

A Companion to the Proof of the New Lower Bound for Erdős' Unit Distance Problem

Rolf Klein

Department of Computer Science I, University of Bonn; retired
rolf.u.klein@gmail.com

Abstract

Recently, an AI model produced a proof showing that there exist an $\epsilon > 0$ and arbitrarily large point sets P in the plane that contain at least $|P|^{1+\epsilon}$ many point pairs of distance 1. A refined exposition was subsequently given by Alon et al. [1]. This result refutes a conjecture of long standing. Its proof employs techniques from algebraic number theory. Since this result illustrates unexpected AI capabilities, it seems desirable that even readers without a background in algebraic number theory should have an opportunity to follow and evaluate the reasoning behind it. To this end, this paper first reviews, in compact form and without proofs, the notions from algebraic number theory and their properties, as they are used herein. Then it gives a compact version of the proof in [1].

Keywords: AI system, Erdős' unit distance problem, algebraic number theory

1. Introduction

P. Erdős [2] raised the question how often the same distance can occur among n points in the plane. An upper bound of $O(n^{3/2})$ was proven and an upper bound of $O(n^{1+o(n)})$ was conjectured for many years. Quite recently, this conjecture was disproven by an internal model at OpenAI that showed that there exist an $\epsilon > 0$ and arbitrarily large point sets P in the plane that contain at least $|P|^{1+\epsilon}$ many point pairs of distance 1.

This result is remarkable for at least two reasons. First, because it has been found by an Artificial Intelligence, which raises questions about the intellectual capabilities of current AI systems. And second, because the proof uses non-trivial results and techniques from algebraic number theory, perhaps at the level of a two terms master course on this topic.

Noga Alon et al. [1] presented a "human-digested, somewhat simplified and somewhat generalized" version of the AI proof, along with historical remarks and comments on the novelty of some of the ideas used in the proof. Also included are the individual views of nine experts in the field who provide valuable insight. Yet, it seems desirable that even readers without a background in algebraic number theory should be enabled

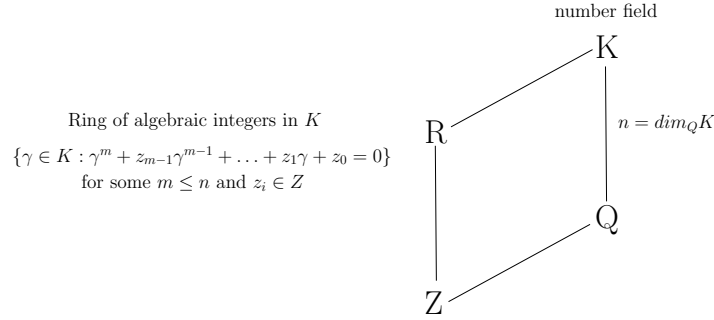


Figure 1: Rings of integers and their quotient fields

to see for themselves how the proof works, in order to evaluate and, perhaps, appreciate, the reasoning behind it.

To this end, this paper first reviews the concepts from algebraic number theory and their properties which are used in the proof, keeping the presentation as lean as possible. For proofs one could consult, e.g., Lang [4] or Lorenz [5]. With these prerequisites in place the proof from [1] is re-formulated in compact and contiguous form. The author acknowledges discussions with ChatGPT+.

2. Some facts from algebraic number theory

2.1. Number fields

A *number field*, K , is an extension of the field of rational numbers, $\mathbb{Q}$, that is of finite dimension n as a vector space over $\mathbb{Q}$; dimension n is called the *degree* of K . Then each element α of K is *algebraic*, i. e., it fulfills a polynomial equation

$$\alpha^m + q_{m-1}\alpha^{m-1} + \dots + q_1\alpha + q_0 = 0$$

with $m \leq n$ and coefficients q_i from $\mathbb{Q}$. The polynomial $f(X) = X^m + q_{m-1}X^{m-1} + \dots + q_1X + q_0$ of smallest possible degree m for which this holds is the *minimal polynomial* of α . The analogue of the ring of integers $\mathbb{Z}$ is the ring R of *algebraic integers* in K , defined as the set of elements $\gamma \in K$ satisfying

$$\gamma^m + z_{m-1}\gamma^{m-1} + \dots + z_1\gamma + z_0 = 0$$

with $m \leq n$ and coefficients z_i from $\mathbb{Z}$. Just as $\mathbb{Q}$ is the field of quotients from $\mathbb{Z}$, every element of K can be written as a quotient of two elements of R . See Figure 1.

2.2. Ideals

Arithmetic in R differs from what is known for $\mathbb{Z}$. In general, prime numbers have to be replaced with *prime ideals*. An *integer ideal* of the ring R is a subset $\mathbf{A}$ that contains 0, is closed under addition, and multiplication with arbitrary elements from R .

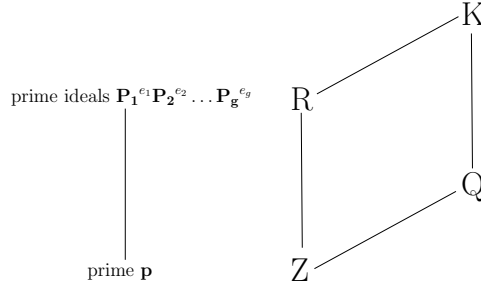


Figure 2: The decomposition of a prime number into prime ideals

Among them are the *principal ideals* $(\gamma) = \gamma \cdot R$. Two ideals $\mathbf{A}, \mathbf{B}$ can be multiplied as follows

$$\mathbf{A} \cdot \mathbf{B} := \{a_1 b_1 + \dots + a_r b_r\} \text{ where } a_i \in \mathbf{A}, b_i \in \mathbf{B} \text{ and } r \geq 1.$$

An ideal $\mathbf{A}$ is said to be a *divisor* of $\mathbf{B}$ iff there exists an ideal $\mathbf{C}$ such that $\mathbf{A} \cdot \mathbf{C} = \mathbf{B}$ holds. Since the ring R is a Dedekind ring, *prime* ideals are exactly the ideals without proper divisors.

Surprisingly, the unique factorization into prime numbers known from $\mathbb{Z}$ carries over to prime ideals $\mathbf{P}$ in R : Each ideal $\mathbf{A}$ different from zero has a unique decomposition

$$\mathbf{A} = \mathbf{P}_1^{e_1} \cdot \mathbf{P}_2^{e_2} \dots \mathbf{P}_g^{e_g}$$

with exponents e_i . Of particular importance is the case where $\mathbf{A}$ equals a principal ideal $(\mathbf{p})$ generated in R by a prime number p from $\mathbb{Z}$; then the exponents e_i are called *ramification indices*. If all of them have value 1 then p is said to be *un-ramified* in K . If some of them are larger than 1 but none is divisible by p , prime p is called *tamely ramified* in K . Moreover, if a prime number p is un-ramified and decomposed into n different prime ideals in R , prime p is said to *split completely* in K . See Figure 2.

The set of integer ideals so far considered can be enlarged by *fractional* ideals. They become integer ideals after multiplication with a suitable element of R . The enlarged set of ideals forms the ideal group $\mathbf{I}(K)$ of K . Taken modulo the subgroup of all principal ideals, the *class group* $\mathbf{CI}(K)$ results, whose order $h(K)$ is called the *class number* of K . The ring R has unique prime number factorization if and only if $h(K) = 1$ holds, i.e., if all ideals of K are principal ideals.

2.3. Embeddings and Galois groups

A number field K of degree n has n different embeddings σ_i into the complex numbers $\mathbb{C}$. Such an embedding is an injective map, compatible with the field operations, that leaves each element of $\mathbb{Q}$ fixed. It is called *real* if $\sigma(K) \subset \mathbb{R}$ holds; otherwise it is called *complex*. Complex embeddings occur in conjugate pairs $\sigma, \bar{\sigma}$. If r_1 denotes the number of real embeddings and r_2 the number of conjugate complex pairs, $r_1 + 2r_2 = n$ holds.

Example. Consider $K = \mathbb{Q}(\sqrt[3]{2})$, the smallest field over $\mathbb{Q}$ containing $\sqrt[3]{2}$. K is of degree 3. The real number $\sqrt[3]{2}$ is a root of the minimal polynomial

$$f(X) = X^3 - 2 = (X - \sqrt[3]{2})(X - \zeta\sqrt[3]{2})(X - \zeta^2\sqrt[3]{2})$$

where $\zeta = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ satisfies $\zeta^2 = \bar{\zeta}$ and $\zeta^3 = 1$. For each embedding σ we have $0 = \sigma(f(\sqrt[3]{2})) = f(\sigma(\sqrt[3]{2}))$. Hence, σ maps $\sqrt[3]{2}$ to a root of $f(X)$. There is one real embedding of K with $\sigma_1(\sqrt[3]{2}) = \sqrt[3]{2}$, and two conjugate complex embeddings satisfying $\sigma_2(\sqrt[3]{2}) = \zeta\sqrt[3]{2}$ and $\sigma_3(\sqrt[3]{2}) = \zeta^2\sqrt[3]{2}$. The product of all three roots equals 2, the negative of the constant coefficient -2 of the minimal polynomial f .

For two number fields $L \supset K$ the extension L/K is called *normal* if, for each $\alpha \in L$ with minimal polynomial $f(X)$ over K , all roots of f are contained in L . If L/K is normal and of degree n , it is *Galois*, and there exists a *Galois group* $G(L/K)$ of n bijective mappings that map L onto itself, leave all elements of K fixed, and are compatible with the field operations. In the above example the extension $K = \mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not normal. But if $\zeta = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is adjoined to $\mathbb{Q}(\sqrt[3]{2})$ then the extension $\mathbb{Q}(\sqrt[3]{2}, \zeta)/\mathbb{Q}$ is Galois of degree 6. Its Galois group consists of the automorphisms $\{\text{id}, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\}$, where $\sigma(\sqrt[3]{2}) = \zeta\sqrt[3]{2}$ and $\sigma(\zeta) = \zeta$, while τ denotes complex conjugation. These mappings induce all possible six permutations of the three roots of $X^3 - 2$.

2.4. The discriminant

Now let $\gamma_1, \dots, \gamma_n$ be an *integral basis* of the ring R of integers over $\mathbb{Z}$, allowing each $\gamma \in R$ to be expressed uniquely as $\gamma = \sum_{i=1}^n z_i \gamma_i$ with coefficients z_i from $\mathbb{Z}$. Using the regular determinant, one defines the *discriminant* d_K of K by

$$d_K := \det\left((\sigma_i \gamma_j)_{1 \leq i, j \leq n}\right)^2$$

While d_K is always in $\mathbb{Z}$ it may well be negative.

The discriminant is a link between different quantities that are going to occur in the proof below. First, it allows to associate with the ring of integers R a *geometric size*. To this end one picks all r_1 real embeddings of K , and one from each pair of conjugate embeddings, to obtain a map σ from K into $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, as shown in Figures 3 and 4. Restricted to the ring R , this map σ generates a *lattice* Λ spanned by the images of the basis vectors γ_i and coefficients from $\mathbb{Z}$. Then the following holds for the volume of its ground mesh $M = \Lambda/\mathbb{Z}^n$:

$$\text{vol}(M) = 2^{-r_2} \sqrt{|d_K|}. \quad (1)$$

Second, discriminant d_K is closely related to the prime numbers that ramify in K :

$$p \text{ is ramified in } K \iff p \mid d_K \quad (2)$$

$$p \text{ is tamely ramified in } K \implies \text{exponent of } p \text{ in } d_K \text{ is } \leq n \quad (3)$$

Third, d_K is related to the class number h_K by

$$h_K \leq |d_K| \text{ if } n \geq 4 \quad (4)$$

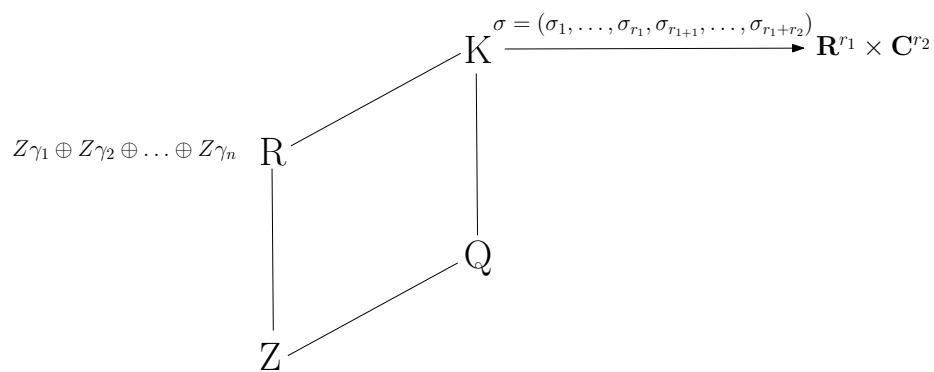


Figure 3: Map σ , to be applied to the integers R of K .

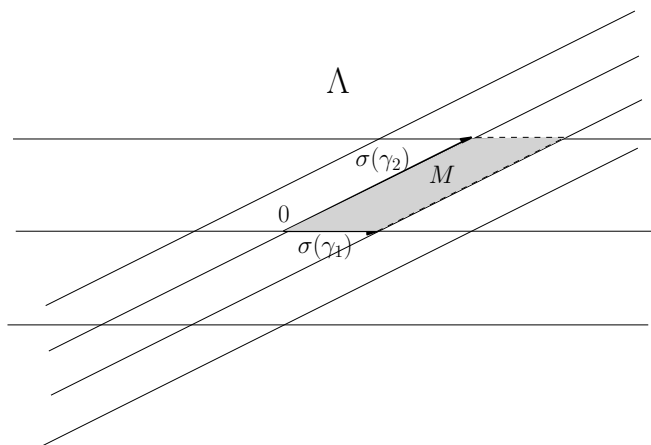


Figure 4: A lattice Λ generated in $\mathbb{R}^2$ and its ground mesh $M = \Lambda/\mathbb{Z}^2$.

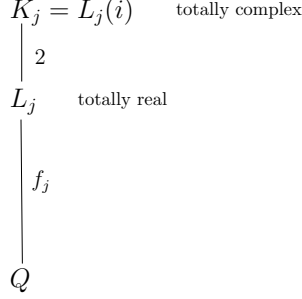


Figure 5: The number fields K_j

3. The lower bound

We want to prove the following result.

Theorem 1. (*OpenAI '26, refined by Alon et al. [1]*) *There exist $\epsilon > 0$ and a sequence of point sets P_i in the plane where $|P_i| \rightarrow \infty$ such that each set P_i contains at least $|P_i|^{1+\epsilon}$ point pairs of distance 1.*

3.1. High degree number fields with prescribed prime behavior

The proof of this theorem is based on the following consequence from deeper results in class field theory whose complete proof would exceed the scope of this write-up. A sketch of the proof will be given in Subsection 3.6, after the proof of Theorem 1.

Lemma 1. *Given the set of prime numbers $T := \{3, 5, 7, 11, 13, 17\}$ and $p := 101$, there exist number fields L_j of arbitrarily large degree f_j where*

1. *only primes from T can ramify in L_j , and only in a tame way*
2. *prime p is completely split in L_j*
3. *all embeddings of L_j are real*

Let K_j be the number field that results from adjoining to L_j the complex number i . Then the degree of K_j over L_j equals 2, and over $\mathbb{Q}$, $2f_j$; see Figure 5. By construction, K_j has only complex embeddings, f_j of which form the lattice Λ_j of the integers R_j in K_j .

By (2), (3) from above and Lemma 1, 1., one has $|d_{L_j}| \leq \prod_{q \in T} q^{f_j}$. In K_j the primes in T remain tamely ramified, and no other prime can ramify in K_j —except 2. In fact, already in $\mathbb{Q}(i)$ one has $(2) = (1 + i)^2$, which is not a tame ramification. Nonetheless, the contribution to d_{K_j} of the prime 2 can be shown to be upper bounded by 2^{2f_j} using the standard formula for discriminants in towers.

Therefore, we obtain

$$|d_{K_j}| \leq r^{2f_j} \text{ where } r := 2 \prod_{q \in T} q, \text{ a constant.} \quad (5)$$

3.2. Finding elements in K_j that are of size 1 in each complex embedding

By Lemma 1, 2., $p = 101$ splits completely in L_j . Since $101 \equiv 1 \pmod{4}$, the prime 101 also splits in $\mathbb{Q}(i)$. Therefore, each of the f_j many primes over 101 in L_j must split into two conjugate pairs after adjoining i . Thus, we have in K_j

$$(101) = \prod_{l=1}^{f_j} \mathbf{P}_l \bar{\mathbf{P}}_l$$

with pairwise different prime ideals. Let k be a constant to be determined later. Of the $(k+1)^{f_j}$ many integral ideals

$$\prod_{l=1}^{f_j} \mathbf{P}_l^{a_l} \bar{\mathbf{P}}_l^{k-a_l}, \text{ where } 0 \leq a_l \leq k,$$

at least

$$\frac{(k+1)^{f_j}}{h(K_j)}$$

must fall into the same class of $\mathbf{CI}(K_j)$, by the pigeonhole principle. The quotient of any two of them is a principal ideal. So, one obtains at least $\frac{(k+1)^{f_j}}{h(K_j)}$ principal ideals of type

$$(\alpha) = \prod_{l=1}^{f_j} \mathbf{P}_l^{a_l - a'_l} \bar{\mathbf{P}}_l^{a'_l - a_l}.$$

Dividing by conjugates results in

$$(\alpha/\bar{\alpha}) = \prod_{l=1}^{f_j} \mathbf{P}_l^{2a_l - 2a'_l} \bar{\mathbf{P}}_l^{2a'_l - 2a_l}.$$

Since none of these exponents is below $-2k$, the ideal $(\mathbf{p})^{2k} (\alpha/\bar{\alpha})$ is integer. In other words, $\alpha/\bar{\alpha}$ is an element of $p^{-2k} R_j$ and will be mapped to the scaled lattice $p^{-2k} \Lambda_j$. Moreover, for each complex embedding σ of K_j , and for each $\gamma \in K_j$, one has $\sigma(\bar{\gamma}) = \overline{\sigma(\gamma)}$, so that

$$|\sigma(\alpha/\bar{\alpha})| = \sigma(\alpha/\bar{\alpha}) \overline{\sigma(\alpha/\bar{\alpha})} = \sigma(\alpha/\bar{\alpha}) \sigma(\bar{\alpha}/\alpha) = 1.$$

To summarize, let $U := \{(x_1, \dots, x_{f_j}) \in p^{-2k} \Lambda_j; |x_i| = 1 \ \forall i\}$. Then,

$$|U| \geq \frac{(k+1)^{f_j}}{h(K_j)} \quad \text{as shown above} \tag{6}$$

$$\geq \frac{(k+1)^{f_j}}{|d_{K_j}|} \quad \text{by (4)} \tag{7}$$

$$\geq \left((k+1)r^{-2}\right)^{f_j} \quad \text{by (5)}. \tag{8}$$

For short, let $\delta := p^{-2k}$, a constant once k has been fixed.

3.3. Constructing a point set P with many unit distance pairs

For $R \geq 2$ let $B_R := \{(x_1, \dots, x_{f_j}) \in \mathbb{C}^{f_j}; |x_i| < R \forall i\}$ be a polydisk. The first step is in translating the lattice $\delta\Lambda_j$ in such a way that many lattice points are contained in B_{R-1} . To this end, the following lemma is helpful.

Lemma 2. *Let Λ be a lattice with ground mesh M and B a compact set, as before. Then there exists an element $a \in M$ such that*

$$|(a + \Lambda) \cap B| \geq \frac{\text{vol}(B)}{\text{vol}(M)} \quad (9)$$

holds.

Proof: For lattice points $\lambda \in \Lambda$ let $B_\lambda := \{a \in M; a + \lambda \in B\}$. Then B equals the union of all sets $B_\lambda + \lambda$ because each point of $b \in B$ can be obtained by adding some $a \in M$ to the local origin λ of a cell containing b . Therefore,

$$\int_M |(a + \Lambda) \cap B| = \int_M \sum_\lambda a \in B_\lambda = \sum_\lambda \int_M a \in B_\lambda = \sum_\lambda \text{vol}(B_\lambda) \geq \text{vol}(B).$$

In the second step, one adds all elements of the set U in (6) to all points of the translated lattice $a + \delta\Lambda_j$ inside B_{R-1} . All resulting points are situated in B_R , because the elements of U are of length 1, and each pair of points can be generated at most twice.

Let $W := (a + \delta\Lambda_j) \cap B_R$. Using the notations in Figure 3, the map σ followed by the s th coordinate map, for $s = 1 \dots f_2$, equals the embedding σ_s of the field K_j into $\mathbb{C}$, which is injective. Therefore, the projection of W results in a point set P in $\mathbb{C} \simeq \mathbb{R}^2$ that has the same cardinality as W .

The number $\nu(P)$ of unit distance pairs of P can now be bounded from below by

$$2\nu(P) \geq ((k+1)r^{-2})^{f_j} \left(\frac{\pi(R-1)^2}{\text{vol}(M)^{1/f_j}} \right)^{f_j} \text{ by (8), (9)} \quad (10)$$

$$\geq \left(\frac{(k+1)\pi R^2}{2\delta^2 r^3} \right)^{f_j} \quad (11)$$

using $R-1 \geq R/2$ and the upper bound to the volume of ground mesh M given by

$$\begin{aligned} \delta^{-2} \text{vol}(M)^{1/f_j} &= \delta^{-2} \left(2^{-f_j} \delta^{2f_j} \sqrt{|d_{K_j}|} \right)^{1/f_j} \text{ by (1), observing that} \\ &\quad \text{stretch factor } \delta \text{ applies to } 2f_j \text{ real coordinates} \\ &= \frac{1}{2} |d_{K_j}|^{1/2f_j} \\ &\leq \frac{1}{2} r \text{ by (5).} \end{aligned}$$

Once the constants k , and thereby, δ , have been fixed, constant R in (10) can be chosen sufficiently large to ensure that $\nu(P)$ and $|P|$ tend to infinity with f_j .

3.4. An upper bound to the size of point set P

Let γ be a nonzero integer in R_j . As in the example in subsection 2.3, the product of the images $\sigma(\gamma)$ taken over the embeddings of K_j equals, up to its sign, the nonzero absolute coefficient of a normed polynomial whose coefficients are in fact integers from $\mathbb{Z}$ because of $\gamma \in R_j$. Therefore, there must be an embedding σ satisfying $|\sigma(\gamma)| \geq 1$. Consequently, each nonzero lattice point of $\delta\Lambda_j$ has at least one coordinate of absolute value at least δ .

In particular, for two distinct $w, w' \in W$ the difference $w - w'$ has such coordinate. This implies that w and w' of W are well separated. Namely, if $w \oplus B_{\delta/2}$ and $w' \oplus B_{\delta/2}$ had nonempty intersection then there would be $b, b' \in B_{\delta/2}$ such that $w + b = w' + b'$ holds, implying

$$|w'_i - w_i| = |b_i - b'_i| \leq |b_i| + |b'_i| < \delta$$

for all coordinates i , a contradiction. Therefore, $|W| \cdot \text{vol}(B_{\delta/2}) \leq \text{vol}(W \oplus B_{\delta/2}) \leq \text{vol}(B_{R+\delta/2}) \leq \text{vol}(B_{3R/2})$ and

$$|P| = |W| \leq \frac{\text{vol}(B_{3R/2})}{\text{vol}(B_{\delta/2})} \quad (12)$$

$$= \frac{(\pi (3R/2)^2)^{f_j}}{(\pi (\delta/2)^2)^{f_j}} = \left(\frac{9R^2}{\delta^2}\right)^{f_j}. \quad (13)$$

3.5. The growth of $\nu(P)$ in terms of $|P|$

From (11) and (13) we obtain the first inequality in

$$\frac{\log 2\nu(P)}{\log |P|} \geq \frac{\log \frac{(k+1)\pi R^2}{2\delta^2 r^3}}{\log \frac{9R^2}{\delta^2}} \geq 1 + 2\epsilon. \quad (14)$$

If we choose $k > 18r^3/\pi - 1$, the second fraction in (14) becomes a constant greater than 1. Thus, we can bound it away from 1 as shown in the second inequality, with some $\epsilon > 0$. This implies

$$\nu(P) > |P|^{1+\epsilon}$$

for $|P|^\epsilon > 2$, thus proving Theorem 1..

3.6. Sketch of the proof of Lemma 1

A Galois extension is called *abelian* if its Galois group is abelian, i.e., if the order of applying its mappings does not matter. One of the main results in class field theory is in establishing a correspondence between all finite abelian extensions of a number field K and certain factor groups of its ideal group $\mathbf{I}(K)$; compare Subsection 2.2.

An interesting special case is the following. Let K be a number field, and let $H(K)$ be the maximal un-ramified abelian extension of K , denoted the *Hilbert class field* of K . Then, by the correspondence, the Galois group $G(H(K)/K)$ is isomorphic to the class group $\mathbf{Cl}(K)$ of K . So, if all ideals of K are principal ideals one has $h(K) = 1$,

hence $H(K) = K$. Otherwise, one can iterate the construction and consider the Hilbert class field $H(H(K))$, and so on. This way, a *class field tower*

$$\mathbb{Q} \subset K \subset H(K) \subset H(H(K)) \subset \dots$$

arises. Only finitely many primes ramify in the starting extension $K/\mathbb{Q}$, while every subsequent extension $H(K)/K, H(H(K))/H(K)$, etc., is un-ramified. Thus, if the tower were infinite, it would satisfy the first part of condition 1 in Lemma 1.

Golod and Šafarevič [3] proved that infinite class field towers exist, for example for the starting field $K = \mathbb{Q}(\sqrt{-3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17})$ (which is not totally real, as required in condition 3 of Lemma 1). Their proof is based on the following fact. Let p be a prime number. A group is called a p -group if each of its elements becomes *id* when raised to a suitable power p^m . The theorem states that in non-trivial *finite* p -groups the minimum number d of generating elements and the minimum number r of relations between these generators satisfy

$$r > \frac{d^2}{4}. \quad (15)$$

Using class field theory one can build a class tower

$$\mathbb{Q} =: L_0 \subset L_1 \subset \dots \subset L_r \subset L_{r+1} \subset \dots \subset \widetilde{\mathbb{Q}} \quad (16)$$

in the following way. Each field L_{r+1} is the maximal abelian extension of L_r of degree a power of 2, un-ramified outside of T , fully split over 101, and totally real.

Every odd prime q that ramifies in some L_r must have ramification indices that divide the degree of L_r over $\mathbb{Q}$, which is a power of 2. Hence, they cannot be divisible by q itself, i.e., the ramification can only be tame. This shows that each field L_r satisfies the conditions in Lemma 1.

If $\widetilde{\mathbb{Q}}/\mathbb{Q}$ were of finite degree, its Galois group $G(\widetilde{\mathbb{Q}}/\mathbb{Q})$, as a finite 2-group, would have to satisfy inequality (15). But a closer analysis of the ranks involved reveals that this is not the case; see [1]. Therefore, the class tower (16) contains fields L_i of arbitrary large degree over $\mathbb{Q}$, and Lemma 1 is proven.

References

- [1] N. Alon, T. F. Bloom, W. T. Gowers, D. Litt, W. Sawin, A. Shankar, J. Tsimerman, V. Wang and M. M. Wood, *Remarks on the disproof of the unit-distance conjecture*, <https://doi.org/10.48550/arXiv.2605.20695>
- [2] P. Erdős, On sets of distances of n points, *American Mathematical Monthly* 53:248–250, 1946
- [3] E. S. Golod and L. R. Šafarevič, *On the class field tower*, *Izv. Akad. Nauk SSSR Ser. Mat.*, 28:261–272, 1964, English translation: *Amer. Math. Soc. Transl. (2)*, 48:91–102, 1965
- [4] S. Lang, *Algebraic number theory*, Graduate Texts in Mathematics, Springer 1994
- [5] F. Lorenz, *Algebraische Zahlentheorie*, BI Wissenschaftsverlag, 1993