Geodesic Flow on Polyhedral Surfaces

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Abstract. On a curved surface the front of a point wave evolves in concentric circles which start to overlap and branch after a certain time. This evolution is described by the geodesic flow and helps us to understand the geometry of surfaces. In this paper we compute the evolution of distance circles on polyhedral surfaces and develop a method to visualize the set of circles, their overlapping, branching, and their temporal evolution simultaneously. We consider the evolution as an interfering wave on the surface, and extend isometric texture maps to efficiently handle the branching and overlapping of the wave.

1 Introduction

The study of geodesics and their behaviour under variations helps us to understand the geometry of curved surfaces. In this paper we try to compute and visualize aspects of the geodesic flow and related terms like injectivity radius and conjugate points, and extend these notions to polyhedral surfaces. By avoiding a formal definition right now some of these terms can be immediately related with the evolution of a wave front. For example, consider the front of a point wave on a surface evolving with unit speed. Then the time of the first hit upon itself is equal to the injectivity radius, unless a previous branching of the front occurs – the branch point is called conjugate point.

We solve two problems: First, to compute the evolution of the front of a point wave on a polyhedral surface. At each time the front is a topological circle on the surface, which may overlap and have singular points resulting from previous branchings of the front. In a numerical step, each point of the front is moved a constant distance in orthogonal direction to the curve, i.e. a constant distance along the geodesic normal to the curve in this point. In section 3 we employ the concept of straightest geodesics on polyhedral surfaces to give a thorough definition of the evolution on polyhedral surface and describe its numerics.

The second problem is to visualize the evolution consisting of a set of concentric, overlapping, and branched circles on a curved surface. Directly drawing such circles as curves would lead to overpainting of previous circles, due to the overlapping, unless transparency is added. As indicated above, the picture of a point wave allows a better visualization of many geometric characteristics as easily perceived phenomena, which includes the overlapping and the temporal evolution. For the visualization of the evolving wave we use isometric texture maps to avoid metric distortions between texture space and the surface, and we

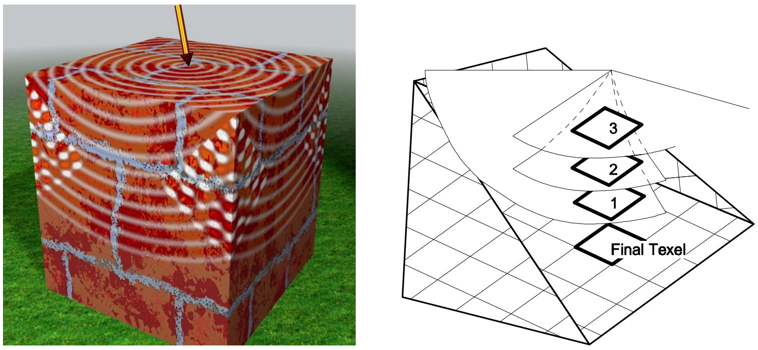

Fig. 1. Point waves on surfaces develop singularities at so-called conjugate points where the wave branches. Right figure shows a stack of abstract texels of the branched texture map, each corresponding to one layer of the wave.

extend these maps to include the multiple covering of some parts of the surface by different layers of the wave. We define branched texture maps which combine the notion of global texture maps that cover the whole surface, and local texture maps, which cover certain polygons. Isometric texture maps on surfaces are used e.g. in the line integral convolution technique in flow visualization [2].

The numerics and visualization ideas in this paper easily extend to other applications besides the geodesic flow. For example, the evolution and interference of other wave fronts over flat or curved surfaces. More abstract even the visualization of a homotopy of a curve, i.e. a one-parameter deformation, may be visualized using the interpretation as an evolving wave front. Waves have been studied in computer graphics from different aspects. In the animation [5] Max used bump mapping to perturb the surface of water for simulating waves viewed from a distance. Fournier and Reeves [4] explicitly model waves using parametric surfaces which allow simulation of detail structure such as waves curling over. The present paper describes ideas used in the video Geodesics and Waves [7].

2 Circles on Surfaces and the Geodesic Flow

A *point wave* in the euclidean plane starts at an origin p and evolves in concentric circles around p. In a particle model all particles of the wave front move with constant unit speed along radial rays away from the origin p if we neglect surface tension. At each time t the outer wave front forms a distance circle $\sigma(t)$ with center p and radius t.

For the construction of concentric circles on curved surfaces we use a similar picture. Particles of a wave front move along geodesic rays emanating from the origin p, therefore the circle at radius t consists of all points at distance t along a geodesic ray from p. It is one purpose of this paper to compute, study, and visualize such distance circles. We start with a review of geodesic curves, their properties, and their extension to polyhedral surfaces.

2.1 Evolution of Distance Circles and Point Waves

Geodesic curves on smooth surfaces are characterized by two equivalent properties, either as locally shortest curves or as straightest curves. This equivalence fails on polyhedral surfaces and leads to different concepts of locally shortest [1][3] and straightest geodesics [6].

Definition 1. *On a smooth surface M a curve γ is a* geodesic curve *if it is not curved within the surface. Formally, a smooth curve $\gamma : [a, b] \to M$ with tangent vector γ', $|\gamma'| = 1$, and surface normal N is* geodesic *if γ'' is parallel to N.*

Since this definition is equivalent to a 2nd-order ordinary differential equation, geodesics are uniquely determined by an initial point and an initial direction. This property allows particles with an initial impulse to move on surfaces along geodesics if there is no tangential acceleration.

Geodesics carry information about the underlying geometry of the surface. We briefly mention two terms, cut locus and conjugate points, since both have a strong relation to distance circles and point waves.

2.2 Cut Locus and Conjugate Points

The set of geodesics emanating from a given point p is conveniently described by the exponential map.

Definition 2. *The* exponential map *at a point p on a smooth surface M associates to each tangent vector $vinthetangentspaceT_pM$ a point on a geodesic γ through $\gamma(0) = p$ with initial direction $\gamma'(0) = v$ as follows:*

$$\begin{array}{c} \exp_p : T_pM \to M \\ \exp_p(v) = \gamma(1) \end{array} . \tag{1}$$

Here $\gamma(1)$ is a point on γ at distance $|v|$ from p since γ runs at speed $|v|$.

The exponential map maps small circles around 0 in the tangent space T_pM to *distance circles* around p on M, i.e. to circular curves on M where all points have the same distance to p. For a given vector $v \in T_pM$ the radial lines $rv \subset T_pM$, $r \in \Re$, are mapped isometrically to a geodesic ray γ.

In this formalism it is straight forward It is now easy to describe distance circles around a point p on a surface M. Let $V(0) \subset T_pM$ be a small neighbourhood of $0 \in T_pM$,then its image under the exponential map is a neighbourhood $U(p) := \exp_p V(0)$ of p. Using polar coordinates (r, φ) in $V(0)$ then each vector $v := r \cdot (\cos\varphi, \sin\varphi)$ is uniquely determined by its coordinates (r, φ). Its image under the exponential map given by $\gamma_\varphi(r) := \exp_p(r \cdot (\cos\varphi, \sin\varphi))$ induces a polar coordinate system on $U(p)$.

Using the above definitions, the particle model of a wave front in the plane extends immediately to curved surfaces M. The particles of a wave front start at $p \in M$ and move with constant speed along geodesics rays emanating at p. If

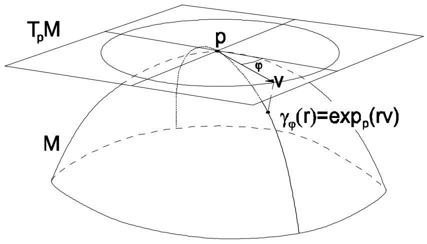

Fig. 2. Exponential map of geodesics emanating at a point.

we normalize the speed to 1 then the wave front at a time t is a distance circle δ_t with radius t given by

$$\begin{aligned} \delta_t &: [0, 2\pi] \rightarrow M \\ \delta_t(\varphi) &:= \gamma_\varphi(t) \end{aligned} \tag{2}$$

In contrast to the euclidean case, the wave front on a curved surface will usually self intersect after some time t_0. There are two possible reasons for the intersection. First of all, if the surface is not simply connected and has a handle like a torus, then for each point p there exist two emanating geodesic rays, γ_{φ_1} and γ_{φ_2}, with $\varphi_1 \neq \varphi_2$ and a time t_0 such that $\gamma_{\varphi_1}(t_0) = \gamma_{\varphi_2}(t_0) =: q$ (which go around a handle of the surface). At q the wave front hits upon itself and starts to interfere. The time of the first hit $t_0 = \min_q dist(p, q)$ is called the *injectivity radius* at p. A second type of intersection occurs at so-called *conjugate points* of p. At conjugate points $q = \gamma_{\varphi_0}(r_0)$ the differential $\nabla \exp_p$ does not have maximal rank, i.e. $\partial/\partial\varphi \exp_p(r_0, \varphi_0) = 0$. Here, the wave front branches and nearby geodesics intersect shortly behind the conjugate point. The branching occurs in the form of a swallow's tail, see figures 3 and 7. For $t > t_0$the polar coordinates fail to be a coordinate chart.

2.3 Review of Straightest Geodesics on Polyhedral Surfaces

Straightest geodesics are introduced in [6] to solve the initial value problem for geodesics on polyhedral surfaces. Since this property is essential for tracing particles we recall the basic definition:

Definition 3. *Let M be a polyhedral surface. A polygonal curve γ on M is a* straightest geodesic *if for each point $p \in \gamma$ the left and right curve angles θ_l and θ_r at p are equal, see figure 8.*

A straightest geodesic in the interior of a face is locally a straight line, and across an edge it has equal angles on opposite sides. The definition of straightest geodesics on faces and through edges is identical to the concept of shortest geodesics, but at vertices the concepts differ.

The most important property of straightest geodesics is the unique solvability of the initial value property which is not available for shortest geodesics.

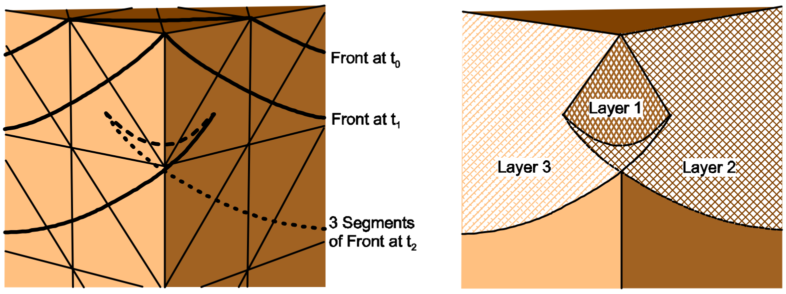

Fig. 3. Front of a point wave branches at conjugates points in the form of a swallow's tail. Behind the vertex of a cube the wave splits in two layers, and a third new layer is generated at the conjugate point. All three layers start to interfere.

Theorem 1 (Discrete Initial Value Problem). *Let M be a polyhedral surface and $p \in M$ a point with polyhedral tangent vector $v \in T_pM$. Then there exists a unique straightest geodesic γ with*

$$\begin{aligned} \gamma(0) &= p \\ \gamma'(0) &= v, \end{aligned} \tag{3}$$

and the geodesic extends to the boundary of M.

3 Computing Discrete Distance Circles

For the computation of distance circles at a point p on a polyhedral surface we start with a topological polygonal circle $\sigma(0)$ such that all of its vertices lie at p. Each vertex $q \in \sigma(0)$ has a unit tangent vector associated to it, and therefore the circle $\sigma(0)$ at time $t_0 = 0$ is completely described by a set of pairs $(q_{i,0}, v_{i,0})\,, i = 1, .., n$. For the numerics, it is essential to distribute the tangent vectors equally spaced in angular direction since they determine the geodesic along which the particles $q_{i,0}$ will move.

In the numerical iteration step from time t_j to t_{j+1}, the circle $\sigma(t_{j+1})$ is obtained by computing the set of vertices and tangent vectors

$$\begin{aligned} q_{i,j+1} &= \gamma_{(q_{i,j}, v_{i,j})}(1) \\ v_{i,j+1} &= \dot\gamma_{(q_{i,j}, v_{i,j})}(1) \end{aligned}, i = 1, .., n, \tag{4}$$

where $\gamma_{(q_{i,j}, v_{i,j})}$ is the straightest geodesic starting at $\gamma_{(q_{i,j}, v_{i,j})}(0) = q_{i,j}$ with initial direction $\dot\gamma_{(q_{i,j}, v_{i,j})}(0) = v_{i,j}$. (Compare with figure 4.) It should be noted that $\gamma_{(q_{i,j}, v_{i,j})}(0) = \gamma_{(p, v_{i,0})}(t_j)$, i.e. for fixed j all points $q_{i,j}$ lie on a distance circle with distance t_j to p.

Equation 4 is essentially the computation of a segment of a straightest geodesic for all vertices on the outer circle $\sigma(t_j)$. On the other hand, the distance

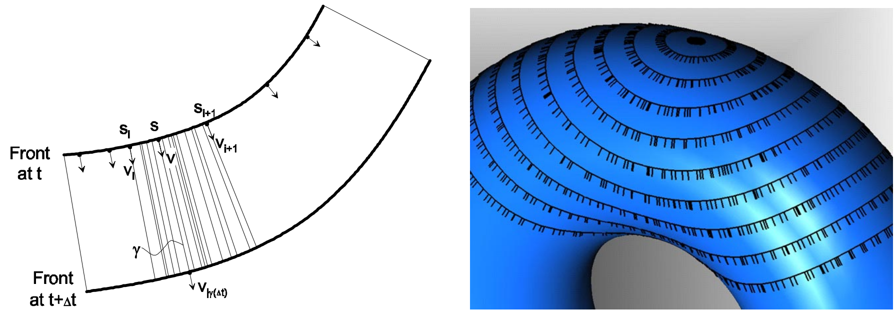

Fig. 4. Set of distance circles with direction of movement (right). Left, the front at time $t + \Delta t$ is generated from the front at time t by computing geodesics with length Δt.

between adjacent vertices on the same circle grows exponentially with the radius. Therefore each timestep includes a refinement and coarsening step to maintain nearly constant distance between adjacent points on each circle. For the insertion of new vertices, say between $q_{i,j}$ and $q_{i+1,j}$, we connect both points by a geodesic segment and insert a new vertex on this geodesic. In fact, for a fixed t_j, the circle $\sigma(t_j)$ is piecewise geodesic - a natural generalization of piecewise linear.

When a curve reaches a conjugate point it starts to form a swallow's tail with sharp edges. This does not irritate the algorithm since each vertex on the curve still has a vector attached which uniquely determines its further movement.

We remark that on a polyhedral surface each positively curved vertex is a conjugate point for all points in a small neighborhood. The resulting branching is studied on the cube in figures 3 and 9. When approximating a smooth geometry with a polyhedral surface, we suppress this type of local branching related to the discretization in favor of the global branching related to the shape of the smooth surface by using a reasonably fine mesh to distinguish between both types of branching.

A direct visualization of the set of circles gives reasonable results only for a small number of circles, see figure 4. In the following section we interpret the set of concentric circles as an evolving wave and use a resolution independent visualization based on texture map techniques.

4 Branched Coverings and Textures

4.1 Isometric Texture Maps

Texture maps of a 2-dimensional texture domain onto a general surface are faced with the problem that the texture images are metrically distorted. These principle difficulties can be avoided in the case of piecewise linear triangulated surfaces where one can choose the texture triangles isometric (up to scaling) to

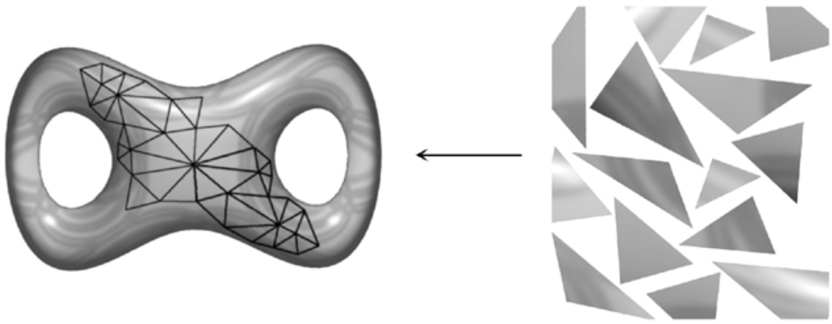

Fig. 5. Locally isometric texture maps allow 2d-texturing of arbitrary surfaces without distortion effects. Corresponding triangles in the texture domain and on the surface are required to be similar up to scaling.

their corresponding surface triangles. Then each texel is isometric to its image on the surface and no distortion of the texture image occurs. Additionally, this concept allows texture maps on arbitrary triangulated surfaces, as shown in figure 5. It is implemented in animation systems like Softimage and employed e.g. in the context of LIC [2].

A major problem of this method arises from the rasterization of the texture domain since adjacent triangles on the surfaces are not adjacent in texture space. The texels along the common edge must be synchronized to avoid aliasing artefact. The use of bilinear texture interpolation even requires synchronization of texels lying outside the triangle as indicated in figure 6. An extrapolation scheme would introduce new discontinuities, therefore we prefer the following direct scheme: let T be a triangle on the polyhedral surface and ϕ_T the isometric map from T to the texture domain. If the point $p \in T$ is hit and assigned a color c, then the corresponding texel $\tau_T(p)$ in $\phi_T(T)$ is assigned the new value

$$F(\tau_T(p)) := \frac{F(\tau_T(p)) + c(p)}{\#Hits(\tau_T(p)) + 1}.$$

If $\phi_T(p)$ is close to the boundary of $\phi_T(T)$ then we color the texture image of the adjacent triangle T_1 in the corresponding texel $\tau_{T_1}(p)$ too. Let (b_1, b_2, b_3) be the barycentric coordinates of p in the triangle $T = (v_1, v_2, v_3)$. The position $\phi_{T_1}(p)$ and the texel $\tau_{T_1}(p)$ are easily computed using the barycentric coordiantes above and the triangle $(\phi_{T_1}(v_1), \phi_{T_1}(v_2), \phi_{T_1}(v_3))$:

$$\phi_{T_1}(p) = \sum_{j=1}^{3} b_j \phi_{T_1}(v_j), \tag{5}$$

see figure 6. To assign multiple texels efficiently we calculate all occuring values $\phi_T(v)$ in a preprocessing step and later only use equation 5.

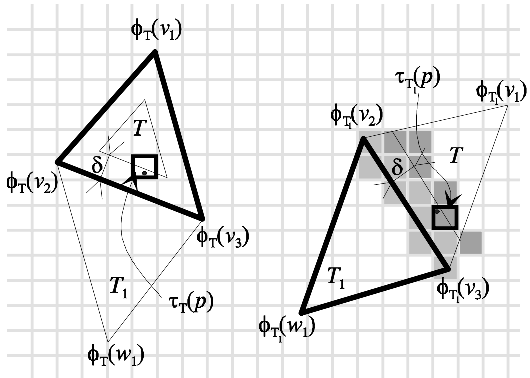

Fig. 6. Texel assignment for isometric texture maps.

4.2 Branched Texture Maps

Two-dimensional textures on surfaces are usually given as a texture map from a two-dimensional image onto the surface. One distinguishes between global texture maps covering the complete surface and local texture maps covering subsets of the surface. A surface may have multiple textures of each kind. During the rendering process, the final texture of a surface is computed by blending the textures associated to a point. The application of local textures requires the explicit specification of a domain on a surface where the local texture shall be applied, and it is therefore conceptually different from the use of global texture maps.

The approach of *branched texture maps* combines global and local textures in one concept and avoids the specification of subdomains for local textures in terms of regions on the surface. We start by covering the complete surface with a set of base texels, e.g. such a covering may be an isometric texture map as discussed in the previous section. At rendering time, these texels will contain the final texture. We associate to each base texel a stack of abstract texels. Each element of the stack corresponds to a texture layer covering a local region of the surface, and two different entries correspond to two different layers. For example, a surface which has two global textures associated would have a stack of constant height 2 at every point on the surface, and in the case of one global and one local texture the height would change between 1 and 2.

In contrast to the use of global and local texture maps, we do not allocate a new global or local texture image for each new layer. Instead whenever the wave fronts reaches a point on the surface, the corresponding texel stack is adjusted only at this base texel if necessary. Figure 1 shows some layers and the corresponding stack at a base texel.

5 Dynamic Computation of a Wave Texture

We divide the simulation of the wave in two major computational steps: first, the computation of the evolution of the wave front, which consists of geometric problems described in section 3 and leads to a static set of concentric circles, i.e. a set of wave fronts. Second, the simulation of the actual flow by animating the set of wave fronts. The animation is not done on the original set of fronts but on the level of textures. From the set of fronts, i.e. a set of geometric curves on the surface, we produce a single branched texture map which associates to each base texel of the surface a stack of abstract texels. The final animation of the wave is obtained from the single branched texture map by imposing a period function, but without any new computation of the wave evolution.

The separation of the numerical step and the use of branched texture maps makes the animation of moving waves a very cheap computation. To produce static pictures it is sufficient to color the base texels directly.

5.1 Generating the Branched Texture Map

The set of wave fronts $\sigma(t_j), j = 1, 2, ...$ computed in section 3 are a discretization of the exponential map from T_pM to M and is now translated into a branched texture map. First, we construct an isometric texture map covering M once with so-called base texels and associate to each base texel of M an empty stack of abstract texels, as shown in figure 1. The height of each stack is not known in advance and will vary from base texel to base texel. Now we analyze the set of curves $\sigma(t_j)$, and whenever the front has flowed over a base texel we add a new abstract texel to the stack of this base texel. Each abstract texel is essentially of the following structure:

`struct {float` α, t`; AbstractTexel next} AbstractTexel.`

The task of generating the stack is simple if the maximal distance between two successive curves is smaller than half the diameter of the smallest texel on the surface, which can be easily controlled for isometric texture maps. Here we sample each wavefront which carries the necessary information about angle α and time t. To obtain smooth results, it is essential to hit texels more frequently, say 4-8 times, and store average values $(\bar{\alpha}, \bar{t})$.

Each sheet of the wave hitting a given base texel corresponds to exactly one abstract texel above the base texel. A serious problem is the detection of the sheet corresponding to the current hit. For polyhedral geometries, we avoid this problem by letting the front detect branch points from the vertex curvature and split. This allows us to assign to each front segment a unique level identifying the sheet.

But, when approximating smooth surfaces, we need to distinguish between the branch points of the smooth geometry and those induced by polyhedral vertices. In this case, we let each base texel reconstruct the necessary information for each circle, resp. geodesic, from the time and angle of the current hit. Let m_T be the midpoint of a base texel T with edge size δ and let each abstract texel have stored average values $(\bar{\alpha}, \bar{t})$. Assume a circle $\sigma(\alpha, t)$ hits the base texel at a point q corresponding to an angle α, then q belongs to the same layer of the abstract texel if

$$dist_M(m_T, q) \approx \sqrt{\frac{d}{d\alpha}\sigma(\alpha,t)^2\,(\alpha-\bar{\alpha})^2 + (t-\bar{t})^2} \leq \frac{\delta}{\sqrt{2}}$$

for a threshold δ depending on the discretization of the flow.

In practice, we have a lower resolution in time direction and compute fewer wave fronts with distance of more than a few triangle diameters, and interpolate between successive fronts as indicated in the left image in figure 4.

6 Summary and Acknowledgments

As an application of the concept of straightest geodesics on polyhedral surfaces, we have computed concentric circles around a given point. Considering the set of circles as an evolving wave front by animating the radius offers a natural visualization approach of the circle homotopy, where geometric properties like injectivity radius and conjugate points are easily perceived as properties of the evolving wave. For the visualization we define branched texture maps, which extend isometric texture maps to (partially) multiply covered surfaces. The visualization methods presented in this paper may be applied to other problems such as propagation of general waves and homotopies of curves.

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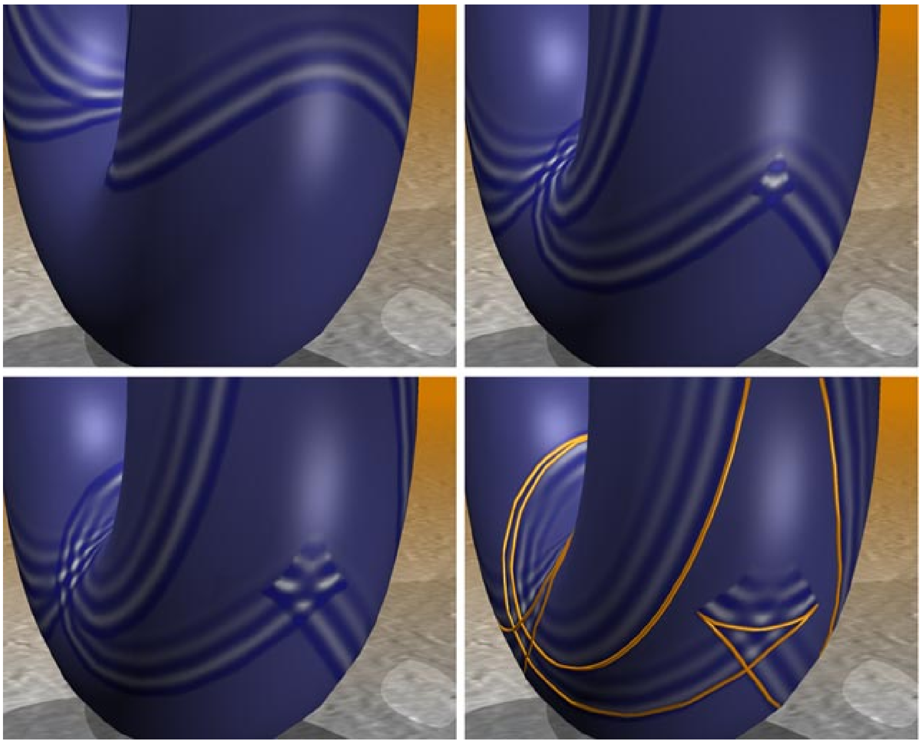

Fig. 7. A point wave on a polyhedrally approximated torus initiated at top branches at the conjugate point in the form of a swallow's tail. Visualization of the interference uses branched texture maps.

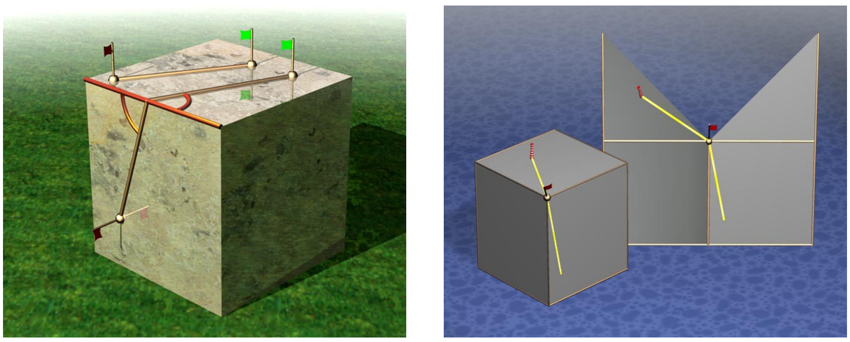

Fig. 8. Straightest geodesics are defined to have equal angle on both sides at each point. On planar faces they are a straight lines, and across edges they have equal angle on opposite sides. Straightest geodesics can be extended through polyhedral vertices, a property not available for shortest geodesics.

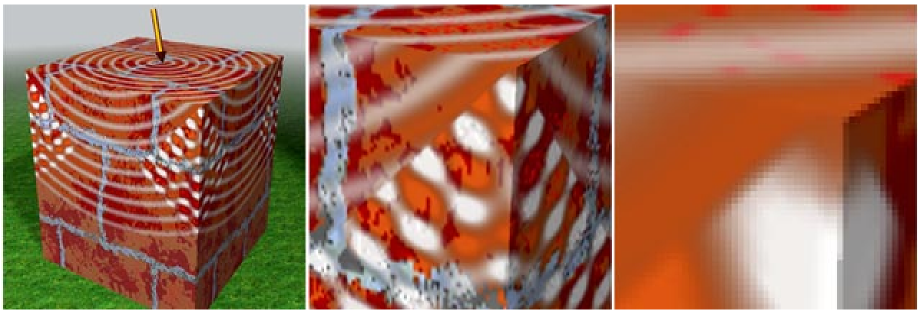

Fig. 9. A point wave branches at the vertices of a cube which are equivalents of conjugate points on smooth surfaces. The section behind a vertex is covered by three interfering texture layers while other parts are covered once. The zoom shows the triangles and texels on the surface.

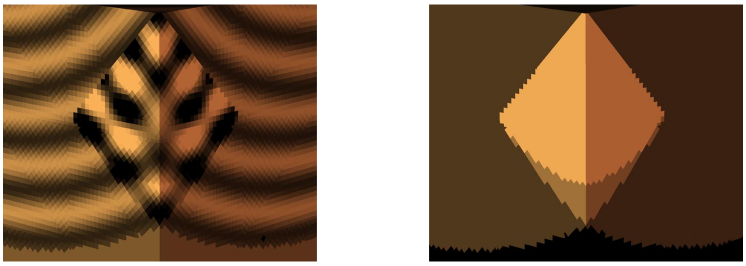

Fig. 10. Interference of a point wave at the vertex of a cube and texture layers. The left picture shows the interference behavior of the wave at the vertex of a cube. The wave is stored as a branched texture map, where to each point on the surface a stack of texels is associated. In the right picture the surface texels are colored according to the height of the texture stack at each point.

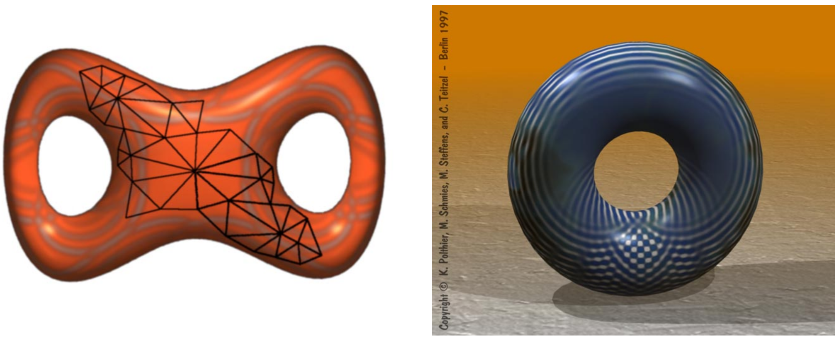

Fig. 11. Evolution of distance circles under the geodesic flow on the (highly discretized) polyhedral model of a pretzel and a torus.