The Quadratic Assignment Problem with a Monotone Anti-Monge and a Symmetric Toeplitz Matrix: Easy and Hard Cases *

Dedicated to the memory of Gene Lawler

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Abstract

This paper investigates a restricted version of the Quadratic Assignment Problem (QAP), where one of the coefficient matrices is an Anti-Monge matrix with non-decreasing rows and columns and the other coefficient matrix is a symmetric Toeplitz matrix. This restricted version is called the Anti-Monge–Toeplitz QAP. There are three well-known combinatorial problems that can be modeled via the Anti-Monge–Toeplitz QAP: (P1) The "Turbine Problem", i. e. the assignment of given masses to the vertices of a regular polygon such that the distance of the center of gravity of the resulting system to the center of the polygon is minimized. (P2) The Traveling Salesman Problem on symmetric Monge distance matrices. (P3) The arrangement of data records with given access probabilities in a linear storage medium in order to minimize the average access time.

We identify conditions on the Toeplitz matrix B that lead to a simple solution for the Anti-Monge–Toeplitz QAP: The optimal permutation can be given in advance without regarding the numerical values of the data. The resulting theorems generalize and unify several known results on problems (P1), (P2), and (P3). We also show that the Turbine Problem is NP-hard and consequently, that the Anti-Monge–Toeplitz QAP is NP-hard in general.

1 Introduction

Given two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, $1 \le i, j \le n$, with real entries, the *Quadratic Assignment Problem* (QAP) in the Koopmans-Beckmann form [13] consists in finding the permutation π which minimizes

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{\pi(i)\pi(j)} b_{ij}.$$

Here, π ranges over the set S_n of all permutations of $\{1, 2, \dots, n\}$. The QAP with matrices A and B will be abbreviated by QAP(A, B). Lawler [15] introduced the Quadratic Assignment Problem with the more general form of the objective function

$$\sum_{i=1}^{n} \sum_{j=1}^{n} d_{\pi(i)\pi(j)ij}$$

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for a given four-dimensional array d_{klij}.

The QAP is an NP-hard problem, since by an appropriate choice of matrix B, the traveling salesman problem becomes one of its special cases. Moreover, the QAP contains many other well studied NP-hard problems as special cases, e.g. *the linear ordering problem, the maximum clique problem, graph packing, subgraph isomorphism, the maximum cut problem* etc. The QAP is one of the most difficult problems in combinatorial optimization; currently, solving general problems of size $n \geq 20$ is still considered intractable (see Clausen and Perregård [9]). The QAP has many applications e.g. in location theory, scheduling, manufacturing, parallel and distributed computing, and statistical data analysis. For more comprehensive information, the reader is referred to the survey papers by Lawler [16], Burkard [3], and Pardalos, Rendl and Wolkowicz [20]. Thus the QAP continues to be interesting and stimulating both from computational and theoretical point of view.

1.1 The Anti-Monge–Toeplitz QAP: Definitions

In this paper we investigate a restricted version of the QAP, the *Anti-Monge–Toeplitz QAP*, where matrix A is restricted to be a *monotone Anti-Monge matrix* and where matrix B is a *symmetric Toeplitz matrix*. Let us recall the definitions of Anti-Monge matrices and Toeplitz matrices as they usually occur in the literature.

Definition 1.1 An $n \times n$ matrix $A = (a_{ij})$ is called an *Anti-Monge matrix* if it satisfies the inequality $a_{ij} + a_{rs} \geq a_{is} + a_{rj}$ for all $1 \leq i < r \leq n$ and $1 \leq j < s \leq n$ (this inequality is called the *Anti-Monge inequality*). Matrix A is called *monotone* if $a_{ij} \leq a_{i,j+1}$ and $a_{ij} \leq a_{i+1,j}$ for all i, j, i. e. the entries in every row and in every column are in non-decreasing order. ■

Example 1.1 The following 3×3 matrix M is Anti-Monge and monotone.

$$M = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 5 \\ 6 & 7 & 9 \end{pmatrix} \qquad ■$$

In the literature, Anti-Monge matrices are sometimes also called *inverse Monge matrices* or *Contra-Monge matrices*, whereas matrices M for which $-M$ is an Anti-Monge matrix are called *Monge matrices*. Monge and Anti-Monge matrices are very important for many fields of applied mathematics and optimization, see for example the survey of Burkard, Klinz and Rudolf [6].

Simple examples of Anti-Monge matrices are *sum matrices* and *product matrices*. A matrix A is a *sum matrix* (respectively, a *product matrix*) if there exist real numbers r_i and c_i, $1 \leq i \leq n$, such that $a_{ij} = r_i + c_j$ (respectively, $a_{ij} = r_i \cdot c_j$) holds for all $1 \leq i, j \leq n$. Every sum matrix is an Anti-Monge matrix, but it is not necessarily a monotone matrix. A product matrix is a monotone Anti-Monge matrix, if the numbers r_i and c_j are non-negative and sorted in increasing order. Sum matrices are the only matrices which are at the same time Monge and Anti-Monge matrices.

Definition 1.2 An $n \times n$ matrix $B = (b_{ij})$ is a *Toeplitz matrix*, if there exists a function $f\colon \{-n+1, \ldots, n-1\} \to \mathbb{R}$ such that $b_{ij} = f(i-j)$, for $1 \leq i, j \leq n$. The Toeplitz matrix B is said to be *generated* by function f. ■

A Toeplitz matrix is completely determined if its first row and first column are known. The function f essentially contains this information.

Example 1.2 The following 4×4 matrix T is a (symmetric) Toeplitz matrix:

$$T = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

T is generated by the function $f\colon \{-3,-2,-1,0,1,2,3\} \to \mathbb{R}$, where $f(k) = \cos(k\pi/2)$. ■

1.2 The Anti-Monge–Toeplitz QAP: Applications

Our interest in the Anti-Monge–Toeplitz QAP arose from the "Turbine Problem", as investigated by Bolotnikov [2], Stoyan, Sokolovskii and Yakovlev [23], Laporte and Mercure [14], and Schlegel [22]. In the manufacturing of turbines, n given positive masses (=blades) m_i, $1 \le i \le n$, have to be placed on the vertices of a regular polygon (=turbine rotor) in a balanced way, which means that the center of gravity of the resulting weighted polygon should be as close to its rotational center (=rotational axis) as possible. Without loss of generality we may assume that $m_1 \le m_2 \le \cdots \le m_n$ holds. It can be shown that this objective leads to the following quadratic assignment problem (see Section 3 for more details).

$$\min_{\phi \in S_n} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{\phi(i)} m_{\phi(j)} \cos \frac{2(i-j)\pi}{n}$$

In this QAP, the matrix $A = (a_{ij}) = (m_i m_j)$ is a *product matrix*, and as we observed above, it is a monotone Anti-Monge matrix. The matrix $B = (b_{ij}) = \left(\cos \frac{2(i-j)\pi}{n}\right)$ is a symmetric Toeplitz matrix generated by the function $f(x) = \cos \frac{2x\pi}{n}$. Thus, the Turbine Problem is a special case of the Anti-Monge–Toeplitz QAP.

There are also two other optimization problems that have been studied in the literature a long time ago and that can be formulated as Anti-Monge–Toeplitz QAPs: The *Travelling Salesman Problem* with a symmetric Monge distance matrix as studied by Supnick [24] and a *Data Arrangement Problem* investigated by Burkov, Rubinstein and Sokolov [7], Timofeev and Litvinov [25], Pratt [21] and Metelski [18]. These two applications of the Anti-Monge–Toeplitz QAP are described in detail in Section 4.

1.3 The Anti-Monge–Toeplitz QAP: Negative Results

Despite its simple and restricted combinatorial structure, the Anti-Monge–Toeplitz QAP can be shown to be NP-hard. In fact, we will provide two NP-hardness results for special cases of the Anti-Monge–Toeplitz QAP which clearly implies NP-hardness of the general problem.

The first result is an NP-hardness proof for the Turbine problem that is presented in Section 3. This NP-hardness proof is interesting in its own, since due to its simple objective function the Turbine problem was conjectured to be an easy problem, and people looked for a polynomial-time solution algorithm. The second result is the NP-hardness of QAP(A, B) with a monotone Anti-Monge matrix A and a symmetric Toeplitz matrix $B = (b_{ij})$ defined by $b_{ij} = (-1)^{i+j}$ which is proved in Subsection 5.2. This result demonstrates NP-hardness of the Anti-Monge–Toeplitz QAP even in the case where the generating function of the Toeplitz matrix B is periodic with only 0-1-values.

1.4 The Anti-Monge–Toeplitz QAP: Positive Results

Since the Anti-Monge–Toeplitz QAP is NP-hard, the search for polynomially solvable special cases arises as a natural question. In this paper we single out two restricted versions which are polynomially solvable. In both cases, the corresponding QAP can be solved very easily: One can give an optimal permutation π even if the corresponding matrices A and B are not explicitly given. The information about the structure of the problem alone suffices to solve it, even without knowing the numerical values of the matrix entries. More generally we introduce the *constant permutation property*.

Definition 1.3 An $n \times n$ matrix B has the *constant permutation property* with respect to a class of $n \times n$ matrices $\mathcal{A}$, if there is a permutation $\pi^B \in S_n$ which solves the problem QAP(A, B) for all matrices $A \in \mathcal{A}$.

A class of matrices $\mathcal{B}$ has the constant permutation property with respect to the class of matrices $\mathcal{A}$ if each matrix $B \in \mathcal{B}$ has the constant permutation property with respect to the class $\mathcal{A}$. ■

Throughout this paper, when matrix B has the constant permutation property with respect to class $\mathcal{A}$, B will be a symmetric Toeplitz matrix and $\mathcal{A}$ will be the class of monotone Anti-Monge matrices. Therefore, we will often simply write "B has the constant permutation property" without specifying the class $\mathcal{A}$ of matrices.

The main goal of our work on polynomially solvable cases of the Anti-Monge–Toeplitz QAP is to identify conditions on the symmetric Toeplitz matrix B which guarantee that B has the constant permutation property. The first of these conditions involves *benevolent* matrices.

Definition 1.4 A function $f\colon \{-n+1, \dots, n-1\} \to \mathbb{R}$ is called *benevolent* if it fulfills the following three properties.

(Ben1) $f(-i) = f(i)$ for all $1 \le i \le n-1$.

(Ben2) $f(i) \le f(i+1)$ for all $1 \le i \le \lfloor \frac{n}{2} \rfloor - 1$.

(Ben3) $f(i) \le f(n-i)$ for all $1 \le i \le \lceil \frac{n}{2} \rceil - 1$.

A matrix is called benevolent if it is a Toeplitz matrix generated by a benevolent function. ■

Example 1.3 Let us give the example of a benevolent function $f\colon \{-7, -6, \dots, 0, 1, \dots 7\}$ defined by

$$f(0) = 1.5, \quad f(1) = 0.5, \quad f(2) = 1, \quad f(3) = 1.25, \quad f(4) = 2,$$

$$f(5) = 2.5, \quad f(6) = 2.5, \quad f(7) = 1.5$$

The graph of this function is shown in Figure 1. Note that the graph of this function for $x \in \{5, 6, 7\} \cup \{-5, -6, -7\}$ lies above the thin lines. This shows that in this case property (Ben3) is satisfied with strict inequality. ■

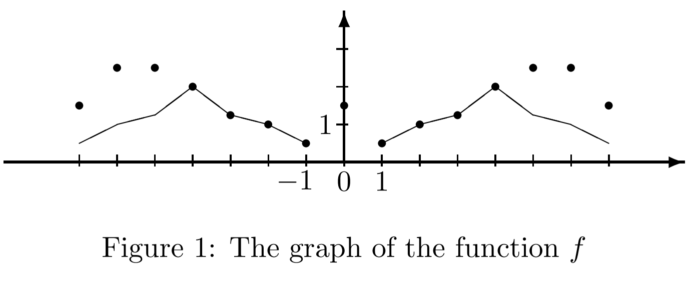

Figure 1: The graph of the function f

By property (Ben1), a benevolent matrix is symmetric.

The permutation π^* which is optimal for every QAP with a benevolent matrix B and a monotone Anti-Monge matrix A is given as follows.

Definition 1.5 The permutation $\pi^* \in S_n$ is defined by $\pi^*(i) = 2i - 1$ for $1 \le i \le \lceil \frac{n}{2} \rceil$, and $\pi^*(n+1-i) = 2i$ for $1 \le i \le \lfloor \frac{n}{2} \rfloor$. ■

Note that π^* starts with the odd numbers in increasing order followed by the even numbers in decreasing order. We adopt the following notation for permutation ϕ:

$$\phi = \langle \phi(1), \phi(2), \ldots, \phi(n) \rangle$$

With this notation we have: $\pi^* = \langle 1, 3, 5, 7, 9, \ldots, 10, 8, 6, 4, 2 \rangle$. The inverse permutation of ϕ is denoted by ϕ^{-1}.

The main result of this paper states that every benevolent matrix B has the constant permutation property with respect to the class of monotone Anti-Monge matrices. The proof of this theorem will be given in Section 2.

Theorem 1.6 (Main theorem)
The permutation π^* solves $\mathrm{QAP}(A, B)$ when A is a monotone Anti-Monge matrix and B is a symmetric Toeplitz matrix which is generated by a benevolent function.

Theorem 1.6 generalizes and unifies a number of known results on the Data Arrangement Problem and on the Travelling Salesman Problem. For example, we give a very simple proof for the polynomial solvability of the TSP on symmetric Monge matrices, whereas the original proof given by Supnick in 1957 [24] is quite involved. Further, the main theorem leads to a polynomially solvable case of the Data Arrangement Problem which contains and implies all polynomial results presented in [7, 18, 21, 25]. Detailed information is given in Section 4. Moreover, if B is the Toeplitz matrix involved in the formulation of the Turbine Problem as an Anti-Monge–Toeplitz QAP, then $-B$ is a benevolent matrix. Hence, Theorem 1.6 yields that the *maximization version of the Turbine Problem* is solvable in polynomial time.

Further, we will investigate certain periodic extensions of benevolent functions which we call *k-benevolent* functions. The symmetric Toeplitz matrices that are generated by k-benevolent functions are called *k-benevolent* matrices and can be shown to fulfill the constant permutation property with respect to monotone Anti-Monge matrices. The corresponding solution permutations may also be considered as periodic extensions of the solution permutation π^* in Theorem 1.6.

1.5 Organization of the paper

In Section 2, the rather technical and complex proof of the main Theorem 1.6 will be given; the proofs of two underlying lemmas are contained in Appendices A and B. Section 3 deals with the Turbine Problem: First, we derive the mathematical formulation of the Turbine Problem as an Anti-Monge–Toeplitz QAP and then we prove that is is NP-hard. Section 4 discusses the applications of the main theorem to the Data Arrangement Problem and to the Travelling Salesman Problem. It also provides some historical information on these problems. In Section 5 the k-benevolent matrices are introduced as a periodic extension of benevolent matrices. It is shown that also these matrices have the constant permutation property with respect to monotone Anti-Monge matrices. Section 6 concludes the paper, with a summary, some conjectures and open questions.

2 Proof of the Main Result

In this section, Theorem 1.6 is proved. Without loss of generality, we assume that all entries in the matrices A and B are nonnegative. Otherwise, we may add a sufficiently large constant to all entries without changing the combinatorial structure of $\mathrm{QAP}(A, B)$. All matrices in this section will be of dimension $n \times n$, for some fixed $n \ge 3$. Notice that, if all diagonal entries of matrix B

are equal to a certain constant, the value of this constant does not influence the optimal solution of $\mathrm{QAP}(A,B)$ but only its optimal value, for any arbitrary matrix A. Thus, as the QAP we are investigating is the Anti-Monge–Toeplitz QAP, we may assume that all Toeplitz matrices in this section have 0-entries on the diagonal. Consequently, the equality $f(0)=0$ holds for any function f generating some Toeplitz matrix in this section. The proof of the theorem is quite involved although it is based on a simple idea. Before going into technicalities we describe the underlying idea.

2.1 A sketch of the proof

We want to proof that π^* solves $\mathrm{QAP}(A,B)$ when A is a non-negative monotone Anti-Monge matrix and B is a Toeplitz matrix generated by a benevolent function f with $f(0)=0$. We will actually prove an even stronger statement. Consider the relaxation of $\mathrm{QAP}(A,B)$ where the columns and the rows of matrix A may be permuted *independently of each other*, the columns according to some permutation ϕ and the rows according to some permutation ψ:

$$\min_{\phi,\psi\in S_n} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{\phi(i)\psi(j)} b_{ij} \tag{1}$$

We call this problem *independent-QAP*(A,B), and we denote its objective function by

$$Z(\phi,\psi,A,B)=\sum_{i=1}^{n}\sum_{j=1}^{n} a_{\phi(i)\psi(j)} b_{ij}.$$

For $\phi=\psi$ we have the objective function of the usual QAP, and we will then also use the simplified notation

$$Z(\phi,A,B):=Z(\phi,\phi,A,B).$$

Even for the relaxed problem, the independent-$\mathrm{QAP}(A,B)$, it will turn out that the double sum in (1) is minimized by $\phi=\psi=\pi^*$. This trivially guarantees that π^* is also an optimal solution of $\mathrm{QAP}(A,B)$. Thus, Theorem 1.6 can be derived as a corollary of the following theorem:

Theorem 2.1 *The pair of permutations (π^*,π^*) solves the independent-*$\mathrm{QAP}(A,B)$ *when A is a non-negative monotone Anti-Monge matrix and B is a symmetric Toeplitz matrix which is generated by a non-negative benevolent function f with $f(0)=0$.*

In the proof of the theorem, we will see that the set $\mathcal{A}$ of non-negative monotone Anti-Monge matrices is a cone, and likewise, the set $\mathcal{B}$ of benevolent matrices with zeros on the diagonal is a cone. It is sufficient to prove the theorem for the extreme rays of these cones, as we sill demonstrate shortly. It turns out that these extreme rays have a simple structure, which makes the proof tractable.

In Appendix A it is shown that the extreme rays of the cone $\mathcal{A}$ are generated by the 0-1-matrices $R^{(pq)}=(r_{ij}^{(pq)})$, $1\le p,q\le n$, defined by $r_{ij}^{(pq)}=1$ for $n-p+1\le i$ and $n-q+1\le j$, and $r_{ij}^{(pq)}=0$ otherwise. In other words, the matrix $R^{(pq)}$ has a $p\times q$ block of one entries in the lower right corner and zero entries elsewhere.

Lemma 2.2 *The monotone Anti-Monge matrices with nonnegative entries form a cone $\mathcal{A}$. The extreme rays of this cone are generated by the matrices $R^{(pq)}$.*

In Appendix B it is shown that extreme rays of the cone $\mathcal{B}$ are the 0-1 Toeplitz matrices generated by certain functions g^α and h^β, which map $\{-n+1,\dots,n-1\}$ to $\{0,1\}$ and which are defined as follows.

Definition 2.3 The functions g^α ($\lfloor \frac{n}{2} \rfloor + 1 \le \alpha \le n-1$) and h^β ($1 \le \beta \le \lfloor \frac{n}{2} \rfloor$) are defined by

$$g^\alpha(x) = \begin{cases} 1, & \text{for } x \in \{-\alpha, \alpha\}, \\ 0, & \text{for } x \notin \{-\alpha, \alpha\}. \end{cases} \qquad h^\beta(x) = \begin{cases} 1, & \text{for } \beta \le |x| \le n-\beta, \\ 0, & \text{otherwise.} \end{cases} \qquad \blacksquare$$

Example 2.1 For $n = 8$ the two functions $g^5, h^2 \colon \{-7, -6, \ldots, 6, 7\} \to \{0, 1\}$ generate the following Toeplitz matrices.

$$B(g^5) = \begin{pmatrix} 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \end{pmatrix} \qquad B(h^2) = \begin{pmatrix} 0&0&1&1&1&1&1&0 \\ 0&0&0&1&1&1&1&1 \\ 1&0&0&0&1&1&1&1 \\ 1&1&0&0&0&1&1&1 \\ 1&1&1&0&0&0&1&1 \\ 1&1&1&1&0&0&0&1 \\ 1&1&1&1&1&0&0&0 \\ 0&1&1&1&1&1&0&0 \end{pmatrix}$$

Observe the circular structure of the second matrix, which is typical for the functions h^β: Going from one row to the next corresponds to a circular right shift, and this remains true when returning from the last row to the first row. The same holds for the columns. $\blacksquare$

Lemma 2.4 *The benevolent functions $f \colon \{-n+1, \ldots, n-1\} \to \mathbb{R}$ with $f(0) = 0$ form a cone. The extreme rays of this cone are the functions g^α and h^β defined in Definition 2.3.*

Accordingly, the benevolent matrices with zeros on the diagonal form a cone $\mathcal{B}$, whose extreme rays are the Toeplitz matrices generated by the functions g^α and h^β.

Let $\mathcal{A}'$ be the set of n^2 matrices $R^{(pq)}$, for $1 \le p, q \le n$, and let $\mathcal{B}'$ be the set of the $n-1$ Toeplitz matrices generated by g^α, $\lfloor \frac{n}{2} \rfloor + 1 \le \alpha \le n-1$, and h^β, $1 \le \beta \le \lfloor \frac{n}{2} \rfloor$.

Now, the proof of Theorem 2.1 can be reduced to proving the optimality of (π^*, π^*) for the independent-QAP(A, B) with $A \in \mathcal{A}'$ and $B \in \mathcal{B}'$. This is seen by the following observation.

Observation 2.5 *Assume that independent-QAP(A_1, B) and independent-QAP(A_2, B) are both solved by the pair of permutations (π_0, ψ_0). Then for any two real numbers $k_1, k_2 \ge 0$, the problem independent-*QAP$(k_1 A_1 + k_2 A_2, B)$ *is also solved by (π_0, ψ_0).*

Proof. This follows from the equation

$$Z(\phi, \psi, k_1 A_1 + k_2 A_2, B) = k_1 Z(\phi, \psi, A_1, B) + k_2 Z(\phi, \psi, A_2, B),$$

which holds for arbitrary permutations ϕ and ψ. $\blacksquare$

Of course, an analogous statement holds for the linear combinations of the second matrix B. Concluding, our job is reduced to the proof of the following lemma:

Lemma 2.6 *If $1 \le p, q \le n$ and B is a Toeplitz matrix generated by g^α, $\lfloor \frac{n}{2} \rfloor + 1 \le \alpha \le n-1$, or h^β, $1 \le \beta \le \lfloor \frac{n}{2} \rfloor$, then the pair of permutations (π^*, π^*) is an optimal solution of the independent-QAP$(R^{(pq)}, B)$.*

2.2 The independent-QAP(A, B) for $A = R^{(pq)}$

In this subsection we reformulate the independent-QAP$(R^{(pq)}, B)$, where A is one of the matrices $R^{(pq)}$, $1 \le p, q \le n$, as a *selection* problem which is more convenient to handle. This reformulation is possible due to the specific structure of the matrices $R^{(pq)}$.

For $A = R^{(pq)}$, the objective function $Z(\phi, \psi, A, B)$ of the independent-QAP$(R^{(pq)}, B)$ is the sum of $p \cdot q$ entries of the matrix B. These entries lie in p rows and q columns. The selected rows are those which are mapped to the last p rows $n-p+1,\ldots,n$ of A by the permutation ϕ, and the selected columns are those which are mapped to the last q columns $n-q+1,\ldots,n-1,n$ of A by the permutation ψ. So the independent-QAP$(R^{(pq)},B)$ amounts to selecting p rows and q columns from the matrix B such that the total sum of all pq selected entries is minimized. Thus, our ultimate goal, translated in the language of the selection problem, is to prove the following lemma:

Lemma 2.7 (The optimal selection lemma)
Let $1 \le p,q \le n$. Let B be a non-negative Toeplitz matrix generated by g^α, $\lfloor \frac{n}{2} \rfloor + 1 \le \alpha \le n-1$, or h^β, $1 \le \beta \le \lfloor \frac{n}{2} \rfloor$. Suppose that p rows and q columns of the matrix B have to be selected such that the total sum of all pq selected entries of B is minimized. Then it is optimal to select the last p elements of the sequence

$$1,\, n,\, 2,\, n-1,\, 3,\, \ldots \tag{2}$$

as row indices and the last q elements of this sequence as column indices.

Claim *Lemmas 2.6 and 2.7 are equivalent.*

Proof. Consider a feasible solution (ϕ,ψ) of the independent-QAP$(R^{(pq)},B)$. Clearly, row i of $R^{(pq)}$ is mapped to row $\phi^{-1}(i)$ of B. Thus, the rows of $R^{(pq)}$ which contain the one-entries are mapped to the last p elements of the sequence

$$\phi^{-1}(1),\, \phi^{-1}(2),\ldots,\, \phi^{-1}(n).$$

This sequence coincides with the sequence given in (2) for all p,q precisely when $\phi=\pi^*$, and hence permuting the rows of $R^{(pq)}$ according to π^* is equivalent to the selection of the p last elements in (2) as row indices. The same argument applies to the columns. ■

So we finally have reduced the proof of Theorem 2.1 to the optimal selection lemma. For definiteness, let us write down the rows and columns to be selected according to the above lemma. The claimed optimal solution selects the p rows from

$$p_1 := \left\lceil \frac{n-p}{2} \right\rceil + 1 \quad \text{to} \quad p_2 := n - \left\lfloor \frac{n-p}{2} \right\rfloor \tag{3}$$

and the q columns from

$$q_1 := \left\lceil \frac{n-q}{2} \right\rceil + 1 \quad \text{to} \quad q_2 := n - \left\lfloor \frac{n-q}{2} \right\rfloor. \tag{4}$$

2.3 Proof of the optimal selection lemma

In this section we prove Lemma 2.7 by splitting it in two parts. Namely, we consider separately the cases where the Toeplitz matrix B is generated by one of the functions g^α or one of the functions h^β. See Appendix A for examples of such Toeplitz matrices. Let us start with Toeplitz matrices generated by functions g^α, $\lfloor \frac{n}{2} \rfloor + 1 \le \alpha \le n-1$.

Lemma 2.8 *For any $1 \le p,q \le n$ and any $\lfloor \frac{n}{2} \rfloor + 1 \le \alpha \le n-1$, the independent-*QAP$(A,B)$ *with $A = R^{(pq)}$ and B the symmetric Toeplitz matrix generated by g^α, is solved to optimality by the pair of permutations (π^*,π^*).*

In other words, selecting p rows and q columns from B according to Lemma 2.7 minimizes the sum of the selected elements.

Proof. We first show that the value of the objective function must be at least $p + q - 2\alpha$, and we then show that our claimed solution achieves the value $\max\{0, p + q - 2\alpha\}$.

Instead of selecting p rows and q columns of the matrix B, let us take the opposite view and delete $n-p$ rows and $n-q$ columns from B. The matrix B originally contains $2(n-\alpha)$ one-entries, and no two of them are in the same row or in the same column. Thus, by deleting $n-p$ rows and $n-q$ columns we may delete at most $(n-p)+(n-q)$ ones, which gives an easy lower bound of $2(n-\alpha) - ((n-p)+(n-q)) = p + q - 2\alpha$ for the remaining number of one-entries.

Let us now compute the objective function for our claimed optimal solution. We have selected columns with indices in the interval $[q_1, q_2]$ where q_1 and q_2 are given by (4). Column j contains at most one 1-entry, namely in row $j+\alpha$ or $j-\alpha$, whenever this index falls in the range $[1, n]$. (At most one of the two cases can occur for a given j.) Thus the selected columns contain 1-entries in the rows with indices in the set

$$([q_1 - \alpha, q_2 - \alpha] \cup [q_1 + \alpha, q_2 + \alpha]) \cap [1, n]$$

We have to intersect this set with the interval $[p_1, p_2]$ of selected rows given by (3) to get the number of selected 1-entries, i. e. the value of the claimed solution:

$$\Big|([q_1-\alpha, q_2-\alpha] \cup [q_1+\alpha, q_2+\alpha]) \cap [p_1, p_2]\Big| = \Big|([q_1-\alpha, q_2-\alpha] \cap [p_1, p_2]) \cup ([q_1+\alpha, q_2+\alpha] \cap [p_1, p_2])\Big|$$

Since $p_1, q_1 \leq \frac{n+1}{2} \leq p_2, q_2$ and $\alpha \geq \frac{n+1}{2}$, the two sets in the last union are disjoint and we may add their cardinalities. Moreover, we know the endpoints of the two intersections in case they are non-empty:

$$\big|[q_1 - \alpha, q_2 - \alpha] \cap [p_1, p_2]\big| = \max\{0,\ q_2 - \alpha - p_1 + 1\} = \max\{0,\ \lfloor \tfrac{p}{2} \rfloor + \lceil \tfrac{q}{2} \rceil - \alpha\}$$

and

$$\big|[q_1 + \alpha, q_2 + \alpha] \cap [p_1, p_2]\big| = \max\{0,\ p_2 - (q_1 + \alpha) + 1\} = \max\{0,\ \lceil \tfrac{p}{2} \rceil + \lfloor \tfrac{q}{2} \rfloor - \alpha\}.$$

For $p+q \leq 2\alpha$ both expressions are 0. For $p+q \geq 2\alpha$ the two expressions that have to be compared with 0 are non-negative and this yields

$$(\lfloor \tfrac{p}{2} \rfloor + \lceil \tfrac{q}{2} \rceil - \alpha) + (\lceil \tfrac{p}{2} \rceil + \lfloor \tfrac{q}{2} \rfloor - \alpha) = p + q - 2\alpha. \qquad \blacksquare$$

Now let us turn to the benevolent functions h^β generating the other extreme rays of the benevolent matrices with zeros on the diagonal:

Lemma 2.9 *For any $1 \leq p, q \leq n$ and any $1 \leq \beta \leq \lfloor \frac{n}{2} \rfloor$, the independent-QAP$(A, B)$ with $A = R^{(pq)}$ and B being the symmetric Toeplitz matrix generated by h^β, is solved to optimality by the pair of permutations (π^*, π^*).*

In other words, selecting p rows and q columns from B according to Lemma 2.7 minimizes the sum of the selected elements.

For the proof it will be convenient to work with a certain quantity $Q(a, m)$. We define it here and list some properties of it.

Definition 2.10 For integers $a, m \geq 0$, the quantity $Q(a, m)$ denotes the sum of the first a terms of the following sum:

$$\min\{1, m\} + \min\{1, m\} + \min\{2, m\} + \min\{2, m\} + \min\{3, m\} + \min\{3, m\} + \cdots \qquad \blacksquare$$

Lemma 2.11 *For fixed m, the difference $Q(a+1,m) - Q(a,m)$ increases with a. Therefore the function $Q(a,m)$ is a convex function in a with $Q(0,m) = 0$, and so we have in particular*

$$Q(a_1, m) + Q(a_2, m) \le Q(a_1 + a_2, m). \tag{5}$$

We also have

$$Q(a+1, m) \le Q(a, m) + m. \tag{6}$$

An explicit representation of $Q(a,m)$ is

$$Q(a,m) = \begin{cases} \left\lfloor \left(\frac{a+1}{2}\right)^2 \right\rfloor, & \text{for } a < 2m-2, \\ m(a+1-m), & \text{for } a \ge 2m-2. \end{cases}$$

Proof. Only the last expression is not obvious, but it can be checked by elementary calculations, which we omit. ■

Through the rest of this section B is a Toeplitz matrix generated by the function h^β for some $1 \le \beta \le \lfloor \frac{n}{2} \rfloor$. Each column of B has a very simple structure. If we arrange the row indices in a circular sequence $1, 2, \dots, n, 1, 2, \dots$, the one-entries in column j form a single circular interval of length $n - 2\beta + 1 =: \gamma$. This circular interval of ones starts at row $j + \beta$ and ends at row $j + n - \beta$, wrapping around from row n to row 1 if necessary. Notice that all indices are taken modulo n throughout this section. Next we present three simple observations to be used in the proof of Lemma 2.9.

Observation 2.12 *Let B be a Toeplitz matrix generated by the function h^β for some $1 \le \beta \le \lfloor \frac{n}{2} \rfloor$. Suppose we have selected a certain set of q columns from the matrix B and denote by x_i, $1 \le i \le n$, the sum of the entries in the i-th row in the selected columns. Then, the numbers (x_i), $1 \le i \le n$, fulfill the following three conditions:*

(i) $0 \le x_i \le q$

(ii) $q + \gamma - n \le x_i \le \gamma$

(iii) $|x_i - x_{i-1}| \le 1$

Moreover, we have $\sum_{i=1}^n x_i = q\gamma$.

Proof. Condition (i) is trivially fulfilled for all $1 \le i \le n$. The upper bound in (ii) follows because γ is the row sum of the whole matrix. Similarly, the lower bound $q + \gamma - n$ can be seen, because we remove $n - q$ columns from the whole matrix. Condition (iii) follows from the fact that there is only one column which has a 0 in row $i - 1$ and and 1 in row i; and there is only one column with a 1 in row $i - 1$ and and 0 in row i, for every i. (Recall that all indices are taken modulo n, and so x_0 denotes the same variable as x_n.) Finally, we have $\sum_{i=1}^n x_i = q\gamma$, because each of the q columns contains γ 1-entries. ■

Let us look at the vector x_i for our proposed optimal solution. Figure 2 shows such a vector for the case $n = 19$, $q = 5$, and $\gamma = 8$. Since we select q "adjacent" columns, the circular sequence of values x_i consists of two horizontal intervals connected by a rising and a falling interval of slope ± 1.

It takes a bit of attention to analyze this example in detail and to make then the following general statements about the two horizontal pieces:

Observation 2.13 *Assume that we have selected p columns and q rows from matrix B, as specified by Lemma 2.7 and let (x_i) be the corresponding vector defined as in Observation 2.12. Then the following four statements hold:*

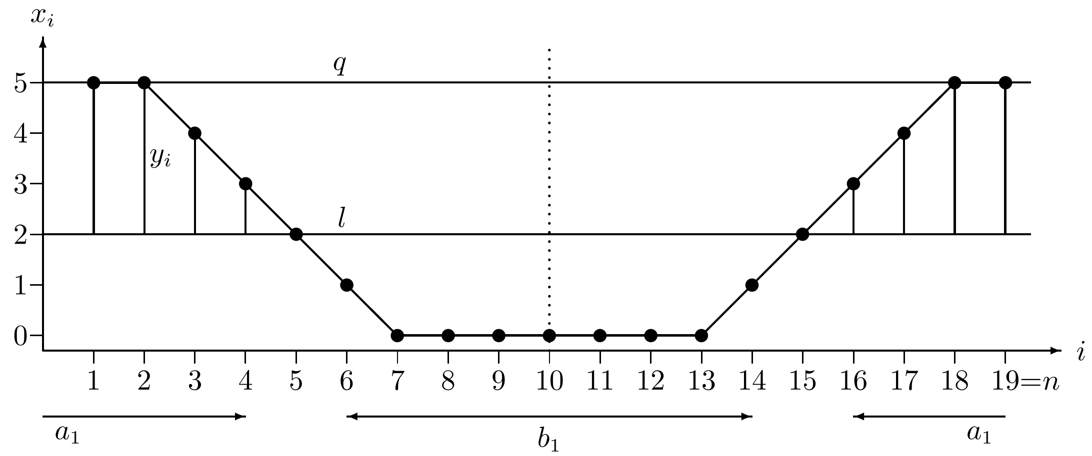

Figure 2: *The vector (x_i) for the optimal selection of columns for an example with $n = 19$, $q = 5$, and $\gamma = 8$ ($\beta = 6$). The indices of the selected columns are 8, 9, 10, 11 and 12. If $p = 10$ or 11, the p-smallest x_i has the value $l = 2$ indicated by the horizontal line.*

(U1) *If $q \geq \gamma$ the maximum value of x_i is γ and it occurs for $q - \gamma + 1$ adjacent positions (rows).*

(U2) *If $q \leq \gamma$ the maximum value of x_i is q, and it occurs for $\gamma - q + 1$ adjacent positions.*

(L1) *If $q + \gamma \leq n$, the minimum value of x_i is 0, and it occurs for $n - q - \gamma + 1$ adjacent positions.*

(L2) *If $q + \gamma \geq n$, the minimum value of x_i is $q + \gamma - n$, and it occurs for $q + \gamma - n + 1$ adjacent positions.* ∎

Let us denote by z^* the sum of the selected entries when the rows and the columns are selected as specified by Lemma 2.7. The value z^* can be computed easily as shown by the following observation:

Lemma 2.14 *Assume that we select p rows and q columns form the Toeplitz matrix B generated by some function h^β in the way described in Lemma 2.7. Then, the sum z^* of the selected entries is given as follows:*

$$z^* = \begin{cases} 0 & \text{if } q+\gamma \leq n \text{ and } p \leq n-q-\gamma+1 \\ p(q+\gamma-n) & \text{if } q+\gamma \geq n \text{ and } p \leq q+\gamma-n+1 \\ (p+q-n)\gamma & \text{if } q \geq \gamma \text{ and } p \geq n-(q-\gamma+1) \\ (p+\gamma-n)q & \text{if } q \leq \gamma \text{ and } p \geq n-(\gamma-q+1) \\ \left\lfloor \left(\frac{p+q+\gamma-n}{2}\right)^2 \right\rfloor & \text{if } |n-q-\gamma|+1 < p < n-|q-\gamma|-1 \end{cases}$$

Proof. The key remark in the proof is that the p rows selected according to Lemma 2.7 are really the rows with the p smallest sums of entries in the selected columns. (It takes only a bit of care to check this fact.) Then, the straightforward but tedious proof is completed by computing z^* in each of the following five cases:

Case 1. $q + \gamma \leq n$ and $p \leq n - q - \gamma + 1$
Case 2. $q + \gamma \geq n$ and $p \leq q + \gamma - n + 1$
Case 3. $q \geq \gamma$ and $p \geq n - (q - \gamma + 1)$
Case 4. $q \leq \gamma$ and $p \geq n - (\gamma - q + 1)$
Case 5. $|n - q - \gamma| + 1 < p < n - |q - \gamma| - 1$

In both Case 1 and Case 2 the minimum values of x_i, $1 \leq i \leq n$, as given in (L1) and (L2), equal 0 and $q + \gamma - n$, respectively. Moreover, these values occur for $n - q - \gamma + 1$ and

$q+\gamma-n+1$ adjacent rows, respectively. Now p has a small enough value ($p \le n-q-\gamma+1$ and $p \le q+\gamma-n+1$, respectively) so that only rows with minimal values of x_i are selected. Thus, $z^* = 0$ and $z^* = p(q+\gamma-n)$, respectively.

Similarly, in both Case 3. and Case 4. the maximum values of x_i, $1 \le i \le n$, as given in (U1) and (U2) equal γ and q, respectively. These maxima occur for $q-\gamma+1$ and $\gamma-q+1$ adjacent rows, respectively. Now p has a large enough value ($p \ge n-(q-\gamma+1)$ and $p \ge n-(\gamma-q+1)$, respectively) so that only rows with maximal values of x_i are not selected. As the sum of all entries in the selected columns is $q\gamma$, we have $z^* = q\gamma-(n-p)\gamma = (p+q-n)\gamma$ in Case 3 and $z^* = q\gamma-(n-p)q = (p+\gamma-n)q$ in Case 4.

In the remaining Case 5 z^* is evaluated as follows: If $q+\gamma \le n$, the minimal value of x_i is 0 and it is taken by $n-q-\gamma+1$ elements x_i. Thus

$$z^* = (n-q-\gamma+1) \times 0 + [1+1+2+2+3+3+\cdots],$$

where $p-(n-q-\gamma+1)$ numbers are taken from the sum in brackets. Now Lemma 2.11 yields

$$z^* = Q(p+q+\gamma-n-1, \infty) = \left\lfloor \left(\frac{p+q+\gamma-n}{2}\right)^2 \right\rfloor. \tag{7}$$

If $q+\gamma \ge n$, every x_i is at least $q+\gamma-n$, according to (L2), and this value is taken by $q+\gamma-n+1$ elements x_i. Summing separately the excess of x_i over $q+\gamma-n$, we can write

$$z^* = p \times (q+\gamma-n) + (q+\gamma-n+1) \times 0 + [1+1+2+2+3+3+\cdots],$$

where $p-(q+\gamma-n+1)$ numbers are taken from the sum in brackets. This yields

$$\begin{aligned} z^* &= p \times (q+\gamma-n) + Q(p-(q+\gamma-n+1), \infty) \\ &= \left\lfloor \frac{4p(q+\gamma-n)+(p-(q+\gamma-n))^2}{4} \right\rfloor = \left\lfloor \frac{(p+q+\gamma-n)^2}{4} \right\rfloor, \end{aligned}$$

which coincides with (7). ∎

Proof of Lemma 2.9. It is sufficient to show that the sum z^* of entries selected as described by Lemma 2.7, is a lower bound for the sum of selected entries in any arbitrary selection.

Let us select q arbitrary columns and consider the corresponding vectors x_i. In order to minimize the sum of the entries, we have to select the p rows with the smallest x_i values. Let z be the sum of the resulting selected entries. By (i) and (ii) in Observation 2.12 we get

$$p \cdot \max\{0, q+\gamma-n\} \le z \le p \cdot \min\{q, \gamma\}.$$

If Case 1 or Case 2 of Lemma 2.14 occurs, we have $z^* = p\max\{0, q+\gamma-n\} \le z$ and thus the claim is true.

Considering the inequalities (i) and (ii) and the equality $\sum_{i=1}^n x_i = q\gamma$ in Observation 2.12 we get

$$z \ge q\gamma - (n-p)\min\{q, \gamma\}$$

For Case 3 or Case 4 of Lemma 2.14, we have $z^* = q\gamma-(n-p)\min\{q,\gamma\} \le z$. Thus, it remains to prove the inequality $z^* \le z$ in Case 5, where $|n-q-\gamma|+1 < p < n-|q-\gamma|-1$.

Let l denote the p-smallest element of the n elements x_i, which is at the same time the $(n-p+1)$-largest element, and let l' denote the $(p+1)$-smallest (the $(n-p)$-largest) of the elements x_i. Let z_{small} denote the sum of the p smallest x_i values, and let $I_{\text{small}} \subseteq \{1, \ldots, n\}$ denote the index set of the p smallest x_i-values. (This set is not uniquely defined if several x_i's are equal to l. We can resolve ties arbitrarily.) Let $I_{\text{large}} = \{1, \dots, n\} - I_{\text{small}}$ be the complementary index set of the $n-p$ largest x_i values, and let $z_{\text{large}} = q\gamma - z_{\text{small}}$ denote the sum of these values. We would like to bound z_{small} from below, or, what amounts to the same thing, to bound z_{large} from above.

Let us look at a maximal block of consecutive elements x_i in the circular sequence which are larger than l:

$$x_{i_0} = l, \quad x_{i_0+1}, x_{i_0+2}, \dots, x_{i_0+a_j} > l, \quad x_{i_0+a_j+1} = l \tag{8}$$

(Recall that all indices are interpreted modulo n. The block length $a_j = n-1$ is permitted.) Let $a_1, a_2, \dots, a_k$ denote the lengths of all these blocks. By definition, all these $a := a_1 + a_2 + \cdots + a_k$ elements belong to I_{large} and so we have $a \le n-p$. For $i \in I_{\text{large}}$ we substitute $y_i = x_i - l \ge 0$. These quantities are indicated by vertical lines in Figure 2. We have

$$z_{\text{large}} = \sum_{i \in I_{\text{large}}} x_i = (n-p)l + \sum_{i \in I_{\text{large}}} y_i.$$

Let us look at one block of length a_j as defined in (8). The maximum possible sum $y_{i_0+1} + y_{i_0+2} + \cdots + y_{i_0+a_j}$ of such a block can be estimated by setting up a linear program with the constraints coming from (i), (ii), and (iii) in Observation 2.12:

$$\max \Big\{ \sum_{i=1}^{a} \hat{y}_i \;\Big|\; \hat{y}_0 = \hat{y}_{a+1} = 0;\ 0 \le \hat{y}_i \le m \text{ and } |\hat{y}_i - \hat{y}_{i-1}| \le 1 \text{ for } 1 \le i \le a+1 \Big\},$$

with $m := \min\{q, \gamma\} - l$. In this linear program, the variables $\hat{y}_i$ represent the possible values for y_{i_0+i}. It is easy to see (cf. Figure 2) that the optimal value is equal to the first a_j terms of the sum:

$$\min\{1,m\} + \min\{1,m\} + \min\{2,m\} + \min\{2,m\} + \min\{3,m\} + \min\{3,m\} + \cdots,$$

which we defined as $Q(a_j, m)$. Thus we get

$$y_{i_0+1} + y_{i_0+2} + \cdots + y_{i_0+a_j} \le Q(a_j, \min\{q,\gamma\} - l).$$

By summing over all k blocks we get the following bound

$$z_{\text{large}} \le (n-p)l + \sum_{j=1}^{k} Q(a_j, \min\{q,\gamma\} - l).$$

By repeatedly using the relation $Q(a_1, m) + Q(a_2, m) \le Q(a_1 + a_2, m)$ from (5), we can simplify this to our first essential bound:

$$z_{\text{large}} \le (n-p)l + Q(a, \min\{q,\gamma\} - l) \le (n-p)l + Q(n-p, \min\{q,\gamma\} - l)$$

We can apply a similar reasoning to l' instead of l. The indices i with $x_i \ge l'$ still include all elements of I_{large}. The only difference is that for one of the $n-p$ elements $i \in I_{\text{large}}$ we must surely have $x_i = l'$, and therefore the number of x_i which are strictly larger than l' is at most $n-p-1$. So we get a second bound

$$z_{\text{large}} \le (n-p)l' + Q(a, \min\{q,\gamma\} - l') \le (n-p)l' + Q(n-p-1, \min\{q,\gamma\} - l').$$

Now we apply analogous considerations to z_{small}. For the p elements $i \in I_{\text{small}}$ we introduce the non-negative quantities $y_i = l - x_i$ or $y_i = l' - x_i$, respectively, and their sum can be bounded in terms of the Q function. The roles of l and l' are now reversed: The number of x_i which are strictly smaller than l is at most $p-1$, and the number of x_i which are strictly smaller than l' is at most p. This gives the following bounds.

$$z_{\text{small}} \geq pl' - Q(p,\, l' - \max\{0, q+\gamma-n\})$$
$$z_{\text{small}} \geq pl - Q(p-1,\, l - \max\{0, q+\gamma-n\})$$

Using the relation $z_{\text{small}} = q\gamma - z_{\text{large}}$ we thus get the following four lower bounds on z.

$$\begin{aligned} U_1^-(l) &= pl - Q(p-1,\, l - \max\{0, q+\gamma-n\}) \\ U_2^-(l') &= pl' - Q(p,\, l' - \max\{0, q+\gamma-n\}) \\ U_1^+(l') &= q\gamma - (n-p)l' - Q(n-p-1,\, \min\{q,\gamma\} - l') \\ U_2^+(l) &= q\gamma - (n-p)l - Q(n-p,\, \min\{q,\gamma\} - l) \end{aligned} \tag{9}$$

These bounds may be combined into one lower bound, which depends on the quantities l and l'. Since these are unknown, we have to minimize over all choices of l and l', subject only to the constraints $l \leq l' \leq l+1$. The inequality $l' \leq l+1$ follows from (iii) in Observation 2.12. Thus,

$$z^* \geq \min_{\substack{l \leq l' \leq l+1 \\ \max\{0,q+\gamma-n\} \leq l \leq l' \leq \min\{q,\gamma\}}} \max\{U^-(l,l'), U^+(l,l')\}, \tag{10}$$

where

$$U^-(l,l') := \max\{U_1^-(l), U_2^-(l')\} \quad \text{and} \quad U^+(l,l') := \max\{U_1^+(l'), U_2^+(l)\}.$$

From $Q(a+1,m) \leq Q(a,m)+m$ (see (6)) it follows that $U_1^-(l)$, $U_2^-(l')$, and hence $U^-(l,l')$, are nondecreasing in l and in l', and similarly $U^+(l,l')$ is nonincreasing in l and l'. Thus, if we consider the possible pairs (l,l') in the order

$$(0,0),\ (0,1),\ (1,1),\ (1,2),\ (2,2),\ (2,3),\ (3,3), \dots,$$

then U^- increases and U^+ decreases. It suffices therefore to exhibit a pair (l,l') for which $U^-(l,l') = U^+(l,l')$ in order to identify the point in (10) where the minimum occurs, and to thus produce a valid lower bound for z. We will show that the pair of values

$$l = \left\lfloor \frac{p+q+\gamma-n}{2} \right\rfloor \text{ and } l' = \left\lceil \frac{p+q+\gamma-n}{2} \right\rceil$$

has this property, and the resulting bound is equal to the value z^* of our proposed optimal solution computed in (7). This will involve some calculations, which will complete the proof. We must distinguish two cases.

Case 1. $p+q+\gamma+n \equiv 0 \pmod 2$. We have $l = l' = (p+q+\gamma-n)/2$. In this case the two expressions $U_1^-(l)$ and $U_2^-(l')$ differ only in the first argument of the function Q. Using the monotonicity of Q in its first argument we conclude that $U^-(l,l') = U_1^-(l)$ and similarly that $U^+(l,l') = U_1^+(l')$.

The expressions which occur as second arguments to Q in (9) can be expressed as follows.

$$m^- := l - \max\{0, q+\gamma-n\} = \frac{p - |q+\gamma-n|}{2} \quad \text{and} \quad m^+ := \min\{q,\gamma\} - l' = \frac{n-p-|q-\gamma|}{2}.$$

These arguments always fall in the range where the function Q is linear. We have to check that $p-1 \geq 2m^- - 2$ and $n-p-1 \geq 2m^+ - 2$. These inequalities are obviously fulfilled. We can therefore use the expression $Q(a,m) = m(a+1-m)$ for computing both $U_1^-(l)$ and $U_1^+(l')$. Substitution of l and m^- into the respective formulas (9) yields

$$\begin{aligned}U^-(l,l') = U_1^-(l) &= \frac{p(p+q+\gamma-n)}{2} - \frac{p-|q+\gamma-n|}{2}\cdot\Big(p - \frac{p-|q+\gamma-n|}{2}\Big)\\ &= \frac{p(p+q+\gamma-n)}{2} - \frac{(p-|q+\gamma-n|)\cdot(p+|q+\gamma-n|)}{4}\\ &= \frac{2p^2+2p(q+\gamma-n)-p^2+(q+\gamma-n)^2}{4} = \Big(\frac{p+q+\gamma-n}{2}\Big)^2\end{aligned}$$

$U^+(l,l') = U_1^+(l')$ can be evaluated in a similar way and it yields the same result. This concludes the proof for the first case.

Case 2. $p+q+\gamma+n \equiv 1 \pmod 2$. We have $l = (p+q+\gamma-n-1)/2$ and $l' = l+1$. Again it can be checked easily that the four expressions $m^- := \frac{p-|q+\gamma-n|-1}{2}$, m^-+1, $m^+ := \frac{n-p-|q-\gamma|+1}{2}$, and m^+-1, which occur as the second arguments to Q in (9), always fall in the range where the function Q is linear.

The two expressions $U_1^-(l)$ and $U_2^-(l')$ can be compared as follows.

$$U_2^-(l') - U_1^-(l) = p - Q(p, m^-+1) + Q(p-1, m^-)$$

Using the equation $Q(p, m^-+1) - Q(p-1, m^-) = p - m$ we conclude that this difference is non-negative, and hence $U^-(l,l') = U_2^-(l')$ and similarly that $U^+(l,l') = U_2^+(l)$. Both expressions can be evaluated just as in case 1 and yield the same result:

$$U^-(l,l') = U^+(l,l') = \frac{(p+q+\gamma-n)^2-1}{4}$$

This concludes the proof for the second case. Since the lower bound for z which we have just proved coincides with the corresponding value of z^* given in (7) the proof of the lemma is complete. ■

A different proof of Lemma 2.9 can be given by using a result of Çela and Woeginger (Theorem 4.1 in [8]). They showed, translated into the language of Lemma 2.7, that there exists an optimal solution of the selection problem which selects a block of p (cyclically) *adjacent* rows and a block of q *adjacent* columns. Their result is formulated in a graph-theoretic setting, and the proof is based on an exchange argument.

Notice that the value of z^*, which is the optimal value of the independent-QAP$(R^{(pq)}, B)$ for a Toeplitz matrix B generated by some function h^β, can be given by a closed-form expression. Let $N := \max\{0,\ p+q+\gamma-n\}$ and $k := \min\{p, q, \gamma, \lfloor N/2\rfloor\}$. Then, it is easy to check that

$$z^* = N\cdot(N-k).$$

It is perhaps astonishing that this formula is completely symmetric in p, q, and γ.

3 The Turbine Problem

Hydraulic turbine runners as used in electricity generation consist of a cylinder around which a number of blades are welded at regular spacings. Due to imprecisions in the manufacturing process, the weights of these blades differ slightly, and it is desirable to locate the blades around the cylinder in such a way that the distance between the center of mass of the blades and the axis of the cylinder is minimized. This problem was initially introduced by Bolotnikov [2] in 1978. who gave a formulation of this problem as a QAP. Without being able to solve the problem optimally, Bolotnikov proposed a heuristic for it. Other heuristics were proposed later on by Stoyan, Sokolovskii and Yakovlev [23]. Apparently, the first occurrence of this problem in the Western literature is due to Mosevich [19] in 1986. Laporte and Mercure [14] observed that this problem can be formulated as a QAP in the following way.

The places at regular spacings on the cylinder are modeled by the vertices $v_1, \dots, v_n$ of a regular n-gon on the unit circle in the Euclidean plane, i. e. the points with coordinates

$$v_i = \Big(\sin\frac{2i\pi}{n}, \cos\frac{2i\pi}{n}\Big),$$

$1 \le i \le n$. The masses of the n blades are given by the positive reals $0 < m_1 \le m_2 \le \cdots \le m_n$. The goal is to assign the n masses to the n vertices in such a way that the center of gravity of the resulting mass system is as close to the origin as possible, i. e. to find a permutation $\phi \in S_n$ that minimizes the Euclidean norm of the vector

$$\sum_{i=1}^{n} m_{\phi(i)} \begin{pmatrix} \sin\frac{2i\pi}{n} \\ \cos\frac{2i\pi}{n} \end{pmatrix}.$$

An easy calculation reveals that minimizing the Euclidean norm of this vector is equivalent to minimizing the expression

$$\sum_{i=1}^{n}\sum_{j=1}^{n} m_{\phi(i)} m_{\phi(j)} \cos\frac{2(i-j)\pi}{n}. \tag{11}$$

This is a quadratic assignment problem QAP(A,B). Note that the matrix $A = (a_{ij})$ defined by $a_{ij} = m_i \cdot m_j$ is a product matrix and therefore a monotone Anti-Monge matrix, since the masses m_i are sorted in increasing order. The matrix $B = (b_{ij})$ defined by $b_{ij} = \cos\frac{2(i-j)\pi}{n}$ is a symmetric Toeplitz matrix. Note that the function $f(i) = \cos(2\pi i/n)$ which generates B is not benevolent, whereas the function $f(i) = -\cos(2\pi i/n)$ which generates $-B$ is benevolent. Therefore the original turbine problem does not fall under Theorem 1.6.

Several heuristics for this problem have been proposed and tested by a number of authors, see the papers [2, 14, 22, 23]. However, no fast (polynomial time) exact solution algorithm has been derived till today. In Theorem 3.4, we show that this is not a coincidence because the problem is in fact NP-hard.

Since, on the other hand, matrix $-B$ is benevolent, we know that it has the constant permutation property, and therefore QAP$(A,-B)$ is easy to solve. This corresponds to the *maximization* of (11). In the context of the Turbine Problem this means that the goal is to get the center of gravity of the mass system as far away from the origin as possible. Thus, Theorem 1.6 implies the following result.

Corollary 3.1 *The maximization version of the Turbine Problem, i. e. maximization of (11) over all permutations $\phi \in S_n$, is solved to optimality by permutation π^*.* ∎

We next show that the original turbine problem, i. e. the minimization of (11), is NP-hard. The following simple result is needed for our proof.

Lemma 3.2 *Let $a_1, \dots, a_{2k}$ and $b_1, \dots, b_{2k}$ be real numbers, where $a_{2i-1}, a_{2i} < a_{2i+1}, a_{2i+2}$ holds for all $1 \le i \le k-1$ and $b_i = b_{2k+1-i}$ and $b_i > b_{i+1}$ for $1 \le i \le k$. Then the set of permutations $\pi \in S_{2k}$ which minimize the expression*

$$\sum_{i=1}^{2k} a_{\pi(i)} b_i$$

contains precisely those permutations for which $\{\pi(2i-1), \pi(2i)\} = \{i, 2k+1-i\}$ holds for all $1 \le i \le k$.

Proof. This statement is an extension of the following well-known result and can be proved very easily, for example by an exchange argument.

Proposition 3.3 (Hardy, Littlewood and Pólya [11, pp. 260])
Let the real numbers $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be given. Then the inequality

$$\sum_{i=1}^{n} a_i b_{n-i+1} \leq \sum_{i=1}^{n} a_i b_{\phi(i)}$$

holds for any permutation $\phi \in S_n$. ∎

Theorem 3.4 *The Turbine Problem, i. e. minimization of* (11) *over all permutations $\phi \in S_n$, is an NP-hard problem.*

Proof. The proof is a reduction from the NP-complete EVEN-ODD PARTITION problem, (cf. Garey and Johnson [10]):

Problem: EVEN-ODD PARTITION

Instance: $2k$ positive integers $x_1, x_2, \ldots, x_{2k}$.

Question: Is there a subset $I \subseteq \{1, 2, \ldots, 2k\}$, $|I| = k$, such that the $|I \cap \{2i-1, 2i\}| = 1$ for every $i = 1, \ldots, k$, and $\sum_{i \in I} x_i = \sum_{i \notin I} x_i$?

Without loss of generality we may assume $x_{2i-1} \leq x_{2i}$ for $1 \leq i \leq k$. For the convenience of presentation, define for $1 \leq i \leq 2k+1$ the numbers

$$\alpha_i := 2i\pi/(2k+1)$$

and for $1 \leq i \leq k$ the numbers y_i by

$$y_{2i-1} = x_{2i-1}/\sin\alpha_i \quad \text{and} \quad y_{2i} = x_{2i}/\sin\alpha_i.$$

Note that $0 < \alpha_i < \pi$ holds for $1 \leq i \leq k$, and thus all numbers y_i are positive, $1 \leq i \leq 2k$. Finally, let $S = \sum_{i=1}^{2k} y_i$.

Consider the following instance of the Turbine Problem. The number n of vertices on the unit circle is $2k+1$. The first $2k$ of the masses are defined by $m_{2i-1} = iS + y_{2i-1}$ and $m_{2i} = iS + y_{2i}$. Observe that by this definition

$$m_1 \leq m_2 < m_3 \leq m_4 < \cdots < m_{2i-1} \leq m_{2i} < m_{2i+1} \leq m_{2i+2} < \cdots < m_{2k-1} \leq m_{2k}$$

The value of mass $m_{2k+1} > 0$ is defined by the equation

$$m_{2k+1} + \sum_{i=1}^{k} (m_{2i-1} + m_{2i}) \cos\alpha_i = 0 \qquad (12)$$

The claim is that the thereby defined instance of the Turbine Problem allows a mass assignment with center of gravity in the origin if and only if the instance of EVEN-ODD PARTITION has answer "Yes".

Without loss of generality assume that mass m_{2k+1} is assigned to vertex $v_{2k+1} = (1, 0)$. This induces a strong momentum towards the positive x-axis. To balance this momentum, we claim that the masses $\{m_{2i-1}, m_{2i}\}$ must be assigned to the two vertices $\{v_i, v_{2k+1-i}\}$ (in any order), for all $1 \le i \le k$. Let us first consider the x-coordinate of the center of gravity. It is given by the formula

$$m_{2k+1} + \sum_{i=1}^{k} (m_{\phi(i)} + m_{\phi(2k+1-i)}) \cos \alpha_i.$$

The condition that this x-coordinate equals 0, together with (12), yields

$$\sum_{i=1}^{k} (m_{2i-1} + m_{2i}) \cos \alpha_i = \sum_{i=1}^{k} (m_{\phi(i)} + m_{\phi(2k+1-i)}) \cos \alpha_i \tag{13}$$

Note that the following relationships hold for the $\cos \alpha_i$, $1 \le i \le 2k$: $\cos \alpha_i = \cos \alpha_{2n+1-i}$ for $1 \le i \le n$, and $\cos \alpha_i > \cos \alpha_{i+1}$ for $1 \le i \le n-1$. Hence, the conditions of Lemma 3.2 are fulfilled. Applying this lemma we conclude that the left side of (13) gives the minimum of the expression on the right side over all $\phi \in S_{2n}$, and moreover, the right side is equal to this minimum if and only if $\{\pi(2i-1), \pi(2i)\} = \{i, 2k+1-i\}$, for all $1 \le i \le k$, i. e. the masses $\{m_{2i-1}, m_{2i}\}$ are assigned to the vertices $\{v_i, v_{2k+1-i}\}$.

Let us now consider the y-coordinate of the center of gravity. The above argument implies that in the corresponding formula the coefficient of masses m_{2i-1}, m_{2i} are either $\sin \alpha_i$, $-\sin \alpha_i$ or $-\sin \alpha_i$, $\sin \alpha_i$, respectively. Hence the total value contributed to the y-coordinate of the center of gravity by these two masses is either

$$\sin \alpha_i m_{2i-1} - \sin \alpha_i m_{2i} = x_{2i-1} - x_{2i}$$

or

$$-\sin \alpha_i m_{2i-1} + \sin \alpha_i m_{2i} = -x_{2i-1} + x_{2i}.$$

With this it is easy to see that the y-coordinate can be zero if and only if there is a solution of EVEN-ODD PARTITION. The set I contains those indices whose corresponding masses are assigned to vertices $v_1, \dots, v_k$.

The above arguments assumed exact calculations with real numbers. To make the proof valid, one has to work with sufficiently precise rational approximations of sines and cosines. The condition in the claim must be modified: Instead of insisting that the center of gravity of a mass assignment lies exactly in the origin, we have to require that its distance from the origin is smaller than some given threshold ε. Since the values of the right-hand side of (13) which are not equal to the minimum can be bounded away from zero, it is possible to work out such a threshold ε and the precision requirement for the computations in polynomial time. We omit the details. ■

4 Two Further Applications of the Main Theorem

This section deals with the Traveling Salesman Problem (P2) on symmetric Monge distance matrices, and with a data arrangement problem (P3). The main theorem can be applied to both problems and yields short proofs for known results on problems (P2) and (P3).

4.1 The TSP on symmetric Monge matrices

The *Traveling Salesman problem* (TSP) consists in finding a shortest closed tour through a set of cities with given distance matrix. This problem is a fundamental problem in combinatorial optimization and well-known to be NP-hard. For more information, the reader is referred to the comprehensive book edited by Lawler, Lenstra, Rinnooy Kan, and Shmoys [17]. Several special cases of the TSP are known to be solvable in polynomial time due to special combinatorial structures in the distance matrix. One of the first results on easy special cases of the TSP was derived by Supnick in 1957. Recall that a matrix D is called a *Monge matrix* if the matrix $-D$ is an Anti-Monge matrix.

Proposition 4.1 (Supnick [24], 1957)
For every instance of the TSP with a symmetric Monge distance matrix $D = (d_{ij})$ an optimal tour is given by

$$\pi^*(1) \to \pi^*(2) \to \cdots \to \pi^*(n) \to \pi^*(1). \tag{14}$$

For the proof we need a definition and some elementary observations. A matrix A is called a *circulant* if there is a function $g\colon \{0, \ldots, n-1\} \to \mathbb{R}$ such that $a_{ij} = g((i-j) \bmod n)$. In other words, it is a Toeplitz matrix generated by a function f with $f(i) = f(i-n)$ for $i = 1, \ldots, n-1$. A circulant matrix is not necessarily symmetric although in this paper we will only use symmetric ones.

Observation 4.2 *For a sum matrix A and a circulant matrix B, the value of $Z(\pi, A, B)$ is independent of the permutation π. In other words,* QAP(A, B) *is solved by* any *permutation $\pi \in S_n$.*

Proof. Each circulant matrix $B = (b_{ij})$ has constant row and column sums $s = \sum_{i=1}^n b_{ij} = \sum_{i=1}^n b_{ji}$, for all $1 \le j \le n$. Next, according to the definition of sum matrices, there are real numbers r_i and c_j, $1 \le i, j \le n$, such that $a_{ij} = r_i + c_j$, $1 \le i, j \le n$. The following chain of equalities completes the proof.

$$\begin{aligned} Z(\pi, A, B) &= \sum_{i=1}^n \sum_{j=1}^n a_{\pi(i)\pi(j)} b_{ij} = \sum_{i=1}^n \sum_{j=1}^n (r_{\pi(i)} + c_{\pi(j)}) b_{ij} = \\ &= \sum_{i=1}^n \Big[r_{\pi(i)} \cdot \sum_{j=1}^n b_{ij} \Big] + \sum_{j=1}^n \Big[c_{\pi(j)} \cdot \sum_{i=1}^n b_{ij} \Big] = s \cdot \Big(\sum_{i=1}^n r_i + \sum_{j=1}^n c_j \Big) \end{aligned}$$

■

The following easy observation is analogous to Observation 2.5.

Observation 4.3 *Assume that* QAP(A_1, B) *and* QAP(A_2, B) *are both solved by permutation π_0. Then for any positive reals $k_1, k_2 \ge 0$, the problem* QAP$(k_1 A_1 + k_2 A_2, B)$ *is also solved by π_0.* ■

Proof of Proposition 4.1. Let $\Delta = 2 \max_{1 \le i,j \le n}\{|d_{ij}|\}$ and define a sum matrix $S = (s_{ij})$ by $s_{ij} = (i+j)\Delta$. Moreover, let us define a symmetric Toeplitz matrix B by its generating benevolent function $f(1) = f(-1) = f(n-1) = f(-n+1) = -1$ and $f(i) = 0$ for $i \notin \{-n+1, -1, 1, n-1\}$. The proof results from the main theorem in several easy steps.

Firstly, QAP$(S - D, B)$ is solved by π^*: Since S and $-D$ both are Anti-Monge matrices, so is $S - D$. Moreover, it is straightforward to verify that the matrix $S - D = (c_{ij})$ is monotone. Since B is a symmetric Toeplitz matrix generated by a benevolent function f, Theorem 1.6 applies.

Secondly, QAP$(-S, B)$ is solved by π^*: Since $-S$ is a sum matrix and B is a circulant, Observation 4.2 implies that QAP$(-S, D)$ is solved by every permutation $\pi \in S_n$.

Finally, by adding $S - D$ and S we get, using Observation 4.3, that QAP$(-D, B)$ is solved by π^*. Since QAP$(-D, B)$ is the same problem as QAP$(D, -B)$, the problem QAP$(D, -B)$ is also solved by π^*.

Now, the matrix $-B$ is the adjacency matrix of an undirected cycle on n vertices. Hence, QAP$(D, -B)$ exactly corresponds to the TSP with distance matrix D and the solution π^* of QAP$(D, -B)$ exactly corresponds to the optimal tour (14) for the TSP. ■

4.2 Data arrangement in a linear storage medium

Consider a set of n records $r_1, \dots, r_n$ which are referenced repetitively, where with probability p_i the reference is to record r_i and where different references are independent. Without loss of generality the records are numbered such that $p_1 \le p_2 \le \cdots \le p_n$. The goal is to place these records into a linear array of storage cells, like a magnetic tape, such that the expected distance between consecutively referenced records is minimized, i. e. one wishes to minimize

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p_{\phi(i)}p_{\phi(j)}d_{ij}$$

where d_{ij} is the distance between the records placed at storage cells i and j respectively. In the late 1960s and early 1970s, much research has been done on the special case of this problem where the distance d_{ij} is given as $d_{ij} = f(|i-j|)$, $f\colon \{0, \dots, n-1\} \to \mathbb{R}$, i. e. d_{ij} only depends on the absolute value of the difference between i and j. The following proposition summarizes three of these results in order of increasing generality.

Proposition 4.4 (a) *If $d_{ij} = |i-j|$, then the data arrangement problem is solved by the permutation π^*.* (Timofeev and Litvinov [25], 1969)

(b) *If $d_{ij} = f(|i-j|)$ with non-decreasing and convex f, then the data arrangement problem is solved by the permutation π^*.* (Burkov, Rubinstein and Sokolov [7], 1969)

(c) *If $d_{ij} = f(|i-j|)$ with non-decreasing f, then the data arrangement problem is solved by the permutation π^*.* (Metelski [18], Pratt [21], 1972) ■

Metelski [18] and Pratt [21] realized that the above results are all contained in the following result due to Hardy, Littlewood and Pólya, here formulated in the language of the QAP and proved by applying Theorem 1.6.

Proposition 4.5 (Hardy, Littlewood and Pólya [12], 1926)
Let the matrix $A = (a_{ij})$ be defined by $a_{ij} = x_i y_j$ for nonnegative real numbers $x_1 \le \cdots \le x_n$ and $y_1 \le \cdots \le y_n$. Let $B = (b_{ij})$ be a symmetric Toeplitz matrix generated by a function f that is non-decreasing on $\{0, \dots, n\}$. Then QAP(A, B) *is solved by π^*.*

Proof. It is easy to verify that the matrix A is a monotone Anti-Monge matrix. Since moreover the matrix B is generated by a benevolent function f, Theorem 1.6 can be applied. ■

Clearly, our main theorem implies an even stronger result. Namely, if the matrix (d_{ij}) is a Toeplitz matrix generated by a benevolent function f, then the data arrangement problem is solved by permutation π^*.

5 Periodic Toeplitz matrices

In this section, we return to well solvable special cases of the Anti-Monge–Toeplitz QAP. First, we give a generalization of our main Theorem 1.6 to matrices B with a certain periodic structure. Then we show that the periodic structure alone is not a sufficiently strong property to make the Anti-Monge–Toeplitz QAP easy to solve.

5.1 Toeplitz matrices generated by k-benevolent functions

In this section we extend the benevolent functions in a periodic way, and we show that the resulting Toeplitz matrices have the constant permutation property. The quantity n' in the following definition can be considered as the block size or the period length, and k denotes the number of periods.

Definition 5.1 Let $k \geq 1$, $n' \geq 2$ and $n = kn'$. A function $f\colon \{-n+1, \ldots, n-1\} \to \mathbb{R}$ is called *k-benevolent* if it fulfills the following four properties.

(i) $f(i) \leq f(i+1)$, for $0 \leq i \leq \lfloor \frac{n'}{2} \rfloor - 1$.

(ii) $f(i) = f(n'-i)$, for $0 \leq i \leq \lceil \frac{n'}{2} \rceil - 1$.

(iii) $f(i) = f(i+jn')$, for $0 \leq i \leq n'-1$, $1 \leq j \leq k-1$.

(iv) $f(-i) = f(i)$, for $0 \leq i \leq n-1$.

A Toeplitz matrix generated by a k-benevolent function is called a k-benevolent matrix. ∎

Properties (i), (ii), and (iv) are the same as the properties of benevolent functions for the range $\{-n'+1, \ldots, n'-1\}$, with two exceptions: The condition (ii) requires that the graph of f restricted on $\{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$ is symmetric with respect to $\lceil \frac{n}{2} \rceil$, and $f(0)$ is involved in (i). Notice, that due to these two exceptions there exist benevolent matrices which are not 1-benevolent. On the other side any 1-benevolent matrix is also a benevolent matrix. Property (iii) provides the periodic continuation with period n'.

Example 5.1 Let $n = 15$, $k = 3$, $n' = 5$. Define a function $f\colon \{-14, -13, \ldots, 0, 1, \ldots, 14\} \to \mathbb{R}$, fulfilling properties (i)–(iv), by the following equalities: $f(0) = 1$, $f(1) = 2$, $f(2) = 3$. Figure 2 represents its graph.

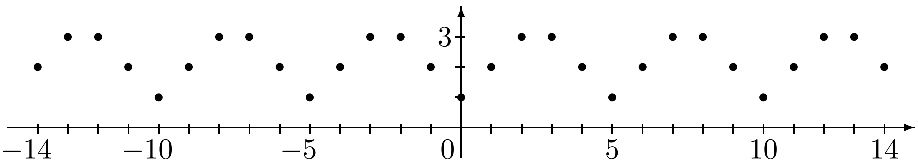

Figure 3: *The graph of the function f in Example 5.1.*

The Toeplitz matrix B generated by this function looks as follows.

$$B = \left(\begin{array}{ccccc:ccccc:ccccc}
1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 \\
2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 \\
3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 \\
3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 \\
2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 \\
\hdashline
1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 \\
2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 \\
3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 \\
3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 \\
2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 \\
\hdashline
1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 \\
2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 \\
3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 \\
3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 \\
2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1
\end{array}\right)$$

One can see clearly that the matrix consists of $k \times k = 9$ identical submatrices of size $n' \times n' = 5 \times 5$ each. Two columns whose indices are congruent modulo $n' = 5$ are identical, and the same holds for the rows. ∎

In general, a k-benevolent Toeplitz matrix consists of $k \times k$ identical submatrices of size $n' \times n'$. Indeed, let us partition the set $\{1, 2, \dots, n\}$ into k blocks

$$N_u := \{(u-1)n' + 1, \dots, un'\}, \quad \text{for } 1 \le u \le k.$$

For $1 \le u, v \le k$, let us denote by B_{uv} the $n' \times n'$ submatrix of B obtained by selecting n' rows with indices in N_u and n' columns with indices in N_v. It is straightforward to check that, as in Example 5.1, the matrices B_{uv}, $1 \le u, v \le k$ are identical, by part (iii) of Definition 5.1. Moreover, the matrices B_{uv} are benevolent matrices. According to the main theorem each of these matrices has the constant permutation property and π^* is an optimal solution of the corresponding QAP. We will show that the k-benevolent matrices have the constant permutation property too. The corresponding optimal solution of the Anti-Monge–Toeplitz QAP with a k-benevolent matrix B is denoted by $\pi^{(k)}$. Recall from Definition 1.5 that $\pi^* = \langle 1, 3, 5, 7, 9, \dots, 8, 6, 4, 2\rangle$, and regard π^* as a permutation of $\{1, \dots, n'\}$ throughout this section. In terms of $\pi^* \in S_{n'}$, the permutation $\pi^{(k)} \in S_n$ (with $n = kn'$) is given as follows

$$\pi^{(k)}((u-1)n' + i) = k\pi^*(i) - (u-1), \qquad \text{for } 1 \le u \le k,\ 1 \le i \le n'. \tag{15}$$

For example, for $k = 4$, $n' = 5$, $n = kn' = 20$,

$$\pi^{(4)} = \langle \underbrace{4, 12, 20, 16, 8}, \underbrace{3, 11, 19, 15, 7}, \underbrace{2, 10, 18, 14, 6}, \underbrace{1, 9, 17, 13, 5} \rangle.$$

The sequence $\langle \pi^{(k)}(1), \pi^{(k)}(2), \dots \rangle$ is naturally divided into $k = 4$ blocks with $n' = 5$ elements each. The first block, corresponding to $u = 1$ in (15), is obtained from $\pi^* = \langle 1, 3, 5, 4, 2\rangle$ by multiplying every element by $k = 4$. Each successive block is obtained from the previous one by subtracting one from each entry. Thus, the numbers in the i-th block are those numbers between 1 and $n = 20$ which are congruent to $-(i-1)$ modulo k. Now we can state the main theorem of this section.

Theorem 5.2 *The permutation $\pi^{(k)}$ solves $\mathrm{QAP}(A, B)$ when A is a monotone Anti-Monge matrix and B is a k-benevolent matrix.*

As previously, we can show the constant permutation property even for the independent-QAP. We can assume non-negativity of A by adding, if necessary, a constant to all entries of A. We can achieve $f(0) = 0$ by subtracting a constant from all values of f. Since $f(0)$ is the smallest value of f, the resulting matrix B will be non-negative. Clearly, these addition and/or subtraction operations do not change the optimal permutation of independent-QAP(A, B). Thus, Theorem 5.2 follows from the following theorem.

Theorem 5.3 *The pair of permutations $(\pi^{(k)}, \pi^{(k)})$ solves independent-*$\mathrm{QAP}(A, B)$ *when A is a non-negative monotone Anti-Monge matrix and B is a k-benevolent matrix with zeros on the main diagonal.*

As in Section 2, we can restrict our attention to the matrices $A = R^{(pq)}$ which are the extreme rays of the cone of non-negative monotone Anti-Monge matrices. It would be easy to find the extreme rays of the cone of non-negative k-benevolent Toeplitz matrices B, but we do not need this because we will rely directly on some lemmas of Section 2. Thus our final goal in this subsection is to prove the following lemma:

Lemma 5.4 *For any $1 \le p, q \le n$, the independent-*$\mathrm{QAP}(A, B)$*, with $A = R^{(pq)}$ and a k-benevolent matrix B with zeros on the main diagonal, is solved to optimality by the pair of permutations $(\pi^{(k)}, \pi^{(k)})$.*

Proof. We know that this problem can be seen as a selecting problem. Namely, select p rows and q columns of the matrix B such that the total sum of all pq selected entries is minimized. Now suppose that some q columns have already been selected and we have to select the rows. Let x_i for $i = 1, \ldots, n$ denote the sum of the selected entries in row i. Clearly, we have to select those p rows with the smallest x_i values. Since rows of B whose indices i are congruent modulo n' are identical, the numbers x_i corresponding to these rows are equal. Therefore, if $v_1 < v_2 < \ldots < v_j$ are the values taken by the elements x_i and $V_t = \{x_i \mid x_i = v_t\}$, then $|V_t|$ is a multiple of k, for any $1 \le t \le j$. Moreover, the elements of V_t "are uniformly distributed in blocks", i. e. there are $|V_t|/k$ elements of V_t belonging to each block, for $1 \le t \le j$. Thus we may impose the following structure on the selected set of rows.

Claim 1 *There is an optimal selection of p rows, where the number p_u of selected rows in each block N_u is either $\lfloor p/k \rfloor$ or $\lceil p/k \rceil$.*

Since we can equally apply the argument to the columns once the rows are selected (in accordance with Claim 1), we also get:

Claim 2 *There is an optimal selection of p rows and q columns, where, in addition to the properties of Claim 1, the number q_v of selected columns in each block N_v is either $\lfloor q/k \rfloor$ or $\lceil q/k \rceil$.*

The entries of B which lie in the selected rows and columns can be summed separately for each block B_{uv}, $(1 \le u, v \le k)$. All blocks B_{uv} are identical to a certain $n' \times n'$ benevolent Toeplitz matrix B'. By Lemma 2.7 we know how to optimally select a given number p' of rows and a given number q' of columns from B' if we want to minimize the overall sum of the selected entries. Let us denote by $z(B', p', q')$ the optimal value of this problem, i. e. the value of independent-QAP$(R^{(p'q')}, B')$. So we get the following lower bound for our problem.

$$Z(\phi, \psi, R^{(pq)}, B) \ge \sum_{u=1}^{k} \sum_{v=1}^{k} z(B', p_u, q_v)$$

Let us denote $R_p := p \bmod k$. Then R_p of the values p_u must be equal to $\lceil p/k \rceil$, and the remaining $k - R_p$ of the values p_u are equal to $\lfloor p/k \rfloor$. Similarly, $R_q := q \bmod k$ of the values q_v are equal to $\lceil q/k \rceil$, and $k - R_q$ of them are equal to $\lfloor q/k \rfloor$.

To finish the proof of the lemma, we have to show that the permutation $\pi^{(k)}$ indeed selects the optimal set of p_u rows out of each block of rows N_u and the optimal set of q_v columns out of each block of columns N_v, as specified by Lemma 2.7. This is easy to check: The selected row and column indices i are those which satisfy $\pi^{(k)}(i) > n - p$ or $\pi^{(k)}(i) > n - q$, respectively. By the way how $\pi^{(k)}$ is constructed, if we look at the indices of selected rows in each block N_u, these are precisely the positions where the p_u largest entries in π^* occur:

$$\pi^{(k)}((u-1)n' + i) > n - p \ \text{ if and only if } \ \pi^*(i) > n' - p_u, \quad \text{for } \ 1 \le u \le k, \ 1 \le i \le n'$$

where

$$p_u = \begin{cases} \lceil p/k \rceil, & \text{for } u = 1, \ldots, R_p, \\ \lfloor p/k \rfloor, & \text{for } u = R_p + 1, \ldots, k. \end{cases}$$

The same situation holds for the columns, and this is just in accordance with Lemma 2.7. ∎

5.2 Toeplitz matrices generated by general periodic functions

The simplest non-trivial periodic functions f for generating a Toeplitz matrix B have period $n' = 2$ and thus only two values: $f(0) = f(i)$ for all even i and $f(1) = f(i)$ for all odd i. These two values form a chess-board pattern in the matrix B. The case $f(0) \leq f(1)$ was treated above. It leads to a k-benevolent function and hence to the constant permutation property. In this section we deal with the other case, $f(0) > f(1)$, and we show that it represents an NP-hard problem. It is no loss of generality to assume $f(0) = 1$ and $f(1) = -1$. In this case $B = (b_{ij})$ can be written as $b_{ij} = (-1)^{i+j}$. The following theorem shows that the Anti-Monge–Toeplitz QAP with a periodic Toeplitz matrix is in general NP-hard.

Theorem 5.5 *The* QAP *is NP-hard even if A is a $(2k) \times (2k)$ monotone Anti-Monge matrix and $B = (b_{ij})$ is a $(2k) \times (2k)$ symmetric Toeplitz matrix with $b_{ij} = (-1)^{i+j}$.*

Proof. The proof is done by a reduction from the NP-complete EQUIPARTITION problem (cf. Garey and Johnson [10]):

Problem: EQUIPARTITION

Instance: $2k$ positive integers $x_1, x_2, \dots, x_{2k}$.

Question: Is there a subset $I \subseteq \{1, 2, \dots, 2k\}$, $|I| = k$, such that $\sum_{i \in I} x_i = \sum_{i \notin I} x_i$ holds?

Without loss of generality suppose that $x_1 \leq x_2 \leq \ldots \leq x_{2k}$. Define the $2k \times 2k$ matrix $A = (a_{ij})$ by $a_{ij} = x_i \cdot x_j$ for $1 \leq i, j \leq 2k$. Obviously, A is a monotone Anti-Monge matrix. Now we consider the QAP instance QAP(B, A) with B as defined in the theorem and show that the instance of EQUIPARTITION has answer "Yes" if and only if QAP(B, A) has optimal value 0. We have

$$Z(\pi, B, A) = \sum_{i=1}^{2k} \sum_{j=1}^{2k} (-1)^{\pi(i)+\pi(j)} x_i x_j = \left(\sum_{i=1}^{2k} (-1)^{\pi(i)} x_i \right)^2 \geq 0$$

Therefore, $Z(\pi, B, A) = 0$ holds if and only if $\sum_{i=1}^{2k} (-1)^{\pi(i)} x_i = 0$, i. e., if $\sum_{\pi(i) \text{ is even}} x_i = \sum_{\pi(i) \text{ is odd}} x_i$. This means that $I := \{ i \mid 1 \leq i \leq 2k,\ \pi(i) \text{ is even} \}$ is a solution of EQUIPARTITION. ■

6 Conclusions

In this paper we investigated the Anti-Monge–Toeplitz QAP, a restricted version of the quadratic assignment problem QAP(A, B) where A is a monotone Anti-Monge matrix and B is a symmetric Toeplitz matrix. We have shown that the TSP on symmetric Monge matrices, the turbine problem and the linear data arrangement problem all are instances of this restricted version of QAP. By proving that the turbine problem of Section 3 is NP-hard, we have shown that even this apparently simple version of the QAP is NP-hard. We conjecture that the Turbine Problem is even *strongly* NP-hard. In particular, we propose the following one-dimensional version of the Turbine Problem.

Let $a_1, a_2, \dots$ be the fixed sequence $1, -1, 2, -2, 3, -3, \dots$.

Problem: WEIGHTED PARTITION

Instance: n positive integers $x_1, x_2, \dots, x_n$.

Question: Is there a permutation $\phi \in S_n$ such that

$$\sum_{i=1}^{n} a_{\phi(i)} x_i = 0? \tag{16}$$

The numbers a_i are the positions to which the masses x_i have to be assigned. Using some ideas from the NP-hardness proof for the Turbine Problem, we can show that this problem is also NP-hard, but we do not know whether it is NP-hard in the strong sense. Of course we may consider other fixed sequences a_i, or we may even allow the sequence a_i to be specified as part of the input. If we use the sequence $a_i = (-1)^i$ we get just the EQUIPARTITION problem of Section 5.2, which can be solved in pseudo-polynomial time. The sequence (a_i) proposed above seems to be the simplest sequence for which the problem is open.

A related problem of optimizing the position of the center of gravity, which arose in the context of loading cargo on a truck or on an airplane, was considered by Amiouny et al. [1]. They gave a heuristic which tries to minimize the absolute value of the expression in (16). The heuristic guarantees that the deviation of this objective function from the minimum is not larger than the largest difference between two adjacent values in the sequence a_i, which is 1 in our case. In fact they considered a more general problem of packing boxes of given lengths and weights inside an interval along a one-dimensional axis. The boxes have to be arranged in a sequence which has the center of gravity close to a given target point. The problem is complicated by the fact that boxes may have different lengths, and thus the position of the i-th box in the sequence depends also on the other boxes, and the sequence a_i is not fixed. For the special case when all boxes have the same length and the target point is in the middle of the interval, we get our WEIGHTED PARTITION problem.

Our main result is the identification of an easy special case of the Anti-Monge–Toeplitz QAP: If A is a *monotone* Anti-Monge matrix and B is a symmetric Toeplitz matrix generated by a *benevolent function*, then a fixed permutation π^* is an optimal solution of QAP(A, B). Thus, in this case QAP(A, B) is trivial in the sense that the optimal solution of an instance of the problem can be given independently on the numerical problem data. As a generalization of this type of result, we introduced matrices with the *constant permutation property*: A Toeplitz matrix B has the constant permutation property with respect to a class of matrices $\mathcal{A}$, if there exists a permutation π^B that solves QAP(A, B) for all matrices $A \in \mathcal{A}$. In particular, we have investigated the constant permutation property with respect to monotone Anti-Monge matrices. *Deriving a characterization of all Toeplitz matrices that have this property* is an open problem whose complete solution is currently out of sight. As a first step toward the solution of this problem, we have identified two classes of Toeplitz matrices which have the constant permutation property with respect to monotone Anti-Monge matrices. These are the *benevolent* and *k-benevolent* matrices. Another class of Toeplitz matrices with the constant permutation property are the Toeplitz matrices with bandwidth 1 as shown in [4], see also [5]. As for Toeplitz matrices of larger bandwidth, it can be shown that they do not have the constant permutation property. However, the computational complexity of the Anti-Monge–Toeplitz QAP with a Toeplitz matrix of limited bandwidth remains an open question.

As a "negative" result, it is shown that the Anti-Monge–Toeplitz QAP remains NP-hard even when considering only Toeplitz matrices generated by a function of period two and with only ± 1 entries.

Thus, there is a "thin" borderline between "easy" and "hard" cases of this restricted version of QAP, as well as between Toeplitz matrices with and without the constant permutation property with respect to monotone Anti-Monge matrices. It is an open question whether these two borderlines coincide.

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A The extreme rays of the non-negative monotone Anti-Monge matrices

In this appendix we prove Lemma 2.2: The non-negative monotone Anti-Monge matrices form a cone, and the 0-1-matrices $R^{(pq)}$ defined before the statement of Lemma 2.2 generate the extreme rays of this cone. We first make the following easy observation.

Observation A.1 (a) *A matrix A is Anti-Monge if and only if*

$$\Delta_{ij} := a_{ij} - a_{i,j-1} - a_{i-1,j} + a_{i-1,j-1} \geq 0, \quad \text{for } 1 < i, j \leq n. \tag{17}$$

(b) *An Anti-Monge matrix is monotone if its first row and its first column are monotone, i. e. if*

$$\Delta_{i1} := a_{i1} - a_{i-1,1} \geq 0, \quad \text{for } 1 < i \leq n, \text{ and} \tag{18}$$

$$\Delta_{1j} := a_{1j} - a_{1,j-1} \geq 0, \quad \text{for } 1 < j \leq n. \tag{19}$$

(c) *A monotone Anti-Monge matrix is non-negative if*

$$\Delta_{11} := a_{11} \geq 0. \tag{20}$$

(d) *Moreover, a matrix A is completely determined by the n^2 values Δ_{ij}, $1 \leq i, j \leq n$.* ∎

Note that a matrix A is a non-negative monotone Anti-Monge matrix if and only if the $(n+1) \times (n+1)$ matrix obtained by bordering A with an additional top row of zeros and an additional left column of zeros is an Anti-Monge matrix. In this way, inequalities (18)–(20) become special cases of (17), and the additional requirements of monotonicity and non-negativity appear natural for Anti-Monge matrices.

For $1 \leq p, q \leq n$, we have defined the matrix $R^{(pq)} = (r_{ij}^{(pq)})$ which has a $p \times q$ block of one entries in the lower right corner and zero entries everywhere else. Formally, $r_{ij}^{(pq)} = 1$ for $n - p + 1 \leq i$ and $n - q + 1 \leq j$, and $r_{ij}^{(pq)} = 0$ otherwise.

Now, Since the nonnegative monotone Anti-Monge matrices are defined by a homogeneous system of linear inequalities (17)–(20), they form a cone. Part (d) of Observation A.1 implies that the mapping from the $(n \times n)$-matrices A to the $(n \times n)$-matrices $\Delta = (\Delta_{ij})$ is a one-to-one linear transformation. In the transformed "Δ-space" the n^2 defining inequalities of Observation A.1 take a very simple form: $\Delta_{ij} \geq 0$ for all $1 \leq i, j \leq n$. Hence the extreme rays of the transformed cone are just the coordinate axes in Δ-space. A unit vector in Δ-space, i. e. a matrix with $\Delta_{st} = 1$ for some pair (s, t) and $\Delta_{ij} = 0$ for all other pairs corresponds just to the matrix $R^{(n-s+1,n-t+1)}$ in the original space. This shows that the matrices $R^{(pq)}$ generate the extreme rays of the cone, concluding the proof of Lemma 2.2. ■

B The extreme rays of the benevolent matrices with zeros on the diagonal

In this appendix we prove Lemma 2.4: The benevolent matrices with zeros on the diagonal form a cone. We will show that each benevolent function f with $f(0) = 0$ can be represented as non-negative linear combination of the "special" benevolent functions g^α and h^β, which were defined in Definition 2.3, and which take only 0-1-values.

Since benevolent functions which map 0 to 0 are defined by linear equations and inequalities with right-hand side equal to zero, they form a cone. The functions g^α and h^β are clearly benevolent. In fact, each of these functions satisfies precisely one of the characterizing inequalities (BEN2), (BEN3), and $f(1) \geq 0$ as a strict inequality and the remaining ones with equality.

We have to show that an arbitrary benevolent function with $f(0) = 0$ is a non-negative linear combination of functions g^α and h^β. For this purpose we define two auxiliary functions

$$f_1(i) = \begin{cases} f(i) & \text{for} \quad |i| \leq \lfloor \frac{n}{2} \rfloor \\ f(n-i) & \text{for} \quad |i| > \lfloor \frac{n}{2} \rfloor \end{cases}$$

and

$$f_2(i) = \begin{cases} 0 & \text{for} \quad |i| \leq \lfloor \frac{n}{2} \rfloor \\ f(i) - f(n-i) & \text{for} \quad |i| > \lfloor \frac{n}{2} \rfloor \end{cases}$$

It is easily seen that $f(i) = f_1(i) + f_2(i)$ holds for all i in $\{-n+1, \ldots, n-1\}$. Finally, observe that

$$f_1 = \sum_{i=1}^{\lfloor n/2 \rfloor} [f(i) - f(i-1)] \cdot h^i \qquad\qquad f_2 = \sum_{i=\lfloor n/2 \rfloor + 1}^{n-1} [f(i) - f(n-i)] \cdot g^i \tag{21}$$

and apply conditions (BEN2) and (BEN3) to see that all coefficients in these expressions are non-negative. Hence, both f_1 and f_2 are nonnegative linear combinations of functions g^α and h^β and this completes the proof of Lemma 2.4. ■