The Degree of Convexity

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Abstract

We measure the degree of convexity of a planar region by the probability that two randomly chosen points see each other inside the polygon. We show that, for a polygonal region with n edges, this measure can be evaluated in polynomial time as a sum of $O(n^9)$ closed-form expressions.

1 Introduction

A set P is convex if, for every two points $u, w \in P$, the whole segment uw belongs to P. If P is not convex, this conclusion will not always be true, and we can get a quantity for the "degree" or "measure" of convexity if we take the probability with which it is fulfilled, for two points u, w selected uniformly at random from P.

More formally, let $|P|$ denote the area of P. Then the *degree of convexity* $C(P)$ is defined as the normalized double area integral

$$C(P) := \frac{1}{|P|^2} \int_{u \in P} \int_{w \in P} [uw \subset P] \, dw du, \qquad (1)$$

using the bracket notation for the characteristic function of a logical expression: $[uw \subset P]$ equals 1 if the condition $uw \subset P$ holds and 0 otherwise.

2 Related Work

Stern [4] has been the first to consider the measure (1), as a simple alternative to another measure he proposed, the *polygonal entropy.* He evaluated $C(P)$ by Monte Carlo estimation. Stern observed that $C(P)$ can equally be expressed as the normalized average visible area, i.e., the expected area of the visibility polygon of a random point, divided by $|P|$.

Various other definitions have been proposed for evaluating a measure of "convexity", primarily in the pattern recognition community, in addition to measuring "circularity", "squareness", "rectangularity" "elongation" etc.

In principle, one can take any quantity that is bounded by 1, for which equality holds (among compact sets) for convex sets only. Some very primitive measures that count the reflex angles, (or sum them up) fulfill this requirement but they are not very sensitive to the shape of P.

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Zunić and Rosin [5] mention, besides $C(P)$, the area of P divided by the area of the convex hull. The complement of this is called the *deficit of convexity* in the textbook [3, p. 35] (p. 23 in the 2008 version). Instead of the area, one can also look at the perimeter.

Boxer [1] considered yet another measure, the *index of non-convexity*, the maximum distance of a pocket from the corresponding convex hull edge, and related measures.

3 Properties of the Degree of Convexity

The following basic properties were already established by Stern [4].

Clearly, $C(P)$ is between 0 and 1, and $C(P) = 1$ if P is convex. For a compact set P which is the closure of its interior, $C(P) = 1$ if and only if P is convex.

$C(P)$ is invariant under affine transformations.

4 Partitioning the Region

Let P be a polygonal region with n boundary edges. Throughout the paper we will assume general position, to keep the discussion simple.

An *internal bitangent* of P with tangency vertices x and y ($x \neq y$) is a line segment $ab \subset P$ whose endpoints a and b lie in the interior of P and which has the two distinct points x and y in common with the boundary of P, see Fig. 1a. Then x and y are necessarily reflex vertices.

We extend the rays from x and y towards the endpoints until they hit the boundary of P, see Fig. 1b–c. We use these extension rays to partition P, as in Fig. 1e. We don't include the segment xy.

We also extend the edges of every reflex vertex, in order to ensure that all cells of resulting partition Z are convex as shown in Fig. 1f.

Finally, we will compute the integral in (1) separately for each pair of cells A, B of Z, and add up the results

$$I(A, B) := \int_{u \in A} \int_{w \in B} [uw \subset P] \, dw du \qquad (2)$$

Remark: If we would extend the segments between any two vertices, like in Fig. 1d, we would get a refinement of Z that corresponds to the combinatorially different visibility polygons of all points in P. [Q: Is there an established name for this refined partition?]

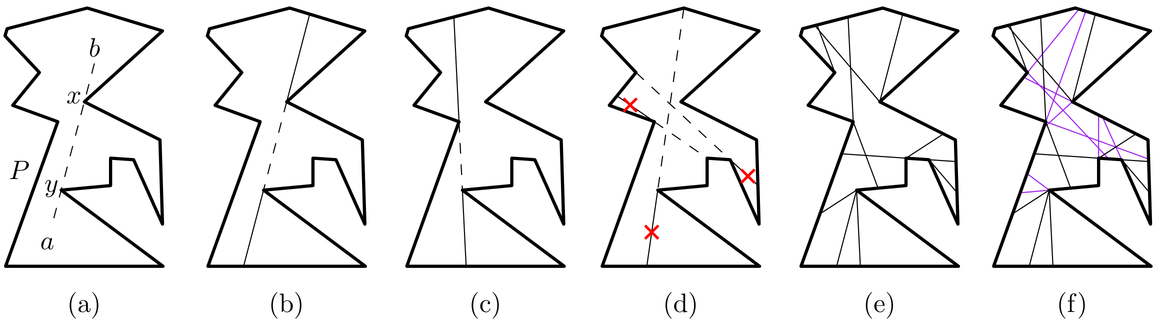

Figure 1: Partitioning P by extensions of bitangents. (a) a bitangent. (b–c) some extensions of bitangents, and (e) the arrangement of all extensions. (d) Not all extensions of visibility edges are used (only bitangents). (f) the final partition Z.

Definition 1 *Consider to disjoint open convex regions $A, B \subset P$. Let $V_1, V_2, \ldots, V_k$ be a set of open segments (i.e., not containing their starting point), disjoint from A and B (see Fig. 2).*

- *We say that* visibility between A and B is determined by the segments $V_1, V_2, \ldots$ *if the following holds: two points $u \in A$ and $w \in B$ can see each other iff the segment uw doesn't intersect any of the segments $V_1, V_2, \ldots$. We refer to these segments as* blocking segments.

- *We say that a set of blocking segments is* mutually exclusive *if no segment uw for $u \in A$ and $w \in B$ can intersect more than one of the blocking segments.*

If we have mutually exclusive blocking segments, we can consider them independently from each other and evaluate the integral (2) as follows

$$I(A,B) = \int_{u\in A}\int_{w\in B}\big(1 - [uw \cap V_1 \neq \emptyset] - \\ - [uw \cap V_2 \neq \emptyset] - \cdots\big)\, dwdu \\ = |A|\cdot|B| - \sum_{i=1}^{k}\int_{u\in A}\int_{w\in B}[uw \cap V_i \neq \emptyset]\, dwdu$$

Lemma 1 *Let P be a polygonal region, possibly with holes. Consider two cells A, B of the partition Z, considered as open sets. Then there are three possibilities.*

1. *All points of A see all points of B.*
2. *No point from the interior of A sees a point from the interior of B.*
3. *Visibility between A and B is determined by some mutually exclusive blocking segments $V_1, \ldots, V_k$.*

For the case of a simple polygon, we can have at most two blocking segments (one blocking from the left and one from the right, when looking from A towards B). If P has h holes, there can be at most h additional blocking segments.

Proof. Suppose that A and B are separated by a vertical line, as in Fig. 2, so that, for a segment uw from A to B, it makes sense to speak of "above uw" or "below uw".

If $u \in A$ and $w \in B$ move, the segment uw will sometimes be contained in P (we call it a *free segment* in this case), and sometimes it will intersect the boundary of P. (Otherwise, we are in Case 1 or 2 and we are done.) When a free segment hits the boundary of P, it will do so at a reflex vertex r of P. We call such a vertex r a *blocking corner*, and it is a *top blocking corner* or a *bottom blocking corner*, when the direction into which uw can freely move is above r or below r. Fig. 2 shows a top blocking corner r. We can rotate uw around r while maintaining $u \in A$ and $w \in B$. The extremes of this rotation are segments where uw becomes tangent to A or B. If, during this rotation, uw would hit another obstacle vertex r' before hitting the extreme directions we would have an extension ray for the bitangent rr' that would cut through A and B, a contradiction to the assumption that A and B are cells of Z. Thus, we can state: *No segment from A to B goes through two blocking corners.*

As a consequence, the blocking corners can be linearly ordered from bottom to top: Choose an arbitrary free segment through each blocking corner. For two corners r and r' with respective segments uw and $u'w'$ through them, r lies above $u'w'$ iff r' lies below uw. Thus, we get a (consistent) linear order (cf. Fig. 3a). In this order, top corners and bottom corners must alternate We create the blocking segments V_i by matching each top blocking corner with the bottom blocking corner immediately below it.

If a bottom corner r at the very top remains unmatched, we attach a sufficiently long upward vertical segment (or effectively, an infinite ray) to r, as shown for segment V_1 in Fig. 2. Similarly, at the bottom,

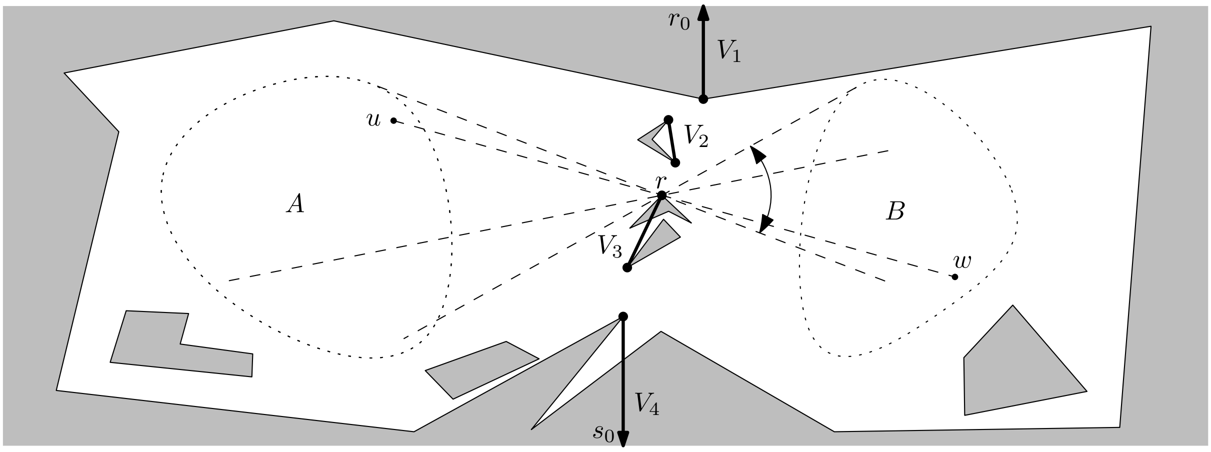

Figure 2: Visibility between two (hypothetical) regions A and B in a polygon with six holes is determined by the mutually exclusive blocking segments V_1, V_2, V_3, V_4.

we attach a downward ray to an unmatched bottom corner (segment V_4 in Fig. 2).

The resulting segments determine the visibility between A and B. □

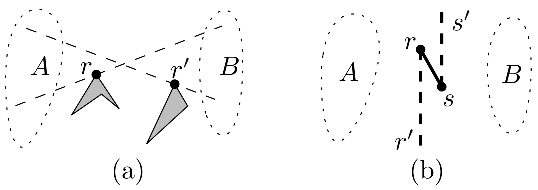

Figure 3: (a) An inconsistent order between blocking vertices cannot happen. (b) Replacing a blocking segment rs by two unbounded blocking rays

Cases 1 and 2 can of course be considered as special cases of Case 3, but they are easy to deal with directly. As mentioned above, in Case 3 the calculation can be reduced to considering the blocking probability of single blocking segments: For such a single blocking segment $V = rs$, we have to integrate over the point pairs $(u, w) \in A \times B$ for which $uw \cap V \neq \emptyset$. This can be further reduced to two integrations over unbounded vertical blocking rays rr' and ss', as shown in Fig. 3b. For the integrand, we have

$$[uw \cap V \neq \emptyset] = [uw \cap rr' \neq \emptyset] + [uw \cap ss' \neq \emptyset] - 1$$

For an integral of the form

$$\int_{u \in A} \int_{w \in B} [uw \cap rr' \neq \emptyset] \, dw du$$

we simply have to test whether uw passes above or below the point r.

5 A Single Blocking Ray

Let us consider the integral

$$\int_{u \in A} \int_{w \in B} [uw \cap V \neq \emptyset] \, dw du \qquad (3)$$

for a single blocking segment V extending from r downward to infinity, see Fig. 4a. We draw a line from

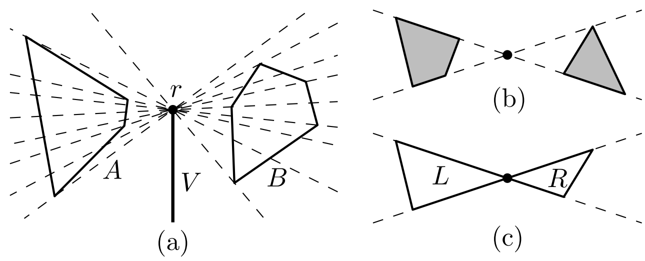

Figure 4: Two cells A and B with a single blocking segment V.

each vertex through r. These lines decompose the problem into double wedges between adjacent lines. Sectors of A and B which are not in the same double-wedge are blocked by V either completely or not at all, and their contribution to (3) is easy to compute. We are left with the situation of regions in two wedges like in Fig. 4b, where each boundary edge extends between the two rays of the wedge. On each side, we can express the region as a difference of two triangles that involve the apex r, as in Fig. 4c. Doing this on each side, the evaluation of the integral for a single double-wedge is reduced to four integrals over triangular regions L and R as in Fig. 4c.

6 The Basic Integral

It is now convenient to revert to a probabilistic interpretation of these integrals. The integral (3), for

$A = L$ and $B = R$, equals $|L| \cdot |R|$ times the probability that a random point $u \in L$ and a random point $w \in R$ form a counterclockwise triangle with r. To evaluate this probability, we make our life easier by transforming the situation to the standard situation shown in Fig. 5. First of all, the problem is invariant under affine transformations. So we assume that the apex r is at the origin O and the left triangle L is bounded by the lines of slope 0 and 1 and the line $x = -1$. Scaling the right triangle R from the origin does not change the probability of a positive orientation for the random triangle uwO. This we can scale R so that the lower corner lies at $\binom{1}{0}$. Then the upper corner lies on the line $y = x$ of slope 1 at some point $\binom{a}{a}$, for $a > 0$. There is only one free parameter, a.

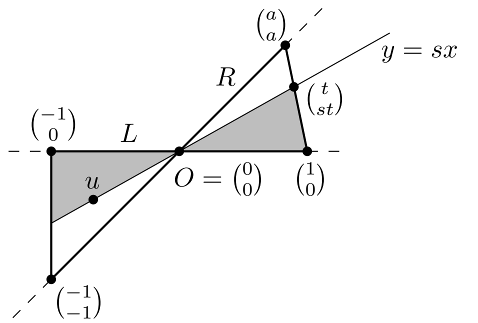

Figure 5: After transforming the problem to the standard situation, for which we evaluate the integral

The area L is $1/2$ and the area of R is $a/2$. If we take a random point $u \in L$, the slope s of the line uO is uniformly distributed in $[0, 1]$. (This follows from the fact that the probability that the slope is smaller than s is the area of the shaded region on the left, divided by $1/2$, which equals s.) Thus, the probability of blocking is the expected value of the region in the triangle on the right side below the line $y = sx$ of slope s, when s chosen uniformly at random, divided by the area $a/2$ of the whole triangle R. This region, which is shaded in Fig. 5, is a triangle with vertices O, $\binom{1}{0}$, and $\binom{t}{st}$, where t can be evaluated as $t = \frac{a}{a+s-as} > 0$. The area of this triangle is $\frac{st}{2} = \frac{1}{2} \cdot \frac{as}{a+s-as}$, and hence the desired probability is

$$\begin{aligned} Q(a) &:= \frac{1}{a} \int_{s=0}^{1} \frac{as}{a+s-as}\, ds = \int_{s=0}^{1} \frac{s}{a+s-as}\, ds \\ &= \left[\frac{s}{1-a} - \frac{a}{(1-a)^2} \ln((1-a)s + a) \right]_{s=0}^{1} \\ &= \frac{1-a+a\ln a}{(1-a)^2} \end{aligned} \qquad (4)$$

The above derivation assumes $a \neq 1$. For the symmetric situation ($a = 1$) we get $Q(1) = \frac{1}{2}$. The formula (4) is numerically unstable near $a = 1$. Near $a = 1$ we might therefore resort to the power series expansion

$$\begin{aligned} Q(a) &= \sum_{i=0}^{\infty} \frac{(1-a)^i}{(i+1)(i+2)} \\ &= \tfrac{1}{2} + \tfrac{1-a}{6} + \tfrac{(1-a)^2}{12} + \tfrac{(1-a)^3}{20} + \tfrac{(1-a)^4}{30} + \cdots \\ &= \tfrac{1}{2} - \tfrac{\ln a}{6} + \tfrac{(\ln a)^3}{180} - \tfrac{(\ln a)^5}{5040} + \tfrac{(\ln a)^7}{151200} - \cdots \end{aligned}$$

7 Complexity and Runtime Analysis

Suppose that P has h holes, and the boundary of P consists of n edges. Then P has at most n vertices. Let n_R be the number of reflex vertices. Let $n_B = O(n_R^2)$ be the number of interior bitangents. The partition Z is generated by $2n_B + 2n_R$ segments, and hence its complexity is $n_Z = O((n_B + n_R)^2 + n) = O(n_R^4 + n) = O(n^4)$. For each of the $n_Z^2 = O(n^8)$ pairs of regions A, B we have to evaluate the integral $I(A, B)$ and sum up the results. There might be $2 + h$ blocking segments, which are reduced to $2 + 2h$ blocking rays, accounting for a factor of $O(1 + h)$. the decomposition in Fig. 4a incurs an overhead proportional to the complexity of A plus B; in total, this is still $O(n_Z^2) = O(n^8)$. Thus, the total number of integrals (3) that we have to compute is $O(n_Z^2(1 + h)) = O((n_B + n_R)^4 + n^2)(1 + h) = O(n_R^8 + n^2)(1 + h) = O(n^8(1 + h)) = O(n^9)$.

Algorithmically, we proceed as follows. The n_B bitangents and extension rays can be computed in $O(n_R n \log n)$ time by a circular sweep arount each reflex vertex. The partition Z can be computed in $O(n_Z \log n)$ time by a plane sweep.

We pick one of the $O(n_Z)$ regions A and compute the visible region from some arbitrary point $u \in A$, in $O(n \log n)$ time. All the potential lower and upper blocking points are now known, including their sorted order around u. For every region B, we can now select the blocking points that lie between A and B (in the convex hull of $A \cup B$). By processing them in sorted order, we can identify the blocking segments. This takes $O(n)$ time per pair A, B.

As mentioned above, the overhead from the decomposition into wedges (Fig. 4a) does not add up to more than $O(n_Z^2) = O(n^8)$. Thus, for the overall running time, we get $O((n_R n + n_Z) \log n + n_Z n \log n + n_Z^2 n) = O(n((n_B + n_R)^4 + n^2)) = O(n(n_R^8 + n^2)) = O(n^9)$. If P is a simple polygon, some simplifications should be possible.

8 A Simplified Partition

We believe that is is not necessary to make the cells of the partition convex as in Fig. 1f. The partition as shown in Fig. 1e should have the essential properties that are necessary for the algorithm. In this case, among other things, one needs to deal with self-blocking *within* one non-convex cell. But this is not difficult since the different dentures (or pockets) of a cell are "mutually exclusive". This might reduce the running time in particular examples, but it does not affect the asymptotic worst-case running time in terms of n.

An alternative, and probably faster approch to computing $C(P)$ might be based on the interpretation as the expected value of the visibility region of a random point $u \in P$.

9 Other Convexity Measures

Stern [4] defined the *polygonal entropy* as the expected value of $|V(u)| \ln \frac{1}{|V(u)|}$, where $V(u)$ denotes the visibility region of a random point $u \in P$, apart from some additive and multiplicative normalization terms which are intended to ensure that the polygonal entropy ranges between 0 and 1. (By contrast, $C(P)$ is the expected value of $|V(u)|/|P|$.) The idea behind this definition is to consider a random variable $p \in P$ whose density is proportional to $|V(p)|$. For a convex set, $|V(p)|$ has a constant value, and therefore the random variable is uniform on P and has the maximum possible (differential) entropy among random variables on P. The article [4] makes an erroneous assumption about the differential entropy being always nonnegative, and therefore one would need a different normalization than the paper proposes.

It is not clear how the polygonal entropy can be evaluated, or even how it can be approximated by Monte Carlo simulation. It would be interesting to find the *distribution* of the random quantity $|V(u)|$.

Another interesting measure would be the average "detour" of the geodesic path between u and w within P (either the *quotient* over the Euclidean distance, or the *difference*, suitably normalized). With quotients, this looks very hard, but computing the excess might be within reach, although I don't even know whether the average Euclidean distance in a polygon can be computed in closed form. Taking *squared* distances might be a way out: the average squared distance is just the variance.

The definition (1) extends to higher dimensions, but the computation of this integral seems to be much harder.

Acknowledgments

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