The Geometric Dilation of Three Points

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Abstract

Given three points in the plane, we construct the plane geometric network of smallest geometric dilation that connects them. The geometric dilation of a plane network is defined as the maximum dilation (distance along the network divided by Euclidean distance) between any two points on its edges. We show that the optimum network is either a line segment, a Steiner tree, or a curve consisting of two straight edges and a segment of a logarithmic spiral.

Keywords: Geometric dilation, geometric network, plane graph, urban street system.

Contents

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1 Introduction

Urban street systems can be modeled by geometric graphs: streets correspond to (possibly curved) edges, and intersections are represented by vertices. In a densely populated area, houses are everywhere along the streets. In this situation, the quality of a street system N can be measured by its *geometric* (or: *point-to-point*) *dilation*, which is defined as follows.

For any two points a and b of N let $d_N(a,b)$ denote the length of a shortest path from a to b in N. Then,

$$\delta_N(a,b) := \frac{d_N(a,b)}{|ab|}$$

is called the dilation of a and b. It measures the detour one encounters in using N, in order to get from a to b, instead of traveling straight; here $|\cdot|$ denotes the Euclidean length. The geometric dilation of N is given by

$$\delta(N) := \sup_{a \neq b \text{ points of } N} \delta_N(a,b).$$

The crucial point is that *all* points a, b of N are considered in this definition, vertices and interior edge points alike. This is quite different from the standard vertex-to-vertex dilation (also known as stretch factor or spanning ratio) of geometric graphs where only the vertices matter, as we shall point out in Section 1.3 below.

1.1 Problem statement

We are given a finite set S of points in the plane. We are interested in a network connecting them whose geometric dilation is as small as possible. Let

$$\Delta(S) := \inf\{\, \delta(N) : N \text{ is a finite plane geometric network containing } S \,\}$$

denote the smallest possible dilation value for point set S. We call $\Delta(S)$ the geometric dilation of the set S. Three questions arise naturally.

1. How large is $\Delta(S)$?
2. Can we find a network N attaining this value?
3. In what time can such a network N be constructed (or closely be approximated)?

When S consists of two points, the obvious answer is the line segment connecting these two points. In this paper we are going to answer these questions for point sets S of cardinality 3. The answer is certainly not easy to guess: The optimum network containing three given points is either a line segment, a tree with a single vertex of degree 3, or a curve consisting of an arc of a logarithmic spiral and two straight edges.

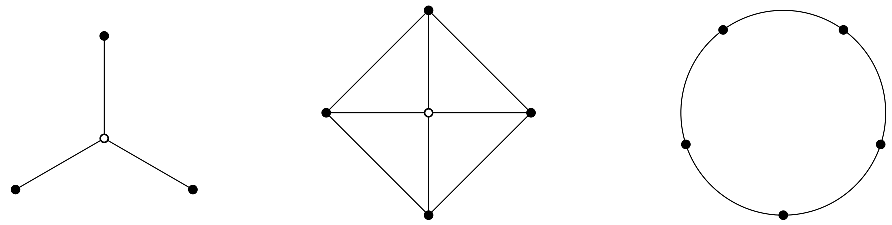
1.2 Previous work

The geometric dilation of finite point sets was first studied by Ebbers-Baumann, Grüne, and Klein [4]. They proved $\Delta(S) \leq 1.678$ for each finite point set S in the plane. Moreover, they computed the geometric dilation, and optimum embeddings, for the sets S_n of n points evenly placed on a circle; see Figure 1. Their results are based on the following facts, whose proofs can be found in [4].

Proposition 1. *1. If a network N contains a vertex v where two straight edges e_1 and e_2 meet at some angle α, then two points $a_1 \in e_1$ and $a_2 \in e_2$ that are placed at equal distance and sufficiently close to v have dilation $\delta_N(a_1, a_2) = 1/\sin\frac{\alpha}{2}$. Thus, $\delta(N) \geq 1/\sin\frac{\alpha}{2}$. This result applies also when the two edges meeting at v are smooth curves. In this case, the angle α is measured between their tangents.*

2. *No tree containing S_n, where $n \geq 5$, can have a geometric dilation $\leq \pi/2$.*

3. *A plane network containing a cycle must have a geometric dilation $\geq \pi/2$, which is attained by a circle.*



$\Delta(S_3) = \sqrt{4/3} \approx 1.1547$ $\quad$ $\Delta(S_4) = \sqrt{2} \approx 1.4142$ $\quad$ $\Delta(S_n) = \pi/2 \approx 1.5708$ for $n \geq 5$

Figure 1: The point sets whose geometric dilation has been known so far.

The lower bound of part 3 has been sharpened by Dumitrescu, Ebbers-Baumann, Grüne, Klein, and Rote [2]. Using a packing theorem of K. Kuperberg, W. Kuperberg, Matoušek, and Valtr [9], they proved that there exists a finite point set whose geometric dilation exceeds $(1 + 10^{-11}) \cdot \frac{\pi}{2}$. But until now, the regular sets S_n shown in Figure 1 were the only point sets whose geometric dilations have been determined exactly.

1.3 Related questions

If one is interested in a network of shortest length that connects a given point set S of size n without using additional points as vertices, one can construct in time $O(n \log n)$ the Euclidean minimum spanning tree of S, cf. [11]. A shorter connecting network is given by the *Steiner tree*, which may use additional vertices. If S contains only three points A, B, C that form a triangle of maximum angle less than 120°, the point F minimizing the sum of distances to A, B, C lies inside the triangle and sees each pair of points at angle exactly 120°. It is called the Fermat–Torricelli point of S. In this case, the Steiner tree of S is given by the star connecting F to A, B and C. If the triangle formed by A, B, C has an angle $\geq 120°$ at B, then B minimizes the sum of distances and the Steiner tree of S is the path from A through B to C.

All additional vertices of a Steiner tree are Fermat-Torricelli points of their three neighbors. Euclidean Steiner trees are NP-hard to compute, but they can be approximated in polynomial time [1].

In the context of spanners [6, 10], one usually studies the vertex-to-vertex dilation of geometric graphs. This approach fits well to railway networks, where access is only possible at the stations. The same questions posed in Section 1.1 for the geometric dilation have been investigated for the vertex-to-vertex dilation, too. Clearly, in this context one needs to consider triangulations only, because the vertex-to-vertex dilation of a plane graph can only decrease by pulling curved edges taught, or by adding straight edges that do not produce crossings.

It has been shown by Ebbers-Baumann, Grüne, Karpinski, Klein, Knauer, and Lingas [5] that each finite point set can be embedded into the vertex set of a finite triangulation of dilation ≤ 1.1247. Only very special point sets are embeddable into a triangulation of vertex-to-vertex dilation equal to 1, and they have been classified by Eppstein [7]. Klein, Kutz, and Penninger [8] have shown that if S is not one of these special sets then there exists a lower bound $\eta > 1$ such that each triangulation whose vertex set contains S has a vertex-to-vertex dilation at least η. But up to now, there is no non-special point set for which the exact lowest dilation value is known. Since all sets S of cardinality ≤ 4 are special, the set S_5 is the simplest open example.

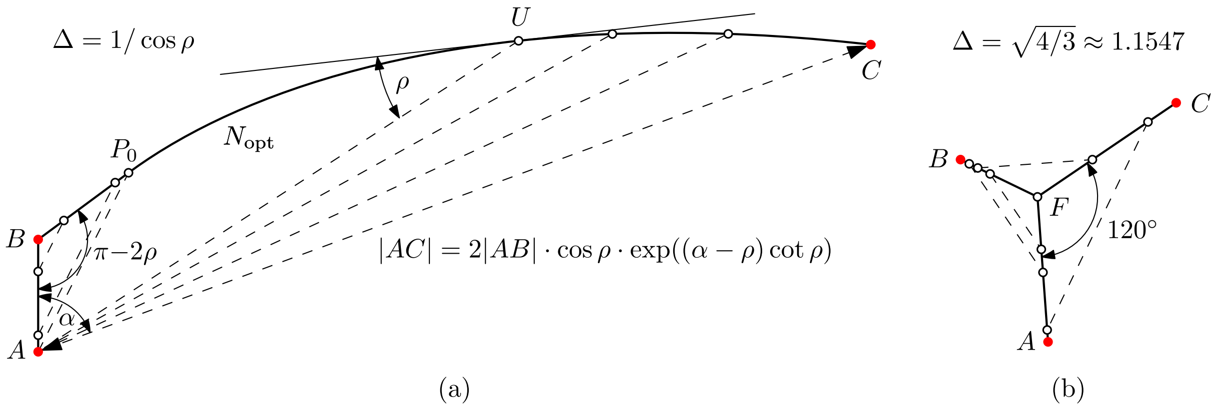


Figure 2: The network of lowest possible geometric dilation that connects three points A, B, C is (a) a path $N_{\text{opt}}(\rho, \alpha)$, or (b) the Steiner tree. The dashed chords connect point pairs where the geometric dilation is attained.

1.4 The result

For the geometric dilation, the smallest non-trivial cardinality equals 3. This case will be completely solved in the present paper by the following result.

Theorem 1. *Let $S = \{A, B, C\}$.*

1. *If the points A, B, C are collinear then $\Delta(S) = 1$, realized by a line segment.*
2. *If the points A, B, C form a proper triangle with edge lengths $|AB| \le |BC| \le |AC|$, then the optimum network has one of the following forms, see Figure 2a–b.*

   a) *It consists of a straight edge AB, followed by another straight edge BP_0 of length $|AB|$ forming an angle $\angle ABP_0 = 180° - 2\rho$ for an appropriate value ρ with $0 < \rho < 90°$. This is followed by an arc of a logarithmic spiral connecting P_0 with C, which is defined by the property that it intersects the rays through A at the constant angle ρ. The value of ρ is determined by these conditions, and it is the solution of the equation*

   $$2 \cos \rho \cdot \exp((\alpha - \rho) \cot \rho) = |AC|/|AB|,$$

   where $\alpha = \angle BAC$. In this case, the dilation $\Delta(S)$ is $1/\cos \rho$.

   b) *It is the Steiner tree of A, B, C: a star whose central vertex F is the Fermat–Torricelli point of S. Every pair of edges forms a $120°$ angle. In this case, the dilation $\Delta(S)$ is $1/\cos 30° = \sqrt{4/3} \approx 1.1547$;*

   The first case is optimal for $\rho \le 30°$, and the second case is optimal for $\rho > 30°$.

The Steiner tree does not always have a degree-3 vertex, and the point F with the claimed properties might not exist, but if $\rho > 30°$, then this is guaranteed.

In case 2.a), when the network is a path, we denote it by $N_{\text{opt}} = N_{\text{opt}}(\rho, \alpha)$, leaving the dependence on A, B, C implicit. $N_{\text{opt}}(\rho, \alpha)$ is defined for the range of parameters $0 < \rho < 90°$ and $\rho \le \alpha \le 180°$. In this network, the geometric dilation $\Delta = 1/\cos \rho$ is attained by all pairs of points on the two straight edges that have the same distance from B, and between A and every point U on the spiral.

Figure 3 shows a classification of the points C in the plane according to the minimum dilation of a network that is formed with two fixed points A, B, according to Theorem 1. The graphic is symmetric both with respect to the line AB and the symmetry axis of A and B. The shaded region is the area where the Steiner tree with a degree-3 vertex is the optimum.

Roughly, one can say that the geometric dilation of A, B, C is close to 1 if either the triangle ABC has an angle close to 180°, or two of the points have a small distance and the third point is very far. We mention without proof that the optimum network is unique if the point set $\{A, B, C\}$ has no symmetries. This follows from our arguments. Some triples of points with a mirror symmetry have two different optimum networks.

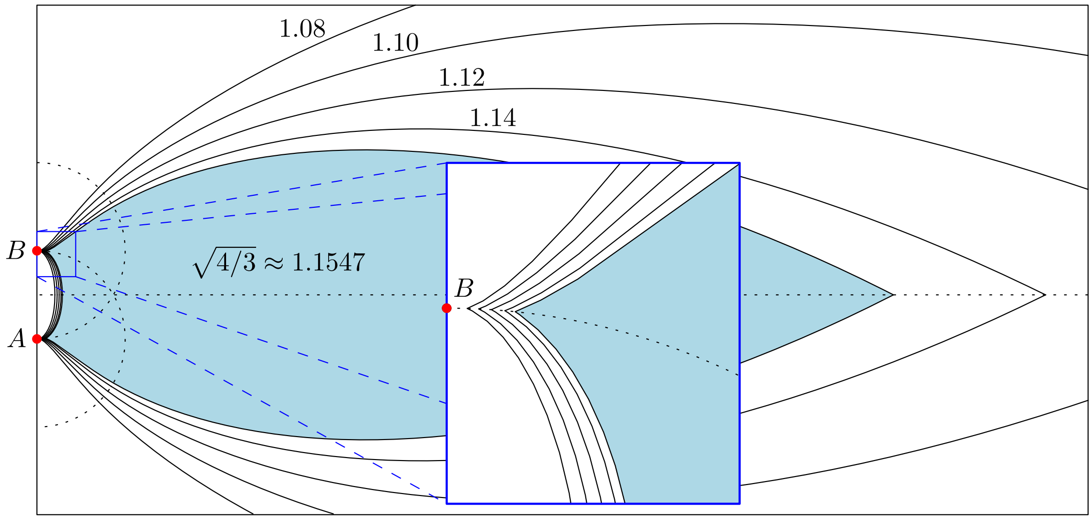


Figure 3: The minimum dilation of two fixed points A, B together with a variable point C. If C lies in the shaded area, the dilation is $\sqrt{4/3}$, and the optimal network is the Steiner tree of A, B, C. The level curves of dilation $1.14, 1.12, 1.10, 1.08$ are also shown. The dotted lines are the boundaries where the order of the lengths $|AC|$, $|BC|$, and $|AB|$ changes. A small rectangular region around B is enlarged in the inset. The situation to the left of the line AB is symmetric.

1.5 Overview of the proof

The rest of the paper is devoted to the proof of Theorem 1. We first sketch the main idea of the argument.

The optimal network can either be a path or a more complicated network. If it is not a path, then it has a vertex of degree ≥ 3, and by Proposition 1.1, the geometric dilation is at least $\sqrt{4/3}$. Now, a geometric dilation of $\leq \sqrt{4/3}$ can always be achieved by taking the Steiner tree. If it contains a Fermat-Torricelli point that sees each pair of A, B, C at angle 120°, its dilation is exactly $\sqrt{4/3}$, and if it is a path leading through a vertex of angle $\geq 120°$, the dilation can only be smaller.

In summary, we know that the optimum geometric dilation is $\leq \sqrt{4/3}$, and if we want to go below this threshold, we have to look only among path networks.

In Section 2, we show that the geometric dilation of $N_{\text{opt}}(\rho, \alpha)$ is indeed equal to $1/\cos\rho$. In Section 3 we claim that N_{opt} is the best path that visits three points X, Y, Z *in the given order* (Lemma 4). To prove optimality, we construct a *forbidden region* R that cannot be entered by any path of given geometric dilation that starts from X and passes through Y (Section 6). We prove this fact by a polygonal discretization of R (Sections 8–9).

2 The Dilation of the Best Path

We now prove that the spiral curve $N_{\text{opt}}(\rho, \alpha)$ has indeed the claimed dilation $1/\cos\rho$. We recall the constraint that $0 < \rho < 90°$, and that the curve sweeps at most an angular range of $180°$ around A, i.e., $\alpha = \angle BAC \le 180°$. In particular, the curve does not wind several times around A. Other than that, we impose no restriction on the parameters in this section. We thus include cases that do not arise in Theorem 1 because the endpoint C is closer to A than to B or because $\rho > 30°$.

Proposition 2. *The spiral path $N_{\text{opt}}(\rho, \alpha)$ lies on the boundary of its convex hull.*

Proof. We assume without loss of generality that the triangle ABC is oriented clockwise, and N_{opt} winds clockwise around A. It is also possible that the angle $\angle ABC = 0°$; in this case, we also assume that N_{opt} winds clockwise around A, covering a $180°$ angle.

We now move the point U on N_{opt} from P_0 to C. As the tangent direction keeps a constant angle with the direction AU, the tangent direction turns clockwise, and hence the curve is convex. When $U = P_0$, the tangent coincides with the edge BP_0. Therefore the convex hull includes the segments AB, BP_0, and AC, and the whole curve lies on the boundary of its convex hull. □

Proposition 3. *The geometric dilation of the spiral path $N_{\text{opt}}(\rho, \alpha)$ is $1/\cos\rho$.*

We need the following auxiliary lemma:

Lemma 1. *Let $S = S(t)$ be a piecewise differentiable curve parameterized by t, and let $S(t_0)$, $S(t_1)$, $S(t_2)$ for $t_0 \le t_1 < t_2$ be points on S, and let ρ be some angle with $0° < \rho < 90°$. Assume that*

- *$t_0 = t_1$, or $\delta_S(S(t_0), S(t_1)) \le 1/\cos\rho$.*
- *For all $t \in [t_1, t_2]$, the angle $\angle\big(S'(t), \overrightarrow{S(t_0)S(t)}\big)$ between the right derivative $S'(t)$ and the vector $\overrightarrow{S(t_0)S(t)}$ is $\le \rho$.*

Then $\delta_S(S(t_0), S(t_2)) \le 1/\cos\rho$.
If equality holds in both assumptions, then $\delta_S(S(t_0), S(t_2)) = 1/\cos\rho$.

Proof. Assume without loss of generality that S is parameterized by arc length. Then

$$\frac{d}{dt}|S(t_0)S(t)| = \cos\angle\big(S'(t), \overrightarrow{S(t_0)S(t)}\big) \ge \cos\rho.$$

By integration, we get

$$|S(t_0)S(t_2)| = |S(t_0)S(t_1)| + \int_{t=t_1}^{t_2} \tfrac{d}{dt}|S(t_0)S(t)|\,dt \ge |S(t_0)S(t_1)| + (t_2 - t_1)\cos\rho, \quad (1)$$

while the distance d_S along the path S grows in accordance with t:

$$d_S(S(t_0), S(t_2)) = d_S(S(t_0), S(t_1)) + (t_2 - t_1) \quad (2)$$

Comparing (1) with (2), the assumption $|S(t_0)S(t_1)| \ge \cos\rho \cdot d_S(S(t_0), S(t_1))$ gives $|S(t_0)S(t_2)| \ge \cos\rho \cdot d_S(S(t_0), S(t_2))$. The equality case is analogous. □

We can now justify the remark after Theorem 1 about the pairs where the dilation is attained, see the dashed chords in Figure 2a: The dilation between A and P_0 is $1/\cos\rho$; and the angle between the ray AU and the tangent at U is ρ, thus, the assumption of Lemma 1 are fulfilled with equality.

Proof of Proposition 3. We have to show that the dilation between any two points is not larger than $1/\cos\rho$. If both points lie on the two arms ABP_0, this is elementary, cf. Proposition 1.1. Otherwise, it is sufficient to consider the dilation between an arbitrary point V and C, where C can in fact be any point on the spiral part of N_{opt}. If V lies on the path ABP_0, we apply the lemma with $N_{\text{opt}}(t_0) = V$, $N_{\text{opt}}(t_1) = P_0$, and $N_{\text{opt}}(t_2) = C$. The first assumption, $\delta(N_{\text{opt}}(t_0), N_{\text{opt}}(t_1)) \le 1/\cos\rho$, has already been established because P_0 is still on the path ABP_0. Moreover, the angle between VU and the curve is less than ρ, because the chord VU lies in the wedge between the chord AU and the tangent at U.

If V lies on the spiral, we apply the lemma with $N_{\text{opt}}(t_0) = N_{\text{opt}}(t_1) = V$ and $N_{\text{opt}}(t_2) = C$. The argument about the bounded angle remains valid. □

3 The Smallest Dilation of a Path

Lemma 2. *The minimum dilation of a path that visits three distinct points X, Y, Z in this order is determined as follows.*

- *Assume $|XY| \le |YZ|$, by swapping X and Z if necessary.*
- *Let $t := |XZ|/|XY|$, and $\xi := \angle YXZ \le 180°$.*
- *If $\xi = 0°$, then the optimum dilation is 1, and it is obtained by the line segment XZ.*

  If $\xi > 0°$, there is a unique angle ρ with $0 < \rho < 90°$ and $\rho \le \xi$ such that

$$2\cos\rho \cdot \exp((\xi - \rho)\cot\rho) = t. \tag{3}$$

  The optimum dilation is $1/\cos\rho$, and it is obtained by the curve $N_{\text{opt}}(\rho, \xi)$.

It is clear where the function in (3)

$$f(\rho, \xi) := 2\cos\rho \cdot \exp((\xi - \rho)\cot\rho),$$

comes from, see Figure 2a, which uses the notations A, B, C, α instead of X, Y, Z, ξ. The distance $|XP_0|$ is $|XY| \cdot 2\cos\rho$. This length is multiplied by the distance gain $\exp((\xi - \rho)\cot\rho)$ of the logarithmic spiral over an angle range of $\xi - \rho$, and hence $f(\rho, \xi)$ should be equal to $|XZ|/|XY| = t$.

We have already seen in Proposition 3 that the path N_{opt} has the claimed geometric dilation. The proof that there is no better path will be given in Section 6. We will first justify the claim that their is always a unique angle ρ that satisfies (3), by studying the monotonicity properties of the involved functions.

4 Monotonicity

The function $f(\rho, \xi)$ is defined on the domain $0° < \rho \le 90°$, $0° < \xi \le 180°$, restricted by the constraint $\rho \le \xi$. In order to justify the claim that Eq. (3), $f(\rho, \xi) = t$, has a unique solution, we describe the monotonicity properties and the range of f:

Proposition 4.
1. *The function $f(\rho, \xi)$ is strictly decreasing in ρ.*
2. *$f(\rho, \xi)$ is strictly increasing in ξ.*
3. *For each fixed value $\xi \in (0, 90°)$, the function $f(\rho, \xi)$, regarded as a function of ρ, is an order-reversing bijection from the interval $(0°, \xi]$ onto $[2\cos\xi, \infty)$.*

   For $\xi \in [90°, 180°)$, the function $f(\rho, \xi)$ is an order-reversing bijection from the interval $(0°, 90°]$ onto $[0, \infty)$. □

Proof. The first claim is easy to see, because all three terms where ρ occurs — $\cos\rho$, $\xi - \rho$, and $\cot\rho$ — are strictly decreasing in ρ, and these terms are combined by monotone operations. The second claim is obvious. The third claim follows by evaluating f at the endpoints of the range. □

By Proposition 4.1, the equation $f(\rho, \xi) = t$ has at most one solution ρ. To show that a solution exists, by Proposition 4.3, it is sufficient to show that

$$t = |XZ|/|XY| \geq 2\cos\xi \tag{4}$$

if $\xi < 90°$. Consider the triangle XYZ, see Figure 4. By assumption, $|XY| \leq |YZ|$, and thus

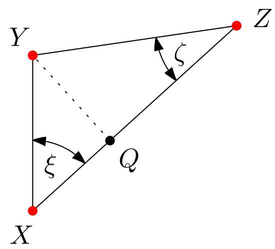


Figure 4: The triangle XYZ

the angle ζ at Z is at most ξ. In our case, this implies that both angles ζ and ξ are acute, and the foot Q of the height through Y lies on the side XZ. We know that $|ZQ| \geq |XQ|$, because $|ZY| \geq |XY|$. Therefore, $|XZ| = |XQ| + |ZQ| \geq 2|XQ|$. Since $|XQ| = |XY|\cos\xi$, the relation (4) follows.

We have thus justified the claim of Lemma 2 that there exist a unique ρ satisfying (3).

Let us define $H(\xi, t)$ as the optimum dilation according to Lemma 2, as a function of ξ and t, where $0 < \xi \leq 180°$ and t is constrained by (4) and $t > 0°$. This function has the following monotonicity properties:

Lemma 3. *The function $H(\xi, t)$ is strictly increasing in ξ. It is strictly decreasing in t.*

Proof. The optimum dilation equals $H(\xi, t) = 1/\cos\rho$, where ρ is the solution of $f(\rho, \xi) = t$. Since the transformation $\rho \mapsto 1/\cos\rho$ is strictly monotone for $0 \leq \rho \leq 90°$, it is sufficient to establish the monotonicity properties for ρ.

We have seen in Proposition 4 that $f(\rho, \xi)$ is strictly decreasing in ρ. Thus, the monotone decreasing dependence of ρ on t follows directly.

On the other hand, $f(\rho, \xi)$ is strictly increasing in ξ. Thus, if we increase ξ and thereby make $f(\rho, \xi)$ larger, this has to be compensated by an increase of ρ in order to maintain the relation $f(\rho, \xi) = t$ when t is fixed. This means that ρ has to increase as ξ increases. □

5 Proof of the Main Theorem

Before giving the proof of Lemma 2, we show how it implies Theorem 1. The case when the points are collinear is trivial, and we know that the dilation is $\sqrt{4/3}$ unless the best network is a path. We only have to figure out the order in which the path should connect the three points A, B, C.

We denote the angles of the triangle ABC at A, B, C by α, β, γ, and the opposite sides by $a = |BC|$, $b = |AC|$, and $c = |AB|$. By the conventions of Theorem 1,

$$c \leq a \leq b \text{ and } \gamma \leq \alpha \leq \beta,$$

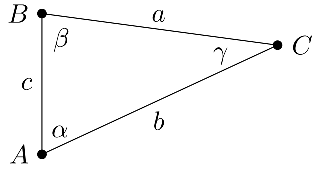


Figure 5: A general triangle ABC

see Figure 5. We have three choices for the order. For easy use, we summarize the essence of Lemma 2: The optimum dilation of a path visiting three points X–Y–Z in the given order is $H(\xi, |XZ|/x)$, where x is the length of the shorter of the two arms YX and YZ, and ξ is the angle in the triangle XYZ at the endpoint of that arm (and $|XZ|$ is the distance between the endpoints). Thus, when we compare the three possibilities of visiting the three points, we get the following dilations:

$$\begin{aligned} A\text{–}B\text{–}C\colon &\quad H(\alpha, b/c) \\ B\text{–}A\text{–}C\colon &\quad H(\beta, a/c) \\ B\text{–}C\text{–}A\colon &\quad H(\beta, c/a) \end{aligned}$$

The monotonicity properties of H in Lemma 3 give $H(\alpha, b/c) \le H(\beta, a/c) \le H(\beta, c/a)$, and thus the first possibility is the best. This concludes the proof of Theorem 1. □

6 The Forbidden Region

In this section, we assume that Y lies vertically above X, and ρ is an angle in the interval $(0°, 90°)$. The forbidden region $R = R(\rho)$ is defined as follows, see Figure 6. We look at the

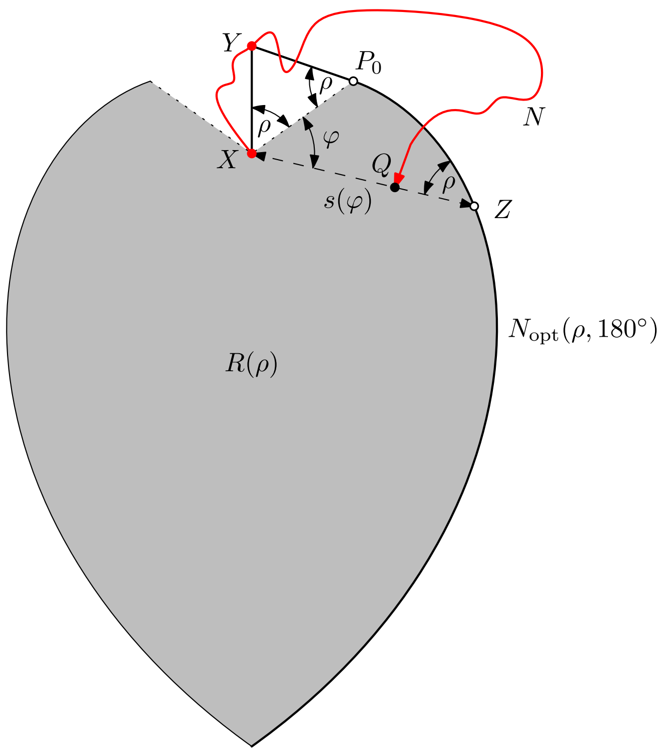


Figure 6: The forbidden region $R(\rho)$ and a hypothetical path N that will be discussed in Section 9.

path $N_{\text{opt}}(\rho, 180°)$ that makes a clockwise turn around X until it hits the line XY below X.

Then $R = R(\rho)$ is the heart-shaped region that bounded by the segment XP_0 and the spiral part of this curve, together with the mirror image at the vertical axis XY.

The points Z on the logarithmic spiral that forms the boundary of R can be specified by parameterizing the radius $|XZ|$ by the angle $\varphi = \angle YXZ - \rho$:

$$|XZ| = s(\varphi) := |XP_0| \cdot \exp(\varphi \cot \rho) = |XY| \cdot f(\rho, \angle YXZ) \tag{5}$$

Here is our main lemma about the forbidden region:

Lemma 4. *A path of geometric dilation $\le \frac{1}{\cos \rho}$ that starts in X and passes through Y can afterwards not enter the interior of the region $R(\rho)$.*

The proof will be given, after some preparations, in Section 9. We show how the lemma implies the optimality of the path $N_{\mathrm{opt}}(\rho, \alpha)$ (Lemma 2): The point Z lies on the boundary of $R(\rho)$ by construction. A path with a smaller dilation would have to avoid the region $R(\rho')$ for some $\rho' < \rho$. The distance from X to the boundary of R along the ray XZ is given by $s(\varphi) = |XY| \cdot f(\rho, \angle YXZ)$ according to (5), and we have seen in Proposition 4.1 that $f(\rho, \angle YXZ)$ increases strictly as ρ decreases. The wedge of opening angle 2ρ around XY with is cut out from the top of R also becomes smaller as ρ decreases. Therefore, Z lies in the interior of $R(\rho')$. This means that a path with smaller dilation than $1/\cos \rho$ cannot reach Z, and this concludes the proof of Lemma 2. □

7 Dilation with a Varying Endpoint on a Ray

We will use the following simple observation:

Lemma 5. *Let S be a path that consists of some fixed curve C from A to P, followed by the straight segment PQ to a variable third point Q moving on a ray $\vec{r}$ through A that makes an angle $0 < \alpha < 180°$ with AP, see Figure 7. Then the geometric dilation $\delta_S(A, Q)$ between A and Q decreases strictly from ∞ to 1 as Q moves away from A along $\vec{r}$.*

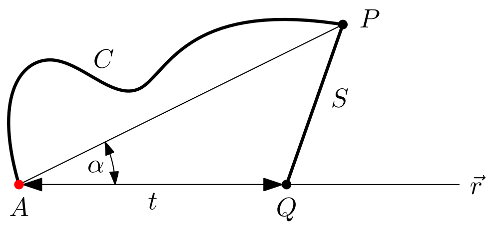


Figure 7: The dilation $\delta_S(A, Q)$ decreases as Q moves away from A.

Proof. With the fixed angle $\alpha = \angle QAP > 0°$, and the variable distance $t = |AQ|$, we apply the cosine law and the substitution $u = |AP|/t$ to express the dilation:

$$\delta_S(A, Q(t)) = \frac{|C| + \sqrt{|AP|^2 - 2t|AP|\cos\alpha + t^2}}{t} = \frac{|C|}{|AP|} \cdot u + \sqrt{u^2 - 2u\cos\alpha + 1}$$

The derivative with respect to u is

$$\frac{|C|}{|AP|} + \frac{2u - 2\cos\alpha}{2\sqrt{u^2 - 2u\cos\alpha + 1}} = \frac{|C|}{|AP|} \pm \frac{\sqrt{u^2 - 2u\cos\alpha + \cos^2\alpha}}{\sqrt{u^2 - 2u\cos\alpha + 1}} > 1 + (-1) = 0.$$

Thus the dilation is strictly increasing in u, and strictly decreasing as a function of t. The limiting values for $t \to 0$ and $t \to \infty$ are straightforward. □

8 A Polygonal Forbidden Region

In the remainder of the paper, we denote the threshold on the dilation by

$$\Delta := 1/\cos\rho.$$

We approach the proof of Lemma 4 by discretizing the boundary of the forbidden shape and approximating it from inside. We construct a polygonal path $XYP_0P_1P_2\dots$ winding clockwise around X. We will then show that its diagonals XP_i cannot be intersected by any path N of geometric dilation $\leq \Delta$; see Figure 8. In the limit, this path will converge to the boundary of the region $R(\rho)$.

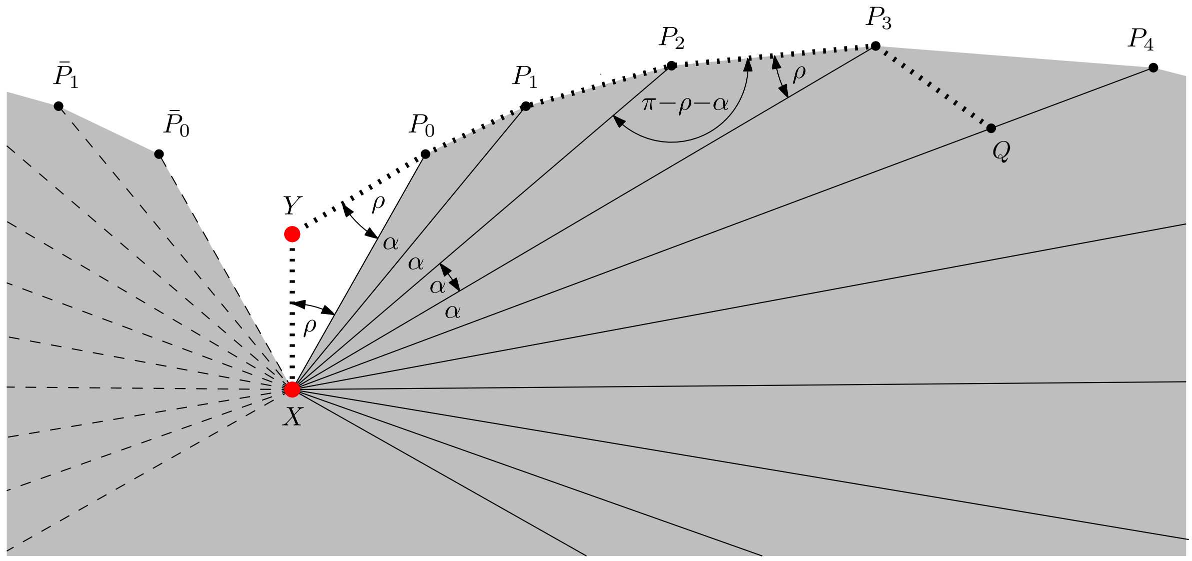


Figure 8: Discretization of the forbidden region, and a dotted path from X via Y to a point Q on the segment XP_4.

The path starts with an isosceles triangle XYP_0 with angle ρ at X and P_0. Let $\alpha > 0$ denote a small angle that will later converge to 0. We add a sequence of similar triangles XP_iP_{i+1} with angle α at X and angle ρ at P_{i+1}. This can be continued as long as the total accumulated angle YXP_{i+1} around X does not exceed 180°. We also construct a symmetric path through points $\bar{P}_i$ that winds counterclockwise around X.

We denote the polygonal path $XYP_0P_1P_2\dots P_i$ by C_i. As a special case, C_{-1} denotes just the edge XY, and accordingly, we set $P_{i-1} := Y$.

Lemma 6. $\delta_{C_i}(X, P_i) \geq \Delta$ *for* $i = 0, 1, 2, \dots$.

Proof. We have to show

$$|C_i| \geq \Delta |XP_i| \tag{6}$$

for all i. We use induction on i. For the path $C_0 = XYP_0$, this is elementary.

For the inductive step with $i \geq 1$, we first establish the inequality

$$|P_{i-1}P_i| \geq \Delta(|XP_i| - |XP_{i-1}|). \tag{7}$$

The three lengths in this relation are the three sides of the triangle $XP_{i-1}P_i$, and hence we can use the sine law to express their ratios in terms of the angles, cf. the triangle XP_2P_3 in Figure 8:

$$\sin\alpha \geq \Delta(\sin(180° - \alpha - \rho) - \sin\rho),$$

or

$$\sin\alpha \ge \tfrac{1}{\cos\rho}(\sin(\alpha+\rho) - \sin\rho).$$

This is expands to

$$\sin\alpha\cos\rho \ge \sin\alpha\cos\rho + \sin\rho\cos\alpha - \sin\rho,$$

which is easily checked to be true. Now we can prove (6), using the induction hypothesis (6) for $i-1$ and (7):

$$|C_i| = |C_{i-1}| + |P_{i-1}P_i| \ge \Delta|XP_{i-1}| + \Delta(|XP_i| - |XP_{i-1}|) = \Delta|XP_i| \qquad \square$$

The "forbidden" character of the construction is expressed in the following statement:

Lemma 7. *A path N of geometric dilation $\le \Delta = \frac{1}{\cos\rho}$ that starts in X and passes through Y can afterwards not reach any point Q on the open segments XP_i or $X\bar{P}_i$, for $i \ge 0$.*

Proof. Consider a path N that reaches a point Q on the open segment XP_i. We first consider the possibility that N reaches Q by winding clockwise around X. We assume by induction that path N must avoid the interior of the segments XP_0, XP_1, …, XP_{i-1}, after passing through Y. The shortest path that avoids these segments is the path C_{i-1} from X to P_{i-1} plus the segment $P_{i-1}Q$. (Note that this statement holds also for $i = 0$.)

According to Lemma 5, the dilation between the endpoints is strictly decreasing as Q moves from X to P_i along XP_i. When Q reaches P_i, we have the path C_i, where the dilation is already $\ge \Delta$ by Lemma 6. Thus, a path to Q cannot have dilation $\le \Delta$.

We still have to consider the possibility that N reaches Q by winding counterclockwise around X. By induction, N must then avoid the interior of the segments $X\bar{P}_0$, $X\bar{P}_1$, …, $X\bar{P}_{i-1}$ after passing through Y, and the above argument proves that N cannot intersect the segment $X\bar{P}_i$ at an interior point. The shortest possible counterclockwise path that avoids these segments is the path $\bar{C}_i$ followed by some path from $\bar{P}_i$ to Q. This path is even longer than C_i and its endpoint Q is closer to X than P_i. Thus, such a path also has dilation $> \Delta$. $\square$

9 Proof that the Forbidden Region Cannot be Entered

In order to prove Lemma 4, we apply Lemma 7 to show that a curve N of dilation $\le \Delta$ that starts in X cannot reach a point Q in the interior of $R(\rho)$ after going through Y. Without loss of generality, assume that Q lies in the right half of $R(\rho)$, see Figure 6. Let $\varphi := \angle YXQ - \rho$. We construct the polygonal forbidden region with $\alpha = \frac{\varphi}{n}$. Then, by construction, the point P_n lies on the ray XQ. We will show that

$$\lim_{n\to\infty} |XP_n| = |XP_0| \cdot \exp(\varphi\cot\rho) \tag{8}$$

This expression is equal to the distance $s(\varphi)$ from X to the boundary of $R(\rho)$ along the ray XQ, according to (5), and this means that, for large enough n, the segment XP_n will cover the point Q in its interior. By Lemma 7, this implies that N cannot reach Q, thus proving Lemma 4.

In order to show (8), we write $|XP_n|$ as follows, using the sine law in the triangles $XP_{i-1}P_i$:

$$|XP_n| = |XP_0| \prod_{i=1}^{n} \frac{|XP_i|}{|XP_{i-1}|} = |XP_0| \left(\frac{\sin(\alpha+\rho)}{\sin\rho}\right)^n,$$

We are therefore done if we can show that

$$\lim_{n\to\infty} \left(\frac{\sin(\frac{\varphi}{n}+\rho)}{\sin\rho}\right)^n = \exp(\varphi\cot\rho)$$

The limit expression is of the form $(a_n)^n$, with a sequence (a_n) that converges to 1. Writing a_n in the form $a_n = 1 + b_n/n$ and using that $\lim(1 + b_n/n)^n = \exp\lim b_n$, we obtain the formula

$$\lim_{n\to\infty} (a_n)^n = \exp\left(\lim_{n\to\infty} n(a_n - 1)\right),$$

if the latter limit exists. By this formula, it is sufficient to show that

$$\lim_{n\to\infty} n \cdot \left(\frac{\sin(\frac{\varphi}{n} + \rho)}{\sin\rho} - 1\right) = \varphi\cot\rho. \tag{9}$$

We expand and simplify this expression:

$$\begin{aligned} n \cdot \left(\frac{\sin(\frac{\varphi}{n} + \rho)}{\sin\rho} - 1\right) &= n \cdot \left(\frac{\sin\frac{\varphi}{n}\cos\rho + \cos\frac{\varphi}{n}\sin\rho}{\sin\rho} - 1\right) \\ &= n\sin\tfrac{\varphi}{n} \cdot \cot\rho + n \cdot (\cos\tfrac{\varphi}{n} - 1) \end{aligned}$$

The term $n\sin\frac{\varphi}{n}$ converges to φ. The second term vanishes in the limit because $\cos\frac{\varphi}{n} = 1 - O(\frac{1}{n^2})$. This establishes (9) and concludes the proof of Lemma 4. □

10 Conclusions

We have constructed planar embeddings of minimum geometric dilation for all point sets of size 3. An obvious challenge is to extend this result to larger point sets. With respect to applications, it would also be interesting to find an upper bound to the total edge length of a plane network that attains, or approximates, the minimum dilation for a given point set.

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