Time-Space Trade-offs for Triangulations and Voronoi Diagrams

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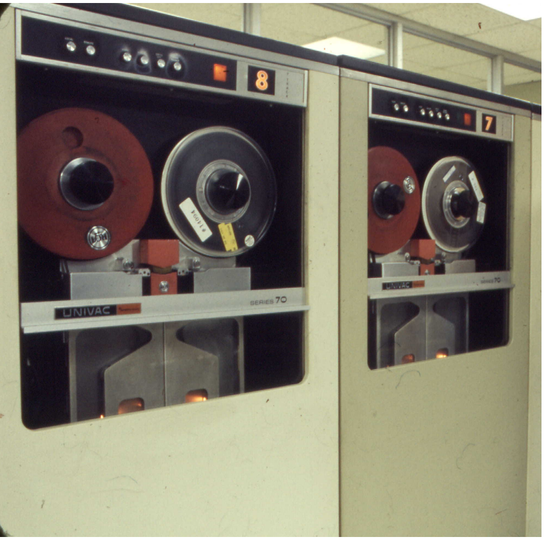
Limited Memory

Started in the 70's



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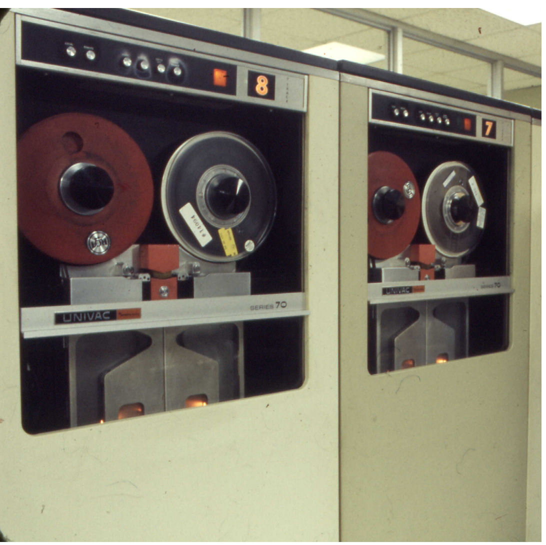
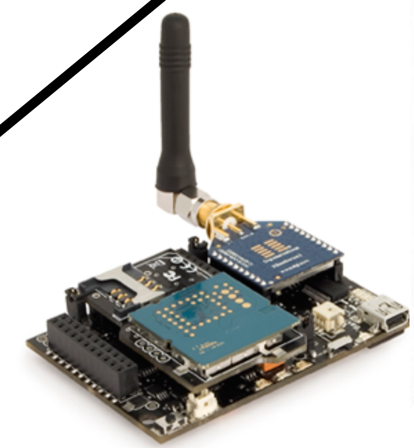
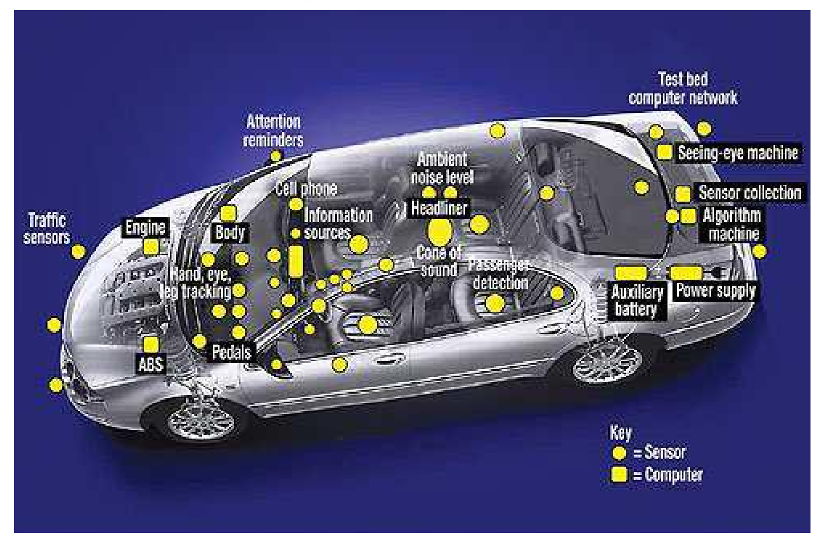
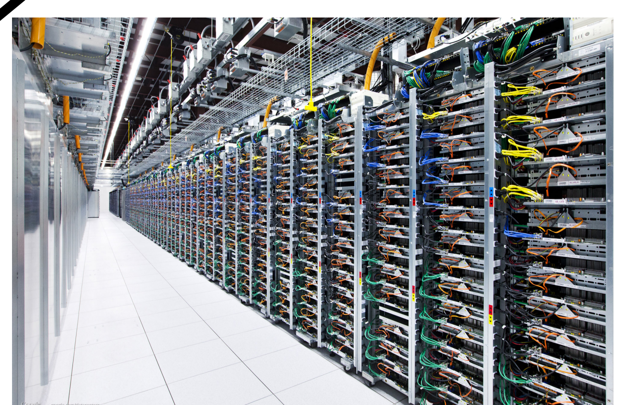




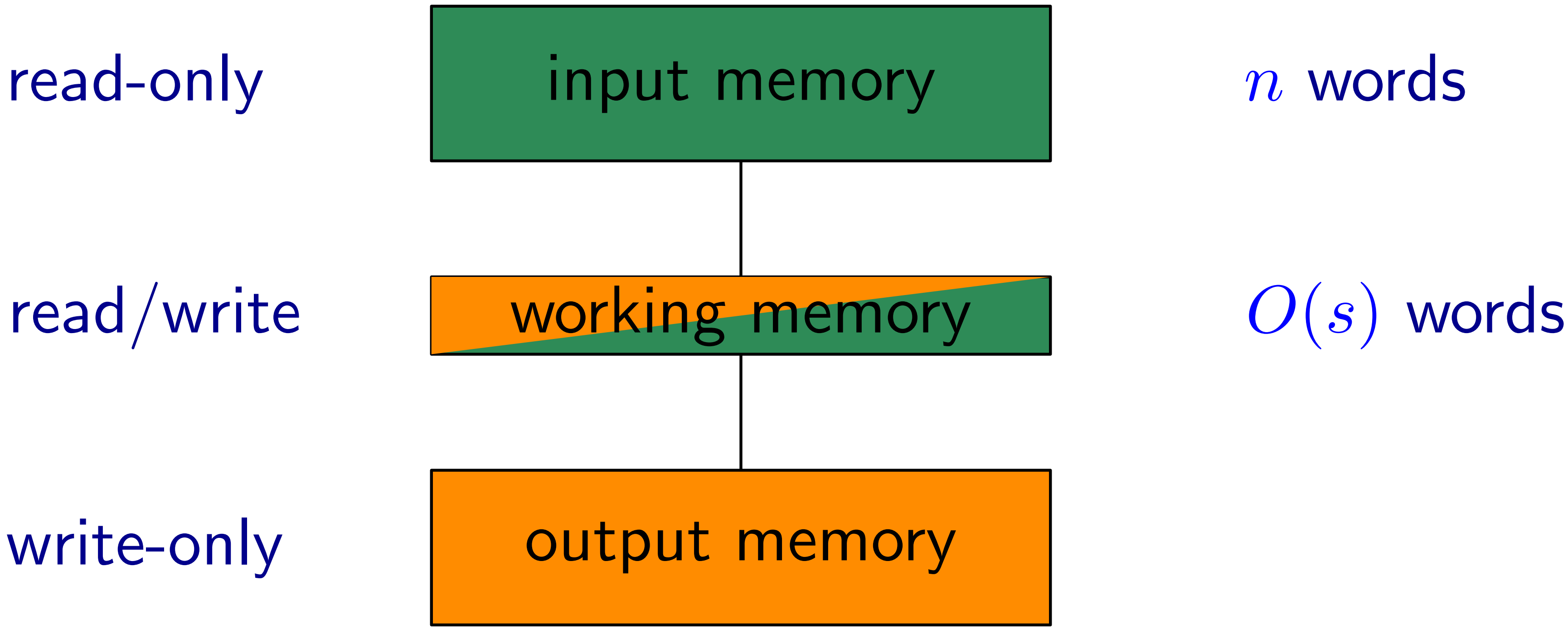




still relevant today

Model

Word RAM with unit costs, parameter s

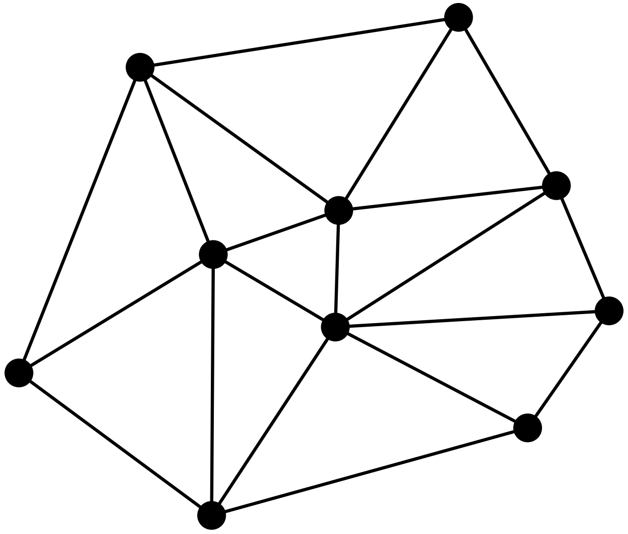


word $= \Omega(\log n)$ bits

Our Results

Input set P of n points in $\mathbb{R}^2$

Output edges of a triangulation in arbitrary order



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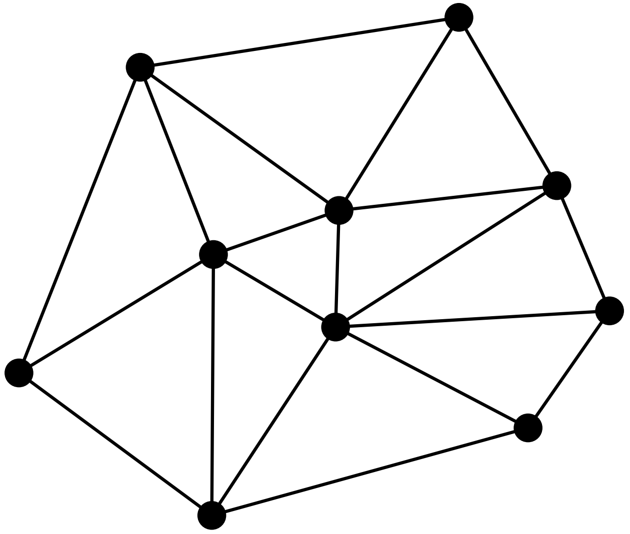

Related Results

- $O(n \log n)$ time with $O(n)$ space
- $O(n^2)$ time with $O(1)$ space [Asano et al. '11]

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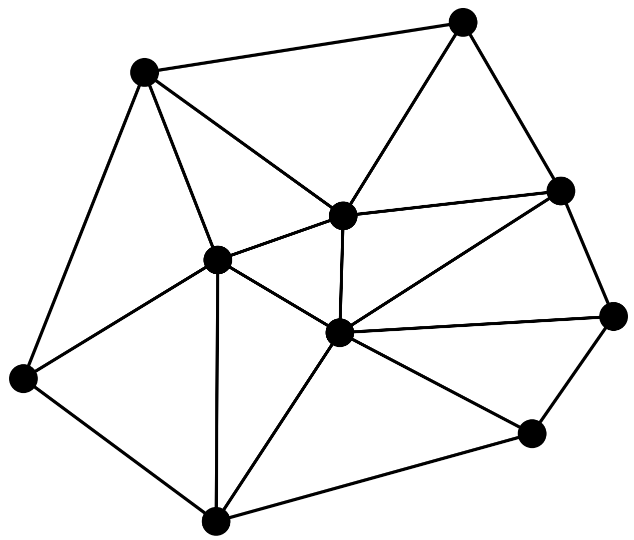

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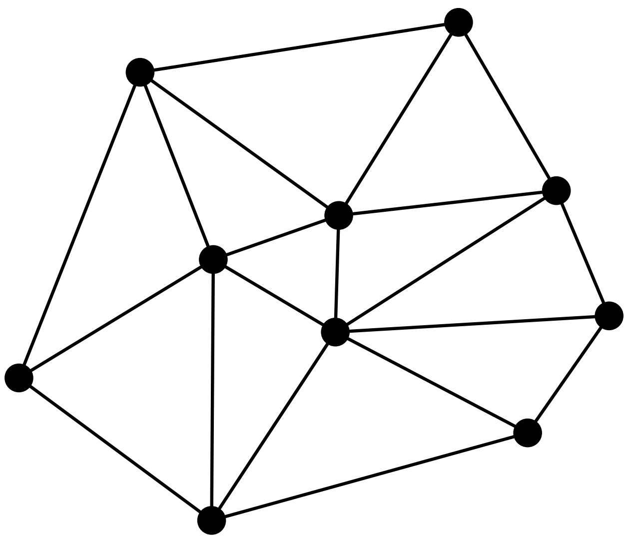

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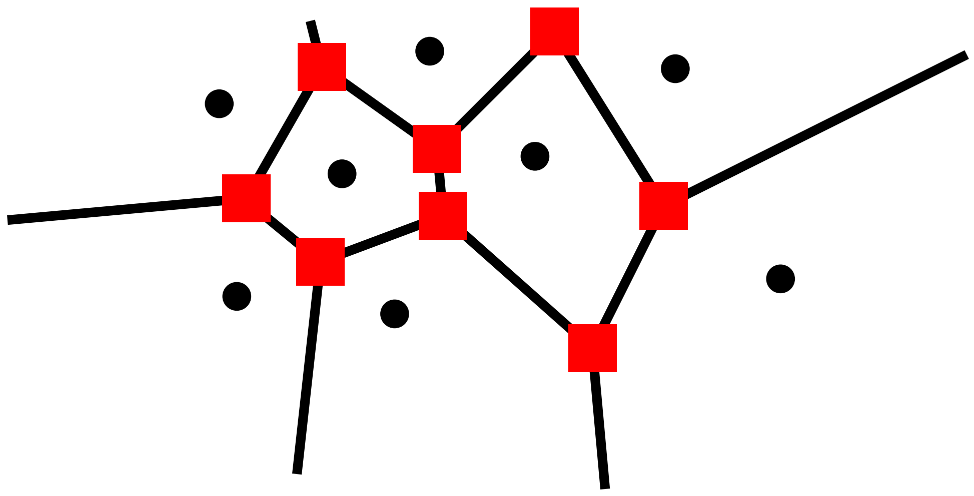

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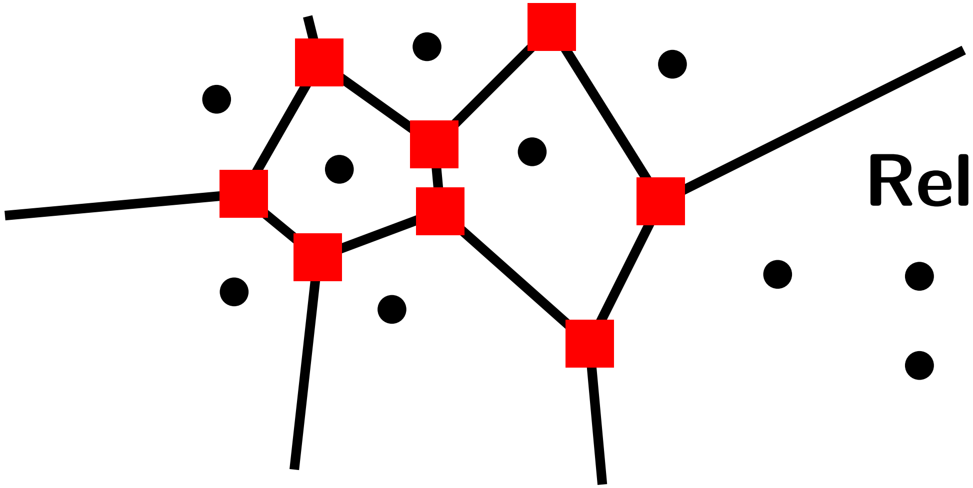

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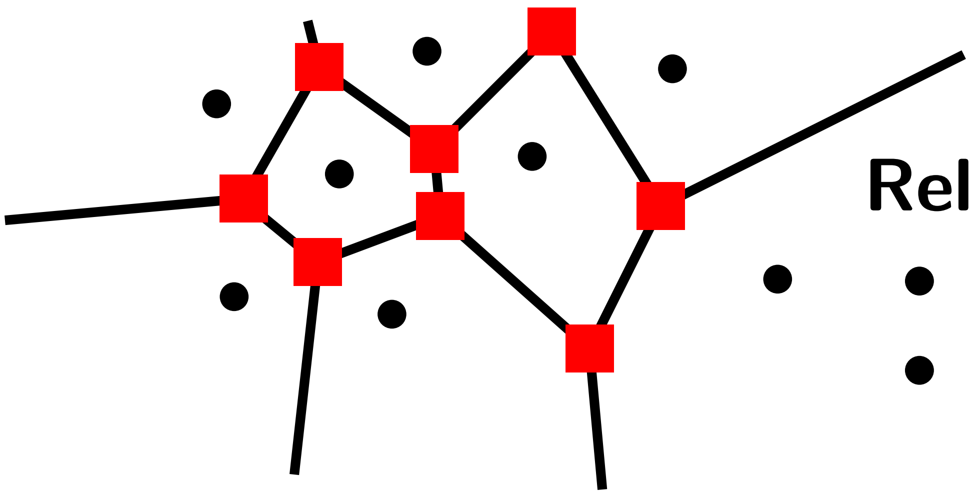

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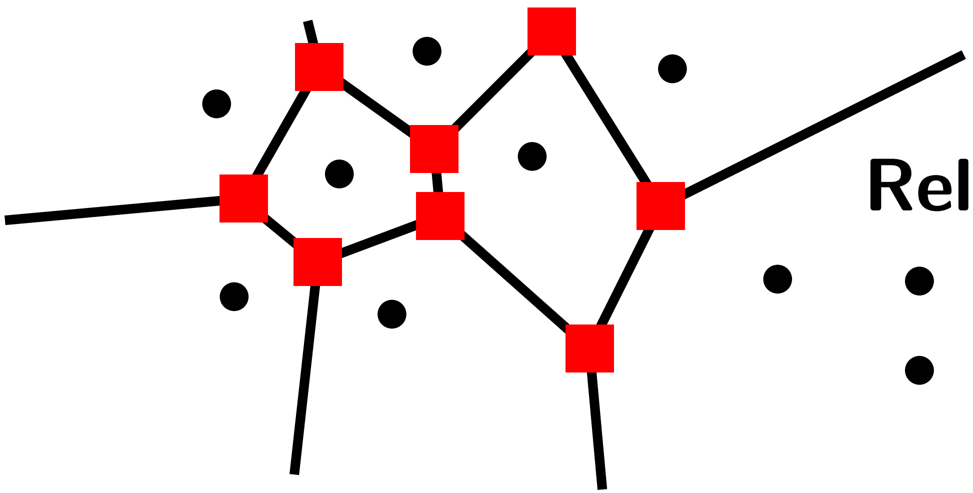

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Computing $\mathrm{VD}(P)$ in $O(s)$ space — Overview

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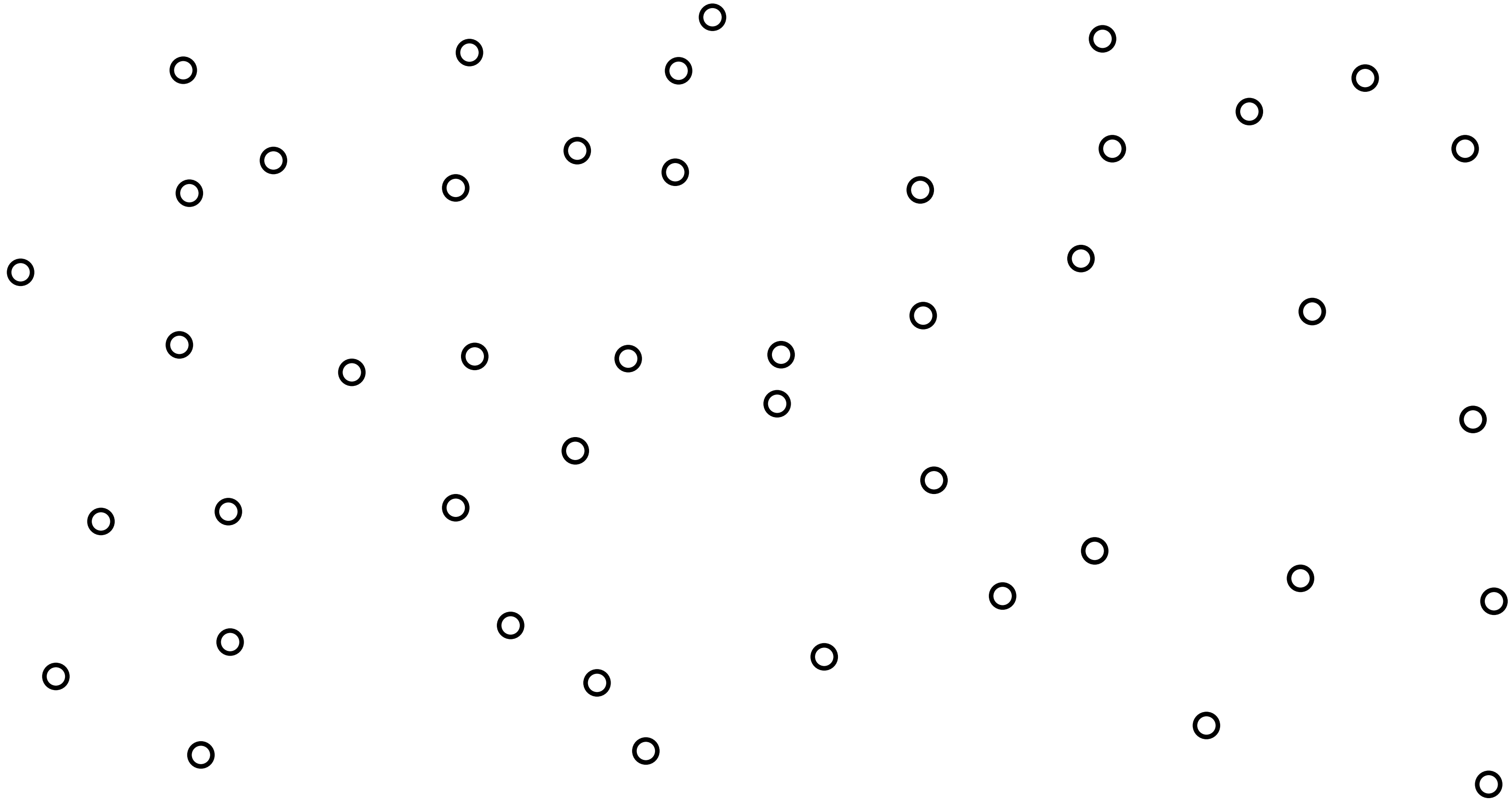
- Take random sample $R \subset P$ of size $O(s)$



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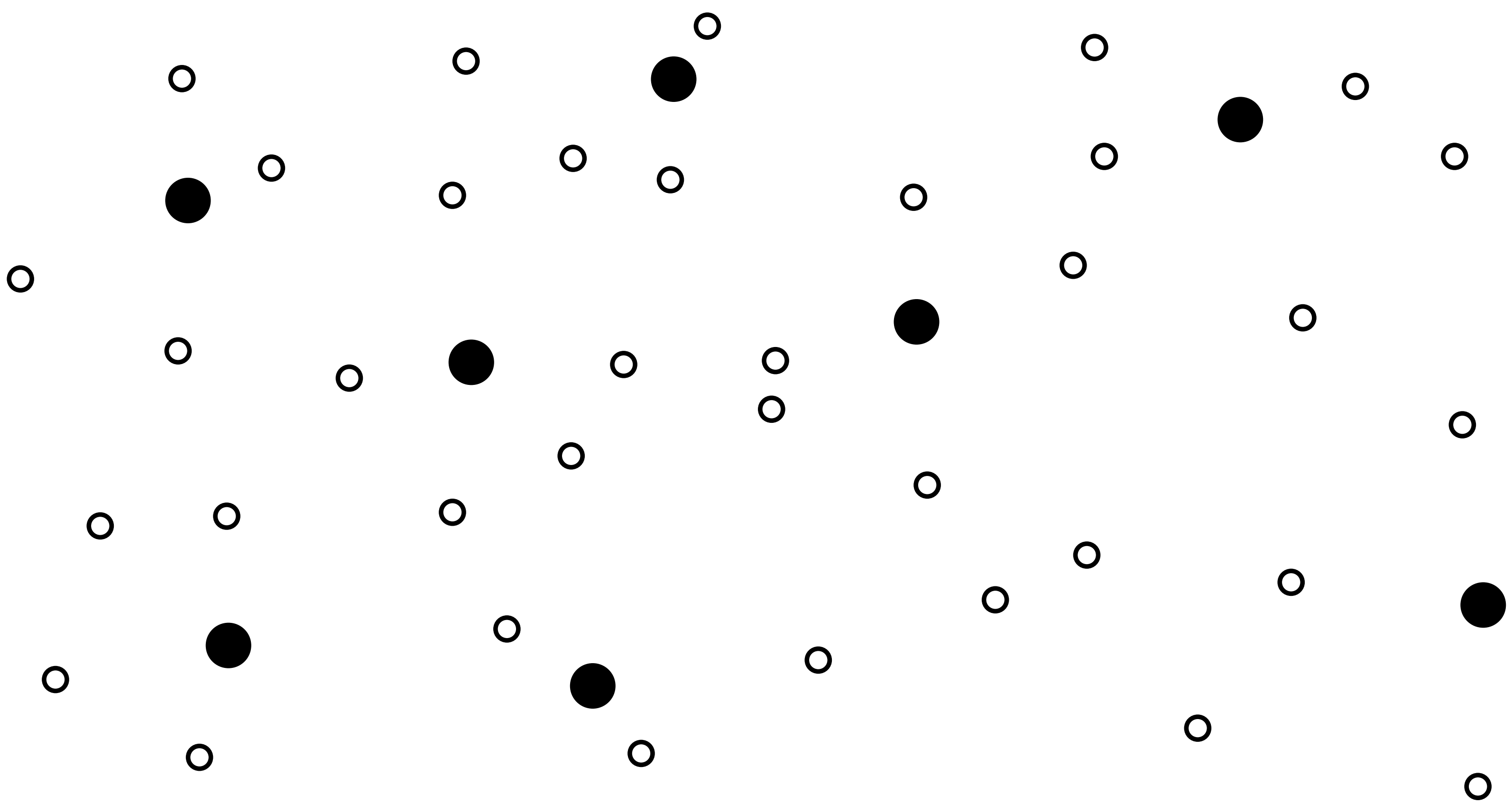

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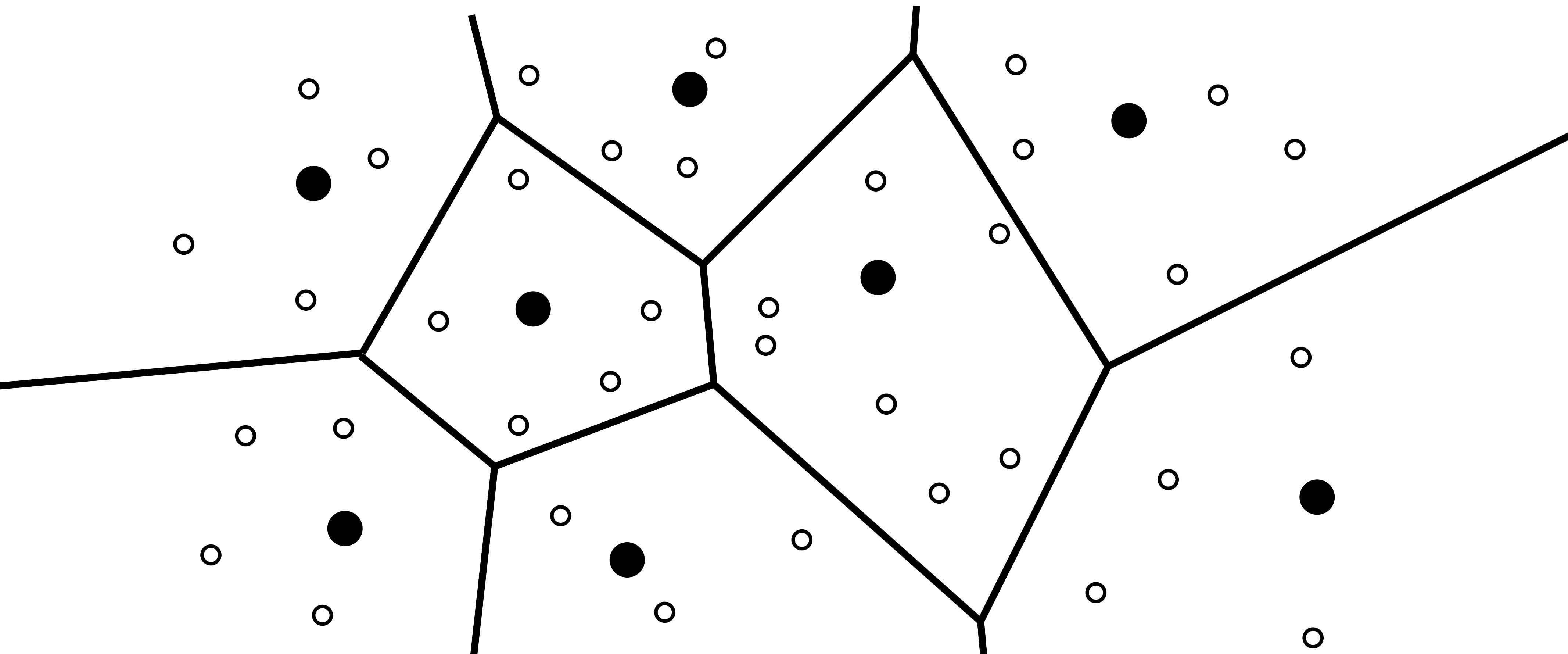

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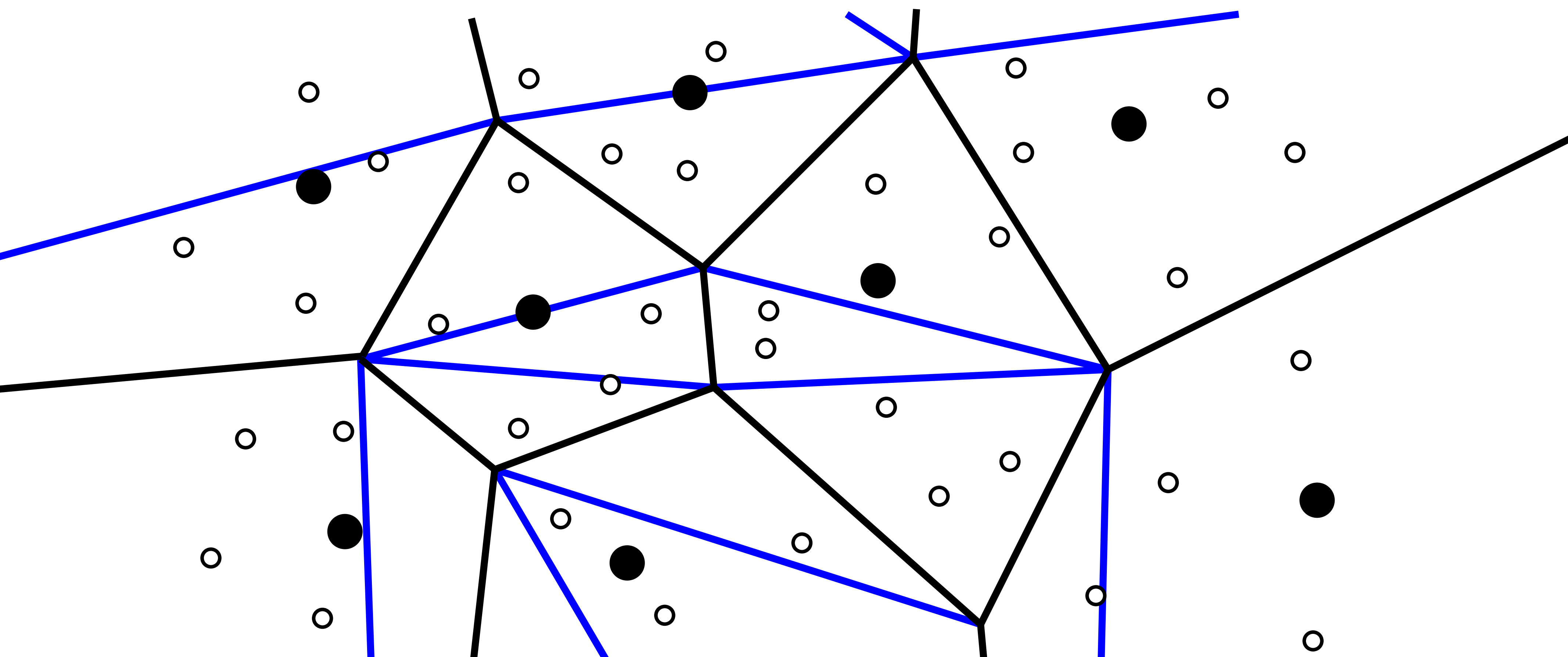

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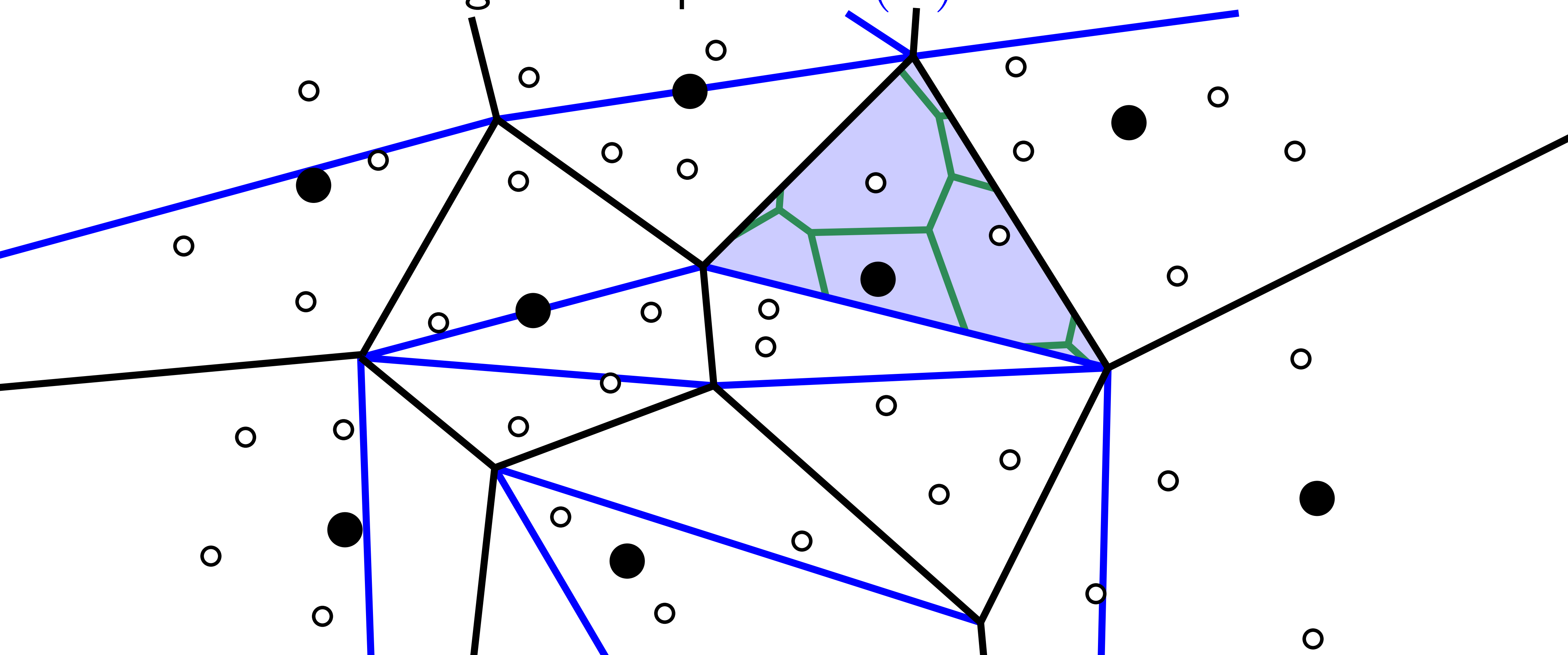
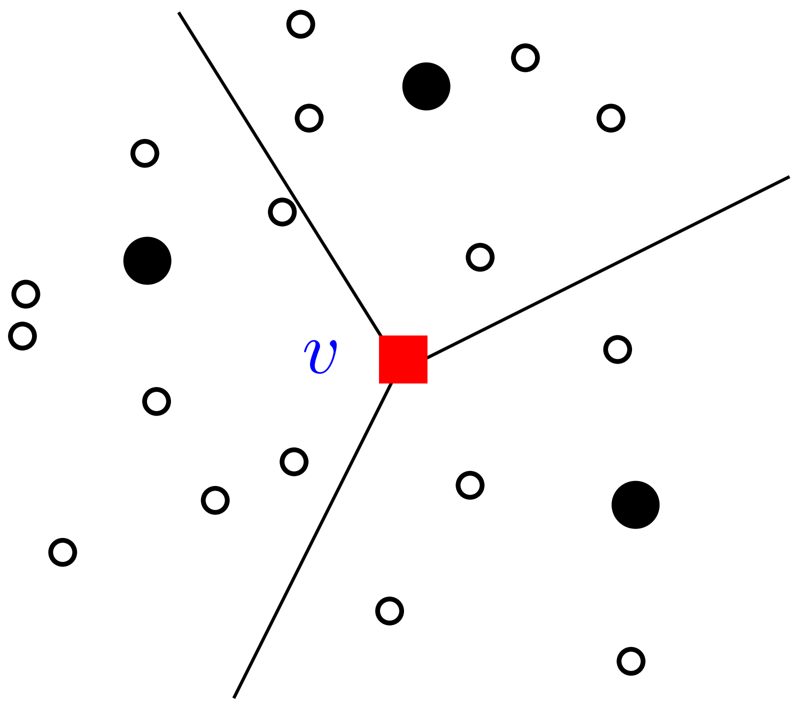
- For each triangle Δ: report $\mathrm{VD}(P) \cap \Delta$



Triangles Can Be Handled Locally

Definition.

$\mathrm{VD}(R)$



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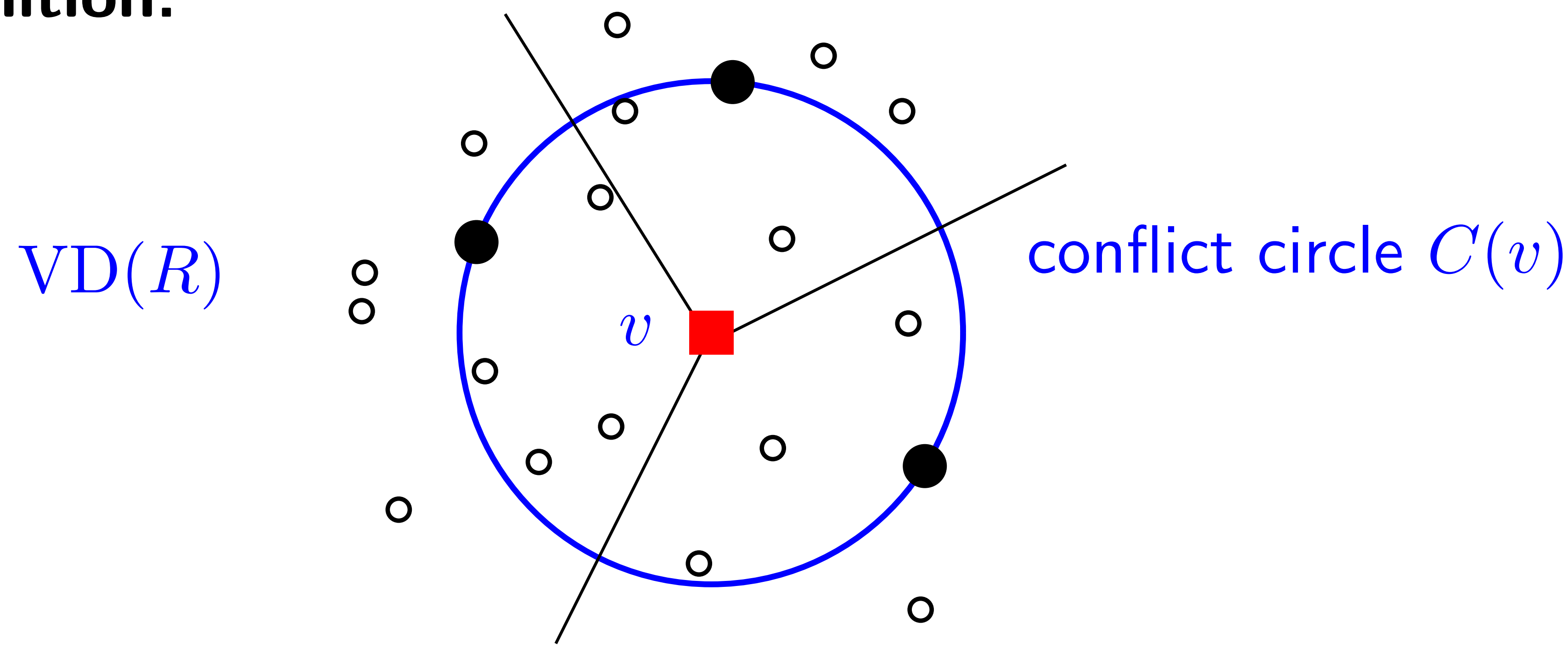

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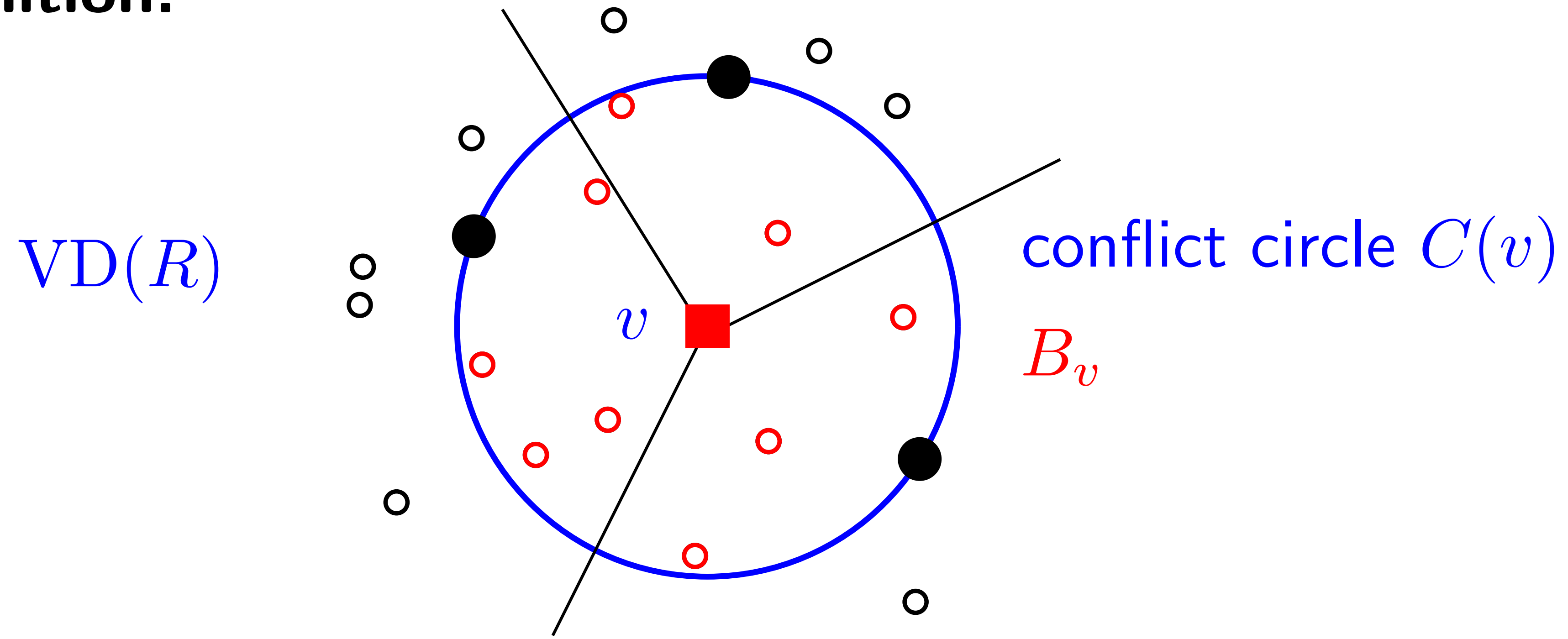

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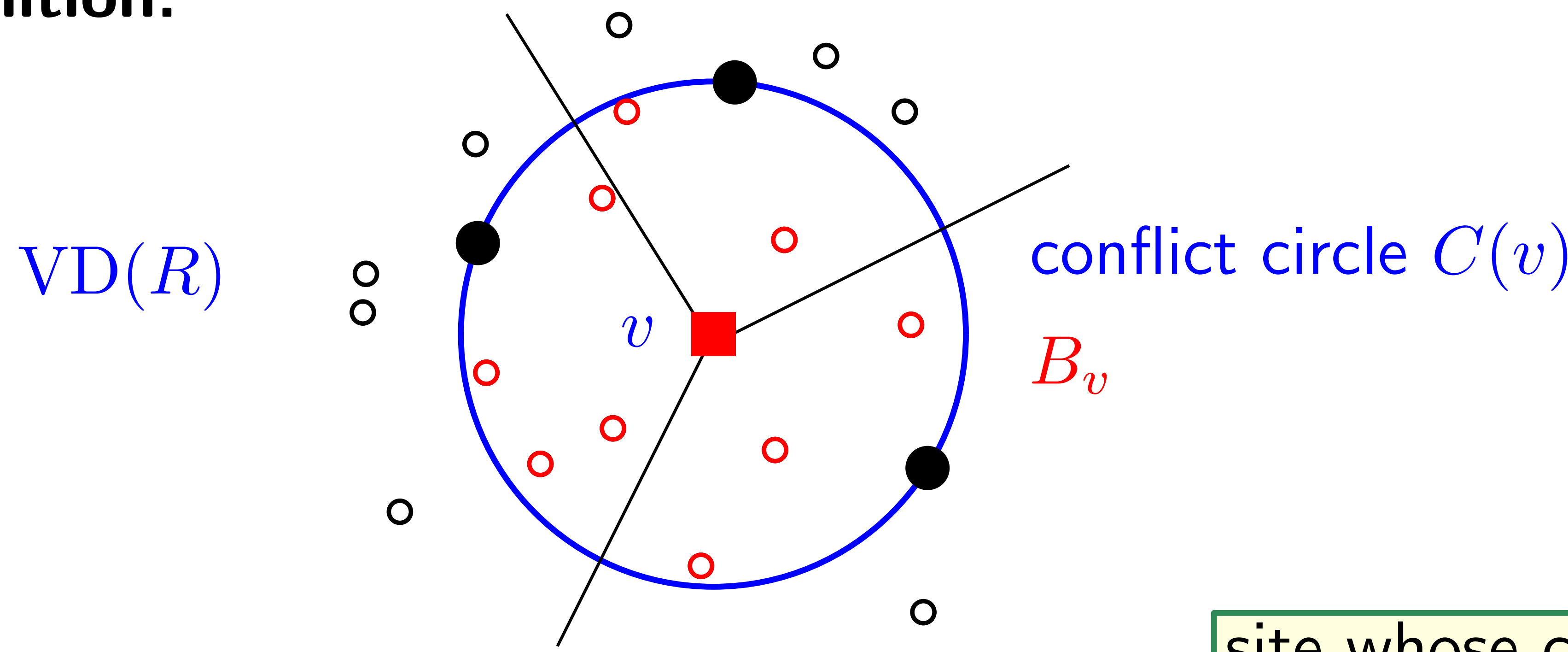

site whose cell contains Δ

Lemma. Let $\Delta = \{v_1, v_2, v_3\}$ be a triangle in the triangulation of $\mathrm{VD}(R)$. Then,

$$\mathrm{VD}(P) \cap \Delta = \mathrm{VD}(\underbrace{B_{v_1} \cup B_{v_2} \cup B_{v_3} \cup \{s\}}_{:=B_\Delta}) \cap \Delta$$

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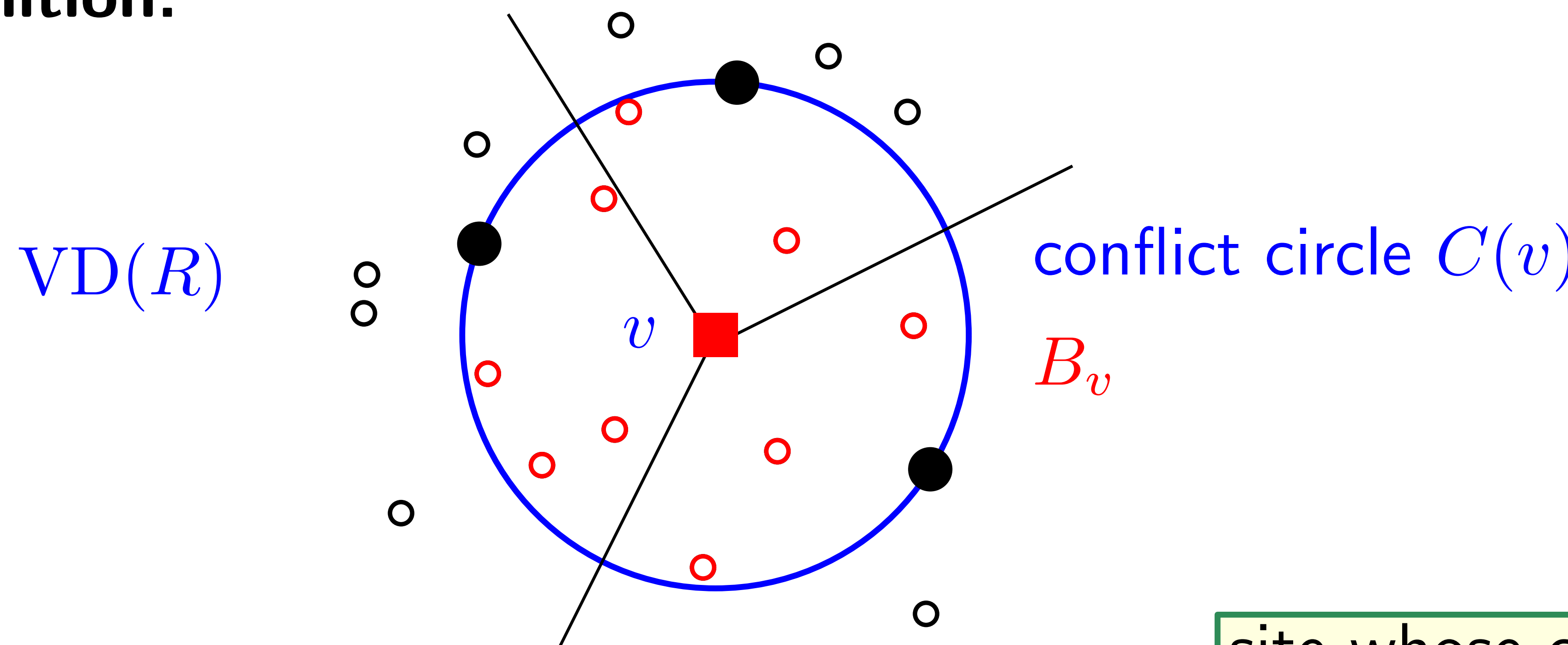

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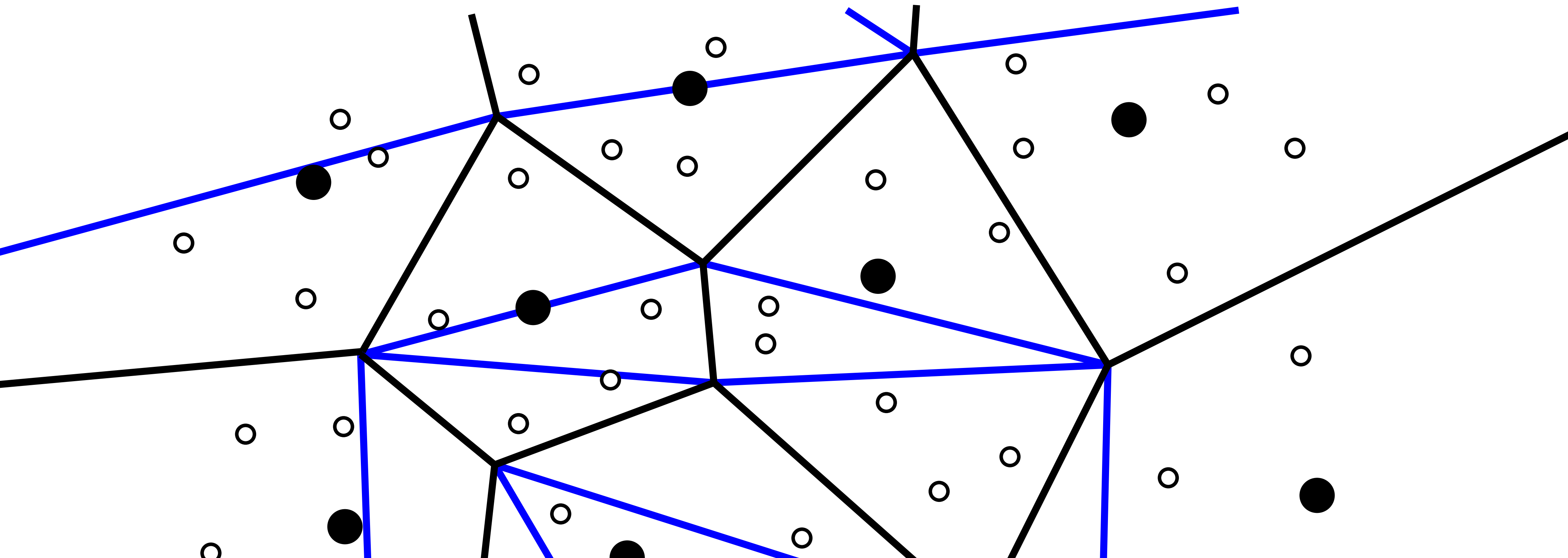
$O(s)$ triangles $\rightarrow$ want $B_\Delta = O(n/s)$ for all triangles Δ

Phase II: Compute $\mathrm{VD}(P)$

Assumption: for each Δ: $B_\Delta = O(n/s)$



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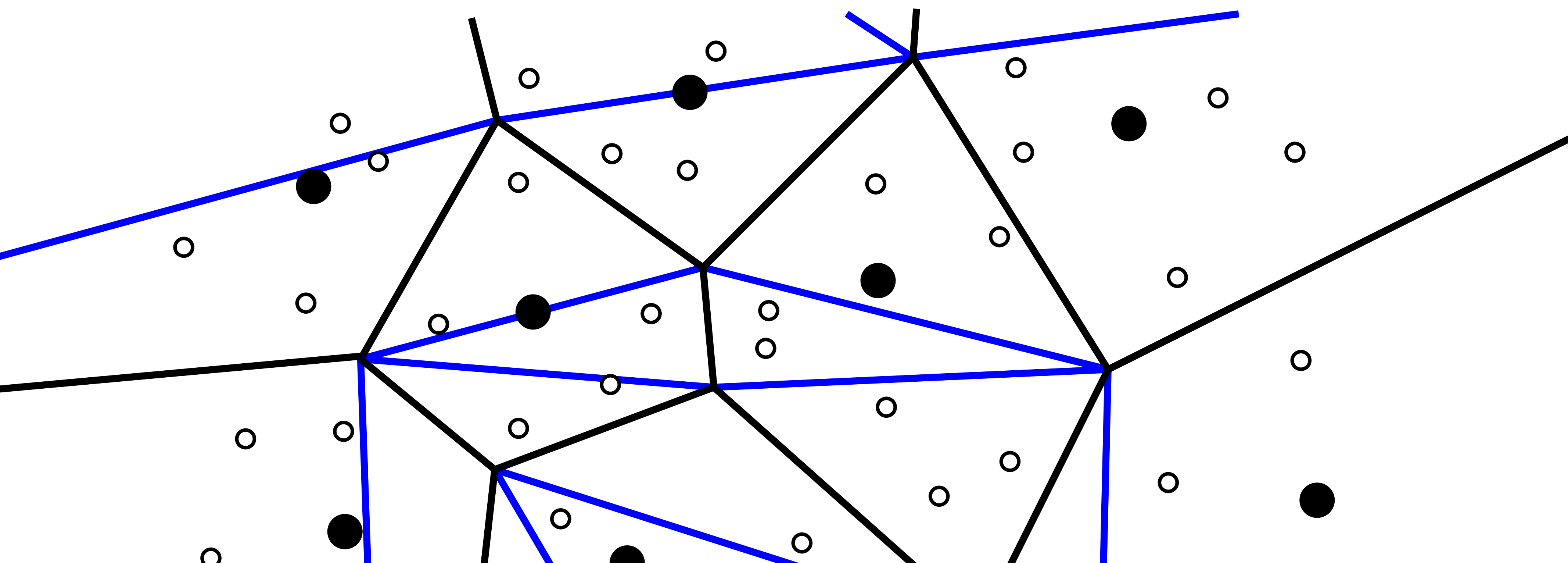
Run in parallel for each Δ the algorithm by Asano et al.

- $O(n^2/s^2)$ time and $O(1)$ space



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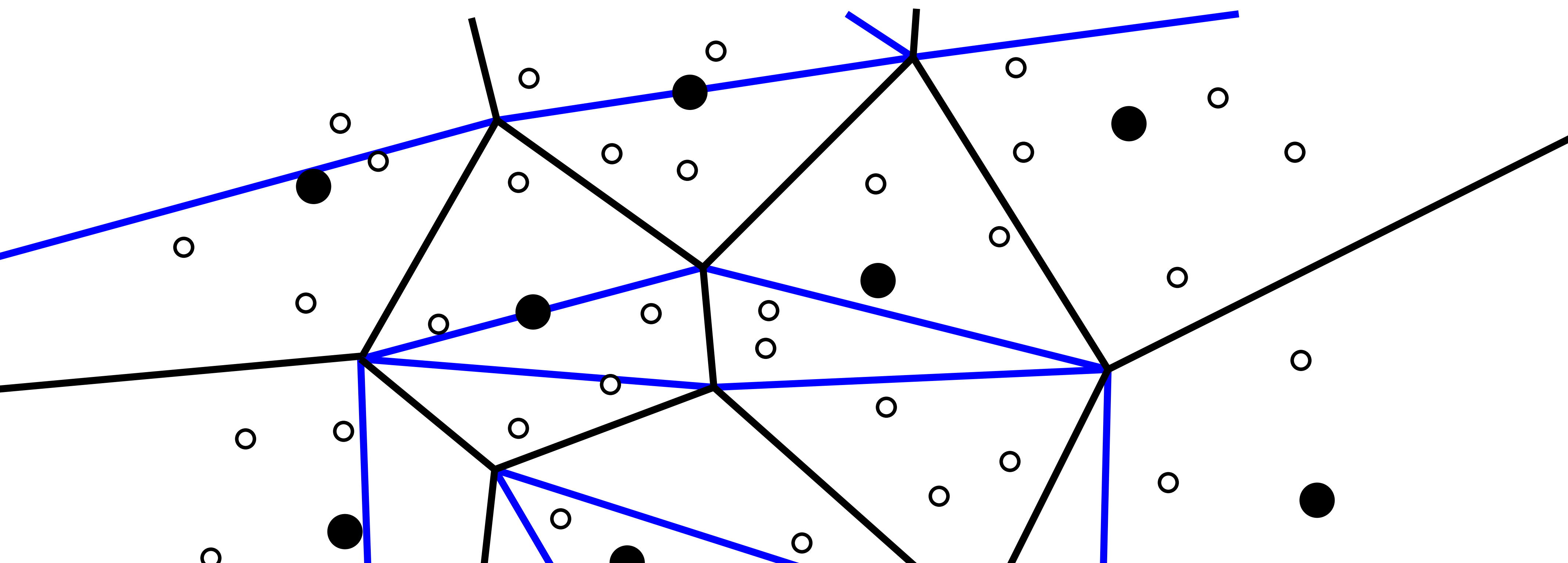

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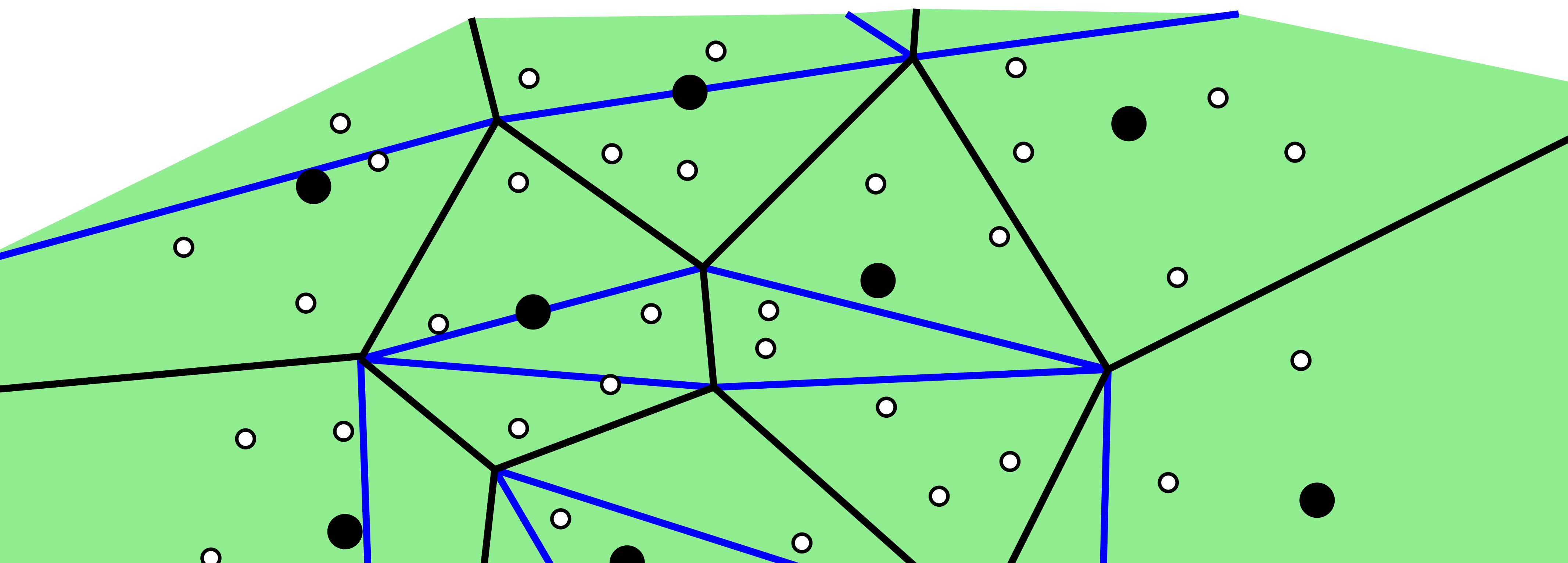

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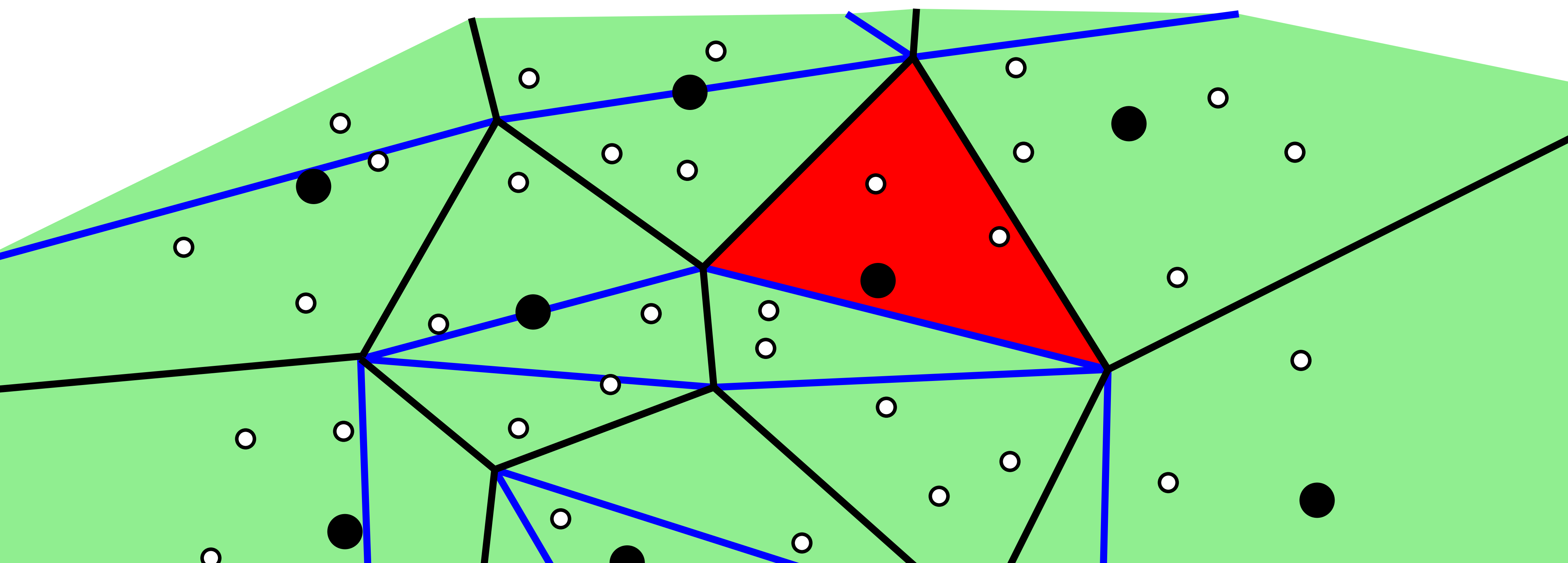

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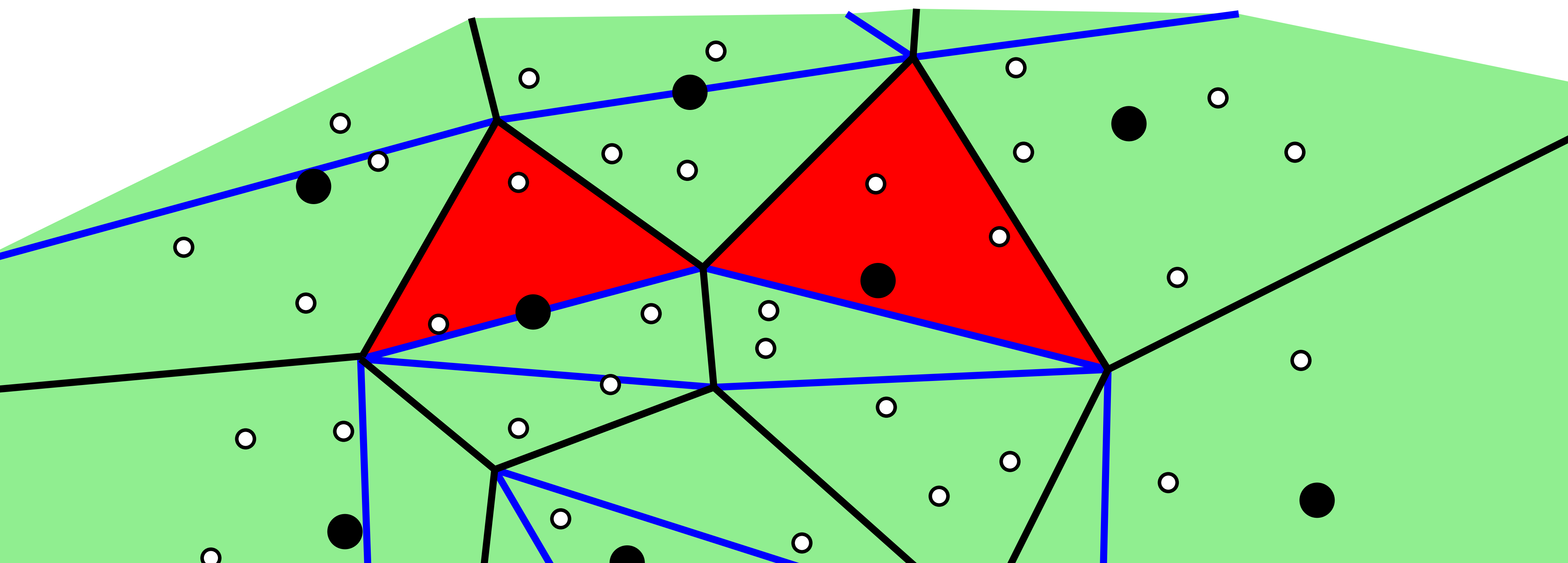

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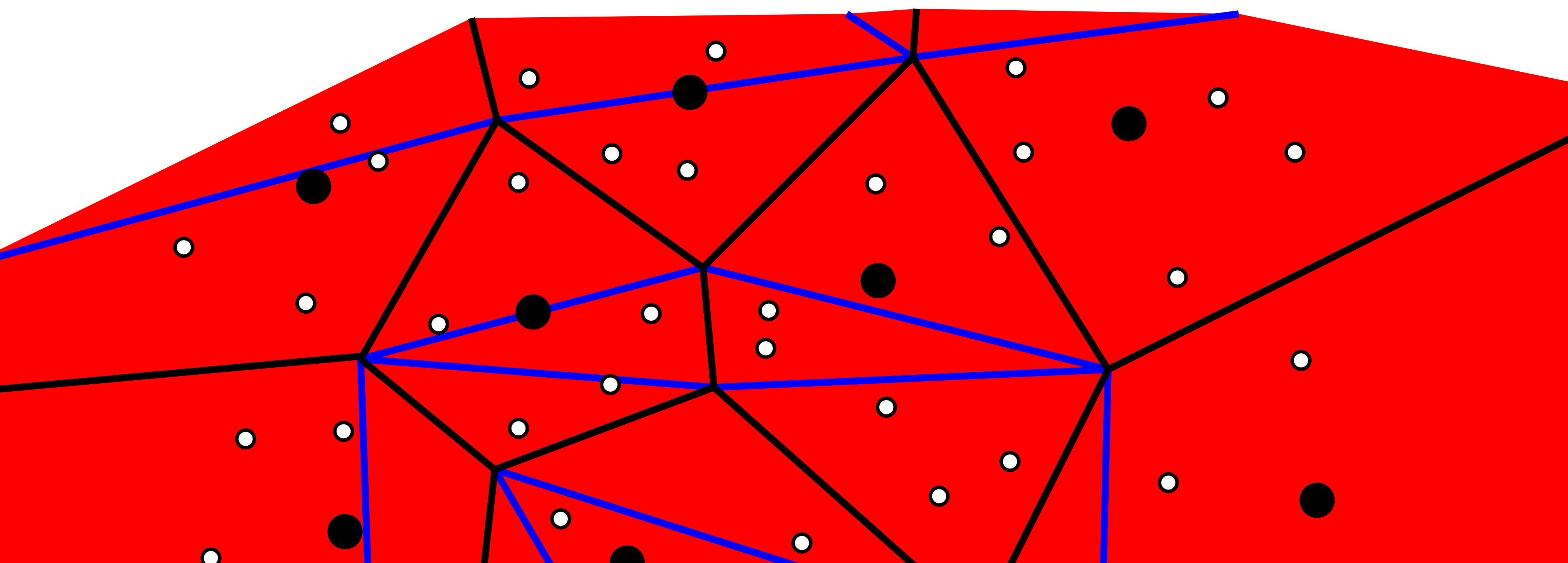

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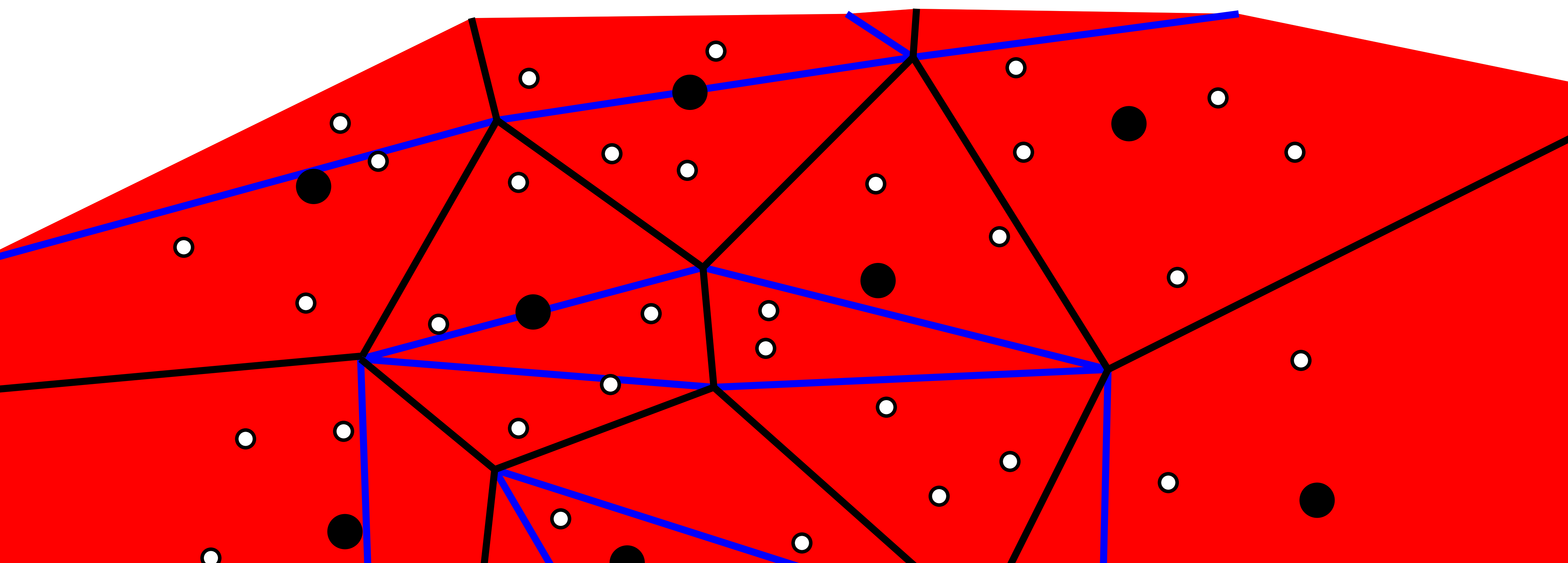

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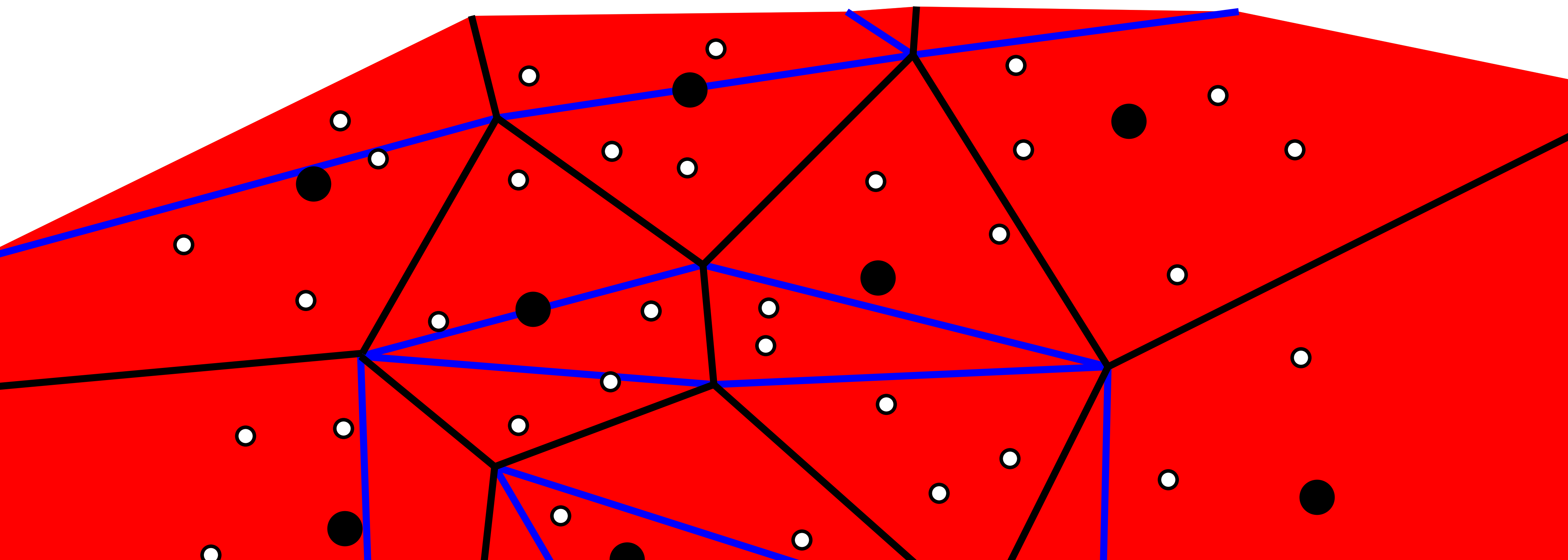

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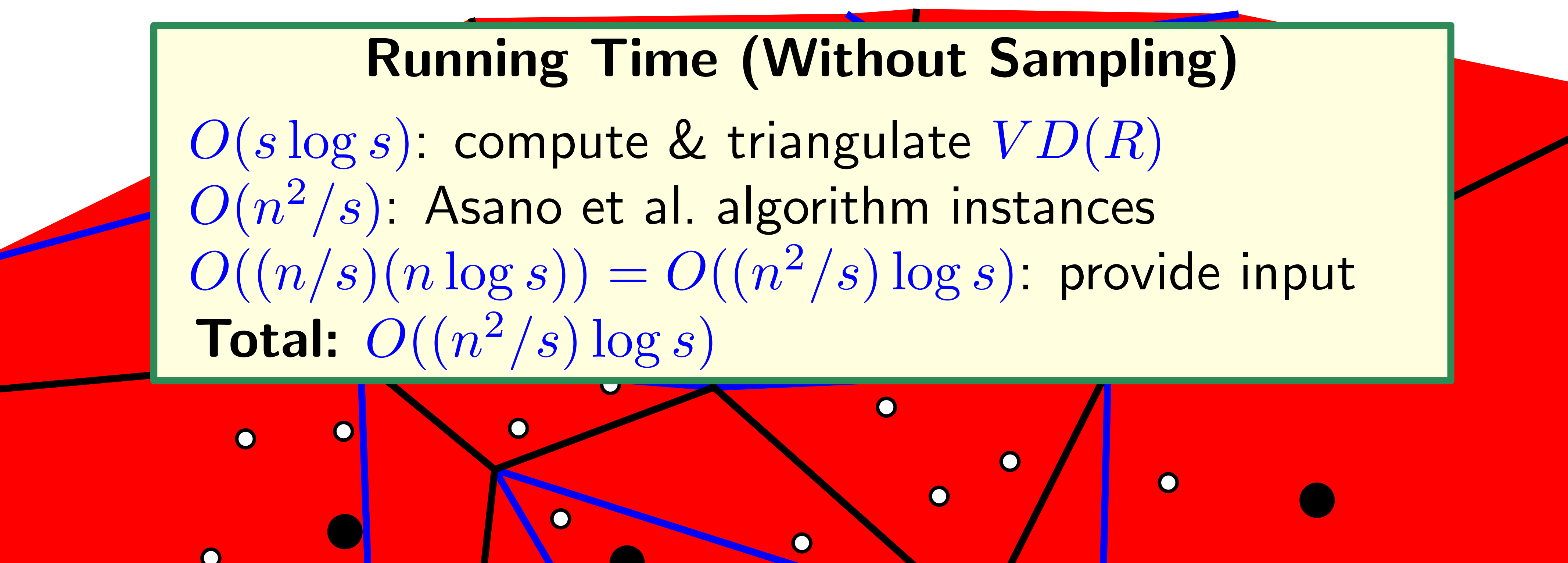

Running Time (Without Sampling)

$O(s \log s)$: compute & triangulate $VD(R)$

$O(n^2/s)$: Asano et al. algorithm instances

$O((n/s)(n \log s)) = O((n^2/s) \log s)$: provide input

Total: $O((n^2/s) \log s)$

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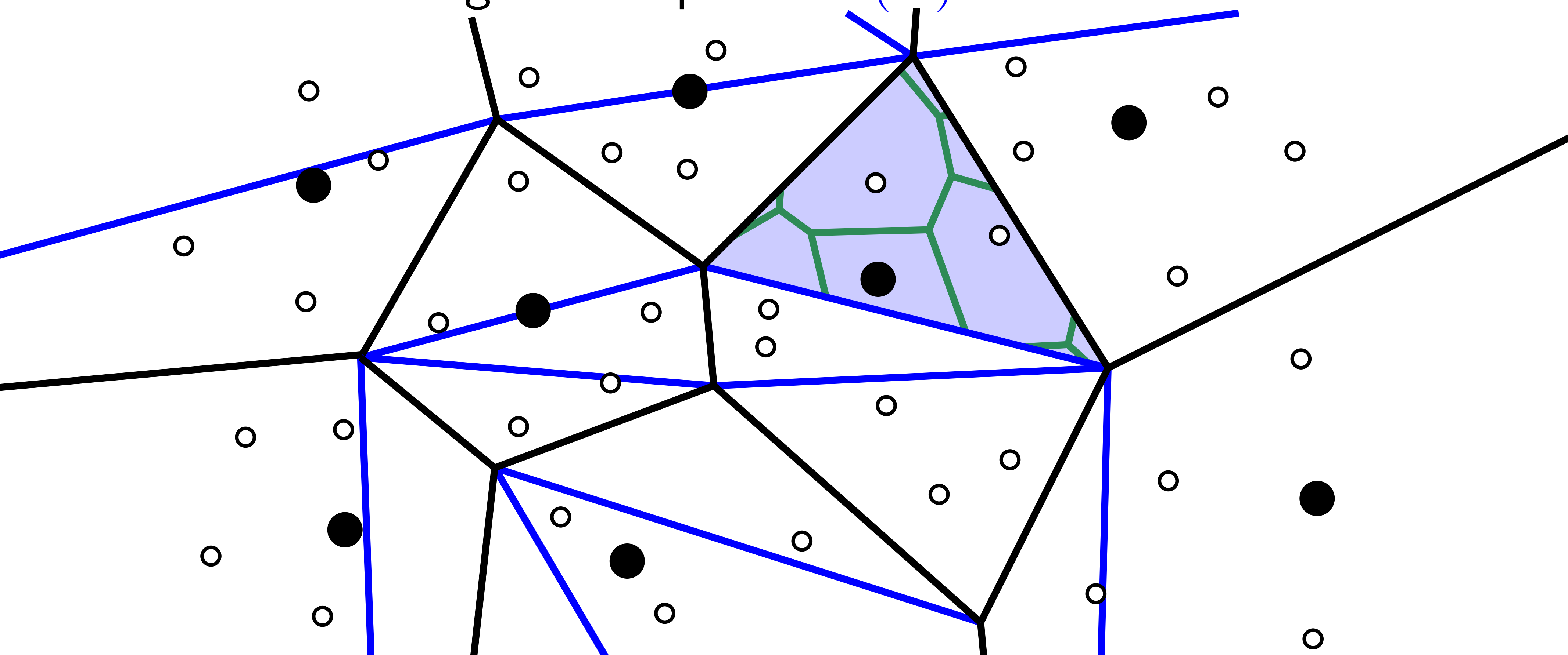

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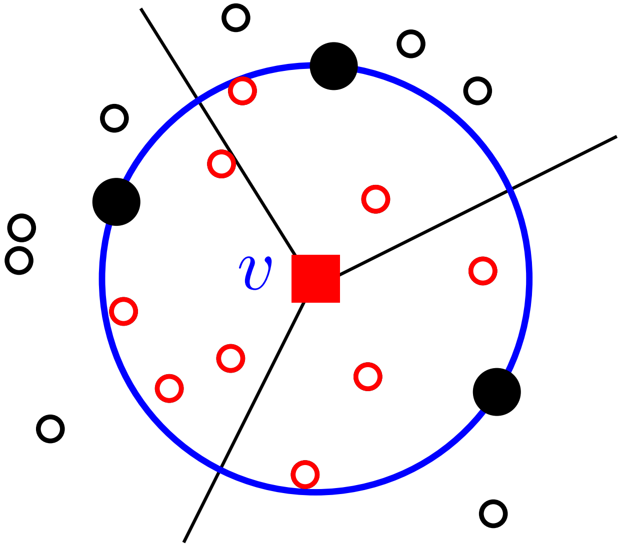
$$t_v = \frac{|B_v|}{n/s}$$



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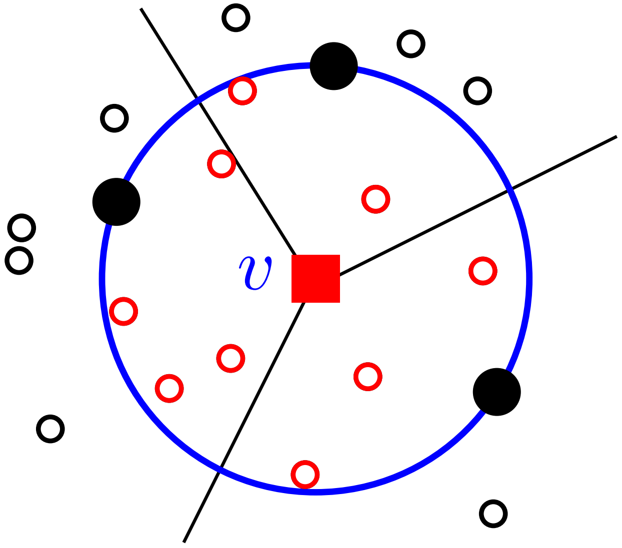

- counter per vertex $v \in \mathrm{VD}(R)$
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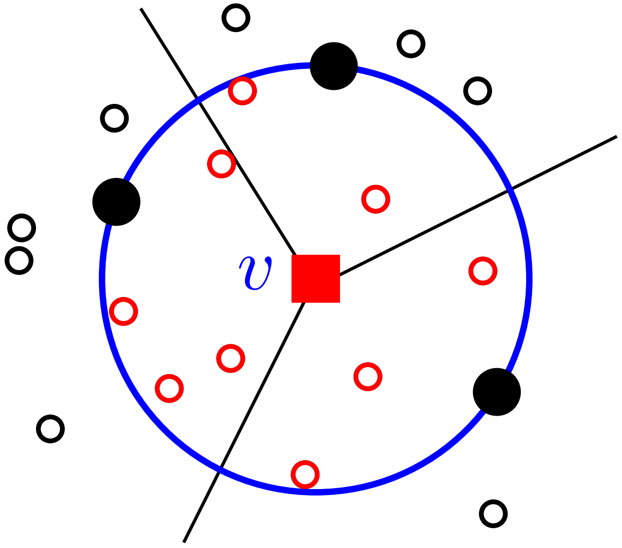

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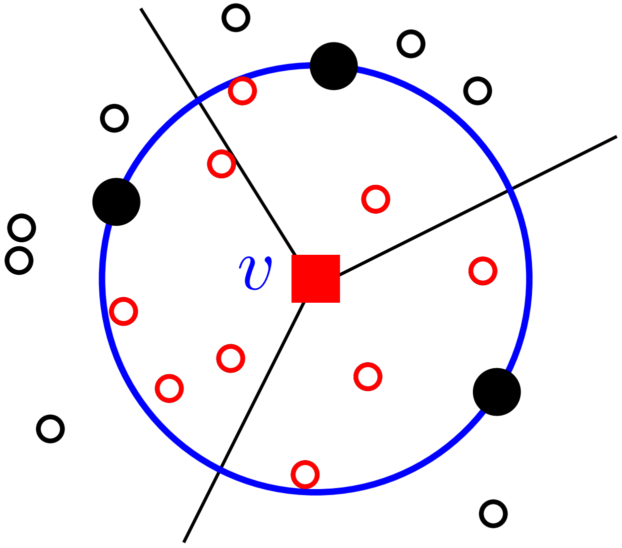

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- For vertices v with large excess:
  - sample additional points from B_v

Sampling from Conflict Sets

- For vertices v with large excess:
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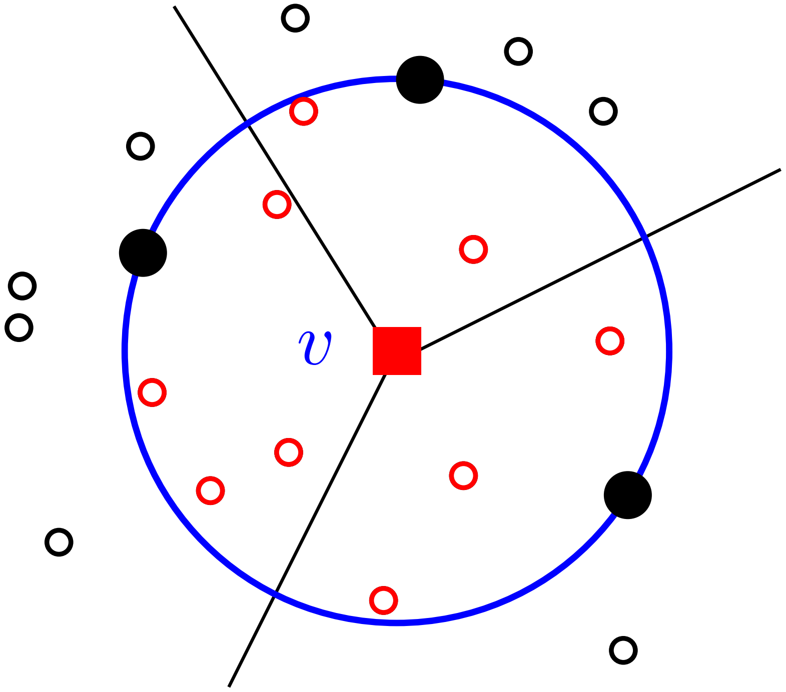

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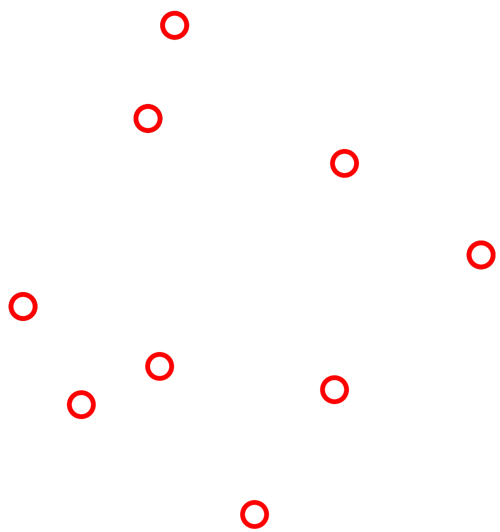

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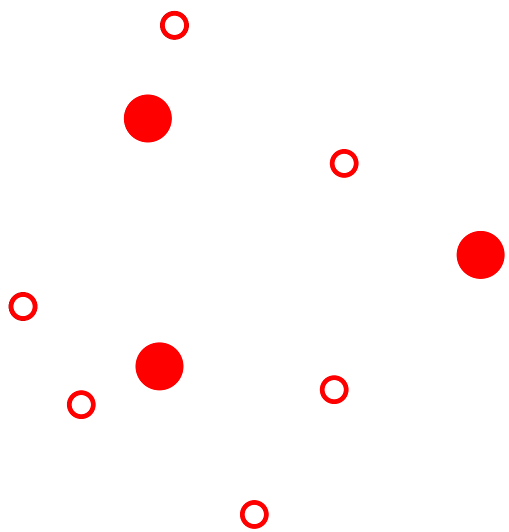
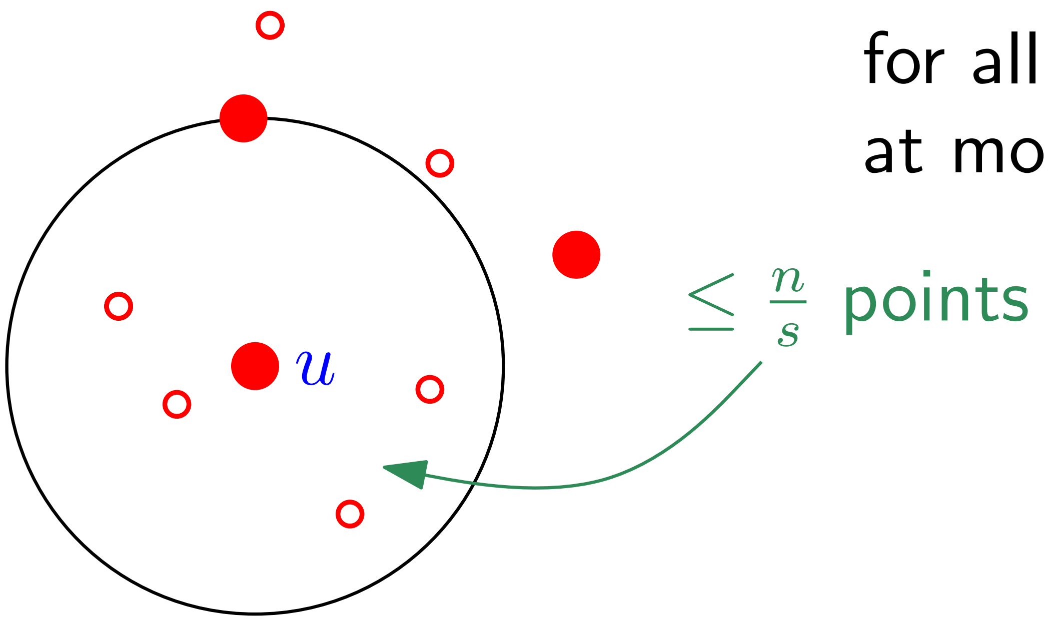


Sampling from Conflict Sets

- For vertices v with large excess: $\rightarrow t_v \geq 2$
  - sample additional $O(t_v \log t_v)$ points R_v from B_v

Call a sample R_v *good* iff for all $u \in R_v$: B_u contains at most $\frac{n}{s}$ points from B_v



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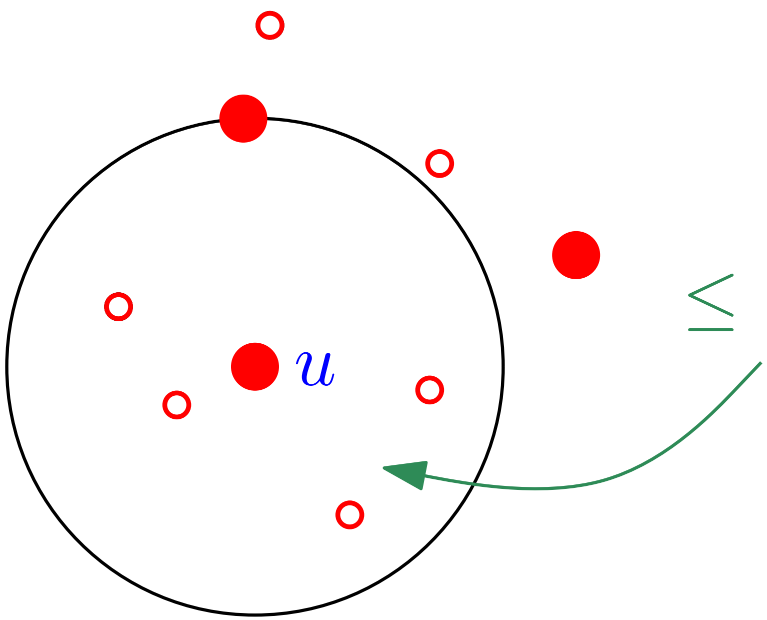

Lemma. Let $R' = R \cup \{R_v \mid t_v \geq 2,\ R_v \text{ is a good sample}\}$. Then, for all triangles Δ in the triangulation of $\mathrm{VD}(R')$ we have $B_\Delta = O(n/s)$.

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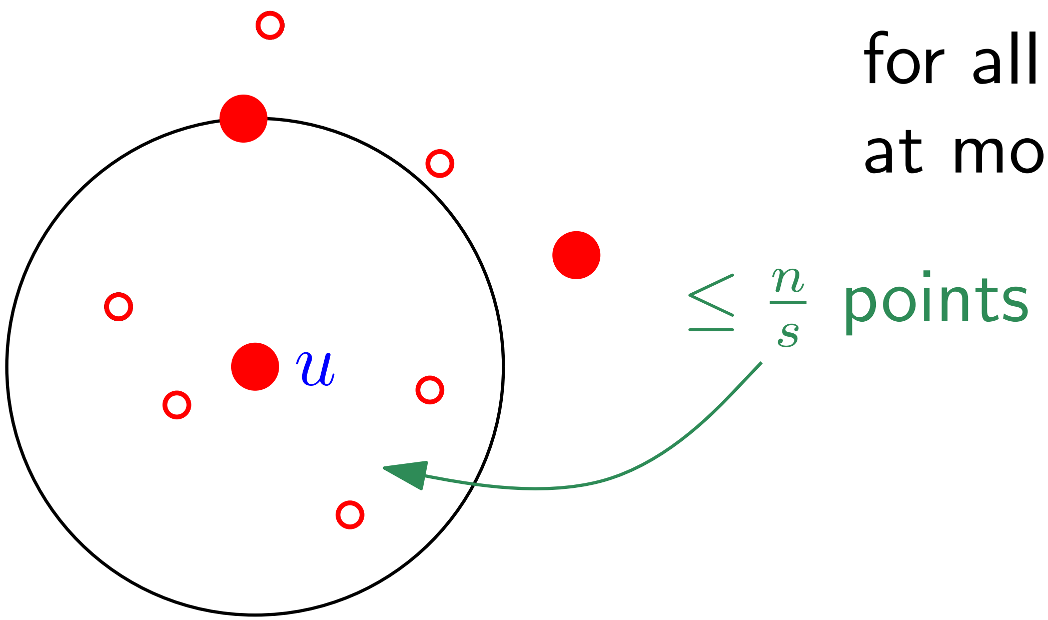

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Lemma. $\mathbf{E}[\sum_{v \in \mathrm{VD}(R)} |R_v|] = O(s)$

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Expected #Rounds: $O(\log^* s)$
One Round: Sampling & Checking Samples $O(n \log s)$
Total: $O(n \log s \log^* s)$

Putting it together

Phase I: Computing R': $O(n \log s \log^* s)$ expected time

Phase II: Computing $VD(P)$: $O((n^2/s) \log s)$ time

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Theorem. Reporting Voronoi diagrams of a set of n points in the plane can be done in $O((n^2/s) \log s + n \log s \log^* s)$ expected time using $O(s)$ space.

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Theorem. Reporting Voronoi diagrams of a set of n points in the plane can be done in $O((n^2/s) \log s + n \log s \log^* s)$ expected time using $O(s)$ space.

Open Problem: Can we do the same in worst-case time?