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THE INFIMUM OF THE VOLUMES OF CONVEX POLYTOPES OF ANY GIVEN FACET AREAS IS 0

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Abstract

We prove the theorem mentioned in the title for $\mathbb{R}^n$ where $n \geqq 3$. The case of the simplex was known previously. Also the case $n = 2$ was settled, but there the infimum was some well-defined function of the side lengths. We also consider the cases of spherical and hyperbolic n-spaces. There we give some necessary conditions for the existence of a convex polytope with given facet areas and some partial results about sufficient conditions for the existence of (convex) tetrahedra.

1. Preliminaries

Minimum-area convex polygons with given side lengths are characterized by the following theorem of Böröczky–Kertész–Makai, Jr.

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Theorem A ([11]). *Let $m \geqq 3$ and $s_m \geqq s_{m-1} \geqq \ldots \geqq s_1 > 0$ and $s_m < s_{m-1} + \cdots + s_1$. Then the infimum of the areas of convex m-gons in $\mathbb{R}^2$ that have side lengths s_i equals the following number A. This number A is the minimum area of all triangles with side lengths $\sum_{i\in I_1} s_i$, $\sum_{i\in I_2} s_i$, $\sum_{i\in I_3} s_i$. The minimum is taken over all partitions $\{I_1, I_2, I_3\}$ of $\{1, \ldots, m\}$ into non-empty parts for which the three resulting side lengths satisfy the non-strict triangle inequality. If the cyclic order of the sides is fixed then an analogous statement holds, where the sides with indices in each of the sets I_1, I_2, I_3 form an arc of the polygonal curve.*

When we investigate simple polygons instead of convex polygons, we have the following result, due to Böröczky–Kertész–Makai, Jr. and Nikonorova.

Theorem B ([11, 42]). *Let $m \geqq 3$ and $s_m \geqq s_{m-1} \geqq \ldots \geqq s_1 > 0$ and $s_m < s_{m-1} + \cdots + s_1$. Then the infimum of the areas of simple m-gons in $\mathbb{R}^2$ that have side lengths s_i equals the following number B. This number B is the minimum area of all triangles with side lengths $\sum_{i\in I_1} \varepsilon_i s_i$, $\sum_{i\in I_2} \varepsilon_i s_i$, $\sum_{i\in I_3} \varepsilon_i s_i$. The minimum is taken over all partitions $\{I_1, I_2, I_3\}$ of $\{1, \ldots, m\}$ into non-empty parts, and all signs $\varepsilon_1, \ldots, \varepsilon_m$, for which the three resulting side lengths are non-negative and satisfy the non-strict triangle inequality.*

Moreover, if this minimum is not 0 then we may additionally suppose the following. For each $j \in \{1, 2, 3\}$ the sum $\sum_{i\in I_j} \varepsilon_i s_i$ cannot be written as $\sum_{i\in I'_j} \varepsilon_i s_i + \sum_{i\in I''_j} \varepsilon_i s_i$ where $\{I'_j, I''_j\}$ is a partition of I_j and where these partial summands are both positive.

We remark that the proofs in the two papers were different. Moreover, in [42] the result is formulated in a special case only, but all ingredients of the proof of the general case are present in [42] as well.

In our paper we write $\mathbb{R}^n$, $\mathbb{H}^n$ and $\mathbb{S}^n$ for the *Euclidean, hyperbolic* and *spherical n-space*, respectively. Theorems A and B extend to $\mathbb{S}^2$ and $\mathbb{H}^2$ as follows.

Theorem C ([11]). *Let $m \geqq 3$ and $s_m \geqq s_{m-1} \geqq \ldots \geqq s_1 > 0$ and $s_m < s_{m-1} + \cdots + s_1$. Rather than $\mathbb{R}^2$ we consider $\mathbb{H}^2$ and $\mathbb{S}^2$, but in case of $\mathbb{S}^2$ we additionally suppose $\sum_{i=1}^m s_i \leqq \pi$. Then in both cases the word-for-word analogues of Theorems* A *and* B *hold for $\mathbb{H}^2$ and $\mathbb{S}^2$.*

In each of these three theorems the question of finding the infimum is reduced to finding the minimum of a set of non-negative numbers whose cardinality is bounded by a function of m. In Theorems A and B, this bound is 3^m and 6^m, respectively. In Theorem A with given cyclic order of the sides, the bound is $\binom{m}{3}$. In Theorem C, the bounds are the same as for Theorems A and B.

Böröczky et al. [11] posed the question whether it is possible to extend these theorems to dimensions $n \geqq 3$. Their conjecture was that, analogously to the two-dimensional case, the solutions would be given as the volumes of some simplices. Unfortunately, they were unaware of the fact that the case of simplices already had long ago been solved, namely in 1938, as we will describe below.

The analogous problem about the maximal volume of simplices with given facet areas (an isoperimetric-type problem) was solved by Lagrange [35] in 1773 for $\mathbb{R}^3$ and by Borchardt [9] in 1866 for $\mathbb{R}^n$. A simplex is called *orthocentric* if it has an orthocentre, i.e., a common point of all altitudes. It can be characterized also as a simplex where any two disjoint edges (or, equivalently, any two disjoint faces of dimension at least 1) are orthogonal. For this reason, an orthocentric simplex is sometimes also called *orthogonal*, although orthocentric is the presently used terminology. For a relatively recent exposition of the above-mentioned facts and some other properties of orthocentric simplices, see for example [24]. See also the recent paper [20], whose first part is a comprehensive survey about orthocentric simplices. There it is also stressed that for many elementary geometrical theorems the objects in $\mathbb{R}^n$ corresponding to triangles are not the general simplices but just the orthocentric ones.

THEOREM D ([9,35]). *Let $n \geqq 3$. Then among the simplices in $\mathbb{R}^n$ with given facet areas $S_{n+1} \geqq \ldots \geqq S_1 > 0$ (if such simplices exist) there exists (up to congruence) exactly one simplex of maximal volume. It is also the (up to congruence) unique orthogonal simplex with these facet areas.*

Unaware of the above-mentioned solution of the maximum problem, A. Narasinga Rao [50] posed the following problem in 1937:

> "The areas of the four facets of a tetrahedron are α, β, γ, δ. Is the volume determinate? If not, between what limits does it lie?"

This problem was soon solved independently by Venkatachaliengar, Iyengar, Auluck, and Iyengar–Iyengar [6,30,31,61]. In fact, under the above hypothesis the volume is not determined (supposing that such tetrahedra exist). Moreover, they reproved that there is up to congruence exactly one tetrahedron of maximal volume with the given facet areas, which is also orthogonal – and they reproved that it is also the unique orthocentric tetrahedron with these facet areas. Moreover, they proved that there exists a tetrahedron with the given face areas that has an arbitrarily small volume. A generalization of their first mentioned result to multi-dimensional Euclidean spaces was obtained in [30,31,61]. We cite only their statement about the infimum of the volumes.

THEOREM E ([6,30,31,61]). *Let $n \geqq 3$ and $S_{n+1} \geqq S_n \geqq \ldots \geqq S_1 > 0$. Then there exists a simplex in $\mathbb{R}^n$ with these $(n-1)$-volumes of the facets if and only if*

$$S_{n+1} < S_1 + S_2 + \cdots + S_n.$$

If this inequality holds, then, for any $\varepsilon > 0$, there is a nondegenerate simplex in $\mathbb{R}^n$ with facet areas $S_1, S_2, \ldots, S_{n+1}$ and volume at most ε.

The proof of Theorem E by Iyengar and Iyengar [31] was based on the following statement, which is valid for simplices only [31, p. 306]. Let T be a simplex with facet areas $S_1, S_2, \ldots, S_{n+1}$ and respective outer unit facet normals $u_1, \ldots, u_{n+1}$. Then we have $\sum_{i=1}^{n+1} S_i u_i = 0$. Let us consider an $(n+1)$-gon $P \subset \mathbb{R}^n$ with side vectors $S_1 u_1, \ldots, S_{n+1} u_{n+1}$. Its convex hull T' is then also a simplex, whose volume is invariant under permutations of the side vectors of P. Moreover, for the volumes of T and T', we have $V(T)^{n-1} = V(T')\big[(n-1)!\big]^2/n^{n-2}$. Based on this relation and some calculations, Iyengar and Iyengar could make $V(T')$ arbitrarily small. However, this can also be done by choosing $u_1, \ldots, u_{n+1}$ in a small neighbourhood of the $x_1 \ldots x_{n-1}$-coordinate hyperplane. See also the first and third proofs of our Theorem 2.

The question of maximal volume of polytopes with given facet areas is much less understood.

For non-degenerate polytopes in $\mathbb{R}^n$ with given facet areas and given facet outer unit normals, we have the following result of Brunn [14], see also [41, §10.5]. The maximal volume is attained for the (up to translation) unique *convex* polytope with these given facet areas and given facet outer unit normals. (For coinciding facet outer unit normals, one has to add their areas.) This result was rediscovered in [10, Theorems 2 and 3] and applied to solve another problem. In crystallography, this set of maximal volume is called the *Wulff shape* [62]. It minimizes total surface energy of the crystal and is always convex. For a nice description of the interplay of mathematics and crystallography see [12, §10.11].

For any given number m of facets and fixed total surface area, the polytope of largest volume has an inball and the facets must touch the inball at their centroids (Lindelöf's theorem, see [56, p. 43] or [22, II.4.3, p. 264; English ed. IX.43, p. 283].

Now let us restrict our attention to $\mathbb{R}^3$. L. Fejes Tóth [21, Theorem 1, p. 175] (see also [22, II.4.3, p. 265, Satz; English ed. IX.43, p. 283, Theorem]) asserts that among (convex) polyhedra with given surface area and $m = 4$, 6, and 12 faces, the largest volume is attained for the regular tetrahedron, cube, and regular dodecahedron, respectively. He gave a bound on the maximum volume [21, p. 175], valid for each $m \geqq 4$, which is also asymptotically sharp for $m \to \infty$. For $m = 5$, the extremal polyhedron is the regular triangular prism that has an inball [56, p. 41]. A recent complete and simple proof of this fact is given in [25, Theorem 5.10]. However, for $m = 8$ and $m = 20$, the extremal polyhedron is not the regular octahedron and icosahedron, respectively [26, p. 234]. For more information about this isoperimetric problem about convex polyhedra in $\mathbb{R}^3$ with given number of faces, see the old survey in [26] or the recent survey in the introduction of [57]. For recent numerical results (examples) about the isoperimetric problem for polyhedra, with large symmetry groups, see [36].

A different problem is to maximize the volume enclosed by a given surface that may be bent (isometrically) but not stretched. A theorem of S. P. Olovianishnikoff says the following. For convex bodies $P, Q \subset \mathbb{R}^3$, where P is a convex polyhedron, any mapping of ∂P to ∂Q preserving the geodesic distance of every pair of points of ∂P (i.e., the length of the shortest arc in ∂P joining these points) extends to an isometry of $\mathbb{R}^3$. See [3, Ch. 3, §3, **2**, p. 150, Satz 1] for a special case, and S. P. Olovianishnikoff [43], p. 441, Theorem for the general case described above. For convex bodies $P, Q \subset \mathbb{R}^3$ where ∂P is of class C^2, the analogous theorem holds. See [2, Ch. 8, §5, p. 337] for a special case and A. V. Pogorelov [46, Introduction, §1, A, p. 8, Theorem 1, and Ch. 3, **3**, p. 66, Theorem 1] for the general case described above.

However, this uniqueness theorem does not say that this unique convex polyhedron would have the maximal volume. The opposite is true: the surface of *every* convex polytope can be isometrically deformed to increase the enclosed volume [44]. For example, the cube can be "blown up": the face centers move outwards and the vertices move closer to the center. The face diagonals maintain their original length, but the original edges of the cube are longer than necessary: they become crumpled, with wrinkles perpendicular to the original edge. Globally, the polyhedron becomes more "ball-like". This volume-increasing phenomenon for convex bodies was first observed by A. V. Pogorelov in the theory of thin shells in mechanics [47,48]. (A short summary of the results of [47] and of some other related results is given in [49].) An animation showing a deformation of the cube with a volume increase by a factor of about 1.2567 has been produced by Buchin and Schulz [15]. The problem of enclosing the largest volume with the surface of a given convex polyhedron, possibly under the constraint of preserving the original symmetries, has been treated in many papers [4,8,16,38,39,44,58,59] ("inextensional" in the title of [58] means "isometric w.r.t. the geodesic distance"). (According to a private communication from the second author of [39], in the tableau summarizing the numerical results in pp. 154 and 181, the values in the middle column for the dodecahedron and the icosahedron are not correct. They are actually smaller than the values in the third column, which are proved in [39], and those are the best published values.) For a recent survey on this and related questions see [51].

Notation. In this paper, $V(\cdot)$ denotes volume of a set, $S(\cdot)$ its surface area, $\mathrm{diam}(\cdot)$ its diameter, $\mathrm{aff}(\cdot)$ its affine hull, $\mathrm{lin}(\cdot)$ its linear hull, and $\partial(\cdot)$ its boundary. If we want to indicate also the dimension n then we will write $V_n(\cdot)$ for the n-volume. Sometimes we will refer to the $(n-1)$-volume in $\mathbb{R}^n$, $\mathbb{H}^n$ or $\mathbb{S}^n$ as *area*. We write κ_n for the volume of the unit ball in $\mathbb{R}^n$. For x, y in $\mathbb{R}^n$, $\mathbb{H}^n$ or $\mathbb{S}^n$, we write $[x, y]$ for the segment and $\ell(x, y)$ for the line joining x and y. On $\mathbb{S}^n$, x and y must not be antipodes, and we mean by $[x, y]$ the minor arc on the great circle through x and y. The line $\ell(x, y)$ is well-defined only for $x \neq y$ – writing $\ell(x, y)$ we suppose $x \neq y$. We denote the distance between x and y by $|xy|$.

For standard facts about convex bodies we refer to [55].

2. New Results

2.1. Euclidean Space

The following theorem can be considered as folklore, but we could not locate a proof. For completeness, we state and prove it.

Theorem 1. *Assume that $m > n \geqq 3$ are integers, and consider any sequence of numbers $S_m \geqq S_{m-1} \geqq \ldots \geqq S_1 > 0$. Then the following statements are equivalent:*

(i) *There exists a non-degenerate polytope $P \subset \mathbb{R}^n$ with m facets and with facet areas $S_1, S_2, \ldots, S_m$.*

(ii) *There exists a non-degenerate convex polytope $P \subset \mathbb{R}^n$ with m facets and with facet areas $S_1, S_2, \ldots, S_m$.*

(iii) *The inequality $S_m < S_1 + S_2 + \cdots + S_{m-1}$ holds.*

If we also allow degenerate polytopes in (i) *or* (ii)*, then they imply, rather than* (iii),

(iii′) *$S_m \leqq S_1 + \cdots + S_{m-1}$ with equality if and only if the polytope degenerates into the doubly counted facet with area S_m.*

Theorem 2. *Let $m > n \geqq 3$ be integers. Let $\varepsilon > 0$ and $S_m \geqq S_{m-1} \geqq \ldots \geqq S_1 > 0$ be a sequence of numbers such that $S_m < S_1 + S_2 + \cdots + S_{m-1}$. Then there exists a non-degenerate convex polytope $P \subset \mathbb{R}^n$ with m facets and with facet areas $S_1, S_2, \ldots, S_m$ and with volume $V(P) \leqq \varepsilon$.*

Remark 1. This theorem shows that for dimension $n \geqq 3$ there are no separate questions for convex and general polytopes. Recall that for dimension $n = 2$ these questions had different answers, see Theorems A and B.

We give three different proofs of Theorem 2. The first one is independent of Theorem D and reproves the case of the simplex. It is an existence proof by contradiction. The second proof uses Theorem D. It reduces the question to the case of simplices. Both proofs rely on delicate convergence arguments (see Sections 3 and 4.0). The third proof is geometric. It constructs examples with small volumes that are like "needles". In particular we will give an explicit upper bound for the volumes of our examples in terms of the "steepness" of their facets (Lemmas 2 and 4). If we consider n, m and the facet areas as fixed then our estimate is sharp up to a constant factor (see Lemma 4).

Note that there is a very interesting dichotomy. In Theorems A and B for $\mathbb{R}^2$ (and also in Theorem C for $\mathbb{H}^2$ and $\mathbb{S}^2$) we have some definite functions of the side lengths as infima. In Theorem 2 for $\mathbb{R}^n$ with $n \geqq 3$ the infimum does not depend at all on the facet areas.

2.2. Hyperbolic Space

For the hyperbolic case we have a word-for-word analog of the implications (ii)$\Longrightarrow$(iii) and (ii)$\Longrightarrow$(iii$'$) from Theorem 1 (under the respective hypotheses).

PROPOSITION 1. *Let $P \subset \mathbb{H}^n$ be a polytope with facet areas $S_m \geqq S_{m-1} \geqq \ldots \geqq S_1 > 0$. Then the inequality $S_m \leqq \sum_{i=1}^{m-1} S_i$ holds, with equality if and only if P degenerates into the doubly counted facet with area S_m.*

Next we give two statements that show the following. The necessary condition in Proposition 1 together with the inequalities $S_i \leqq \pi$ is not sufficient even for the existence of a tetrahedron in $\mathbb{H}^3$ with these facet areas. That is, there are some further necessary conditions. Recall that the area of a simple k-gon in $\mathbb{H}^2$ is bounded by $(k-2)\pi$.

PROPOSITION 2. *Let us admit polyhedra in $\mathbb{H}^3$ whose vertices are all distinct but which possibly have some infinite vertices. Then a polyhedron with facet areas $S_m, S_{m-1}, \ldots, S_3$ maximal (i.e., $(k-2)\pi$ for a k-gonal face) but with facet areas S_2, S_1 not maximal does not exist.*

Proposition 2 would suggest that for polyhedra in $\mathbb{H}^3$, if all facets but two have areas nearly maximal (i.e., close to $(k-2)\pi$ for a k-gonal face) then the same statement would hold for the remaining two facets as well. However, this is not true. Even in the convex case, these two facets can have areas close to 0, as shown by the following example. Consider a very large circle in $\mathbb{H}^2 \subset \mathbb{H}^3$ and a regular l-gon $p_1 \ldots p_l$ inscribed in it ($l \geqq 3$). Choose p_{l+1} on our circle with $|p_l p_{l+1}| = \varepsilon$. Then all triangles with vertices among the p_i's have areas close to π except those that contain both p_l and p_{l+1}, and those have very small areas. Now perturb these points p_i a little bit in $\mathbb{H}^3$ so that no four lie in a plane. Then their convex hull is a triangle-faced convex polyhedron, and the perturbation of the segment $[p_l, p_{l+1}]$ is an edge of it. (To see this, use the collinear model. For any convex polygon with strictly convex angles, its edges will remain edges of the convex hull after a sufficiently small perturbation.) The two facets of our polyhedron incident to this edge have very small areas while all other facets have areas close to π, i.e., are nearly maximal.

However, an analogous statement for all but one facets will be shown in the convex case.

PROPOSITION 3. *Assume that we have a convex polyhedron in $\mathbb{H}^3$ with infinite vertices admitted. Suppose its m facets are a k_m-gon, $\ldots$, k_1-gon and*

have respective areas $S_m \geqq \ldots \geqq S_2 \geqq S_1 > 0$. Then for any $i \in \{1, \ldots, m\}$ we have

$$(k_i - 2)\pi - S_i \leqq \sum_{\substack{1 \leqq j \leqq m \\ j \neq i}} \big((k_j - 2)\pi - S_j\big).$$

If there is a finite vertex whose incident edges do not lie in a plane, then the above inequality is strict.

In §6 Remark 9, it will be explained that, in a sense, there are no interesting analogues of Proposition 3 for $\mathbb{R}^3$ and $\mathbb{S}^3$.

Now we turn to sufficient conditions for the existence of hyperbolic tetrahedra.

THEOREM 3. *Assume that $\pi/2 > S_4 \geqq S_3 \geqq S_2 \geqq S_1 > 0$, $S_4 < S_1 + S_2 + S_3$, and one of the inequalities*

$$\tan(S_1/2) > \frac{1 - \cos S_4}{2\sqrt{\cos S_4}}, \tag{1}$$

or

$$S_4 \geqq S_3 + S_2 \tag{2}$$

holds. Then there exists a non-degenerate tetrahedron $T \subset \mathbb{H}^3$ with facet areas S_1, S_2, S_3, S_4.

2.3. Spherical Space

For the spherical case, we give some necessary and some sufficient conditions for the existence.

We say that a set $X \subset \mathbb{S}^n$ (for $n \geqq 2$) is *convex* if, for any two non-antipodal $x, y \in X$, the connecting minor great-$\mathbb{S}^1$ arc $[x, y]$ also belongs to X. This definition classifies an antipodal pair of points as a convex set. But these are the only convex sets which are disconnected, and since the sets we consider contain non-trivial arcs, these exceptional cases play no role for us. By a *nondegenerate simplex in* $\mathbb{S}^n$ we mean the set of those points of $\mathbb{S}^n$ that have non-negative coordinates in some (non-orthogonal) coordinate system with origin at 0, with its usual face lattice. A *simplex in* $\mathbb{S}^n$ is a nondegenerate simplex, or a limiting position of nondegenerate simplices. Thus, for example, we will not consider concave spherical triangles or spherical triangles with sides $3\pi/2$, $\pi/4$, $\pi/4$ or 2π, 0, 0, but a spherical triangle with angles π, $\pi/5$, $\pi/5$ and sides π, $\pi/3$, $2\pi/3$ is a (degenerate) simplex. As a point set, this simplex is indistinguishable from a digon. A different division of the digon side, like π, $\pi/3$, $3\pi/4$, is regarded as a different simplex. To emphasize the fact that we do not just regard a simplex as a point set but we consider its face structure, we will often refer to it as a *combinatorial simplex.* All simplices in $\mathbb{S}^n$, as well as in $\mathbb{R}^n$ and $\mathbb{H}^n$, are convex. A simplex in an open half-$\mathbb{S}^n$ is always nondegenerate. (Observe that an open half-$\mathbb{S}^n$ also has a collinear model in $\mathbb{R}^n$ that also respects convexity. For the open southern half-$\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ consider the central projection to the tangent space $\mathbb{R}^n$ at the South Pole.)

PROPOSITION 4. *Let $P \subset \mathbb{S}^n$ be a polytope with facet areas $S_m \geqq \ldots \geqq S_1 > 0$, such that each facet lies in some closed half-$\mathbb{S}^{n-1}$. Then*

$$S_m \leqq S_1 + \cdots + S_{m-1}.$$

Here strict inequality holds if P is contained in an open half-$\mathbb{S}^n$ and does not degenerate into the doubly-counted facet with area S_m.

If P is a convex polytope contained in some closed half-$\mathbb{S}^n$, then

$$\sum_{i=1}^{m} S_i \leqq V_{n-1}(\mathbb{S}^{n-1}).$$

Here strict inequality holds if P is contained in an open half-$\mathbb{S}^n$.

REMARK 2. Clearly, in the first part of Proposition 4, the hypothesis that each facet lies in some closed half-$\mathbb{S}^n$ cannot be dispensed. Already for $n = 2$, we may even have a degenerate combinatorial simplex lying in some great-$\mathbb{S}^{n-1}$ with one facet strictly containing a half-$\mathbb{S}^{n-1}$. In the second part of Proposition 4, if P is contained in a closed half-$\mathbb{S}^n$ but not in an open half-$\mathbb{S}^n$, then equality can occur: P can degenerate so that one facet is a closed half-$\mathbb{S}^{n-1}$, and the union of the other facets is this closed half-$\mathbb{S}^{n-1}$ or the closure of its complement in this $\mathbb{S}^{n-1}$.

REMARK 3. We do not know how to algorithmically decide whether a simplex with given facet areas S_i in $\mathbb{H}^n$ or $\mathbb{S}^n$ exists, for $n \geqq 3$. The main difficulty are the transcendental functions that enter into the calculation of volumes. In $\mathbb{H}^3$ and $\mathbb{S}^3$, however, we have a positive answer to a slightly modified question. The question whether there is a tetrahedron (for $\mathbb{S}^3$ in the sense described above) with facet areas S_1, S_2, S_3, S_4, is decidable if we are given $\tan(S_1/2), \ldots, \tan(S_4/2)$ as inputs.

We model this question by setting up a system of equations and inequalities in the unknown coordinates (x_{ij}) of the four vertices. (For $\mathbb{S}^3$ we use its standard embedding into $\mathbb{R}^4$, while for $\mathbb{H}^3$ we use the hyperboloid model in $\mathbb{R}^4$.) The equations express the condition that the vertices lie on $\mathbb{S}^3$ or $\mathbb{H}^3$, and that the facet areas of the corresponding tetrahedron should be S_1, S_2, S_3, S_4. Further inequalities are necessary for $\mathbb{S}^3$ to ensure our definition of simplices. We are interested in the set of 4-tuples (S_1, S_2, S_3, S_4) for which *there exist* coordinate vectors (x_{ij}) that fulfill the conditions. These conditions turn out to be polynomial equations and inequalities (these polynomials having rational coefficients) in $\tan(S_1/2), \ldots, \tan(S_4/2)$ and in the coordinates x_{ij}. By a fundamental result of Tarski [60], this existence question is therefore (in principle) decidable (we can eliminate the variables x_{ij}). More specifically, the set of quadruples $\big(\tan(S_1/2), \ldots, \tan(S_4/2)\big)$ for all tetrahedra in $\mathbb{H}^3$ or $\mathbb{S}^3$ can be described by a finite number of polynomial equalities and inequalities, these polynomials having rational coefficients, also using the usual logical connectives "and", "or", "not". In other words, this set forms a *semi-algebraic set.*

REMARK 4. For simplices in $\mathbb{S}^n$ and $\mathbb{H}^n$ (with finite vertices) the case $S_1 = 0$ and all other S_i's positive and sufficiently small can be described. The description is: there exists a partition of the other facets into two classes such that for the two classes the sums of the facet areas are equal. For this we have to use index considerations, like later in the Proofs of Theorems 3 and 4.

THEOREM 4. *Assume that $\pi/2 > S_4 \geqq S_3 \geqq S_2 \geqq S_1 > 0$, $S_4 < S_1 + S_2 + S_3$, and one of the inequalities*

$$\tan(S_1/2) \geqq \frac{1 - \cos S_4}{2\sqrt{\cos S_4}}, \tag{3}$$

or

$$S_4 \geqq S_3 + S_2 \tag{4}$$

holds. Then there exists a non-degenerate (*convex*) *tetrahedron $T \subset \mathbb{S}^3$ with facet areas S_1, S_2, S_3, S_4.*

Now we turn to sufficient conditions for the existence of spherical polyhedral complexes. The second statement of Proposition 5 says the following. For combinatorial simplices contained in some closed half-$\mathbb{S}^n$, the two necessary conditions from Proposition 4 are also sufficient for their existence.

PROPOSITION 5. (i) *Let $n \geqq 2$ and $m \geqq 3$ be integers and let $S_m \geqq \ldots \geqq S_1 > 0$ and $S_m \leqq S_1 + \cdots + S_{m-1}$ and $S_1 + \cdots + S_m \leqq V_{n-1}(\mathbb{S}^{n-1})$. Then there exists a convex n-dimensional polyhedral complex in $\mathbb{S}^n$ that lies in a closed half-$\mathbb{S}^n$ and has facet areas $S_1, \ldots, S_m$. All of its facets have two $(n-2)$-faces. If $S_m < S_1 + \cdots + S_{m-1}$ and $S_1 + \cdots + S_m < V_{n-1}(\mathbb{S}^{n-1})$ then all its dihedral angles are less than π.*

(ii) *Let $n \geqq 2$ and assume $S_{n+1} \geqq \ldots \geqq S_1 > 0$, $S_{n+1} \leqq S_1 + \cdots + S_n$, and $S_1 + \cdots + S_{n+1} \leqq V_{n-1}(\mathbb{S}^{n-1})$. Then there exists a convex polyhedral complex in $\mathbb{S}^n$ lying in a closed half-$\mathbb{S}^n$ that is a combinatorial n-simplex with facet areas $S_1, \ldots, S_{n+1}$. Its faces of any dimension* (*including the complex itself*) *have their dihedral angles at most π, and are thus convex, but some of their dihedral angles are equal to π for $n \geqq 3$.*

3. Tools for the Euclidean case: Minkowski's theorems

We recall some classical concepts and theorems, which are in essence due to Minkowski, but got their final form by A. D. Aleksandrov [1] and W. Fenchel and B. Jessen [23]. We state the results first for arbitrary convex bodies, and then we restrict them to convex polytopes. We will actually need general convex bodies when considering convergent sequences of convex polytopes in our first two proofs of Theorem 2 (Sections 4.3 and 4.4). The third proof uses only Minkowski's Theorem about convex polytopes (Theorem F′). The reader may want to skip directly to Theorem F′.

A *convex body in* $\mathbb{R}^n$ is a compact convex set $K \subset \mathbb{R}^n$ with interior points. For $x \in \partial K$ we say that $u \in \mathbb{S}^{n-1}$ is an *outer unit normal vector for* K *at* x if $\max\{\langle k,u\rangle \mid k \in K\} = \langle x,u\rangle$. In this section we assume $n \geqq 2$ although the theorems of this section will be applied later for $n \geqq 3$ only.

Definition 1 (Minkowski, Aleksandrov [1], Fenchel–Jessen [23], see also [55, p. 207, (4.2.24) (with $\tau(K,\omega)$ defined on p. 77)]). Let $K \subset \mathbb{R}^n$ be a convex body. The *surface area measure* μ_K *of* K is a finite Borel measure on $\mathbb{S}^{n-1}$ defined as follows. For a Borel set $B \subset \mathbb{S}^{n-1}$, $\mu_K(B)$ is the $(n-1)$-dimensional Hausdorff measure of the set $\{x \in \partial K \mid$ there is an outer unit normal vector u to K at x such that $u \in B\}$.

Thus, μ_K is an element of $C(\mathbb{S}^{d-1})^*$, the dual space of the space of real-valued continuous functions $C(\mathbb{S}^{d-1})$ on $\mathbb{S}^{d-1}$, i.e., the finite signed Borel measures on $\mathbb{S}^{d-1}$. We will use the weak* topology of $C(\mathbb{S}^{d-1})^*$ as the topology for the finite (signed) Borel measures μ_K. That is, convergence of a sequence (or more generally of a net) of finite signed Borel measures $\mu_\alpha \in C(\mathbb{S}^{d-1})^*$ to a finite signed Borel measure $\mu \in C(\mathbb{S}^{d-1})^*$ means the following. For each $f \in C(\mathbb{S}^{n-1})$, we have $\int_{\mathbb{S}^{n-1}} f(u)d\mu_\alpha(u) \to \int_{\mathbb{S}^{n-1}} f(u)d\mu(u)$. Moreover, since $\mathbb{S}^{n-1}$ is a compact metric space, the space $C(\mathbb{S}^{n-1})$ is separable, and hence the weak* topology of $C(\mathbb{S}^{n-1})^*$ is metrizable. Therefore, it suffices to give the convergent sequences in it (i.e., it is not necessary to consider nets).

For these elementary concepts and facts from functional analysis, we refer to [19].

Theorem F (Minkowski, Aleksandrov [1], Fenchel–Jessen [23], see also [55, p. 389, (7.1.1), pp. 389–390, p. 392, Theorem 7.1.2., p. 397, Theorem 7.2.1]). *Let* $n \geqq 2$ *be an integer and* $K \subset \mathbb{R}^n$ *a convex body. The measure* μ_K *defined in Definition 1 is invariant under translations of* K *and has the following properties.*

(i) $\int_{\mathbb{S}^{n-1}} u\, d\mu_K(u) = 0$, *and*

(ii) μ_K *is not concentrated on any great-*$\mathbb{S}^{n-2}$ *of* $\mathbb{S}^{n-1}$.

Conversely, for any finite Borel measure μ *on* $\mathbb{S}^{n-1}$ *satisfying* (i) *and* (ii), *there exists a convex body* K *such that* $\mu_K = \mu$. *Moreover, this convex body* K *is unique up to translations.*

Thus, we can consider the map $K \mapsto \mu_K$ also as a map $\{$translates of $K\} \mapsto \mu_K$.

Theorem G (Minkowski, Aleksandrov [1], Fenchel–Jessen [23], see also [55, p. 198, Theorem 4.1.1, p. 205, pp. 392–393, proof of Theorem 7.1.2]). *Let* $n \geqq 2$ *be an integer. Then the mapping* $\{$*translates of* $K\} \mapsto \mu_K$ *defined in Definition 1 and just before this theorem is a homeomorphism between its domain and its range. Its domain is the quotient topology of the topology on the convex bodies induced by the Hausdorff metric with respect to the equivalence relation of being translates. Its range is the set of finite Borel measures on* $\mathbb{S}^{n-1}$ *satisfying* (i) *and* (ii) *of Theorem F with the subspace topology of the weak** *topology on* $C(\mathbb{S}^{n-1})^*$.

We have to remark that the cited sources, [55, pp. 392–393, proof of Theorem 7.1.2], as well as [1, proof of the theorem on p. 36, on p. 38], contain explicitly only the proof of the continuity of the bijection $\{$translates of $K\} \mapsto \mu_K$. However, also the continuity of the inverse map is proved at both places although not explicitly stated. In fact, as kindly pointed out to the authors by R. Schneider, one has to make the following addition to his book [55, proof of Theorem 7.1.2]. Let the sequence of surface area measures μ_{K_i} of some convex bodies $K_i \subset \mathbb{R}^n$ converge to the surface area measure μ_K of some convex body $K \subset \mathbb{R}^n$ in the weak* topology. Then all K_i's have a bounded diameter. This is stated there for polytopes only, but the given proof is valid for all convex bodies. By a translation one can achieve that all K_i's and also K are contained in a fixed ball. Let their barycentres be at 0. Recall that the set of non-empty compact closed sets contained in some closed ball is compact in their usual topology (i.e., that of the Hausdorff metric). Therefore, we can choose a convergent subsequence K_{i_j} of K_i with limit K', say. Then the surface area measure $\mu_{K'}$ of K' is the weak* limit of the $\mu_{K_{i_j}}$'s, i.e., it equals the originally considered μ_K. By continuity of the barycentre, also the barycentre of K' is 0, as well as the barycentre of K. Hence we have $K' = K$. Then the entire sequence K_i converges to K. Otherwise, we could choose another subsequence K_{i_k} converging to another convex body K'', also with barycentre at 0, and with $\mu_{K''} = \mu_K$. This is a contradiction.

It was also proved by Minkowski that a convex body K is a convex polytope if and only if μ_K (that satisfies (i) and (ii) of Theorem F) has a finite support [55, p. 390, Theorem 7.1.1, also considering p. 397, Theorem 7.2.1].

If the support is $\{u_1,\dots,u_m\}$, we may write

$$\mu_K=\sum_{i=1}^{m}\mu_K\big(\{u_i\}\big)\,\delta(u_i),$$

where $\delta(u_i)$ is the *Dirac measure concentrated at* u_i. (I.e., for a Borel set $B\subset\mathbb{S}^{n-1}$ we have $\delta(u_i)(B)=0\iff u_i\notin B$ and $\delta(u_i)(B)=1\iff u_i\in B$.) When we write such an equation, we always assume that $\mu_K\big(\{u_i\}\big)\neq 0$ for all $i\in\{1,\dots,m\}$. (Thus, the empty sum means the 0 (finite signed Borel) measure; although for a convex body K, we have $\mu_K\neq 0$.) The weak* topology restricted to the finite signed Borel measures of finite support, where the support has at most m elements, is the following. (We will use only the case when we have a finite Borel measure and (i) and (ii) of Theorem F hold.) For $u_\alpha,u\in\mathbb{S}^{d-1}$ with $u_\alpha\to u$ and for $c_\alpha,c\in\mathbb{R}\setminus\{0\}$ with $c_\alpha\to c$, where the u_α's and c_α's are nets indexed by α's from the same index set, we have $c_\alpha\delta(u_\alpha)\to c\delta(u)$. Moreover, for arbitrary $u_\alpha\in\mathbb{S}^{n-1}$ and $c_\alpha\to 0$, we have $c_\alpha\delta(u_\alpha)\to 0$. Thus, the convergence is defined for finite signed Borel measures whose supports have at most one point. Then the convergence is defined for finite sums of such sequences as well (and in fact, only for these, see the formal definition in the next paragraph).

More exactly, a sequence (or more generally, a net)

$$\mu_\alpha=\sum_{i=1}^{m_\alpha}\mu_\alpha\big(\{u_i\}\big)\,\delta(u_i)$$

of finite signed Borel measures on $\mathbb{S}^{d-1}$ with $m_\alpha\leqq m$ can converge only to a finite signed Borel measure of support of at most m points. Moreover, μ_α tends to a finite signed Borel measure $\mu=\sum_{i=1}^{m'}\mu\big(\{u_i\}\big)\,\delta(u_i)$ on $\mathbb{S}^{n-1}$ with $1\leqq m'\leqq m$ if and only if the following holds. For each α, there exists a partition of $\{1,\dots,m_\alpha\}$ of cardinality m', say $\{P_{\alpha 1},\dots,P_{\alpha m'}\}$ (where each $P_{\alpha j}$ is non-empty), such that

(A) for any $j\in\{1,\dots,m'\}$, the sets $P_{\alpha j}$ converge to u_j (i.e., for any neighbourhood U_j of u_j and for all sufficiently large α, we have $P_{\alpha j}\subset U_j$) and

(B) for any $j\in\{1,\dots,m'\}$, the sum $\sum\Big\{\mu_\alpha\big(\{u_i\}\big)\mid i\in P_{\alpha j}\Big\}$ converges to $\mu\big(\{u_j\}\big)$.

The same sequence (or more generally a net) μ_α tends to the 0 (finite signed Borel) measure if and only if

(C) $\sum_{i=1}^{m_\alpha}\Big|\mu_\alpha\big(\{u_i\}\big)\Big|\to 0$. (This corresponds to the case $m'=0$, and also here, an empty sum means 0.)

For convex polytopes, Theorem F can be rewritten for

$$\mu_K=\sum_{i=1}^{m}\mu_K\big(\{u_i\}\big)\,\delta(u_i)$$

as follows.

THEOREM F′ (Minkowski, see also [55, p. 389, (7.1.1), pp. 389–390, p. 390, Theorem 7.1.1, p. 397, Theorem 7.2.1]). *Let $m>n\geqq 2$ be integers, let $S_1,\dots,S_m>0$, and let $u_1,\dots,u_m\in\mathbb{S}^{n-1}$. Then there exists a non-degenerate convex polytope P having m facets with facet areas $S_1,\dots,S_m$ and respective facet outer unit normals $u_1,\dots,u_m$ if and only if*

(i) $\sum_{1=1}^{m}S_iu_i=0$, *and*

(ii) $u_1,\dots,u_m$ *do not lie in a linear $(n-1)$-subspace of $\mathbb{R}^n$.*

Moreover, if P exists, it is unique up to translations.

For convex polytopes with at most m facets, Theorem G can be rewritten for $\mu_K=\sum_{i=1}^{m}\mu_K\big(\{u_i\}\big)\,\delta(u_i)$ as follows.

THEOREM G′ (Minkowski, see also [55, p. 198, Theorem 4.1.1, p. 205, pp. 392–393, proof of Theorem 7.1.2] and the addition after our Theorem G). *Let $m>n\geqq 2$ be integers. Then the mapping $\{$translates of $K\}\mapsto\mu_K$ defined in Definition 1 and after Theorem F is a homeomorphism between its domain and its range. Its domain is the subspace corresponding to the non-degenerate convex polytopes with at most m facets of the quotient topology of the topology on the convex bodies (induced by the Hausdorff metric) with respect to the equivalence relation of being translates. Its range is the set of finite Borel measures on $\mathbb{S}^{n-1}$ with supports of at most m points satisfying (i) and (ii) of Theorem F′, with the subspace topology of the weak* topology on $C(\mathbb{S}^{d-1})^*$. This subspace topology is described in more explicit form before Theorem F′.*

4. Proofs for the Euclidean case

Essentially the following proposition was used in [31] without explicitly stating and proving it. It can be considered as folklore (as part of the proof of the folklore Theorem 1), but we state and prove it for completeness.

PROPOSITION 6. *Let $m>n\geqq 3$ be integers. Let $\delta>0$ and let $S_m\geqq S_{m-1}\geqq\dots\geqq S_1>0$ be numbers such that $S_m<S_1+S_2+\dots+S_{m-1}$. Then there are pairwise distinct unit vectors $v_1,v_2,\dots,v_m\in\mathbb{S}^{n-1}$ with the following properties:*

(i) *they lie in the open δ-neighbourhood of the x_1x_2-coordinate plane,*

(ii) *they do not lie in a linear $(n-1)$-subspace of $\mathbb{R}^n$, and*

(iii) $S_1v_1 + S_2v_2 + \cdots + S_mv_m = 0$.

PROOF. Let P be the x_1x_2-coordinate plane in $\mathbb{R}^n$. Since $S_m < S_1 + S_2 + \cdots + S_{m-1}$, there exists a convex polygon $A_1A_2\ldots A_m$ (with angles strictly smaller than π) in P such that $|A_iA_{i+1}| = S_i$ for $i = 1, \ldots, m$ (indices considered modulo m), see [32, p. 44], [34, pp. 53–54]. Then the edge directions $u_i := \overrightarrow{A_iA_{i+1}}/|A_iA_{i+1}| \in \mathbb{S}^{n-1} \cap P$ are distinct unit vectors. We will perturb $A_1A_2\ldots A_m$ to a spatial polygon $B_1B_2\ldots B_m$, keeping the side lengths equal: $|B_iB_{i+1}| = |A_iA_{i+1}| = S_i$. The unit vectors $v_i := \overrightarrow{B_iB_{i+1}}/S_i$ will then fulfill (iii) by construction.

Clearly, for $\|v_i - u_i\| < \delta$, the vector v_i lies in the open δ-neighbourhood of P, hence (i) is satisfied. Further, for δ sufficiently small, the vectors $v_1, \ldots, v_m$ are also pairwise distinct.

Let k denote the largest integer such that there are arbitrarily small perturbations B_i of our original points A_i that satisfy the following: all edges have the right length $|B_iB_{i+1}| = S_i$, and the dimension of the affine hull of $B_1, \ldots, B_m$ has dimension k.

Assume for contradition that $k < n$. Then, by $m \geqq n+1 \geqq k+2$, there is an affine dependence among the B_i's. Let, for example, B_m lie in the affine hull H of $B_1, \ldots, B_{m-1}$. Then, fixing $|B_{m-1}B_m|$ and $|B_mB_1|$, the point B_m can move on an $(n-2)$-sphere around the axis $B_{m-1}B_1$ in a hyperplane perpendicular to H. Hence there is an arbitrarily small perturbation of B_m lying outside H, while $\operatorname{aff}\{B_1, \ldots, B_{m-1}\}$ already spans H. Thus we have obtained a contradiction to the choice of k.

This proves $k = n$ and thus (ii). □

4.1. Proof of Theorem 1

The implication (ii) $\Longrightarrow$ (i) is evident.

The implication (i) $\Longrightarrow$ (iii) is well-known, but we give the proof for completeness. Using the notations from Theorem F′, we have $S_m = \|S_mu_m\| = \left\| \sum_{i=1}^{m-1} S_iu_i \right\| \leqq \sum_{i=1}^{m-1} S_i$. The only case of equality is the degenerate case given in condition (iii′) of the theorem.

Finally, (iii) $\Longrightarrow$ (ii) follows from Proposition 6 and Minkowski's Theorem F′.

The degenerate case, with (iii′), follows from the above considerations. □

4.2. Proofs for Theorem 2

We need the following relation between the surface area, diameter and volume of a convex body. Here κ_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$.

PROPOSITION 7 (Gritzmann, Wills and Wrase [27]). *Let $K \subset \mathbb{R}^n$ be a convex body. Then the inequality $S(K)^{n-1} > \kappa_{n-1} \cdot \operatorname{diam}(K) \cdot \big(nV(K)\big)^{n-2}$ holds and this inequality is sharp.* □

We will construct the polytope for Theorem 2 by Minkowski's Theorem F′. We need to choose only an appropriate surface area measure. For a convex polytope, this finite Borel measure is concentrated in finitely many points. Assume that for given facet areas we are far from the degenerate case where this measure is concentrated in a great-$\mathbb{S}^{n-2}$. Then by compactness, the volume of the convex polytope is bounded from below. Therefore, to get an arbitrarily small volume, we must approach the degenerate case. This will be done in the following proof. Recall also the paragraph after Theorem E citing [31] (p. 3), where also the degenerate case was approximated – however, for simplices only.

4.3. First proof of Theorem 2

By Proposition 6, for any $\delta > 0$, there are pairwise distinct vectors $v_1, v_2, \ldots, v_m \in \mathbb{S}^{n-1}$ in the open δ-neighbourhood of the x_1x_2-coordinate plane, satisfying the following. They do not lie in a linear $(n-1)$-subspace of $\mathbb{R}^n$, and $S_1v_1 + S_2v_2 + \cdots + S_mv_m = 0$. By Minkowski's Theorem F′ there exists a non-degenerate convex polytope $P = P(\delta)$ in $\mathbb{R}^n$ having m facets with areas $S_1, \ldots, S_m$ and unit outer normals $v_1, \ldots, v_m$.

Let us consider the sequence of polytopes $P_k = P(1/k)$ for $k = 1, 2, \ldots$. We will show that $V(P_k) \to 0$ as $k \to \infty$. Assume the contrary. Then (possibly passing to a subsequence), we may assume without loss of generality that $V(P_k) \geqq \alpha > 0$.

By Proposition 7, we get the inequality

$$S(P_k)^{n-1} > \kappa_{n-1} \cdot \operatorname{diam}(P_k) \cdot \big(nV(P_k)\big)^{n-2} \geqq \kappa_{n-1} \cdot \operatorname{diam}(P_k) \cdot (n\alpha)^{n-2},$$

where $S(P_k) = \sum_{i=1}^m S_i$ is a constant. From this inequality, we conclude that $\operatorname{diam}(P_k)$ is bounded by some constant D for all k.

By applying translations, we may assume without loss of generality that all P_k's have a common point. Therefore, all polytopes P_k lie in a ball of radius D. Using compactness (possibly passing to a subsequence), we may assume even more. The sequence P_k tends (in the Hausdorff metric) to a non-empty compact convex set P_0 as $k \to \infty$ [55, p. 50, Theorem 1.8.6]. Therefore, $V(P_0) \geqq \alpha > 0$, and hence P_0 is a convex body. Moreover, by Minkowski's Theorem G′ (actually only by the continuity of the bijection in that theorem), P_0 is a convex polytope having m facets with all facet outer unit normals in the x_1x_2-coordinate plane. However, this is a contradiction to condition (ii) of Theorem F′. □

Remark 5. Instead of Proposition 7, where the multiplicative constant is sharp, we could have used a consequence of the Aleksandrov–Fenchel inequality [55, p. 327, Theorem 6.3.1] to show that the diameter is bounded. Namely: the quermassintegrals $W_i(K)$ ([55, p. 209]) for $0 \leqq i \leqq n$ form a logarithmically concave sequence. Here, $W_0(K) = V(K)$ and for fixed n, $W_1(K)$ is proportional to $S(K)$ and $W_{n-1}(K)$ is proportional to the mean width of K ([55, p. 210, p. 291, (5.3.12)]). Then apply this logarithmic convexity for volume, constant times surface area and constant times mean width. Finally, use the fact that the quotient of the diameter and the mean width is between two positive numbers (depending only on n). This yields the inequality of Proposition 7 with a weaker constant.

Example 1. We give an example of a family of tetrahedra ($n = 3$ and $m = 4$) with constant facet areas and arbitrarily small volume. The tetrahedra look like thin vertical needles and have vertices

$$\big(\pm\varepsilon, 0, -(1/\varepsilon)\sqrt{1-\varepsilon^4/4}\big) \quad \text{and} \quad \big(0, \pm\varepsilon, (1/\varepsilon)\sqrt{1-\varepsilon^4/4}\big).$$

All facets have area 2, and the volume is $(4\varepsilon/3)\sqrt{1-\varepsilon^4/4}$, which tends to zero as $\varepsilon \to 0$.

4.4. Second proof of Theorem 2

1. First we will construct a partition $\mathcal{P} = \{P_1, \dots, P_{n+1}\}$ of the index set $\{1, \dots, m\}$ into $n+1$ classes. We will achieve that the $n+1$ numbers $\sum_{i\in P_j} S_i$ (for $1 \leqq j \leqq n+1$) have the property that

($*$) the largest of these numbers is smaller than the sum of all others.

We start with the partition into m singleton classes. Suppose that we already have constructed a partition $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ such that

($**$) the largest of the numbers $T_j := \sum_{i\in Q_j} S_i$, where $1 \leqq j \leqq k$, is smaller than the sum of all the other numbers T_j.

If $k = n+1$, then we stop. If $k > n+1$, then let us assume $T_1 \leqq T_2 \leqq \dots \leqq T_k$. Now we take the two classes Q_1 and Q_2 with the two smallest sums and form their union while the other classes Q_j are kept. In the new partition, the partition class that has maximal sum T_j can be either the same partition class as in the preceding step or the newly constructed union. In the first case, ($**$) is evident. In the second case, we have $T_1 + T_2 \leqq T_{k-1} + T_k < T_3 + \dots + T_{k-1} + T_k$ before taking the union since $k \geqq n+2 \geqq 5$. Thus $T_{\text{new}} := T_1 + T_2 < T_3 + \dots + T_{k-1} + T_k$. Therefore, ($**$) holds in this case as well.

This proves that ($*$) holds for the final partition.

2. We consider the partition $\mathcal{P} = \{P_1, \dots, P_{n+1}\}$ constructed above. For the sums $R_j := \sum_{i\in P_j} S_i$, we use Theorem E to construct a non-degenerate

simplex S that has these facet areas and has an arbitrarily small volume. Then, for the respective outer unit normals u_j of the facets of this simplex, we have $\sum_{j=1}^{n+1} R_j u_j = 0$.

3. We will now split each facet of the simplex into almost parallel facets to get the desired polytope with m facets. Let $\varepsilon > 0$ be small. For each $1 \leqq j \leqq n+1$, choose a linear 2-subspace X_j containing u_j. Choose a vector $u_{ji} \in X_j$ for each $i \in P_j$ such that the vectors $-(1-\varepsilon)\big(\sum_{i\in P_j} S_i\big)u_j$ and $S_i u_{ji}$ (for $i \in P_j$) are the side vectors of a convex polygon in X_j. Since the length of the first vector is almost equal to the sum of the others, all u_{ji} are close to u_j for ε sufficiently small. Then all vectors $S_i u_{ji}$ for $1 \leqq j \leqq n+1$ and $i \in P_j$ linearly span $\mathbb{R}^n$, and their sum is

$$\sum_{1\leqq j\leqq n+1}\Big(\sum_{i\in P_j} S_i u_{ji}\Big) = \sum_{1\leqq j\leqq n+1}(1-\varepsilon)\Big(\sum_{i\in P_j} S_i\Big)u_j$$
$$= (1-\varepsilon)\sum_{1\leqq j\leqq n+1} R_j u_j = 0.$$

4. By Minkowski's Theorem F$'$, there exists a non-degenerate convex polytope with facet outer unit normals u_{ji} and facet areas S_i (for all $1 \leqq j \leqq n+1$ and all $i \in P_j$).

Observe that we have changed in the course of the proof the surface area measure only a little bit (in the weak*-topology of $C(\mathbb{S}^{n-1})^*$). Therefore, after a suitable translation, the obtained convex polytope is arbitrarily close to the original simplex S by Theorem G$'$ (actually only by the continuity of the inverse of the bijection in that theorem). Since the simplex had an arbitrarily small volume, our convex polytope also has an arbitrarily small volume. □

4.5. Third proof of Theorem 2

The first two proofs of Theorem 2 did not give geometric information about the constructed polytopes. (The first proof used an argument by contradiction and the second proof used the examples of the simplices.) Now we give a third proof that is more quantitative and will give also geometric information. This proof constructs "needle-like" polytopes, as in Example 1. See also the paragraph following the statement of Theorem 2 (p. 6).

First we give the proof for $n = 3$ dimensions. We begin with an elementary lemma. It shows that a convex polytope in $\mathbb{R}^3$ that has steep (almost vertical) facets must have steep edges, as long as the angles between the normal vectors of different facets are bounded away from 0 and π, Lemma 1. *Consider two planes in* $\mathbb{R}^3$ *with unit normals* u^+ *and* u^-. *Assume that* u^+ *and* u^- *enclose an angle at most* $\varepsilon \in (0, \pi/2)$ *with the xy-plane, and the angle between them lies in* $[\beta, \pi-\beta]$, *where* $0 < \beta \leqq \pi/2$. *Then their intersection line encloses an angle at most*

$$\delta := \arcsin \frac{\sin\varepsilon}{\sin(\beta/2)}$$

with the z-axis, provided that $\varepsilon \leqq \beta/2$. *This inequality is sharp.*

Proof. We choose a new coordinate system in the following way. The intersection line becomes the vertical axis, and the two normal vectors $u^+, u^- \in \mathbb{S}^2$ lie in the horizontal plane, enclosing an angle $\beta' \in [\beta, \pi-\beta]$ with each other. In the new coordinate system, the original North Pole becomes $n = (n_1, n_2, n_3) \in \mathbb{S}^2$.

By hypothesis,

$$\langle n, u^-\rangle, \langle n, u^+\rangle \in [-\sin\varepsilon, \sin\varepsilon]. \tag{5}$$

We want to conclude that

$$\bigl|\bigl\langle (0,0,1), n\bigr\rangle\bigr| = |n_3| \geqq \cos\delta, \tag{6}$$

i.e., that

$$\sqrt{n_1^2 + n_2^2} \leqq \sin\delta. \tag{7}$$

The points $(n_1, n_2) \in \mathbb{R}^2$ (projections of n to the xy-plane) for n satisfying (5) form a rhomb of height $2\sin\varepsilon$ and angles $\beta', \pi-\beta'$. A farthest point of this rhomb from $(0,0)$ is one of the vertices and its distance from $(0,0)$ is $\max\bigl\{(\sin\varepsilon)/\sin(\beta'/2), (\sin\varepsilon)/\cos(\beta'/2)\bigr\} \leqq (\sin\varepsilon)/\sin(\beta/2) = \sin\delta$. That is, (7), or equivalently, (6) holds and both are sharp inequalities. Hence, the inequality of the lemma holds and it is sharp. □

Lemma 2. *Consider a convex polyhedron* $P \subset \mathbb{R}^3$ *with facet areas* $S_1, \dots, S_m$. *Assume that its facet outer normals enclose an angle at most* ε *with the xy-plane and the angle between any two of them lies in* $[\beta, \pi-\beta]$, *where* $0 < \beta \leqq \pi/2$. *Then its volume is bounded by*

$$V(P) \leqq 2^{-1/4}\pi^{-1} \cdot \left(\sum_{i=1}^m S_i^{3/4}\right)^2 \cdot \left(\frac{\sin\varepsilon}{\sin(\beta/2)}\right)^{1/2},$$

if $(\sin\varepsilon)/\sin(\beta/2) \leqq 1/\sqrt{2}$.

Proof. We denote by $s_i(z)$ the length of the horizontal cross-section of the i-th facet at height z, and by $s_i^{\max}$ the maximum length of such a horizontal cross-section. Let h_i be the "height" of the i-th face: the difference between the maximum and the minimum z-coordinates of its points. Let h_i' be the "tilted height" of this facet in its own plane, i.e., the height when the plane is rotated into vertical position about one of its horizontal cross-sections.

Since $(\sin\varepsilon)/\sin(\beta/2) < 1$, we have by Lemma 1 that P has no horizontal edges. Therefore, using the quantity δ introduced in Lemma 1, we get

$$s_i^{\max} \leqq h_i \cdot \tan\delta. \tag{8}$$

Namely, from the minimal z-coordinate – where $s_i(z) = 0$ – $s_i(z)$ can increase only with a speed at most $2\tan\delta$ ($< \infty$) to reach its maximal value $s_i^{\max}$. This is clear for a vertical face, and for a nonvertical face the speed is even smaller. Observe that the i-th facet lies in an upwards circular cone with vertex the lowest point of the i-th facet and directrices enclosing an angle δ with the z-axis. From the maximal value it must decrease again with speed at most $2\tan\delta$ till 0 at the maximal z-coordinate.

Therefore, using for (10) inequality (8),

$$S_i \geqq s_i^{\max} h_i'/2 \geqq s_i^{\max} h_i/2 \tag{9}$$

$$\geqq (s_i^{\max})^2/(2\tan\delta). \tag{10}$$

This gives

$$s_i^{\max} \leqq \sqrt{2S_i \tan\delta}. \tag{11}$$

These relations allow us to bound the volume $V(P)$ as follows, by using the isoperimetric inequality on each horizontal slice.

$$V(P) = \int_{-\infty}^{\infty} (\text{area of cross-section of } P \text{ at height } z)\, dz$$

$$\leqq \int_{-\infty}^{\infty} \left(\sum_{i=1}^m s_i(z)\right)^2 dz/(4\pi)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{-\infty}^{\infty} s_i(z) s_j(z)\, dz/(4\pi)$$

$$\leqq \sum_{i=1}^{m} \sum_{j=1}^{m} s_i^{\max} s_j^{\max} \min\{h_i, h_j\}/(4\pi) \tag{12}$$

$$\leqq \sum_{i=1}^{m} \sum_{j=1}^{m} s_i^{\max} s_j^{\max} \sqrt{h_i h_j}/(4\pi)$$

$$= \left(\sum_{i=1}^{m} s_i^{\max} \sqrt{h_i} \right)^2 /(4\pi)$$

$$= \left(\sum_{i=1}^{m} \sqrt{s_i^{\max}} \sqrt{s_i^{\max} h_i} \right)^2 /(4\pi)$$

$$\leqq \left(\sum_{i=1}^{m} (2S_i \tan\delta)^{1/4} \sqrt{2S_i} \right)^2 /(4\pi) \tag{13}$$

$$= \frac{\left(\sum_{i=1}^{m} S_i^{3/4}\right)^2}{\sqrt{2}\pi} \cdot \sqrt{\frac{(\sin\varepsilon)/\sin(\beta/2)}{\sqrt{1-(\sin^2\varepsilon)/\sin^2(\beta/2)}}} \tag{14}$$

$$\leqq \frac{\left(\sum_{i=1}^{m} S_i^{3/4}\right)^2}{\sqrt{2}\pi} \cdot \sqrt{\frac{\sqrt{2}\sin\varepsilon}{\sin(\beta/2)}}.$$

The first inequality uses the isoperimetric inequality. The second inequality (12) bounds the integral by an upper bound of the non-negative integrand times the length of the interval where the integrand is positive. For (13), we have used (9) and (11). To obtain (14), we have used Lemma 1. The last inequality simplifies the denominator under the assumption $(\sin\varepsilon)/\sin(\beta/2) \leqq 1/\sqrt{2}$ of the lemma. $\square$

4.6. Third proof of Theorem 2 for $n = 3$ dimensions

As in the first proof, we use Minkowski's Theorem F$'$. We want to apply Lemma 2, making ε small. Thus, we must let the normal vectors with given lengths S_i converge to the xy-plane, keeping their sum to be 0. Moreover, the linear span of the outer unit facet normals should be $\mathbb{R}^3$. Then we apply Minkowski's Theorem F$'$. In the limiting configuration the normals will lie

in the xy-plane. They must form angles in $[\beta, \pi-\beta]$ (with $\beta \in (0, \pi/2]$) with each other in order that Lemma 2 should work. Thus we must avoid parallel sides.

Will show that there is only one exceptional case in which parallel sides cannot be avoided. Consider a planar convex m-gon M with sides S_i that has the minimum number of parallel pairs of sides. Let us assume that M has a side such that the sum of the two incident angles is different from π. Then by a small length-preserving motion of this side and the neighbouring two sides, one can achieve the following. This side changes its direction while new parallel pairs of sides are not created. Therefore, M can have a parallel pair of sides only if, for each of these sides, the sum of the incident angles is π. That is, we have four vertices that determine two parallel sides and whose outer angles (i.e., π minus the inner angles) have sum 2π. Since the sum of all outer angles is 2π, there are no more vertices and M must be a parallelogram. If its sides are not equal then we rearrange the side vectors so as to obtain a (convex) deltoid that is not a parallelogram. So the only remaining case is when $m = 4$ and $S_1 = S_2 = S_3 = S_4$. However, this case has been treated in Example 1: a tetrahedron with four faces of equal areas and having an arbitrarily small positive volume. Suitable inflations provide examples for all values of S_i.

Disregarding this exceptional case, we have now a strictly convex polygon in the xy-plane without parallel sides. Assume that the angle between any two edges is in the range $[\beta_1, \pi-\beta_1]$ for some $\beta_1 > 0$. We still need to perturb the sides so that the edge vectors span $\mathbb{R}^3$. Consider the first three consecutive vertices A_1, A_2, A_3 of M. Let us fix A_1 and A_3. Rotate the two sides $[A_1, A_2]$ and $[A_2, A_3]$ about the line through A_1 and A_3 through a small angle $\alpha > 0$ while keeping their lengths fixed. The $m-2 \geqq 2$ remaining side vectors span the xy-plane since they are not parallel. At the same time, the vector $\overrightarrow{A_1A_2}$ points out of the xy-plane and therefore the edge vectors span $\mathbb{R}^3$.

By making the angle of rotation α small enough, we can ensure the following. The angle between all edge vectors of the perturbed polygon $M(\alpha)$ is still in the range $[\beta_2, \pi-\beta_2]$ for some fixed $\beta_2 > 0$. Moreover, the angle ε of the side vectors with the xy-plane can be made arbitrarily small. We use the edge vectors $\overrightarrow{S_i}$ of $M(\alpha)$ as outer normals and construct the polytope P by Minkowski's Theorem F$'$, with $S_i := \|\overrightarrow{S_i}\|$ and $u_i := \overrightarrow{S_i}/S_i$. By Lemma 2, the volume can be made arbitrarily small. $\square$

Remark 6. In the polytope that we have constructed, all facets except two are vertical. By going through the proof of Lemma 2, one can see the following. It would have been sufficient to assume the constraint $[\beta, \pi-\beta]$ on the angles for those pairs of facet normals that involve at least one of the two nonvertical facets.

EXAMPLE 2. For odd dimension $n = 2k+1$, there is a higher-dimensional generalization of Example 1. Consider a large regular k-simplex of edge length $a := 1/\varepsilon$ in the $x_{k+2} \dots x_n$-coordinate plane. It has $k+1$ vertices $v_1, \dots, v_{k+1}$. At each vertex v_i, we draw a short segment of length $b := \varepsilon$ centred at v_i in the direction of the x_i-axis. The convex hull of the union of these segments is an n-simplex with congruent facets. The facet areas are $\sim \text{const} \cdot a^k b^k = \text{const}$, while the volume is $\text{const} \cdot a^k b^{k+1} = \text{const} \cdot \varepsilon$, which becomes arbitrarily small as $\varepsilon \to 0$.

4.7. Third proof of Theorem 2 for n > 3 dimensions

4.7.1. Construction of an almost flat spatial polygon. As for $n = 3$, we start with a planar convex m-gon M in the x_1x_2-coordinate plane, where $m \geqq n+1$. It has side vectors $\overrightarrow{S_i}$ (this notation will be preserved also after perturbations) with side lengths S_i, for $1 \leqq i \leqq m$.

M is contained in the $x_1 \dots x_{n-1}$-coordinate hyperplane X. By small perturbations of the closed polygon M in X that preserve the side lengths S_i, we want to achieve that

(∗) the perturbed (skew) closed polygon $M \subset X$ has no $n-1$ side vectors lying in an $(n-2)$-dimensional linear subspace of X.

Initially, M lies in a 2-dimensional plane. We will fulfill (∗) by following the proof of Proposition 6. Our desired conclusion is slightly stronger than in Proposition 6: there we excluded only the case that *all* vectors lie in a lower-dimensional subspace.

Assume that some $i \leqq n-1$ side vectors lie in a linear subspace of dimension less than i, where i is the smallest number with this property. We will eliminate these linear dependencies iteratively. We have already seen how we can avoid parallel edges ($i = 2$). The only case where parallel sides could not be avoided was $m = 4$ and $S_1 = S_2 = S_3 = S_4$ (a rhomb) and this happens only for $n \leqq m-1 = 3$. Therefore, we can assume $i \geqq 3$. Observe also that by small perturbations the different vertices remain distinct.

We may any time rearrange the cyclic order of side vectors of M as we want. So we assume that the first i side vectors $\overrightarrow{S_1}, \dots, \overrightarrow{S_i}$ are linearly dependent, and any $i-1$ side vectors are linearly independent. Number the vertices A_j so that $\overrightarrow{S_j}$ goes from vertex A_j to A_{j+1} (indices taken modulo m). We want remove this linear dependence by perturbing the vertex A_i. For a technical reason, we have to first refine the order of the side vectors even further. Let $\sum_{j=1}^{i} \lambda_j \overrightarrow{S_j} = 0$ be the (unique, up to a scalar factor) linear dependence. The λ_j's cannot be all equal, since $\sum_{j=1}^{i} \overrightarrow{S_j} = \overrightarrow{A_1 A_{i+1}} = 0$ would imply that A_1 and A_{i+1} are equal points. The polygon has $m > n > i$ sides,

and hence this is excluded. Therefore, by permuting the side vectors if necessary, we can assume that $\lambda_{i-1} \neq \lambda_i$. Since all $(i-1)$-subsets are linearly independent, we have $\lambda_j \neq 0$ for all j, and thus we can assume without loss of generality that $\lambda_i = -1$. In other words, $\overrightarrow{S_i} = \sum_{j=1}^{i-1} \lambda_j \overrightarrow{S_j}$, with $\lambda_{i-1} \neq -1$.

Now, fixing A_{i-1}, A_{i+1}, $\|\overrightarrow{S_{i-1}}\|$, and $\|\overrightarrow{S_i}\|$, we can perturb A_i as follows. The point A_i moves on an $(n-3)$-dimensional sphere in a hyperplane within X with affine hull orthogonal to the segment $[A_{i-1}, A_{i+1}]$. There is a small motion that moves A_i out of the subspace $H := \operatorname{aff}\{A_1, A_2, \dots, A_{i-1}, A_{i+1}\}$ of X.

Now we show that the dimension of this subspace H is in fact $i-1$, which implies that the dimension of $\operatorname{aff}\{A_1, A_2, \dots, A_i, A_{i+1}\}$ increases by 1 and $\overrightarrow{S_1}, \dots, \overrightarrow{S_i}$ become linearly independent. Clearly, $\dim H$ cannot be greater than $i-1 = \dim \operatorname{aff}\{A_1, A_2, \dots, A_{i-1}, A_{i+1}\}$. To see that $\dim H = i-1$, we note that the vectors $\overrightarrow{A_1A_2} = \overrightarrow{S_1}$, $\overrightarrow{A_2A_3} = \overrightarrow{S_2}, \dots, \overrightarrow{A_{i-2}A_{i-1}} = \overrightarrow{S_{i-2}}$, $\overrightarrow{A_{i-1}A_{i+1}} = \overrightarrow{S_{i-1}} + \sum_{j=1}^{i-1} \lambda_j \overrightarrow{S_j}$ are linearly independent, since $\lambda_{i-1} \neq -1$, and hence, their linear span has dimension $i-1$.

Thus, we have established that, by perturbing A_i, the vectors $\overrightarrow{S_1}, \dots, \overrightarrow{S_i}$ become linearly independent. If the perturbation is small enough, then every set of side vectors that was linearly independent before the motion remains linearly independent. Therefore, the number of linearly dependent i-tuples of side vectors of M decreases. A finite number of iterations eliminates all linearly dependent i-tuples, and i can be increased (till $n-1$), until (∗) is eventually established. This concludes the construction of the polygon M.

Condition (∗) can be rephrased in the following way. The determinant of any $n-1$ normed side vectors $\overrightarrow{S_i}/S_i$ of M (i.e., the signed volume of the parallelepiped spanned by them) is nonzero. We denote by $b > 0$ the smallest absolute value of these determinants. This bound will play the role of the sine of the angle bound β in Lemmas 1 and 2.

4.7.2. Steep facets imply steep edges. We generalize Lemma 1 to higher dimensions:

LEMMA 3. *Let $n \geqq 3$, and consider $n-1$ hyperplanes in $\mathbb{R}^n$ making an angle at most $\varepsilon < \pi/2$ with the vertical axis (the x_n-axis). If their unit normal vectors span an $(n-1)$-parallelotope of volume at least b (> 0) then they intersect in a line. The angle between this line and the vertical direction is bounded by*

$$\delta := \arcsin \frac{(n-1)^{3/2} \sin \varepsilon}{b},$$

provided that $(n-1)^{3/2} \sin \varepsilon \leqq b$.

For fixed n, the order of magnitude of this bound on δ as a function of ε and b is optimal. More precisely, for any ε and b, where $0 < \varepsilon < \pi/2$ and $0 < b \leqq 1$, there are instances with

$$\sin\delta = \min\left\{1, (\sin\varepsilon)/\sin\big((\arcsin b)/2\big)\right\}.$$

Proof. Since the unit normal vectors $v_1, \dots, v_{n-1}$ are linearly independent, the intersection of the hyperplanes is a line ℓ. Let us choose a new orthonormal coordinate system where ℓ is the last coordinate axis. Then the last coordinate of the vectors v_i is zero, and we may write these vectors as $v_i = \binom{v_i'}{0}$ with $v_i' \in \mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$. By assumption, the $(n-1)\times(n-1)$ matrix $V = (v_1', \dots, v_{n-1}')$ has determinant of absolute value $|\det V| \geqq b$.

Let $p = \binom{p'}{p_n}$, with $p' \in \mathbb{R}^{n-1}$, be the unit vector of the original positive x_n-direction in the new coordinate system. Its angle $\delta \in [0, \pi/2]$ with the line ℓ satisfies $\cos\delta = |p_n|$ and $\sin\delta = \|p'\|$, and thus our goal is to show that

$$\|p'\| \leqq (n-1)^{3/2}\frac{\sin\varepsilon}{b}\,. \tag{15}$$

Let α_i denote the angle between p and the normal v_i. By the angle assumption on the hyperplanes, we have $\pi/2 - \varepsilon \leqq \alpha_i \leqq \pi/2 + \varepsilon$. Therefore, with $r_i := \cos\alpha_i = \langle p, v_i\rangle = \langle p', v_i'\rangle$, we have $|r_i| \leqq \sin\varepsilon$.

The $n-1$ equations $\langle p', v_i'\rangle = r_i$ form a linear system $(p')^T V = (r_1, \dots, r_{n-1})$ (the column vectors of V being the v_i's), i.e., $V^T p' = (r_1, \dots, r_{n-1})^T$, which determines p' uniquely:

$$p' = \left(V^T\right)^{-1}(r_1, \dots, r_{n-1})^T. \tag{16}$$

We write $\operatorname{adj}\left(V^T\right)$ for the transpose of the matrix whose entries are the signed cofactors of the respective entries of V^T. By the formula $\left(V^T\right)^{-1} = \operatorname{adj}\left(V^T\right)/\det\left(V^T\right)$, each entry of $\left(V^T\right)^{-1}$ is an $(n-2)\times(n-2)$ subdeterminant of V^T divided by $\pm\det V^T$. The rows of the submatrices of V^T are vectors of length at most 1 and therefore these subdeterminants are bounded in absolute value by 1. It follows that the entries of $\left(V^T\right)^{-1}$ are bounded in absolute value by $1/b$. Since the $|r_i|$'s are at most $\sin\varepsilon$, we get from (16) that the $n-1$ entries of p' are bounded by $(n-1)(\sin\varepsilon)/b$ in absolute value. Hence we have proved (15).

To establish the lower bound, we can lift the tight three-dimensional example from Lemma 1 to n dimensions. For $\varepsilon \leqq \beta/2$, the enclosed angle will be the same as in three dimensions, namely $\arcsin\big((\sin\varepsilon)/\sin(\beta/2)\big)$, where $\sin\beta = b$. For $\varepsilon > \beta/2$, we use the example with $\varepsilon = \beta/2$. We embed the 3-dimensional example into $\mathbb{R}^n$ by a linear isometry that maps the positive x, y, z-axes to the positive x_1, x_2, x_n-coordinate axes of $\mathbb{R}^n$. (The "vertical" direction is now the direction of the x_n-axis.) The two 2-planes of the three-dimensional example are turned into hyperplanes as follows. We replace them by their inverse images under the orthogonal projection of $\mathbb{R}^n$ to the $x_1x_2x_n$-coordinate subspace. Simultaneously, we add the hyperplanes with equations $x_3 = 0, \dots, x_{n-1} = 0$. □

4.7.3. Polytopes with steep facets have small volume.

Lemma 4. *Let $n > 3$ be an integer. Assume that a convex polytope $P \subset \mathbb{R}^n$ has facet areas $S_1, \dots, S_m$. Moreover, its outer unit facet normals enclose an angle at most $\varepsilon \in (0, \pi/2)$ with the $x_1 \dots x_{n-1}$-plane. Also the volume of the $(n-1)$-parallelepiped spanned by any $n-1$ unit facet normals of P is at least $b > 0$. Then its volume is bounded by*

$$V(P) \leqq \mathrm{const}_n \cdot \left(\sum_{i=1}^m S_i^{n/(2n-2)}\right)^2 \cdot \left(\frac{\sin\varepsilon}{b}\right)^{1/(n-1)},$$

if $\sin^2\varepsilon \leqq b^2/\left[2(n-1)^3\right]$.

On the other hand, for $n \geqq 3$ and any $m \geqq 2n$, there exists a suitable $\varepsilon_0 \in (0, \pi/4)$, such that the following holds. For any $\varepsilon \in (0, \varepsilon_0)$, there exists a convex polytope $P(\varepsilon) \subset \mathbb{R}^n$, with m facets, with the following properties. It satisfies the hypotheses of this lemma (except the one about the facet areas), with b depending only on n and m, such that

$$\begin{aligned} V\big(P(\varepsilon)\big) &\geqq \mathrm{const}_n' \cdot S\big(P(\varepsilon)\big)^{n/(n-1)} \cdot (\tan\varepsilon)^{1/(n-1)} \\ &\geqq \mathrm{const}_n' \cdot m^{-(n-2)/(n-1)} \cdot \left(\sum_{i=1}^m S_i(\varepsilon)^{n/(2n-2)}\right)^2 \cdot (\tan\varepsilon)^{1/(n-1)}. \end{aligned}$$

Here, $S_1(\varepsilon), \dots, S_m(\varepsilon)$ are the areas of the facets of $P(\varepsilon)$. In particular, in the inequalities of Lemma 2 and this lemma, the order of magnitude as a function of ε is optimal.

Proof. We begin with the proof of the upper estimate. We denote by $s_i(x_n)$ the $(n-2)$-volume of the horizontal cross-section of the i-th facet at height x_n. Moreover, we denote by $s_i^{\max}$ the maximum $(n-2)$-volume of such a horizontal cross-section. Let h_i be the "height" of the i-th facet: the difference between the maximal and the minimal x_n-coordinates of its points. Let h_i' be the "tilted height" of this facet in its own hyperplane. That is, the height when the hyperplane is rotated into vertical position about one of its horizontal cross-sections.

Now, since $\sin^2\varepsilon \leqq b^2/\big[2(n-1)^3\big]$, the angle δ from Lemma 3 lies in $(0,\pi/2)$. Hence, by Lemma 3, P has no horizontal edges, and thus, also no horizontal k-faces for any $k\in\{1,\dots,n-2\}$. Therefore, once more by Lemma 3, we know that every facet is contained in two rotationally symmetric cones with $(n-1)$-balls as bases. One cone has its apex at the unique lowest point of this facet and extends upwards from there. Its axis is vertical (parallel to the x_n-direction), and the directrices enclose an angle δ with the x_n-axis. The other cone extends downwards from the highest point of the facet and has a vertical axis and directrices enclosing an angle δ with the x_n-axis. We use the upwards cone from the minimal height till the arithmetic mean of the minimal and maximal heights. We use the downward cone for the other half of the vertical extent of the facet. By this argument, we can bound the maximum cross-section area $s_i^{\max}$ of the i-th facet as follows.

$$s_i^{\max} \leqq \big((h_i/2)\cdot\tan\delta\big)^{n-2}\cdot\kappa_{n-2}. \tag{17}$$

(From the minimal height till the arithmetic mean of the minimal and maximal heights we have the following. Any horizontal cross-section of the cone is contained in some $(n-1)$-ball of radius at most $R:=(h_i/2)\cdot\tan\delta$. Thus, any horizontal cross-section of the facet lies inside the intersection of its own affine hull with the upwards cone. That is, it lies in the intersection of an $(n-2)$-dimensional affine subspace with a cone whose base is an $(n-1)$-ball of radius at most R. Hence, this horizontal cross-section lies inside some $(n-2)$-ball of radius at most R. A similar argument holds for the downward cone.) Moreover, we also have

$$S_i \geqq s_i^{\max}h_i'/(n-1) \geqq s_i^{\max}h_i/(n-1). \tag{18}$$

Let us rewrite (17) and (18) as follows.

$$h_i^{-(n-2)}\cdot s_i^{\max} \leqq \big((\tan\delta)/2\big)^{n-2}\cdot\kappa_{n-2} \tag{19}$$

$$h_i\cdot s_i^{\max} \leqq (n-1)S_i. \tag{20}$$

We multiply the $1/\big[(2n-2)(n-2)\big]$-th power of (19) with the $n/(2n-2)$-th power of (20) to get an inequality that we will need.

$$\begin{aligned}(s_i^{\max})^{(n-1)/(2n-4)}&\sqrt{h_i}\\ &\leqq \big((\tan\delta)/2\big)^{1/(2n-2)}\cdot(\kappa_{n-2})^{1/[(2n-2)(n-2)]}\cdot\big((n-1)S_i\big)^{n/(2n-2)}.\end{aligned} \tag{21}$$

Let $K:=\big[(n-1)^{n-1}\kappa_{n-1}\big]^{-1/(n-2)}$ denote the constant of the isoperimetric inequality in $n-1$ dimensions:

$$V_{n-1}(C)\leqq K\cdot\big(V_{n-2}(\partial C)\big)^{(n-1)/(n-2)} \tag{22}$$

(for $C\subset\mathbb{R}^{n-1}$). Now we can bound the volume as follows.

$$\begin{aligned}
V(P) &= \int_{-\infty}^{\infty}\big[(n-1)\text{-volume of the cross-section of } P \text{ at height } x_n\big]\,dx_n\\
&\leqq \int_{-\infty}^{\infty}\left[\left(\sum_{i=1}^m s_i(x_n)\right)^{(n-1)/(2n-4)}\right]^2 dx_n\cdot K\\
&\leqq \int_{-\infty}^{\infty}\left[\sum_{i=1}^m s_i(x_n)^{(n-1)/(2n-4)}\right]^2 dx_n\cdot K\\
&= \int_{-\infty}^{\infty}\sum_{i=1}^m\sum_{j=1}^m s_i(x_n)^{(n-1)/(2n-4)}s_j(x_n)^{(n-1)/(2n-4)}\,dx_n\cdot K\\
&= \sum_{i=1}^m\sum_{j=1}^m\int_{-\infty}^{\infty} s_i(x_n)^{(n-1)/(2n-4)}s_j(x_n)^{(n-1)/(2n-4)}\,dx_n\cdot K\\
&\leqq \sum_{i=1}^m\sum_{j=1}^m (s_i^{\max})^{(n-1)/(2n-4)}(s_j^{\max})^{(n-1)/(2n-4)}\min\{h_i,h_j\}\cdot K \qquad (23)\\
&\leqq \sum_{i=1}^m\sum_{j=1}^m (s_i^{\max})^{(n-1)/(2n-4)}(s_j^{\max})^{(n-1)/(2n-4)}\sqrt{h_ih_j}\cdot K\\
&\leqq (\tan\delta)^{1/(n-1)}\cdot 2^{-1/(n-1)}\cdot(\kappa_{n-2})^{1/[(n-1)(n-2)]}\cdot(n-1)^{n/(n-1)} \qquad (24)\\
&\quad\cdot\left(\sum_{i=1}^m S_i^{n/(2n-2)}\right)^2\cdot K\\
&= \mathrm{const}_n\cdot\left(\sum_{i=1}^m S_i^{n/(2n-2)}\right)^2\left(\frac{(n-1)^{3/2}(\sin\varepsilon)/b}{\sqrt{1-(n-1)^3(\sin^2\varepsilon)/b^2}}\right)^{1/(n-1)} \qquad (25)
\end{aligned}$$

$$\leqq \mathrm{const}'_n \cdot \left(\sum_{i=1}^{m} S_i^{n/(2n-2)} \right)^2 \cdot \left(\frac{\sin \varepsilon}{b} \right)^{1/(n-1)} .$$

The first inequality uses the isoperimetric inequality (22). The second inequality uses the concavity of the function $t^{(n-1)/(2n-4)}$ for $t \in [0, \infty)$ and its vanishing at $t = 0$. (Observe that $0 < (n-1)/(2n-4) \leqq 1$.) Inequality (23), as in (12), bounds the integral of a non-negative function by an upper bound of the integrand times the length of the interval where the integrand is positive. For (24), we have used the bound (21) that we derived above. Inequality (25) uses the bound δ from Lemma 3. Finally, by hypothesis, the expression under the square root in the denominator of (25) is bounded below by $1 - (n-1)^3 (\sin^2 \varepsilon)/b^2 \geqq 1/2$. We have therefore established the claimed upper bound.

Now we give the example for the lower bound for $n \geqq 3$ and $m \geqq 2n$. Let $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 \in (0, \pi/4)$ will be chosen later. Let us write $\mathbb{R}^n = \mathbb{R}^{n-1} \oplus \mathbb{R}$. Let $T^+, T^- \subset \mathbb{R}^{n-1}$ be regular $(n-1)$-simplices circumscribed about the unit ball B^{n-1} of $\mathbb{R}^{n-1}$. Put them in such a general position w.r.t. each other so that any $n-1$ of their altogether $2n$ facet outer normals linearly span $\mathbb{R}^{n-1}$. Let $n \leqq m^+, m^-$ and $m = m^+ + m^-$. Let $R^\pm$ be obtained from $T^\pm$ by intersecting it still with $m^\pm - n$ closed halfspaces in $\mathbb{R}^{n-1}$, all containing B^{n-1}, with their boundaries touching B^{n-1}. Then

$$B^{n-1} \subset R^\pm \subset T^\pm \subset (n-1)B^{n-1}.$$

Let the altogether $m = m^+ + m^-$ facet outer unit normals of R^+ and R^- satisfy the same condition of general position as above. Namely, any $n-1$ of them linearly span $\mathbb{R}^{n-1}$. Let $b > 0$ be the minimum of the $(n-1)$-volumes of the $(n-1)$-parallelotopes spanned by any $n-1$ of these altogether m facet outer unit normals.

Observe that for $n = 3$ and $m \geqq 2$, the largest value of b is $\sin(\pi/m)$ – if we do not begin the construction with two regular triangles but allow any m facet outer unit normals in $\mathbb{S}^{n-2} = \mathbb{S}^1$. For $n > 3$, the maximal value of b can be bounded from above as follows – again not beginning with two regular simplices, but allowing any m facet outer unit normals in $\mathbb{S}^{n-2}$. Let us choose altogether $n-1$ outer unit normal vectors of R^+ and R^-, say, $u_1, \dots, u_{n-1} \in \mathbb{S}^{n-2}$. We have

$$\frac{\left| \det(u_1, \dots, u_{n-1}) \right|}{(n-1)!}$$
$$= V_{n-2}\big(\mathrm{conv}\{u_1, \dots, u_{n-1}\}\big) \cdot \mathrm{dist}\left(0, \mathrm{aff}\{u_1, \dots, u_{n-1}\}\right)/(n-1)$$
$$\leqq V_{n-2}\big(\mathrm{conv}\{u_1, \dots, u_{n-1}\}\big)/(n-1).$$

Here, $\mathrm{dist}(\cdot, \cdot)$ denotes distance. Thus, it suffices to bound

$$V_{n-2}\big(\mathrm{conv}\{u_1, \dots, u_{n-1}\}\big)$$

from above. This is the spherical analogue – for $\mathbb{S}^{n-2}$ – of the celebrated Heilbronn problem. This problem asks about the maximum of the minimal n-volume of n-simplices spanned by any m points in $[0,1]^n$. This problem is poorly understood. For an extensive literature on this problem, see [13, Ch. 11.2]. Unfortunately, this spherical variant cannot be reduced to the case of $[0,1]^{n-2}$ by taking the projection of, say, the intersection of $\mathbb{S}^{n-2}$ with each orthant to the tangent $\mathbb{R}^{n-2}$ at its centre. Namely, the area of $\mathrm{conv}\{u_1, \dots, u_{n-1}\}$ can be large even if its projection has a small area. In one direction, we have an implication: large projection areas imply large areas – but large areas still do not imply large values of $\left| \det(u_1, \dots, u_{n-1}) \right|$. However, this spherical variant is a special case of the $(n-1)$-dimensional Heilbronn problem for $[0,1]^{n-1}$. Namely, we can just add to any set of $(n-1)$-dimensional vectors in $\mathbb{S}^{n-2}$ the single vector 0 – but probably we loose an essential part of the information in this way.

Let $P^\pm(\varepsilon)$ be the half-infinite pyramid with vertex $(0, \dots, 0, \pm \tan \varepsilon)$ and base $R^\pm$. Then

$$C_i^\pm(\varepsilon) \subset P^\pm(\varepsilon) \subset C_o^\pm(\varepsilon),$$

where $C_i^\pm(\varepsilon)$ and $C_o^\pm(\varepsilon)$ is a half-infinite cone with vertex $(0, \dots, 0, \pm \tan \varepsilon)$ and base B^{n-1} and $(n-1)B^{n-1}$, respectively. Therefore,

$$C_i(\varepsilon) := C_i^+(\varepsilon) \cap C_i^-(\varepsilon) \subset P(\varepsilon) := P^+(\varepsilon) \cap P^-(\varepsilon) \subset C_o(\varepsilon) := C_o^+(\varepsilon) \cap C_o^-(\varepsilon).$$

Here, $C_i(\varepsilon)$ and $C_o(\varepsilon)$ are double cones over B^{n-1} and $(n-1)B^{n-1}$, respectively, with vertices $(0, \dots, 0, \pm \tan \varepsilon)$. Moreover, $P(\varepsilon)$ is a convex polytope with m facets, all facet outer unit normals enclosing an angle ε with the $x_1 \dots x_{n-1}$-hyperplane. (Actually they enclose an angle ε with the respective facet outer unit normal of R^+ or R^- in $\mathbb{R}^{n-1}$.) If ε_0 and thus also ε is sufficiently small then still any $n-1$ facet outer unit normals of $P(\varepsilon)$ span an $(n-1)$-parallelotope of volume at least some $b' \in (0, b)$.

A routine calculation gives

$$\frac{V\big(P(\varepsilon)\big)}{S\big(P(\varepsilon)\big)^{n/(n-1)}} \geqq \frac{V\big(C_i(\varepsilon)\big)}{S\big(C_o(\varepsilon)\big)^{n/(n-1)}}$$
$$= \frac{(\tan \varepsilon)^{1/(n-1)}}{n(2\kappa_{n-1})^{1/(n-1)} \big[1 + (n-1)\tan^2 \varepsilon\big]^{n/(2n-2)}}.$$

Therefore,

$$V\big(P(\varepsilon)\big) \geqq \frac{S\big(P(\varepsilon)\big)^{n/(n-1)} \cdot (\tan\varepsilon)^{1/(n-1)}}{n(2\kappa_{n-1})^{1/(n-1)}\big[1+(n-1)\tan^2\varepsilon_0\big]^{n/(2n-2)}}$$

$$\geqq \frac{m^{-(n-2)/(n-1)}\Big(\sum_{i=1}^m S_i(\varepsilon)^{n/(2n-2)}\Big)^2 \cdot (\tan\varepsilon)^{1/(n-1)}}{n(2\kappa_{n-1})^{1/(n-1)}\big[1+(n-1)\tan^2\varepsilon_0\big]^{n/(2n-2)}}.$$

Here, $S_1(\varepsilon),\dots,S_m(\varepsilon)$ are the areas of the facets of $P(\varepsilon)$. The second inequality is equivalent to Hölder's inequality for the numbers $S_i(\varepsilon)$, between their arithmetic mean and their power mean with exponent $n/(2n-2) \in (0,1)$. Finally, observe that $\tan\varepsilon_0 \in (0,1)$. □

4.7.4. Conclusion of the proof. Now we can finish the third proof of Theorem 2 for $n > 3$. We proceed as for $n = 3$ but instead of Lemma 2 we use Lemma 4. In Section 4.7.1 (see its last paragraph), we have constructed a closed polygon M in the $(n-1)$-dimensional subspace X with the following property. Any $n-1$ normed side vectors $\overrightarrow{S_i}/S_i$ span a parallelotope of volume at least b. We follow the third proof of Theorem 2 for $n = 3$. We take the first three consecutive vertices A_1, A_2, A_3 of M and "rotate" A_2 out of the subspace X, keeping A_1, A_3 and the lengths $|A_1A_2|$ and $|A_2A_3|$ fixed. We have a whole $(n-2)$-dimensional sphere on which A_2 can move, which intersects X orthogonally. By bounding the distance by which A_2 moves by a suitable threshold we can ensure the following. Any $n-1$ normed side vectors $\overrightarrow{S_i}/S_i$ still span a parallelotope of volume at least b' with some weaker bound $b' > 0$. The angle between $[A_1, A_2]$ or $[A_2, A_3]$ and the "horizontal" hyperplane X can be made arbitrarily small. Thus, Lemma 4 guarantees that the volume tends to zero as well. □

5. Proofs for the hyperbolic case

For general concepts in hyperbolic geometry, we refer to [5, 7, 18, 37, 40, 45]. In particular, a *Lambert quadrilateral in* $\mathbb{H}^2$ is a quadrilateral that has three right angles.

5.1. Proof of Proposition 1

Let H be the hyperplane of the facet F_m of P of area S_m. Let $p\colon \mathbb{H}^n \to H$ be the orthogonal projection of $\mathbb{H}^n$ to H. The image by p of the union of the $m-1$ facets different from F_m contains F_m.

Let dS be a surface element at a point $x \in \mathbb{H}^n$. Let its image by p be the surface element dS' at $p(x)$. Clearly, it suffices to show that $dS' \leqq dS$. We may assume that dS is an (infinitesimal) $(n-1)$-ball of radius dr in the tangent space $T_x(\mathbb{H}^n)$ of $\mathbb{H}^n$ at x.

First we deal with the case when dS is orthogonal to the line $\ell\big(x, p(x)\big)$. (For $x \in H$, we mean by $\ell\big(x, p(x)\big)$ the line containing x and orthogonal to H.) Then dS' is an infinitesimal $(n-1)$-ball in $T_{p(x)}(\mathbb{H}^n)$ of some radius dr'. By the trigonometric formulas of Lambert quadrilaterals in $\mathbb{H}^2$ (see [45, §29, (V)] or [17, Theorem 2.3.1]), we have $1 \leqq \cosh|xp(x)| = \big(\tanh(dr)\big)/\tanh(dr')$. Hence, $dr' \leqq dr$ and therefore, $dS' \leqq dS$.

Now we extend this analysis to the case when dS is not orthogonal to the line $\ell\big(x, p(x)\big)$. Then the image by p of the infinitesimal $(n-1)$-ball dS in $T_x(\mathbb{H}^n)$ is an infinitesimal $(n-1)$-ellipsoid in $T_{p(x)}(\mathbb{H}^n)$. It has $n-2$ semiaxes equal to dr' and the $(n-1)$-st semiaxis smaller than dr'. Hence, $dS' < dS$ in this case.

The case of equality is clear: the polytope must degenerate to the doubly counted facet F_m. □

5.2. Proof of Proposition 2

From maximality of $S_m, S_{m-1}, \dots, S_3$, it follows that all vertices lie at infinity. Namely, a vertex cannot be incident only to the facets of areas S_2, S_1. Hence, also S_2, S_1 are maximal. □

5.3. Proof of Proposition 3

Let P be a convex polyhedron as in the proposition with respective facets $F_1, \dots, F_m$. Let us consider any vertex v of some facet F_i. In the facets incident to v, the angle of F_i at v is at most the sum of the angles of all other facets incident to v. To see this, we intersect P with an infinitesimally small sphere with centre at this vertex (in the conformal model). We obtain a convex spherical polygon whose side lengths are the (convex) angles of the facets incident to v at v, all these angles being in $[0, \pi)$.

Summing these inequalities over all vertices v of F_i we obtain the following. The sum t_1 of the angles of F_i is at most the sum t_2 of the angles of all other facets at the vertices of F_i. The sum t_2 is bounded above by the sum t_3 of *all angles of all facets different from* F_i. The resulting inequality $t_1 \leqq t_3$ is equivalent to the inequality to be proved.

Clearly, if we have at least one finite vertex with incident edges not in a plane, then we have at least one strict inequality among the summed inequalities. So in this case, we have strict inequality in the proposition. □

The inequality $t_1 \leqq t_3$ from this proof is discussed for the spaces $\mathbb{R}^3$ and $\mathbb{S}^3$ in Remark 9 in Section 6.1.

5.4. Proof of Theorem 3

PROPOSITION 8 ([5, p. 127], [28, Theorem 1, Proposition 2]). *For $n \geqq 2$, a simplex in $\mathbb{H}^n$ (with vertices at infinity admitted) is of maximal volume if and only if all its vertices are at infinity and it is regular. It has a finite volume.*

Let v_n be the maximal volume of a simplex in $\mathbb{H}^n$. For instance, $v_2 = \pi$ and $v_3 = -3\int_0^{\pi/3} \log|2\sin u|\,du = 1.0149416\ldots$ ([40, p. 20], [5, p. 127]). Obviously, the facet areas S_i of a compact simplex in $\mathbb{H}^n$ are smaller than v_{n-1}.

LEMMA 5. *The area S of a right triangle $\triangle ABC \subset \mathbb{H}^2$ with angle $\angle ACB = \pi/2$ and side lengths $|AC| = b$ and $|BC| = a$ fulfills the equation*

$$\tan S = \frac{\sinh a \cdot \sinh b}{\cosh a\ \ + \cosh b}\,.$$

PROOF. This is a routine consequence of the trigonometric formulas for a right triangle in $\mathbb{H}^2$. We use $S = \pi/2 - \alpha - \beta$, $\tan\angle CBA = (\tanh b)/\sinh a$, $\tan\angle CAB = (\tanh a)/\sinh b$ [18, p. 238], and $\tanh x = (\sinh x)/\cosh x$. □

LEMMA 6. *Let $d > 0$. Assume that $\triangle ABC \subset \mathbb{H}^2$ is a triangle such that $|AB| \leqq d$ and $|AC| \leqq d$. Then the area S of this triangle is bounded by the inequality*

$$S \leqq 2\arctan\frac{\cosh d - 1}{2\sqrt{\cosh d}}\,.$$

PROOF. Without loss of generality we may assume that $|AB| = |AC| = d$. Let H be the orthogonal projection of A to the line $\ell(B, C)$. The segment AH cuts the triangle ABC into two congruent right triangles. With $x = \cosh|AH|$ and $y = \cosh|BH| = \cosh|CH|$, we have $x, y \geqq 1$ and $xy = \cosh d$. Let S be the area of $\triangle ABC$. Lemma 5 gives

$$\tan^2(S/2) = \frac{(x^2-1)(y^2-1)}{(x+y)^2}.$$

Looking for the maximum of the numerator and the minimum of the denominator subject to the constraints $x, y > 0$ and $xy = \cosh d$, we see that the maximal value of S is attained for $x = y = \sqrt{\cosh d}$. This proves the lemma. □

To show that a tetrahedron with given facet areas exists, we will use a topological argument, which is encapsulated in the following lemma. The lemma guarantees the existence of a zero of a function under certain conditions on the boundary.

LEMMA 7. *Let $F = (f_1, f_2)\colon P \to \mathbb{R}^2$ be a continuous function defined on a rectangular domain $P = [0, a] \times [0, b]$, where $a, b > 0$. Assume that there are $u_1, u_2, v_1, v_2 \in \mathbb{R}$ such that $u_1 > u_2$ and $v_1 < v_2$ and*

$$f_1(x,0) + f_2(x,0) \leqq 0, \quad f_1(x,b) + f_2(x,b) \geqq 0, \quad \textit{and}$$
$$u_1 f_1(0,y) + u_2 f_2(0,y) \leqq 0, \quad v_1 f_1(a,y) + v_2 f_2(a,y) \leqq 0$$

for every $0 \leqq x \leqq a$ and every $0 \leqq y \leqq b$. Then there exists a point $(c, d) \in P$ such that $F(c, d) = (0, 0)$.

PROOF. The conclusion clearly holds if F vanishes at some point of the boundary ∂P of P. If F has no zero on ∂P, then it is sufficient to establish that the index of the vector field F on the curve ∂P is 1. This implies that F has a zero in the interior of P [29, p. 98, proof of Theorem VI.12, sufficiency].

To determine the index of F, we define the auxiliary function $F_0\colon \partial P \to \mathbb{S}^1$ as follows. On the vertical boundaries of P, we let $F_0(0, y) = A_L := \big(-1/\sqrt{2}, 1/\sqrt{2}\big)$ and $F_0(a, y) = A_R := -A_L = \big(1/\sqrt{2}, -1/\sqrt{2}\big)$ for $0 \leqq y \leqq b$. On the lower boundary, $F_0(x, 0) = (\xi, \eta)$ turns counterclockwise in the half-plane $\xi + \eta \leqq 0$ with constant angular velocity from A_L to A_R as x varies from 0 to b. The upper boundary is similar, but there $F_0(x, b)$ changes clockwise in the half-plane $\xi + \eta \geqq 0$. Then it follows from the assumptions that, for $(x, y) \in \partial P$, $F(x, y)$ and $F_0(x, y)$ never point to opposite directions. Hence, $F(x, y)/\big\|F(x, y)\big\|$, $F_0(x, y)\colon \partial P \to \mathbb{S}^1$ are homotopic. Therefore, the index of F equals the index of F_0, namely 1. □

We still need two lemmas that together form a sharpening of two lemmas from [11].

LEMMA 8 ([11, Lemmas 1 and 2]). *Consider a (possibly degenerate) triangle A in $\mathbb{S}^2$, $\mathbb{R}^2$ or $\mathbb{H}^2$ with sides a, b, x, where $a, b > 0$. For the case of $\mathbb{S}^2$, we additionally assume $a + b \leqq \pi$. Then, for a, b fixed and $|a - b| \leqq x \leqq a + b$, the area $\overline{A}$ of this triangle is a concave function of x. (For $x = a + b = \pi$ on $\mathbb{S}^2$, we define $\overline{A}$ by a limit procedure: namely, fixing a, b, we let $x \to a + b = \pi$. Accordingly, we set $\overline{A} = \pi$. Observe that for $a + b = \pi$, the area $\overline{A}$ is half the area of a digon with sides containing the* *sides a, b of A.) In addition, the area is strictly concave for $\mathbb{R}^2$ and $\mathbb{H}^2$ and, under the additional constraint $a+b<\pi$, also for $\mathbb{S}^2$.* $\square$

We calculate more precise details about this concave function and the value of its maximum.

LEMMA 9. *We use the notations of Lemma 8 and denote by γ the angle between the sides a, b. For $\mathbb{S}^2$, let us additionally assume $a+b<\pi$. Then $\overline{A}$ equals 0 for $x=|a-b|$ and $x=a+b$, and it has a unique maximum for some value $x=x_{\max}$, with corresponding angle $\gamma=\gamma_{\max}$.*

For $\mathbb{H}^2$, we have

$$\cosh(x_{\max}/2)=\sqrt{(\cosh a\ +\cosh b)/2},$$

$$\cos\gamma_{\max}=\tanh(a/2)\cdot\tanh(b/2),$$

and the value of the maximal area is

$$\pi-2\arcsin\frac{\sinh(a/2)}{\sinh r}-2\arccos\frac{\tanh(a/2)}{\tanh r}+$$

$$\pi-2\arcsin\frac{\sinh(b/2)}{\sinh r}-2\arccos\frac{\tanh(b/2)}{\tanh r},$$

where $\cosh r=\sqrt{(\cosh a\ +\cosh b)/2}$.

For $\mathbb{R}^2$, we have

$$x_{\max}^2=a^2+b^2,\quad \gamma_{\max}=\pi/2,$$

and the maximal area is $ab/2$.

For $\mathbb{S}^2$, we have

$$\cos(x_{\max}/2)=\sqrt{(\cos a\ +\cos b)/2},$$

$$\cos\gamma_{\max}=-\tan(a/2)\cdot\tan(b/2),$$

and the maximal area is

$$2\arcsin\frac{\sin(a/2)}{\sin r}+2\arccos\frac{\tan(a/2)}{\tan r}-\pi+$$

$$2\arcsin\frac{\sin(b/2)}{\sin r}+2\arccos\frac{\tan(b/2)}{\tan r}-\pi,$$

where $\cos r=\sqrt{(\cos a\ +\cos b)/2}$.

Moreover, letting $y/2$ be the distance between the midpoint of the side x and the common vertex of the sides a and b, we have the following equivalences:

$$\gamma\in[0,\gamma_{\max})\Longleftrightarrow x<y,\quad \textit{and}\quad \gamma=\gamma_{\max}\Longleftrightarrow x=y,\quad \textit{and}$$

$$\gamma\in(\gamma_{\max},\pi]\Longleftrightarrow x>y.$$

PROOF. For $\mathbb{R}^2$, the statement is elementary. Therefore, we investigate only the cases of $\mathbb{H}^2$ and $\mathbb{S}^2$.

Denote the vertices of the triangle opposite to the sides a, b and x, by A, B and C, respectively. Let D be the mirror image of C with respect to the midpoint of the side x. Then the quadrilateral $ABCD$ is centrally symmetric with respect to the intersection O of its diagonals BC (of length x) and AD (of length y). Its area is $2\overline{A}$, so it suffices to investigate its area.

We recall the isoperimetric property of the circle in $\mathbb{R}^2$, $\mathbb{H}^2$, and on $\mathbb{S}^2$ – but in the last case of radius $r<(a+b)/2<\pi/2$ – among sets of equal perimeter. Namely, that the maximum area is attained for the circle. For $\mathbb{S}^2$, one must restrict the candidate to (closed) sets contained in some open half-$\mathbb{S}^2$ [52, Ch. 18, §6, **2**, (18.39)]. Observe that a piecewise C^1 closed curve on $\mathbb{S}^2$ with length less than 2π lies in some open half-$\mathbb{S}^2$, by elementary integral-geometric considerations [52, Ch. 7, §2, (7.11) and Ch. 18, §6, **1**, (18.37)]. (A very detailed exposition of the isoperimetric inequality in spaces of constant curvature, i.e., in $\mathbb{R}^n$, $\mathbb{H}^n$, and $\mathbb{S}^n$, can be found in [53]. See [54] for further details.)

For $\gamma=0$, we have $x=|a-b|<a+b=y$, while for $\gamma=\pi$, we have $x=a+b>|a-b|=y$. Therefore, for some $\gamma\in(0,\pi)$, we have $x=y$. This implies that, for this γ, i.e., for this x, $ABCD$ is inscribed in a circle of centre O and radius $r:=x/2=y/2$. By the isoperimetric property of the circle – on $\mathbb{S}^2$ of radius $r<(a+b)/2<\pi/2$, in the sense described above – this value of γ must therefore be $\gamma_{\max}$, and this x is $x_{\max}$, see [33, p. 63, Problem 21], [32, §5, Problem 63], [34, p. 52]. (These references deal with the case of $\mathbb{R}^2$. However, their well-known proof carries over to $\mathbb{H}^2$ and $\mathbb{S}^2$ if we use the isoperimetric property of the circle – on $\mathbb{S}^2$ of radius $r<(a+b)/2<\pi/2$, in the sense described above.)

We determine the radius of this circle. We use the law of cosines for the triangles $\triangle AOC$, $\triangle DOC$ and write $\varphi:=\angle BOC$. For $\mathbb{H}^2$, we have

$$\cosh a=\cosh(x/2)\cdot\cosh(y/2)-\sinh(x/2)\cdot\sinh(y/2)\cdot\cos\varphi,$$

and

$$\cosh b=\cosh(x/2)\cdot\cosh(y/2)+\sinh(x/2)\cdot\sinh(y/2)\cdot\cos\varphi.$$

Adding these, we obtain

$$\cosh a + \cosh b = 2\cosh(x/2)\cdot\cosh(y/2). \tag{26}$$

Analogously, for $\mathbb{S}^2$, we obtain

$$\cos a + \cos b = 2\cos(x/2)\cdot\cos(y/2).$$

(These are the analogues of the parallelogram law in $\mathbb{R}^2$.) Thus, for $\mathbb{H}^2$, we have

$$\cosh a + \cosh b = 2\cosh^2 r = 2\cosh^2(x_{\max}/2),$$

and, for $\mathbb{S}^2$, we have

$$\cos a + \cos b = 2\cos^2 r = 2\cos^2(x_{\max}/2)$$

in the range $0 < r \leqq (a+b)/2 < \pi/2$.

Furthermore, for $\mathbb{H}^2$, x is a strictly increasing function of γ and, by (26), y is a strictly decreasing function of x. Hence, for $x = 2r$ we have $y = 2r$, for $|a-b| \leqq x < 2r$ we have $2r < y \leqq a+b$, and similarly, for $2r < x \leqq a+b$ we have $|a-b| \leqq y < 2r$. These imply the last equivalences in the lemma for $\mathbb{H}^2$.

Next we determine $\cos\gamma_{\max}$ for $\mathbb{H}^2$. The law of cosines for the triangle $\triangle ABC$ gives

$$\cos\gamma_{\max} = \frac{\cosh a\cdot\cosh b - \cosh(2r)}{\sinh a\cdot\sinh b}.$$

In this equation, we have $\cosh(2r) = 2\cosh^2 r - 1 = \cosh a + \cosh b - 1$, and this implies the formula in the lemma. (Observe that $0 < a, b$ implies $\cos\gamma_{\max} = \tanh(a/2)\cdot\tanh(b/2) \in (0,1)$.)

For $\mathbb{S}^2$, the proof of the last equivalences in the lemma and the calculation of $\cos\gamma_{\max}$ are analogous. (Observe that now $0 < a, b$ and $a/2 + b/2 < \pi/2$ imply $\cos\gamma_{\max} = -\tan(a/2)\cdot\tan(b/2) \in (-1,0)$.)

Finally, the value of the maximum follows from the trigonometric formulas for a right triangle in $\mathbb{H}^2$ and $\mathbb{S}^2$. □

We prove Theorem 3 with the following

CONSTRUCTION 1. Consider a number

$$S \in (0,\pi/2) \tag{27}$$

and a number $t > 0$ such that

$$2\sinh(t/2) > \tan S.$$

(Later, S will be the area of a compact right triangle, which explains condition (27). At the same time, this explains the hypothesis $0 < S_4 < \pi/2$ of Theorem 3, since in the proof, S will be chosen for example as S_4.)

Now we define a function

$$f_{t,S}\colon [0,t] \to \mathbb{R}$$

as follows. For any $x \in [0,t]$, consider the function

$$g_x(y) := \arctan\frac{\sinh x\cdot\sinh y}{\cosh x + \cosh y} + \arctan\frac{\sinh(t-x)\cdot\sinh y}{\cosh(t-x) + \cosh y},$$

where $y \in [0,\infty)$. It is easy to see that $(d/dy)g_x(y) > 0$ for $y \in [0,\infty)$, and $g_x(0) = 0$, and

$$\lim_{y\to\infty} g_x(y) = \arctan(\sinh x) + \arctan\big(\sinh(t-x)\big)$$
$$\geqq \arctan\big(\sinh x + \sinh(t-x)\big) \geqq \arctan\big(2\sinh(t/2)\big) > S$$

for all $x \in [0,t]$. Here, at the first inequality, we used concavity of the function $\arctan y$ on $[0,\infty)$ and $\arctan 0 = 0$. At the second inequality, we used convexity of the function $\sinh x$ on the interval $[0,t]$. Therefore, there is a unique $\tilde{y} \in (0,\infty)$ such that $g_x(\tilde{y}) = S$. We put

$$f_{t,S}(x) := \tilde{y} \in (0,\infty). \tag{28}$$

Now we investigate some properties of this function. Obviously, $f_{t,S}$ is continuous on $[0,t]$, (moreover, it is C^1 on $(0,t)$), and $f_{t,S}(x) = f_{t,S}(t-x)$.

Here is the geometric interpretation of $f_{t,S}$. Consider a triangle $\triangle ABC \subset \mathbb{H}^2$ with the following properties:

(1) $|AB| = t$,

(2) the area of $\triangle ABC$ is S,

(3) if H is the orthogonal projection of C to the line $\ell(A,B)$, then $H \in [A,B]$ and $|AH| = x \in [0,t]$.

Then it is easy to see (using Lemma 5) that $|CH| = f_{t,S}(x)$. It is also easy to see that for $0 < \tilde{S} < S$ and for every $x \in [0,t]$, we have $f_{t,\tilde{S}}(x) < f_{t,S}(x)$.

In what follows we determine the number

$$h_{t,S} := f_{t,S}(0) = f_{t,S}(t). \tag{29}$$

By Lemma 5, we have

$$\tan S = \frac{\sinh t\cdot\sinh h_{t,S}}{\cosh t + \cosh h_{t,S}}.$$

Solving this equation for $\cosh h_{t,S}$, we get

$$\cosh h_{t,S} = \frac{\tan^2 S \cdot \cosh t + \sqrt{1+\tan^2 S} \cdot \sinh^2 t}{\sinh^2 t - \tan^2 S}\,.$$

(Observe that $\sinh t > 2\sinh(t/2) > \tan S$ by the strict convexity of the function $\sinh t$ on $[0,\infty)$ and $\sinh 0 = 0$. Therefore, the denominator in this formula is positive.) From this, we see that $\cosh h_{t,S} \to 1/\cos S$ for $t \to \infty$.

Proof of Theorem 3. **1.** First we consider the case when hypothesis (1) of Theorem 3 holds.

Let us take a $t > 0$ such that

$$2\sinh(t/2) > \tan S_4, \tag{30}$$

and for the number $h_{t,S_4} := f_{t,S_4}(0) = f_{t,S_4}(t)$ defined in (28) and (29), we have

$$\frac{\cosh h_{t,S_4} - 1}{2\sqrt{\cosh h_{t,S_4}}} < \tan(S_1/2). \tag{31}$$

Such a t exists since $\cosh h_{t,S_4} \to 1/\cos S_4$ for $t \to \infty$, and

$$\lim_{t\to\infty} \frac{\cosh h_{t,S_4} - 1}{2\sqrt{\cosh h_{t,S_4}}} = \frac{1-\cos S_4}{2\sqrt{\cos S_4}} < \tan(S_1/2),$$

by hypothesis (1) of the theorem.

Consider any plane σ in $\mathbb{H}^3$. Take points $A_1, A_2 \in \sigma$ such that $|A_1A_2| = t$, where t has been chosen above. Let σ^+ and σ^- be the two half-planes bounded by the line $\ell(A_1, A_2)$ in σ.

For a given $x \in [0,t]$, we take the point $H = H(x)$ on the segment $[A_1, A_2]$ satisfying $|A_1H| = x$. Consider the half-line l_x from H in σ^+ that is orthogonal to the line $\ell(A_1, A_2)$. Now let $\tilde{A}_4 = \tilde{A}_4(x)$ be the point on l_x satisfying $|\tilde{A}_4H| = f_{t,S_3}(x)$, and let $\tilde{A}_3 = \tilde{A}_3(x)$ be the point on l_x satisfying $|\tilde{A}_3H| = f_{t,S_4}(x)$. By $S_4 \geqq S_3$, we have $\tilde{A}_4 \in [\tilde{A}_3, H]$, and thus also $|H\tilde{A}_4| \leqq |H\tilde{A}_3|$.

Let $\{\gamma(\varphi)\}$ be the one-parameter group of rotations about the line $\ell(A_1, A_2)$ through the angles φ, in some definite sense of rotation, with $\gamma(0)$ being the identity. For $\varphi \in [0,\pi]$ we consider the point $A_3 = A_3(x,\varphi) := \gamma(\varphi)\big(\tilde{A}_3(x)\big)$. Note that $A_3(x,\pi) \in \sigma^-$. Consider also $A_4(x,\varphi) := \tilde{A}_4(x)$, $A_1(x,\varphi) := A_1$ and $A_2(x,\varphi) := A_2$.

We are going to prove that there exists an $(x,\varphi) \in [0,t] \times [0,\pi]$ such that the (possibly degenerate) tetrahedron $T = T(x,\varphi) := A_1(x,\varphi)A_2(x,\varphi)$

$A_3(x,\varphi)A_4(x,\varphi)$ has facet areas S_1, S_2, S_3, S_4, where S_i is the area of the facet opposite to $A_i(x,\varphi)$.

Let $s_i(x,\varphi)$ be the area of the facet of $T(x,\varphi)$ that is opposite to the vertex $A_i = A_i(x,\varphi)$. By our construction, we obviously have $s_4(x,\varphi) = S_4$ and $s_3(x,\varphi) = S_3$.

Let us define functions $f_1, f_2 \colon [0,t] \times [0,\pi] \to \mathbb{R}$ as follows.

$$f_1(x,\varphi) = s_2(x,\varphi) - S_2, \quad f_2(x,\varphi) = s_1(x,\varphi) - S_1. \tag{32}$$

It is easy to see that

$$f_1(x,0) + f_2(x,0) = S_4 - S_1 - S_2 - S_3 < 0, \tag{33}$$

$$f_1(x,\pi) + f_2(x,\pi) = S_3 + S_4 - S_1 - S_2 \geqq 0. \tag{34}$$

Now we check that

$$f_1(0,\varphi) < 0, \quad f_2(t,\varphi) < 0, \tag{35}$$

for all $\varphi \in [0,\pi]$.

For the first inequality in (35), we note that the point $\tilde{A}_3(0)$ satisfies $|\tilde{A}_3(0)A_1| = f_{t,S_4}(0) = h_{t,S_4}$, and the point $\tilde{A}_4(0)$ satisfies $|\tilde{A}_4(0)A_1| = f_{t,S_3}(0) =: h_{t,S_3} \leqq h_{t,S_4}$. Therefore, for every $\varphi \in [0,\pi]$, the triangle $\triangle A_1A_3(0,\varphi)A_4(0,\varphi)$ satisfies $|A_1A_3(0,\varphi)| \leqq h_{t,S_4}$ and $|A_1A_4(0,\varphi)| \leqq h_{t,S_4}$. By Lemma 6 and (31), we get

$$s_2(0,\varphi) \leqq 2\arctan\frac{\cosh h_{t,S_4} - 1}{2\sqrt{\cosh h_{t,S_4}}} < S_1 \leqq S_2.$$

Therefore, $f_1(0,\varphi) = s_2(0,\varphi) - S_2 \leqq s_2(0,\varphi) - S_1 < 0$ for all $\varphi \in [0,\pi]$.

For the second inequality of (35), we replace in the above argument $A_1, \tilde{A}_3(0)$, and $\tilde{A}_4(0)$ by $A_2, \tilde{A}_3(t)$, and $\tilde{A}_4(t)$, respectively. We get $f_2(t,\varphi) = s_1(t,\varphi) - S_1 < 0$ for all $\varphi \in [0,\pi]$.

Taking into account the inequalities (33–35), by applying Lemma 7 with $(u_1,u_2,v_1,v_2) := (1,0,0,1)$, we find an $(x,\varphi) \in [0,t] \times [0,\pi]$ such that $f_1(x,\varphi) = f_2(x,\varphi) = 0$. This means that $s_1(x,\varphi) = S_1$ and $s_2(x,\varphi) = S_2$ for the corresponding (possibly degenerate) tetrahedron T.

2. Now we consider the case when hypothesis (2) of Theorem 3 holds. We use the same construction of the tetrahedron T as in the first case. For the functions f_1 and f_2 defined by (32), we get the inequalities (33) and (34). Now we check that

$$f_2(0,\varphi) \geqq 0, \quad f_1(t,\varphi) \geqq 0 \tag{36}$$

for all $\varphi \in [0, \pi]$. For this we note that

$$s_1(0, \varphi) \geqq s_1(0, 0) = S_4 - S_3 \geqq S_2 \geqq S_1 \tag{37}$$

and

$$s_2(t, \varphi) \geqq s_2(t, 0) = S_4 - S_3 \geqq S_2 \tag{38}$$

for all $\varphi \in [0, \pi]$ provided t is sufficiently large, as we will prove. Of course, we have to prove only the first inequalities in (37) and (38).

We will investigate $s_1(0, \varphi)$. (The case of $s_2(t, \varphi)$ is analogous.) Recall that $|H\tilde{A}_4| \leqq |H\tilde{A}_3|$, which implies

$$h_{0,S_3} = \left| A_1(0,\varphi)A_4(0,\varphi) \right| \leqq \left| A_1(0,\varphi)A_3(0,\varphi) \right| = h_{0,S_4}.$$

For t fixed but $\varphi \in [0, \pi]$ variable, the length of the third side of the triangle $\triangle A_1(0,\varphi)A_3(0,\varphi)A_4(0,\varphi)$ lies in the range

$$\left| A_3(0,\varphi)A_4(0,\varphi) \right| \in [h_{0,S_4} - h_{0,S_3}, h_{0,S_4} + h_{0,S_3}].$$

Therefore, to show

$$s_1(0, \varphi) \geqq s_1(0, 0), \tag{39}$$

we must show the following. Let

$$a := \left| A_2(0,\varphi)A_4(0,\varphi) \right| \quad \text{and} \quad b := \left| A_2(0,\varphi)A_3(0,\varphi) \right|.$$

Then $a \leqq b$, since

$$\begin{aligned} \cosh a &= \cosh t \cdot \cosh \left| A_1(0,\varphi)A_4(0,\varphi) \right| \\ &\leqq \cosh t \cdot \cosh \left| A_1(0,\varphi)A_3(0,\varphi) \right| = \cosh b. \end{aligned}$$

Let $c := \left| A_1(0,\varphi)A_3(0,\varphi) \right| - \left| A_1(0,\varphi)A_4(0,\varphi) \right| = h_{0,S_4} - h_{0,S_3}$. Then the area of the triangle with sides a, b, c is less than or equal to the area of the triangle with the same first two sides a, b and with third side in the interval

$$[h_{0,S_4} - h_{0,S_3}, h_{0,S_4} + h_{0,S_3}] \subset [h_{0,S_4} - h_{0,S_3}, 2h_{0,S_4}] \subset [h_{0,S_4} - h_{0,S_3}, \mathrm{const}].$$

For the last inclusion observe the following. By the geometric interpretation, if S_4 is fixed and t is above the bound $2 \operatorname{arsinh} \left[(\tan S_4)/2 \right]$ from (30) and increases, then h_{0,S_4} decreases. Therefore, h_{0,S_4} *remains bounded for fixed* S_4 *if* t *increases from its originally chosen value* t_0*, say, to infinity.*

Inequality (39) is proved if we show the following monotonicity property. Fixing the first two sides a, b and varying the third side x in the interval $[h_{0,S_4} - h_{0,S_3}, \mathrm{const}]$, the area is a monotonically increasing function of x.

Now we apply Lemmas 8 and 9 to the triangle with sides a, b, x. We need to show that its area is increasing for $x \in [b - a, \mathrm{const}]$, where we know from the preceding considerations that $0 \leqq b - a \leqq \mathrm{const}$. By these Lemmas, this area-increasing property is satisfied for $x \in [b - a, x_{\max}]$, where $x_{\max}$ is defined by $\cosh^2(x_{\max}/2) = (\cosh a + \cosh b)/2$. Thus, to complete the argument, it suffices to show that $x_{\max} \geqq \mathrm{const}$, i.e., that $x_{\max} \to \infty$ for $t \to \infty$.

We estimate $x_{\max}$ from below. We have

$$\cosh^2(x_{\max}/2) = (\cosh a + \cosh b)/2 \geqq \cosh a > e^a/2,$$

hence

$$\left(e^{x_{\max}/2}\right)^2 > \cosh^2(x_{\max}/2) > e^a/2,$$

and hence

$$x_{\max} > a - \log 2 \geqq t - \log 2 \to \infty,$$

as we wanted to show. Thus, (36) is proved.

Taking in account inequalities (33), (34), and (36), we can apply Lemma 7 with $(u_1, u_2, v_1, v_2) = (0, -1, -1, 0)$ to find a point $(x, \varphi) \in [0, t] \times [0, \pi]$ such that $f_1(x, \varphi) = f_2(x, \varphi) = 0$. This means that $s_1(x, \varphi) = S_1$ and $s_2(x, \varphi) = S_2$ for the corresponding (possibly degenerate) tetrahedron T.

3. It remains to exclude degeneration of our tetrahedron. Our construction yields degenerate tetrahedra only for $\varphi = 0$ and $\varphi = \pi$. In the first case, $S_4 = S_1 + S_2 + S_3$, which contradicts our hypotheses. In the second case, $S_4 + S_3 = S_2 + S_1$, which implies $\pi > S_4 = S_3 = S_2 = S_1 > 0$. (By the way, this can occur only for case (1) of the theorem.) Then a suitable regular tetrahedron satisfies the conclusion of the theorem. □

Remark 7. Let us apply the construction in the proof of Theorem 3 to the numbers $S_i\varepsilon^2$ and $t\varepsilon$ rather than S_i and t, where $\varepsilon \to 0$. Then, for sufficiently small $\varepsilon > 0$, hypothesis (1) from Theorem 3 holds, and as an analogue of (30) we have $2\sinh(t\varepsilon/2) > \tan(S_4\varepsilon^2)$. In the limit, we obtain a Euclidean tetrahedron with facet areas S_i and one edge of length t. Letting $t \to \infty$ gives another proof for the last statement of Theorem E for $\mathbb{R}^3$ (existence of tetrahedra of arbitrarily small positive volume). Namely, the heights of the two facets meeting at the edge of length t, corresponding to this edge, are $O(1/t)$. Thus, the tetrahedron is included in a right circular cylinder of height t and radius $O(1/t)$. Hence, the volume of the tetrahedron is $O(1/t)$. Degeneration is excluded as in Step **3** of the above proof of Theorem 3.

6. Proofs for the spherical case

Recall our convention about the notion of *simplices in* $\mathbb{S}^n$ at the beginning of Section 2.3.

6.1. Proof of Proposition 4

1. We begin with the proof of the first inequality. Let the facets of P be F_i. Their areas satisfy the equation

$$S_i = \mathrm{const}_n \cdot \int |F_i \cap \mathbb{S}^1| \, d\mathbb{S}^1. \tag{40}$$

Here, $|\cdot|$ denotes cardinality, and $\mathrm{const}_n > 0$. The integration is taken with respect to the unique $O(n+1)$-invariant probability Borel measure (for the standard embedding $\mathbb{S}^n \subset \mathbb{R}^{n+1}$) on the manifold of all great-$\mathbb{S}^1$'s in $\mathbb{S}^n$ [52, Ch. 18, §6, 1].

In the integration, we may disregard those $\mathbb{S}^1$'s that lie in the great-$\mathbb{S}^{n-1}$'s spanned by the facets $F_1, \ldots, F_m$, since they have measure 0. By the same reason, we may disregard those $\mathbb{S}^1$'s that pass through the relative boundary of F_i (in the great-$\mathbb{S}^{n-1}$ spanned by it) for all $i \in \{1, \ldots, m\}$ simultaneously. If an $\mathbb{S}^1$ does not lie in the above hyperplane and does not intersect the above relative boundaries then it cannot contain two opposite points of any F_i. Namely, since F_i lies in a closed half-$\mathbb{S}^{n-1}$, both of these points would otherwise lie in the relative boundary of F_i taken with respect to the great-$\mathbb{S}^{n-1}$ spanned by it. If such an $\mathbb{S}^1$ enters P at some point $p \in F_m$ it must also leave P, through some other facet (since this $\mathbb{S}^1$ does not contain two opposite points of F_m) till it comes back to p. This holds even in the degenerate case, that is, when some portion of F_m is a doubly counted boundary of P either as a "flat" piece of P or as bounded from both sides by the interior of P.

These considerations imply that the integral (40) for $i = m$ is at most the sum of the integrals for all $1 \leqq i \leqq m-1$. Thus, (40) gives our inequality.

Now assume that P lies in an open half-$\mathbb{S}^n$ (in the open northern hemisphere, say) but does not degenerate to the doubly-counted facet F_m.

Let $\mathbb{S}_m$ be the great-$\mathbb{S}^{n-1}$ spanned by F_m. If $\cup_{i=1}^{n-1} F_i \not\subset \mathbb{S}_m$, then there exists an $x \in \cup_{i=1}^{m-1} \operatorname{rel\,int} F_i$ such that $x \notin \mathbb{S}_m$. Also, there exists a $y \in \mathbb{S}_m \setminus F_m$ that also lies in the open northern hemisphere. Then $x \neq y$, and since both x and y lie in the open northern hemisphere, the great-$\mathbb{S}^1$ xy through these two points exists.

The set of $\mathbb{S}^1$'s transversally intersecting $\cup_{i=1}^{m-1} \operatorname{rel\,int} F_i$ but not intersecting F_m contains some neighbourhood of the great-$\mathbb{S}^1$ xy in the set of

all great-$\mathbb{S}^1$'s and has therefore positive measure. This implies the strict inequality in this case.

If $\cup_{i=1}^{n-1} F_i \subset \mathbb{S}_m$, then in both cases $\cup_{i=1}^{n-1} F_i \not\subset F_m$ and $\cup_{i=1}^{n-1} F_i \subset F_m$, we have strict inequality unless P degenerates to the doubly counted facet F_m. However, this degeneration was excluded.

2. We turn to the proof of the second inequality. We again use the formula (40). Again we disregard those $\mathbb{S}^1$'s that lie in the great-$\mathbb{S}^{n-1}$'s spanned by any facet F_i of P, as well as those $\mathbb{S}^1$'s that pass through the relative boundary of any facet F_i of P. We compare the sum of the right hand sides of (40) for all $1 \leqq i \leqq m$ with the analogous integral when in the right hand side of (40) we take a great-$\mathbb{S}^{n-1}$ rather than F_i.

Clearly, for a great-$\mathbb{S}^{n-1}$, the cardinality of its intersection with a great-$\mathbb{S}^1$ is almost always 2. For the $\mathbb{S}^1$'s that were not disregarded and for any i, the cardinality $|F_i \cap \mathbb{S}^1|$ is at most 1, since a great-$\mathbb{S}^1$ cannot contain two opposite points of F_i (see part **1** of this proof). If P is not degenerate, one great-$\mathbb{S}^1$ cannot transversally intersect the interiors of three facets F_i. Namely, at each of these intersection points, it passes either into P or out of P (with some definite orientation of our $\mathbb{S}^1$). Thus, there would be at least four points of intersection, and the intersection of P and this great-$\mathbb{S}^1$ would be the union of at least two disjoint non-trivial arcs. However, this contradicts convexity of P. Hence, the sum of the integrands in (40) over all facets $i = 1, \ldots, m$ is at most 2. For degenerate P, the same inequality holds. This implies the second inequality of the proposition.

If P lies in an open half-$\mathbb{S}^n$, then the set of $\mathbb{S}^1$'s intersecting the boundary of the open half-$\mathbb{S}^n$ but not intersecting $\cup_{i=1}^m F_i$ has a positive measure. (Namely, any great-$\mathbb{S}^1$ sufficiently close to the boundary of the open half-$\mathbb{S}^n$ has this property.) This implies the strict inequality in this case. □

REMARK 8. Part **1** of the above proof of Proposition 4 extends also for $\mathbb{H}^n$ and yields Proposition 1, however without the case of equality. We have to use also [52, Ch. 18, §6, 1], and instead of $\mathbb{S}^1$, we have to take a segment of a fixed positive length t and then let t tend to infinity. However, we preferred to give the elementary proof for Proposition 1.

REMARK 9. Clearly, the argument for the inequality $t_1 \leqq t_3$ in the proof of Proposition 3 is valid also for $\mathbb{R}^3$ and $\mathbb{S}^3$. However, for $\mathbb{R}^3$, this inequality is easy to show, see below. It is also easy to show for $\mathbb{S}^3$, provided that each facet is contained in a closed half-$\mathbb{S}^2$ and has at least three sides – in particular if the polyhedron is contained in an open half-$\mathbb{S}^3$ – see below. For $\mathbb{R}^3$, the sum of the angles of the k_i-gon F_i is $t_1 = (k_i - 2)\pi$. Also, F_i has k_i neighbouring faces, each of which has angle sum at least π, so $t_1 = (k_i - 2)\pi < k_i \pi \leqq t_3$. Similarly, for $\mathbb{S}^3$, with the above hypotheses, the sum of the angles of F_i is $t_1 = S_i + (k_i - 2)\pi$, while every other facet F_j has an angle sum $S_j + (k_j - 2)\pi \geqq S_j + \pi$, since $k_j \geqq 3$. So the sum of the angles of the other facets F_j is at least $\sum_{j \neq i}(S_j + \pi)$, and hence, $\sum_{j \neq i} S_j + k_i \pi \leqq t_3$. Therefore, $t_1 = S_i + (k_i - 2)\pi < S_i + k_i \pi \leqq (\sum_{j \neq i} S_j) + k_i \pi \leqq t_3$. Here, we used the first inequality of Proposition 4, which implies $S_i \leqq \sum_{j\neq i} S_j$, provided every facet lies in some closed half-$\mathbb{S}^2$.

6.2. Proof of Theorem 4

Now we give the spherical analogues of Lemmas 5 and 6.

LEMMA 10. *The area S of a right triangle $\triangle ABC \subset \mathbb{S}^2$ with angle $\angle ACB = \pi/2$ and side lengths $|AC| = b$ and $|BC| = a$ (where $0 < a, b \leqq \pi$) fulfills the equation*

$$\tan S = \frac{\sin a \cdot \sin b}{\cos a + \cos b}$$

if $a + b \neq \pi$. For $a + b = \pi$, the area is $S = \pi/2$.

LEMMA 11. *Let $0 < d \leqq \pi/2$. Assume that $\triangle ABC \subset \mathbb{S}^2$ is a triangle such that $|AB| \leqq d$ and $|AC| \leqq d$. Then the area S of this triangle is bounded above by the inequality*

$$S \leqq 2 \arctan \frac{1 - \cos d}{2\sqrt{\cos d}}$$

if $d \neq \pi/2$. For $d = \pi/2$, we have $S \leqq \pi$.

PROOF OF LEMMAS 10 AND 11. The case $a+b = \pi$ of Lemma 10 and the case $d = \pi/2$ of Lemma 11 is elementary. In the remaining cases $\cos d > 0$ for Lemma 11, and we proceed analogously as in Lemmas 5 and 6. □

To prove Theorem 4, we use an analogous construction as for Theorem 3.

CONSTRUCTION 2. Let

$$S \in (0, \pi/2], \tag{41}$$

and choose

$$t = \pi/2.$$

This choice is motivated as follows. We will apply Lemma 8 to triangles with sides $a, b \leqq \pi/2$, but Lemma 8 does not hold for $a = b \in (t, \pi) = (\pi/2, \pi)$. Thus, $t = \pi/2$ is the largest value for which our proof applies. Also, S will be the area of a spherical triangle contained in a spherical triangle with three sides $\pi/2$, which explains the constraint (41). Now, we define a function

$$f_{t,S}\colon [0, t] \to \mathbb{R}$$

as follows. For any $x \in [0, t]$, consider the function

$$g_x(y) := \arctan \frac{\sin x \cdot \sin y}{\cos x + \cos y} + \arctan \frac{\sin(t-x) \cdot \sin y}{\cos(t-x) + \cos y},$$

defined for $y \in [0, \pi/2]$. It is easy to see that $(d/dy) g_x(y) > 0$ for $y \in [0, \pi/2)$, $g_x(0) = 0$, and

$$g_x(\pi/2) = \pi/2 \geqq S,$$

for all $x \in [0, t]$. Therefore, there exists a unique $\tilde{y} \in (0, \pi/2]$ such that $g_x(\tilde{y}) = S$. We put

$$f_{t,S}(x) := \tilde{y} \in (0, \pi/2].$$

Now we investigate some properties of this function. Obviously $f_{t,S}$ is continuous on $[0, t]$, (moreover, is C^1 on $(0, t)$), and $f_{t,S}(x) = f_{t,S}(t - x)$.

Here is the geometric interpretation of $f_{t,S}$. Consider a triangle $\triangle ABC \subset \mathbb{S}^2$ with the following properties.

(1) $|AB| = t$,

(2) the area of $\triangle ABC$ is S,

(3) C has an orthogonal projection H to the line $\ell(A, B)$ such that H lies in the segment $[A, B]$ and $|AH| = x \in [0, t]$. (Observe that there are at least two orthogonal projections of C to $\ell(A, B)$.)

Then it is easy to see (using Lemma 10) that $|CH| = f_{t,S}(x)$. It is also easy to see that for $0 < \tilde{S} < S$ and for every $x \in [0, t]$, we have $f_{t,\tilde{S}}(x) < f_{t,S}(x)$.

The boundary values

$$h_{t,S} := f_{t,S}(0) = f_{t,S}(t)$$

are easy to determine. By the geometric interpretation, this is the third side of a spherical triangle with two other sides of length $\pi/2$ and area S, i.e.,

$$h_{t,S} = S.$$

PROOF OF THEOREM 4. **1.** First we consider the case when hypothesis (3) of Theorem 4 holds. We roughly follow the lines of the proof of Theorem 3 for the analogous case when hypothesis (1) holds.

We have

$$\frac{1 - \cos h_{t,S_4}}{2\sqrt{\cos h_{t,S_4}}} = \frac{1 - \cos S_4}{2\sqrt{\cos S_4}} \leqq \tan(S_1/2), \tag{42}$$

by the hypothesis of the theorem. From this point onwards, the construction is the same as in Theorem 3.

We still have to show that our tetrahedron T satisfies our convention about simplices in $\mathbb{S}^n$ (see the beginning of Section 2.3). A convex combinatorial simplex in an *open* half-$\mathbb{S}^n$ is always considered as a simplex in $\mathbb{S}^n$. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ denote the usual unit basis vectors, and let x_1, x_2, x_3, x_4 denote the corresponding coordinates. We set $A_1 := \mathbf{e}_1$ and $A_2 := \mathbf{e}_2$. The rotation about $\ell(A_1, A_2)$ in $\mathbb{S}^3$ maps $A_3 := \mathbf{e}_3$ to $\mathbf{e}_3 \cos\varphi + \mathbf{e}_4 \sin\varphi$, say. Then T is in the closed half-$\mathbb{S}^3$ defined by the inequality $x_1 + x_2 \geqq 0$. Moreover, if $S_3 \leqq S_4 < \pi/2$, then T is contained in the open half-$\mathbb{S}^3$ defined by the inequality $x_1 + x_2 > 0$, and we are done. If $S_3 \lneqq S_4 = \pi/2$, a slight perturbation of the open half-$\mathbb{S}^3$ given by $x_1 + x_2 > 0$ contains T for all $\varphi \in [0, \pi]$, and we are done. If $S_3 = S_4 = \pi/2$, and $0 \leqq \varphi < \pi/2$ is *fixed,* then also a slight perturbation of the open half-$\mathbb{S}^3$ given by $x_1 + x_2 > 0$ contains T and then we are also done. The case $S_3 = S_4 = \varphi = \pi/2$ will be treated in part **3** below.

We have to observe that from the construction, we have

$$\left|A_3(x,\varphi)A_4(x,\varphi)\right| \leqq f_{t,S_3}(x) + f_{t,S_4}(x) \leqq 2f_{t,S_4}(x) \leqq \pi.$$

Thus, the edge $\left[A_3(x,\varphi), A_4(x,\varphi)\right]$ of our tetrahedron is in the closed angular domain swept by $\gamma(\varphi)\sigma^+$ for $\varphi \in [0, \pi]$, as in the hyperbolic case. (This explains the inequality $S_4 \leqq \pi/2$ of the theorem – and thus also the inequality $S \leqq \pi/2$ in the construction: without this inequality of the theorem, the last sentence would not be valid. Moreover, let $S_3 = S_4 \in (\pi/2, \pi)$. Then consider $\left|A_3(x,\varphi)A_4(x,\varphi)\right|$ defined not as a distance but defined by analytic continuation from φ's close to 0, i.e., by retaining the geometry of the figure. Then for $\varphi = \pi$ we would have $\left|A_3(x,\varphi)A_4(x,\varphi)\right| = 2f_{t,S_4}(x) \in (\pi, 2\pi)$. This would imply that the tetrahedron T defined by the same analytic continuation, i.e., by retaining the geometry of the figure, would not be convex.)

We define $s_i(x,\varphi)$ (for $1 \leqq i \leqq 4$) and $f_i(x,\varphi)$ (for $i = 1, 2$) as in the proof of Theorem 3. The formulas

$$f_1(x,0) + f_2(x,0) = S_4 - S_1 - S_2 - S_3 < 0, \tag{43}$$

$$f_1(x,\pi) + f_2(x,\pi) = S_3 + S_4 - S_1 - S_2 \geqq 0 \tag{44}$$

follow like (33) and (34) in the hyperbolic case. The formulas

$$f_1(0,\varphi) \leqq 0, \quad f_2(t,\varphi) \leqq 0,$$

for all $\varphi \in [0, \pi]$ follow from Lemma 11 similarly as in the hyperbolic case from Lemma 6. (Observe that here we have non-strict inequalities. Namely, in hypothesis (3) of Theorem 4 and in (42), we have non-strict inequalities, whereas for the hyperbolic case, we had strict inequalities in hypothesis (1) of Theorem 3 and in (31).) Then, we choose $(u_1, u_2, v_1, v_2) := (1, 0, 0, 1)$ and

finish the proof of case **1** as in the hyperbolic case. Also here, the tetrahedron T is possibly degenerate. (Observe that allowing $S_4 > \pi/2$, we could have $h_{t,S_4} > \pi/2$. Then $\left|A_1A_3(0,\varphi)\right|, \left|A_1A_4(0,\varphi)\right| \leqq h_{t,S_4}$ makes it impossible to apply Lemma 11. This explains once more the inequality $S_4 \leqq \pi/2$ of the theorem – and thus also the inequality $S \leqq \pi/2$ in the construction – for case **1**.)

2. Now we consider the case when hypothesis (4) of Theorem 4 holds. We roughly follow the lines of the proof of Theorem 3 when hypothesis (2) holds.

As in Step **1** of this proof, our tetrahedron satisfies our convention about the notion of a simplex in $\mathbb{S}^n$ (defined the beginning of Section 2.3), unless $S_3 = S_4 = \varphi = \pi/2$. This last case will be handled below in part **3** of this proof.

We obtain the inequalities (43) and (44) similarly to (33) and (34) in the hyperbolic case.

Now we check that

$$f_2(0,\varphi) \geqq 0, \quad f_1(t,\varphi) \geqq 0,$$

for all $\varphi \in [0, \pi]$. As in the hyperbolic case, this reduces to showing that

$$s_1(0,\varphi) \geqq s_1(0,0), \quad s_2(t,\varphi) \geqq s_2(t,0). \tag{45}$$

We will investigate $s_1(0,\varphi)$. (The case of $s_2(t,\varphi)$ is analogous.) Observe that the distance $\left|A_3(x,\varphi)A_4(x,\varphi)\right|$ is a strictly increasing function of $\varphi \in [0, \pi]$, with

$$\left|A_3(x,0)A_4(x,0)\right| = h_{t,S_4} - h_{t,S_3} = S_4 - S_3,$$

and

$$\left|A_3(x,\pi)A_4(x,\pi)\right| = h_{t,S_4} + h_{t,S_3} = S_4 + S_3.$$

By the geometric interpretation (observe that $\left|A_2(0,\varphi)A_1(0,\varphi)\right| = \left|A_2(0,\varphi)A_3(0,\varphi)\right| = \left|A_2(0,\varphi)A_4(0,\varphi)\right| = \pi/2$), we have

$$s_1(0,\varphi) = \angle A_3(0,\varphi)A_2(0,\varphi)A_4(0,\varphi) = \left|A_3(0,\varphi)A_4(0,\varphi)\right|,$$

where the last term is strictly increasing for $\varphi \in [0, \pi]$. This shows (45). Then, as in the hyperbolic case, we choose $(u_1, u_2, v_1, v_2) := (0, -1, -1, 0)$ and finish the proof of **2**. Also here, the tetrahedron T can be degenerate.

3. It remains

1) to exclude degeneration of our tetrahedron, and

2) to verify that our tetrahedron satisfies our convention about the notion of a simplex in $\mathbb{S}^n$ (see the beginning of Section 2.3).

1) is done exactly as in the hyperbolic case in Theorem 3. Here we can even have $\pi \geqq S_4 = S_3 = S_2 = S_1 > 0$.

For 2) we have to handle the case $S_3 = S_4 = \varphi = \pi/2$ only. From Construction 2 for any $x \in [0,t] = [0,\pi/2]$ we have $S \leqq \pi/2$, where equality can be attained for any $x \in [0,t]$. The case of equality is independent of $x \in [0,t]$: namely it is a regular spherical triangle with angles and sides $\pi/2$. Then the fact that the angle of the facets $A_1(x,\varphi)A_2(x,\varphi)A_3(x,\varphi)$ and $A_1(x,\varphi)A_2(x,\varphi)A_4(x,\varphi)$ is $\varphi = \pi/2$ uniquely determines our simplex: it is a regular simplex in $\mathbb{S}^3$ of edge $\pi/2$ (thus we have also $S_1 = S_2 = \pi/2$ – the vertices can be $\mathbf{e}_1, \ldots, \mathbf{e}_4$). This lies in some open half-$\mathbb{S}^3$, therefore is among the simplices that we considered as simplices in $\mathbb{S}^3$. □

6.3. Proof of Proposition 5

For part (i), we start with $n = 2$ dimensions. Here, one has a convex m-gon in a closed half-$\mathbb{S}^2$, with sides $S_1, \ldots, S_m$. In fact, it lies in an open half-$\mathbb{S}^2$, has strictly convex angles, and is non-degenerate if $S_m < S_1 + \cdots + S_{m-1}$ and $S_1 + \cdots + S_m < 2\pi$. For the degenerate cases, i.e., when $S_m = S_1 + \cdots + S_{m-1}$ or $S_1 + \cdots + S_m = 2\pi$, we have a doubly counted segment or a great-$\mathbb{S}^1$, respectively. If both equations hold, then we have also a digon, with sides subdivided to m_1 and m_2 sides, where $m_1 + m_2 = m$. If both inequalities are strict, we can copy the well-known proof in [34, pp. 53–54] – given there for the case of $\mathbb{R}^2$. Thus, we obtain the existence of such a convex m-gon. Actually, one gets such a convex m-gon that is inscribed in a circle of radius less than $\pi/2$.

Now we show how this construction can be lifted to higher dimensions. For $\mathbb{S}^3$, we embed the above m-gon in its equator, which is an $\mathbb{S}^2$. Each side of this polygon is then replaced by a facet that is the union of all meridians (whose lengths are π) meeting that side. The vertices are replaced similarly by edges that are meridians meeting these vertices. Additionally, there are two new vertices at the North and South Poles. Then the ratio of the areas of the spherical digons and the lengths of the corresponding edges of our polygon is $V_2(\mathbb{S}^2)/V_1(\mathbb{S}^1)$, where V_i denotes i-volume. Moreover, the dihedral angles are the same as for the spherical m-gon in $\mathbb{S}^2$.

The inductive step is performed analogously for all $n > 3$. The other stated properties are obvious.

Part (ii) about simplices is proved by induction on n. For $n = 2$, we have a spherical triangle, and we have the same degenerate cases as in part (i) for $m = n+1 = 3$ (with $\{m_1, m_2\} = \{1, 2\}$). Let $n \geqq 3$, and assume that the statement of the theorem holds for $n-1$. With the factor $\alpha := V_{n-2}(\mathbb{S}^{n-2})/V_{n-1}(\mathbb{S}^{n-1})$, the numbers $\alpha(S_1+S_2), \alpha S_3, \ldots, \alpha S_{n+1}$ satisfy the hypotheses of the proposition for $n-1$. Arguing as in the second proof of Theorem 2 in Section 4.4 part **1**, we establish that $S_{\mathrm{new}} := S_1 + S_2 \leqq S_3 + \cdots + S_{n+1}$, since $n+1 \geqq 4$, and, for $j \geqq 3$, $S_j \leqq S_{\mathrm{new}} + S_3 + \ldots + S_{j-1} + S_{j+1} + \ldots + S_{n+1}$.

Therefore, we have on $\mathbb{S}^{n-1}$ a polyhedral complex that is a combinatorial simplex with these facet areas. Again we consider $\mathbb{S}^{n-1}$ as the equator of $\mathbb{S}^n$. We replace each facet and each lower-dimensional face of this polyhedral complex on $\mathbb{S}^{n-1}$ by the union of all meridians (whose lengths are π) meeting it. The resulting facets and also lower-dimensional faces are of of one dimension higher than the original ones. Additionally, there are two new vertices at the North and South Poles. Thus, we have obtained a polyhedral complex on $\mathbb{S}^n$ with facet areas $S_1 + S_2, S_3, \ldots, S_{n+1}$.

This polyhedral complex has only two vertices at the two poles, and n edges joining them. Its n facets are obviously not simplices. The facet of area $S_1 + S_2$ has $n-1$ edges. On each of these $n-1$ edges, we add an extra vertex at the same geographic latitude. Also we add an extra (convex) simplicial $(n-2)$-face with these vertices, together with its faces of lower dimensions, that subdivides the facet of area $S_1 + S_2$. For a suitable choice of the latitude, the facet of area $S_1 + S_2$ is subdivided into two $(n-1)$-dimensional simplicial facets of areas S_1 and S_2. These two facets with all their lower-dimensional faces are added as well. In each of the other facets (of areas $S_3, \ldots, S_{n+1}$), one $(n-2)$-face has been subdivided into two simplicial $(n-2)$-faces. Thus, these other facets also become combinatorial $(n-1)$-simplices, by induction with respect to n.

The other stated properties follow by the construction. □

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