Dissecting a Square into an Odd Number of Triangles of Almost Equal Area

Jean-Philippe Labbé, Günter Rote, Günter M. Ziegler
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\( n = 9 \) triangles

root-mean-square (RMS) error: 0.0002737

The optimum among dissections with at most 8 nodes.
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THEOREM (P. Monsky, 1970)
If $n$ is odd, there is no dissection of the square into $n$ triangles of equal area.
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If \( n \) is odd, there is no dissection of the square into \( n \) triangles of equal area.

dissection \( \neq \) triangulation
Measuring Area Deviation

areas $a_1, \ldots, a_n$, target area $= \frac{1}{n}$

- Root-mean-square error (RMS, standard deviation):

\[
RMS := \sqrt{\frac{1}{n} \cdot \sum_{i=1}^{n} (a_i - \frac{1}{n})^2}
\]

- Range:

\[
\text{range} := \max_{1 \leq i \leq n} a_i - \min_{1 \leq i \leq n} a_i
\]

\[
\frac{\text{range}}{2\sqrt{n}} \leq RMS \leq \text{range}
\]
Lower and Upper Bounds

\[ \text{range} \geq \frac{1}{2^{2^{O(n)}}} \text{ (doubly-exponential)} \]

Proof: Gap theorems from real algebraic geometry

A family of dissections for every \( n \) with

\[ \text{range} \leq \frac{1}{n^{\log_2 n - 5}} = \frac{1}{2^{\Omega(\log^2 n)}} \text{ (superpolynomial)} \]

Previous results:

- Numerical experiments, exhaustive enumeration for small \( n \) (Katja Mansow, 2003)
- A family of triangulations with range \( \leq 1/n^3 \) (Bernd Schulze, 2011)
Overview

- Introduction: Problem statement and results
- Review of Monsky’s proof (2-adic valuation, Sperner’s lemma)
- Modeling the problem
- Lower bound via a gap theorem
- Numerical experiments
- Systematic construction (Thue-Morse sequence)
- More numerical experiments
- Speculations
Monsky: 3-coloring of the plane

2-adic valuation of $\mathbb{R}$
$\to$ coloring of $\mathbb{R} \times \mathbb{R}$ with three colors $A$, $B$, $C$

Crucial property:
A rainbow triangle cannot have area $0$ or $\frac{1}{n}$ for odd $n$. 
Monsky: 3-coloring of the plane

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Crucial property:
A rainbow triangle cannot have area 0 or $\frac{1}{n}$ for odd $n$.

Parity argument like for Sperner’s lemma:
If the boundary of a polygon has an odd number of $AB$-colored edges,
then every dissection has an odd number of rainbow triangles.
Lower Bound: Modeling Collinearity

- Lock at all maximal line segments
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Lower Bound: Modeling Collinearity

- Lock at all maximal line segments
- Open them up
- Triangulate them arbitrarily.
  \[\rightarrow\] combinatorial triangulation of a 4-gon, with additional zero-area triangles \(Z\)
Lower Bound: Modeling Collinearity

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Lower Bound: Modeling Collinearity

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  → combinatorial triangulation of a 4-gon, with additional zero-area triangles $Z$

Area 0 does not enforce collinearity!
$n$ triangles, areas $a_1, \ldots, a_n$

$\nu$ unknown vertex positions (apart from the 4 fixed corners of the square)
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$v$ unknown vertex positions (apart from the 4 fixed corners of the square)
$n$ triangles, areas $a_1, \ldots, a_n$

$v$ unknown vertex positions (apart from the 4 fixed corners of the square)

$$T(\vec{x}) = \sum_{i=1}^{n} (a_i(\vec{x}) - \frac{1}{n})^2, \quad \vec{x} \in \mathbb{R}^{2v}$$

... a degree-4 polynomial, RMS = $\sqrt{T(\vec{x})/n}$
Area Deviation Polynomial

\[ n \text{ triangles, areas } a_1, \ldots, a_n \]

\[ v \text{ unknown vertex positions (apart from the 4 fixed corners of the square)} \]

\[ T(\vec{x}) = \sum_{i=1}^{n} (a_i(\vec{x}) - \frac{1}{n})^2, \quad \vec{x} \in \mathbb{R}^{2v} \]

\[ \ldots \text{ a degree-4 polynomial, RMS } = \sqrt{T(\vec{x})/n} \]

\[ z \text{ zero-area triangles, areas } b_1, \ldots, b_z \]

\[ Z(\vec{x}) = \sum_{j=1}^{z} (b_j(\vec{x}) - 0)^2 \]
Area Deviation Polynomial

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\[ Z(\vec{x}) = \sum_{j=1}^{z} (b_j(\vec{x}) - 0)^2 \]

\[ T(\vec{x}) + Z(\vec{x}) \rightarrow \min!, \quad \vec{x} \in \mathbb{R}^{2v} \]
Lower-Bound Argument

\[
\min \{ \text{RMS}^2 \cdot n \mid \text{dissection} \} \\
= \min \{ T(\vec{x}) \mid \vec{x} \in \mathbb{R}^{2v}, \text{ } \vec{x} \text{ is a dissection} \} \\
\geq \min \{ T(\vec{x}) \mid \vec{x} \in \mathbb{R}^{2v}, Z(\vec{x}) = 0 \} \\
\geq \min \{ T(\vec{x}) + Z(\vec{x}) \mid \vec{x} \in \mathbb{R}^{2v}, Z(\vec{x}) = 0 \} \\
\geq \min \{ T(\vec{x}) + Z(\vec{x}) \mid \vec{x} \in \mathbb{R}^{2v} \} \\
> 0 \quad \text{(à la Sperner and Monsky)}
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Gap Theorems

“An algebraic number $\alpha \neq 0$ cannot be arbitrarily close to 0.”
(depending on the degree and the size of the coefficients)
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"An algebraic number $\alpha \neq 0$ cannot be arbitarily close to 0."
(depending on the degree and the size of the coefficients)

The DMM bound ("Davenport–Mahler–Mignotte")
[ Emiris–Mourrain–Tsigaridas, 2010 ]

- polynomial $f(\vec{x})$ of degree $d$ in $k$ variables
- integer coefficients with $\leq \tau$ bits
- $f(x) > 0$ on the unit simplex in $\mathbb{R}^k$

$\Rightarrow \min\{ f(x) \mid x \in \text{unit simplex} \} \geq m_{\text{DMM}}$

$$\frac{1}{m_{\text{DMM}}} = 2^{d(d-1)(k-1)}((d \log_2 k + \tau + 1)(k+1) + (k^2 + 3k + 1) \log_2 d + d + 2k + 1) \times 2^{(k^2 + k) \log_2 \sqrt{d}}$$
Gap Theorems

"An algebraic number $\alpha \neq 0$ cannot be arbitrarily close to 0."
(depending on the degree and the size of the coefficients)

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$$\implies \min \{ f(x) \mid x \in \text{unit simplex} \} \geq m_{\text{DMM}}$$

$$\frac{1}{m_{\text{DMM}}} = 2^{d(d-1)(k-1)} \left( (d \log_2 k + \tau + 1)(k+1) + (k^2 + 3k + 1) \log_2 d + d + 2k + 1 \right) \times 2^{k^2 + k} \log_2 \sqrt{d}$$

THEOREM: If the unit square is dissected into an odd number $n$ of triangles, the range of areas is at least $1/2^{2^{\Omega(n)}}$. 
Computer Experiments

1. Generate all combinatorial types of triangulations/dissections
   \[ \text{plantri by Brinkmann and McKay} \]

2a. [Katja Mansow 2003] for triangulations:
    Minimize the range numerically
    \[ \text{minmax command of MATLAB} \]

2b. For dissections: Minimize the squared error (RMS):
    Find critical points of \( T(\vec{x}) + \lambda Z(\vec{x}) \).
    \( \rightarrow \) system of polynomial equations
    \[ \text{Bertini of Bates, Hauenstein, Sommese, Wampler} \]
Computer Experiments

\( n = 3, \text{ RMS } = 0.11786, \text{ range } = 0.25 \)
Computer Experiments

$n = 5$, RMS = 0.01030

← the best triangulation found by Mansow: range = 0.0225
Computer Experiments


$n = 7$, $\text{RMS} = 0.000778$

the RMS-optimal solutions with at most 8 vertices:

the best triangulation found by Mansow:

range $= 0.0031$
Computer Experiments

\[ n = 9, \text{RMS} = 0.000274 \]
the RMS-optimal solution with at most 8 vertices:

best triangulation found by Mansow:
9 vertices, range = 0.00014 !

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Discrete Geometry Fest, Budapest, May 15–19, 2017
A Systematic Construction

\[
(0, 1) \quad \ldots \quad (1, 1 - \frac{2}{n}) \quad (1, 0)
\]

range = \( O(1/n^3) \)
A Systematic Construction

\[ \begin{align*}
(0, 1) & \quad (1, 1) \\
(1, 1 - \frac{2}{n}) & \quad (0, 0) \\
(1, 0) &
\end{align*} \]

\[ n \equiv 1 \pmod{4} \]

\[ \text{range} = O\left(\frac{1}{n^5}\right) \]

A Systematic Construction

\[ n - 2^{\lfloor \log_2 n \rfloor} - 1 \text{ filler triangles} \]

\[ 2^{\lfloor \log_2 n \rfloor} \text{ triangles} \]

use the Thue-Morse sequence \( s_1 s_2 s_3 \ldots = +--+-++---++----+++--+-++---++----+++--+-\ldots \)

THEOREM:

\[ \text{range} \leq \frac{8n^2}{n^{\log_2 n}} (1 + O\left(\frac{\log n}{n}\right)) \]
Estimating the error

\[ \frac{1}{n} \]

\[ \frac{2}{n} \]

\[ a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_{n-1} \]

\[ O \]
Estimating the error

\[ \frac{1}{n} \]

\[ \frac{2}{n} \]

\[ a_1 \quad a_2 \quad a_3 \quad \ldots \quad a_i \quad a_{i+1} \quad a_{n-1} \]

\[ O \]
Estimating the error

\[ \prod_{i=1}^{n-1} \left( \frac{1 - iU}{1 - (i - 1)U} \right)^{s_i} \approx 1, \quad U := \frac{4}{n^2} \]
Estimating the error

\[
\begin{align*}
W &:= \sum_{i=1}^{n-1} s_i \ln \frac{1 - iU}{1 - (i-1)U} \approx 0 \\
U &:= \frac{4}{n^2} \\
\prod_{i=1}^{n-1} \left( \frac{1 - iU}{1 - (i - 1)U} \right)^{s_i} &\approx 1,
\end{align*}
\]
Estimating the error

\[ W := \sum_{i=1}^{n-1} s_i \ln \frac{1 - iU}{1 - (i - 1)U} \approx 0 \]

\[ \sum_{i=1}^{n-1} s_i \ln \frac{1 - iU \pm \varepsilon}{1 - (i - 1)U \pm \varepsilon} \neq 0 \]

\[ W \text{ small} \rightarrow \varepsilon \text{ small.} \implies \text{Concentrate on small } W! \]

\[ \sum_{i=1}^{n-1} s_i \ln(1 - iU) \]

\[ = \sum_{i=1}^{n-1} s_i \left( -iU - \frac{i^2}{2}U^2 - \frac{i^3}{3}U^3 - \cdots \right) \]

Try to cancel the first powers of \( U \)
Annihilate Powers

\[
1^0 - 2^0 - 3^0 + 4^0 - 5^0 + 6^0 + 7^0 - 8^0 - 9^0 + 10^0 + 11^0 - 12^0 + 13^0 - 14^0 - 15^0 + 16^0 = \\
1 - 2 - 3 + 4 - 5 + 6 + 7 - 8 - 9 + 10 + 11 - 12 + 13 - 14 - 15 + 16 = \\
1^2 - 2^2 - 3^2 + 4^2 - 5^2 + 6^2 + 7^2 - 8^2 - 9^2 + 10^2 + 11^2 - 12^2 + 13^2 - 14^2 - 15^2 + 16^2 = \\
1^3 - 2^3 - 3^3 + 4^3 - 5^3 + 6^3 + 7^3 - 8^3 - 9^3 + 10^3 + 11^3 - 12^3 + 13^3 - 14^3 - 15^3 + 16^3 = \\
1^4 - 2^4 - 3^4 + 4^4 - 5^4 + 6^4 + 7^4 - 8^4 - 9^4 + 10^4 + 11^4 - 12^4 + 13^4 - 14^4 - 15^4 + 16^4 = \\
\]

Theorem (E. Prouhet 1851)

If \( f \) is a polynomial of degree \( < k \) then

\[
\sum_{i=1}^{2^k} s_i \cdot f(i) = 0,
\]

for the Thue-Morse sequence \( s_1, s_2, s_3, \ldots \).
Upper bound

THEOREM:
For every $n$, there is a dissection with

$$\text{range} \leq \frac{8n^2}{n \log_2 n} \cdot (1 + O\left(\frac{\log n}{n}\right))$$
Systematic is not Always Best

<table>
<thead>
<tr>
<th>$n$</th>
<th>optimal sign sequence $s$</th>
<th>$\varepsilon = \pm \text{range}/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3^*$</td>
<td>++</td>
<td>$-0.16667$</td>
</tr>
<tr>
<td>$5^*$</td>
<td>+++</td>
<td>$+0.01250$</td>
</tr>
<tr>
<td>7</td>
<td>++++</td>
<td>$-0.00010248$</td>
</tr>
<tr>
<td>$9^*$</td>
<td>++++++++</td>
<td>$-0.00016360$</td>
</tr>
<tr>
<td>11</td>
<td>++++++++</td>
<td>$-4.1201 \times 10^{-6}$</td>
</tr>
<tr>
<td>13</td>
<td>++++++++</td>
<td>$+5.9928 \times 10^{-6}$</td>
</tr>
<tr>
<td>15</td>
<td>++++++++</td>
<td>$-5.2871 \times 10^{-7}$</td>
</tr>
<tr>
<td>$17^*$</td>
<td>++++++++</td>
<td>$-3.4708 \times 10^{-8}$</td>
</tr>
<tr>
<td>19</td>
<td>++++++++</td>
<td>$+4.2052 \times 10^{-8}$</td>
</tr>
<tr>
<td>21</td>
<td>++++++++</td>
<td>$-5.5778 \times 10^{-9}$</td>
</tr>
<tr>
<td>23</td>
<td>++++++++</td>
<td>$+3.5359 \times 10^{-9}$</td>
</tr>
<tr>
<td>25</td>
<td>++++++++</td>
<td>$-7.457 \times 10^{-10}$</td>
</tr>
<tr>
<td>27</td>
<td>++++++++</td>
<td>$-1.266 \times 10^{-10}$</td>
</tr>
<tr>
<td>29</td>
<td>++++++++</td>
<td>$+9.026 \times 10^{-12}$</td>
</tr>
<tr>
<td>31</td>
<td>++++++++</td>
<td>$+2.446 \times 10^{-12}$</td>
</tr>
<tr>
<td>$33^*$</td>
<td>++++++++</td>
<td>$-1.423 \times 10^{-12}$</td>
</tr>
<tr>
<td>35</td>
<td>++++++++</td>
<td>$+1.777 \times 10^{-13}$</td>
</tr>
<tr>
<td>37</td>
<td>++++++++</td>
<td>$+1.199 \times 10^{-14}$</td>
</tr>
</tbody>
</table>
Systematic is not Always Best

Example: \( n = 33 \)

Best sequence:

\[
+\ldots-\ldots+-\ldots+\ldots++++
\]

\( \varepsilon = -1.4232 \times 10^{-12} \)

Thue-Morse:

\[
+\ldots-\ldots+-\ldots+\ldots++++
\]

\( \varepsilon = 1.0615 \times 10^{-10} \)

Guarantee from theorem:

\( |\varepsilon| \leq 6.6565 \times 10^{-5} \)
Heuristic Explanation

\[ W := \sum_{i=1}^{n-1} s_i \cdot \ln \frac{1 - iU}{1 - (i - 1)U} \approx 0 \]

RANDOM \( s_i = \pm 1 \)

\( W \): approximately Gaussian with \( \mu = 0 \) and \( \sigma \approx U \sqrt{n} \sim n^{-3/2} \).

Take \( N = 2^{n-1} \) random samples from this distribution.
Heuristic Explanation

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\( W \): approximately Gaussian with \( \mu = 0 \) and \( \sigma \approx U \sqrt{n} \sim n^{-3/2} \).

Take \( N = 2^{n-1} \) random samples from this distribution.

Near \( x = 0 \), these samples are like a Poisson distribution with density
\[ \lambda = N \cdot f(0) \sim 2^n / n^{3/2} \]

\[ \rightarrow \text{Smallest absolute value} = 1/2\lambda \sim n^{3/2} / 2^n \]
Triangulations?

So far: Ideas for systematic computer experiments.
No general analysis.
The Tarry-Escott Problem

\[ \sum_{i=1}^{2^k} s_i \cdot i^d = 0, \text{ for } d = 0, 1, \ldots, k - 1 \]

Can you annihilate the first $k$ powers with a SHORTER sign sequence?

The Tarry-Escott Problem: Find two distinct sets of integers $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ such that

\[ \alpha_1^d + \cdots + \alpha_n^d = \beta_1^d + \cdots + \beta_n^d, \text{ for all } d = 0, 1, 2, \ldots, k - 1 \]

Try to make $n$ as small as possible.