varepsilon} \]

\[ |(x - \varepsilon)y| = 4\varepsilon \]

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Maximize the Area of Planar Triangles

Optimal Triangulation ➔ Interpolating Approximation
Maximize the Area of Planar Triangles

Optimal Triangulation ➔ Interpolating Approximation
Optimize the Shape of Planar Triangles

Secondary criterion: Maximize the smallest angle
Optimize the Shape of Planar Triangles

Secondary criterion: Maximize the smallest angle
Triangulate the Saddle

Lift the planar triangulation to the surface
Can We Do Better?

What do we have?

Given an $\varepsilon > 0$ and a saddle surface $S$, we can find a family $\mathcal{T}$ of triangles which interpolate the surface and

- have maximum area,
- maintain $\text{dist}_V (S, T) \leq \varepsilon$ for all $T \in \mathcal{T}$.

Question...

Can this be improved by allowing non-interpolating triangles?
Pottmann et al. (2000) conjectured NO.

This question is easy for convex approximation.
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Given an $\varepsilon > 0$ and a saddle surface $S$, we can find a family $\mathcal{T}$ of triangles which interpolate the surface and

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- Pottmann et al. (2000) conjectured NO.

This question is easy for *convex* approximation.
A $\lambda$-pseudo Euclidean map is given by:

$$(x, y) \mapsto (\lambda x, \frac{1}{\lambda} y)$$

- Vertical distance is preserved.
- Area (projected) is preserved.
- Surface $S = \{ z = xy \}$ is preserved.
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Fact

The area of the (interpolating) optimal triangles in the plane is $2\sqrt{5}\varepsilon$. 

\[ \text{above } S \quad \text{below } S \]
In the Plane

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\[ p_0 = (s, m) \]
\[ p_1 = (m, s) \]
\[ p_2 = (\eta, \xi) \]
Fact

The area of the (interpolating) optimal triangles in the plane is $2\sqrt{5}\epsilon$. 

\[p_0 = (s, m) \quad p_1 = (\xi, \eta) \quad p_2 = (\eta, \xi)\]
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$p_0 = (s,m)$ 

$p_1 = (\xi, \eta)$ 

$p_2 = (\eta, \xi)$
The area of the (interpolating) optimal triangles in the plane is $2\sqrt{5}\varepsilon$.

- one-parameter family of area preserving triangles
In the Plane

**Fact**

The area of the (interpolating) optimal triangles in the plane is $2\sqrt{5}\epsilon$.

- one-parameter family of area preserving triangles
- How should they be lifted?

\[ p_0 \quad p_1 = (\xi, \eta) \quad p_2 = (\eta, \xi) \]
Lift the triangle vertically such that the distance to $S$ is minimized.

Lift vertices off the surface by $g$:

$$S_g = (x, y, z) : z = xy + g$$

Vertical distance is attained at midpoints.

$\begin{align*}
p_0 &= (s, m) \\
p_1 &= (\xi, \eta) \\
p_2 &= (\eta, \xi)
\end{align*}$
Vertical Perturbed Lifting

▶ Lift the triangle vertically such that the distance to $S$ is minimized.

▶ Lift vertices off the surface by $\alpha$:

$$S_\alpha = \{ (x, y, z) : z = xy + \alpha \}$$
Lift the triangle vertically such that the distance to $S$ is minimized.

Lift vertices off the surface by $\alpha$:

$$S_\alpha = \{ (x, y, z) : z = xy + \alpha \}$$

Vertical distance is attained at midpoints.
Vertical distances from edges to $S$ are

$$\frac{\xi \eta}{4} + \alpha > 0$$

$$\frac{1}{4} (\xi - \eta)^2 - \alpha > 0$$

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$$\frac{1}{4}(\xi - \eta)^2 - \alpha > 0$$

and have to be equal.

$$\alpha = \frac{1}{8}(\xi^2 - 3\xi\eta + \eta^2)$$
The vertical distance is

\[
\left| \frac{1}{8} (\xi^2 - \xi \eta + \eta^2) \right|
\]

Minimum is attained for

\[
s_0 = s_2^{p_5} + p_3^{p_3}
\]

and the vertical distance is

\[
p_1 \approx 0.968246
\]

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The vertical distance is

\[
\frac{1}{8} (\xi^2 - \xi \eta + \eta^2)
\]

Minimum is attained for

\[
\xi_0 = \sqrt{\frac{2 \sqrt{5} \varepsilon 2 + \sqrt{3}}{\sqrt{3}}}
\]

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The vertical distance is

\[ \left| \frac{1}{8} (\xi^2 - \xi \eta + \eta^2) \right| \]

Minimum is attained for

\[ \xi_0 = \sqrt{2 \sqrt{5} \varepsilon \frac{2 + \sqrt{3}}{\sqrt{3}}} \]

and the vertical distance is

\[ \frac{\sqrt{15}}{4} \varepsilon \approx 0.968246 \varepsilon \]
Picture in Space
Pseudo-euclidean motions give a one-parameter family of optimal triangles.
The Planar Super-Optimal Triangle

- Pseudo-euclidean motions give a one-parameter family of optimal triangles.
- Note the tangency property.
The Planar Super-Optimal Triangle

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OPEN:
Lift vertices by *different* amounts?