On the Convergence of Metric and Geometric Properties of Polyhedral Surfaces *

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Abstract

We provide conditions for convergence of polyhedral surfaces and their discrete geometric properties to smooth surfaces embedded in Euclidean 3-space. Under the assumption of convergence of surfaces in Hausdorff distance, we show that convergence of the following properties are equivalent: surface normals, surface area, metric tensors, and Laplace-Beltrami operators. Additionally, we derive convergence of minimizing geodesics, mean curvature vectors, and solutions to the Dirichlet problem.

1 Introduction

Discrete differential geometry on polyhedral surfaces is concerned with discrete analogues of smooth differential geometric concepts. It is a theory *sui iuris* where discrete operators take the place of smooth ones, relying solely on the information inherent to the underlying polyhedral surface. Here we provide a proof of convergence of various discrete notions to their smooth counterparts. In particular, we ask: if a sequence of triangulated polyhedral surfaces, isometrically embedded into $\mathbb{R}^3$, converges to a smooth surface, *under what conditions do metric and geometric properties such as intrinsic distance, area, mean curvature, geodesics, and Laplace-Beltrami operators converge*, too? Pointwise convergence of surfaces is certainly not sufficient to obtain convergence of these properties; the Schwarz lantern [33] provides an informative counterexample to convergence of surface area. Convergence fails since the normal fields of the approximating sequence diverge. In Theorem 2 we prove: if a sequence of polyhedral surfaces $\{M_n\}$ converges to a smooth surface M in Hausdorff distance then the following conditions are equivalent:

i convergence of normal fields,

ii convergence of metric tensors,

iii convergence of area,

iv convergence of Laplace-Beltrami operators.

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The insight that *convergence of normals* is required in addition to pointwise convergence of surfaces in order to obtain convergence of area and intrinsic distance is not new. For example, Morvan and Thibert [23] recently gave quantitative estimates for the distortion of area and length in terms of deviations of normals and pointwise distance between surfaces. We extend these results to the above *equivalent conditions*.

The methodology of this paper is to adopt the point of view of global analysis in the sense of analyzing Riemannian manifolds by means of operators acting between function spaces. The Laplace-Beltrami operator, for example, is regarded as a bounded linear map

$$\Delta : \mathcal{H}_0^1(M) \longrightarrow \mathcal{H}^{-1}(M)$$

from the Sobolev space $\mathcal{H}_0^1(M)$ (the space of weakly differentiable functions on M vanishing along the boundary) to its dual space, $\mathcal{H}^{-1}(M)$. Convergence of these operators is treated in operator norm. Similarly, since the mean curvature vector of an isometrically embedded surface can be written as

$$H = \Delta I,$$

where $I : M \to \mathbb{R}^3$ denotes an isometric embedding, we treat convergence of mean curvature vectors in the sense of distributions, i.e. as elements of $\mathcal{H}^{-1}(M)$. This view is motivated by the observation that mean curvature vectors on polyhedral surfaces have *only distributional components* because ΔI vanishes in the interior of (flat) triangles. Let us briefly recall the meaning of convergence in the sense of distributions. Assume we were dealing with a sequence $\{M_n\}$ of C^2-surfaces converging to a C^2-surface M. Then mean curvature vectors would be continuous, and their convergence, $H_n \to H$ in $\mathcal{H}^{-1}(M)$, would imply

$$\int_M H_n \cdot \phi \longrightarrow \int_M H \cdot \phi \quad \text{for all fixed} \quad \phi \in C_0^1(M),$$

where each H_n is pulled back from M_n to the limit surface M. In other words, we obtain what could be called *convergence of integrated quantities*. This interpretation carries over to polyhedral surfaces (which are only of class $C^{0,1}$) in the sense that $\|H_n - H\|_{\mathcal{H}^{-1}} \to 0$ implies that

$$\langle H_n|\phi\rangle \longrightarrow \langle H|\phi\rangle \quad \text{for all fixed} \quad \phi \in \mathcal{H}_0^1(M),$$

where $\langle\cdot|\cdot\rangle$ is the dual pairing between $\mathcal{H}_0^1(M)$ and its dual, $\mathcal{H}^{-1}(M)$. Convergence in the sense of distributions is similar in spirit to *convergence in measure* [8]. It is an interesting problem to find the precise relationship between these approaches.

Let us emphasize two aspects of our approach.

i Laplace-Beltrami operators and mean curvature vectors are shown to converge in an appropriate *norm*. This does not imply convergence pointwise almost everywhere—we provide a counterexample to $\mathcal{L}^2$-convergence.

ii Estimates are made *explicit* in terms of pointwise distance, deviation of normals, curvature properties of the smooth limit surface, and (where appropriate) shapes of the triangles in the sequence of polyhedral surfaces.

Convergence of operators in norm has to be contrasted with recent work on pointwise convergence treated by means of asymptotic analysis (see e.g. Meek and Walton [20], Xu [38, 39], and references therein). In a sense, this work may be summarized by a negative result: if normals could be approximated to order $\mathcal{O}(h^2)$ then Laplacians and curvatures could be approximated to order $\mathcal{O}(h)$[1] (since curvatures correspond to normal derivatives); however, unless one imposes extra assumptions on the underlying discrete data, normals are in general only known to order $\mathcal{O}(h)$, so that *pointwise convergence of curvatures cannot in general be expected*. Here we show that convergence of surfaces in Hausdorff distance together with convergence of their normals in $\mathcal{L}^\infty$ suffices to ensure convergence of curvature operators in (an appropriately chosen) norm[2].

Several applications of convergence of polyhedral surfaces to a smooth limit surface in Hausdorff distance together with convergence of their normals are derived in Section 4. We show uniform convergence of geodesics on compact sets (Section 4.1), convergence of solutions to the Dirichlet problem (Section 4.2), and convergence of mean curvature vectors (Section 4.3). As mentioned before, we show convergence of mean curvature vectors in the sense of distributions (Theorem 6) and give a counterexample to $\mathcal{L}^2$-convergence.

Discrete minimal surfaces in the sense of [25] are polyhedral surfaces which are stationary points for the area functional within the class of piecewise linear surfaces having the same underlying simplicial complex, and the same piecewise linear boundary. An equivalent definition can be given in terms of finite elements. Let $S_{h,0} \subset \mathcal{H}_0^1$ be the space of continuous piecewise linear functions with zero boundary condition on a polyhedral surface. We define the *piecewise linear mean curvature vector*, $H_h \in (S_{h,0})^3$, by

$$(H_h, u_h)_{\mathcal{L}^2} = \langle H | u_h \rangle \quad \forall\, u_h \in S_{h,0}.$$

In other words, H_h is the piecewise linear *function* corresponding to the action of the mean curvature *functional* $H = \Delta I \in \mathcal{H}^{-1}$ on the finite element space $S_{h,0}$. In Section 4.3.2 we show that $\mathcal{L}^2$-*convergence may fail* for H_h. However, for the special case of *discrete minimal surfaces*, which corresponds to

$$H_h = 0,$$

we prove the following: if a sequence of discrete minimal surfaces converges to a smooth limit surface in Hausdorff distance such that their normal fields converge, too; then the limit surface is minimal in the classical sense (Theorem 7).

Although existence and regularity of minimal surfaces spanning a given boundary is a well-studied problem, it remains a challenge to explicitly construct minimal surfaces with prescribed boundary data. Pinkall and the second author [25] suggested an algorithm for numerically approximating area-minimizing polyhedral surfaces by sequentially solving the Dirichlet problem with respect to the metric of the current iterate. For the same purpose Dziuk [11] used a discretization of the mean curvature flow. Similarly, Ken Brakke's "Surface Evolver" [5] produces numerical approximations of area-minimizing surfaces.

[1] As usual, h denotes *mesh size* (for example, length of longest edge) of M_h.
[2] It appears that Gauss curvature cannot be treated in our sense.

Later, various examples of explicitly computable discrete minimal surfaces were discovered [16, 19, 26, 27, 28, 31]. However, it is an open problem whether one can find discrete minimal surfaces arbitrarily close to a given smooth (and possibly unstable) minimal surface. Dziuk and Hutchinson [12] give a positive answer to this problem for the case of minimal surfaces having the topology of a disk, and Pozzi [29] extends their result to annuli[3]. Related to the approximation of smooth minimal surfaces by discrete minimal ones is the problem of designing a converging refinement scheme of discrete minimal surfaces yielding a smooth minimal surface in the limit. The convergence result of Theorem 7 provides a step into this direction.

Große-Brauckmann and the second author [18] constructed examples of compact constant mean curvature (CMC) surfaces of low genus numerically, based on a discrete version of the conjugate surface construction [24]. It is an interesting question whether the convergence results of the current paper can help to prove that these numerical examples yield smooth CMC surfaces.

The Dirichlet problem is treated in Section 4.2. We prove that the solutions u_n to the Dirichlet problems $\Delta_n u_n = f$ on M_n converge to the solution u of the Dirichlet problem $\Delta u = f$ on M in $\mathcal{H}_0^1(M)$ (Theorem 4). Here Δ_n is the Laplace-Beltrami operator of the approximating polyhedral surface M_n, and Δ is the Laplacian of the smooth limit surface M. A slight extension of an argument of Dziuk [10] then shows convergence of the associated finite element discretization.

Related work. It is impossible to do justice to the various results concerning discretization of smooth differential geometry here. One central question of such a discrete theory is: what is the best way to describe discrete geometric objects? Certainly, a good discrete theory is one which preserves essential properties of the continuous theory. Interesting connections in this direction have recently been explored by Bobenko, Hoffmann, Mercat, Pinkall, Springborn, and Suris, see e.g. [3, 4, 21]. There are several other approaches towards such a discrete theory: Alexandrov [1] and Reshetnyak [30] developed the theory of manifolds of bounded curvature. Thurston [35] and Schramm [32] used circle packings to approximate holomorphic maps and proved a discrete Riemann mapping theorem. Federer [13] and Fu [14] developed geometric measure theory, Banchoff [2] studied discrete Morse theory, Stone [34] related global topology of PL-manifolds to their local geometry, Brehm and Kühnel [6] treated approximations of polyhedra by smooth surfaces, Cheeger, Müller and Schrader [7] employed Lipschitz-Killing curvatures, and Morvan and Cohen-Steiner [8, 22] recently used the normal cycle to construct a discrete shape operator.

[3] Recently Bobenko et al. [3] provided a different ('non-linear') view of discrete mean curvature. They show that their approach allows for finding discrete minimal surfaces arbitrarily close to smooth ones.

2 Approximating smooth surfaces

2.1 Polyhedral surfaces

By a *polyhedral surface* M_h we mean a geometric simplicial complex having the topology of a 2-manifold and consisting of flat Euclidean triangles which are isometrically glued along their common edges. For technical reasons we only consider finite triangulations as well as orientable surfaces in this article; we also request that $M_h \subset \mathbb{R}^3$ be isometrically embedded. If $\gamma : [a,b] \to M_h$ is a continuous curve, then the *length* of γ is the supremum over all partitions $Z = \{t_0 = a \leq t_1 \leq ... \leq t_n = b\}$ of $[a,b]$:

$$l(\gamma) = \sup_Z \sum_{i=1}^{n} d_{\mathbb{R}^3}(\gamma(t_{i-1}), \gamma(t_i)),$$

where $d_{\mathbb{R}^3}$ denotes the Euclidean distance of $\mathbb{R}^3$. Let p and q be two points in M_h. Then the distance $d(p,q)$ between p and q is defined as

$$d(p,q) := \inf_\gamma l(\gamma), \tag{1}$$

the infimum taken over all continuous curves $\gamma : [a,b] \to M_h$. Following Gromov [17], M_h equipped with this metric is called a *length space*. On individual triangles the length metric coincides with the induced flat metric from ambient $\mathbb{R}^3$. Across an edge of two adjacent triangles this metric is still flat since one can rotate those triangles about their common edge until they become coplanar. The situation changes at vertices where the metric exhibits *cone points*, cf. [36].

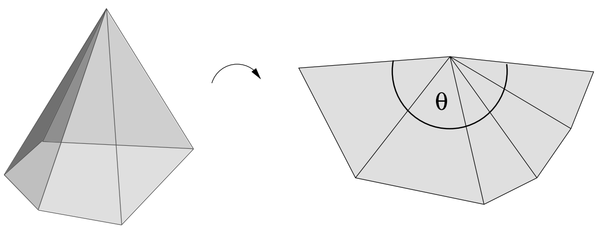

Figure 1: A neighborhood of a vertex with total vertex angle θ equipped with the length metric is isometric to a metric cone with cone angle θ.

Definition 1 (metric cone)**.** The set $C_\theta := \{(r,\varphi) |\, 0 \leq r;\ \varphi \in \mathbb{R}/\theta\mathbb{Z}\}/_\sim$ together with the (infinitesimal) metric

$$\mathrm{d}s = \sqrt{\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2} \tag{2}$$

is called a *metric cone* with cone angle θ. Here $(0,\varphi_1) \sim (0,\varphi_2)$ for any pair (φ_1,φ_2). The *cone point* is the coset consisting of all points $(0,\varphi) \in C_\theta$.

Henceforth we will refer to the cone metric induced by (2) as g_{M_h} in order to indicate its dependence on the polyhedral surface M_h. The metric g_{M_h} is the infinitesimal version of the length metric defined by (1).

2.2 Normal graph and shortest distance map

This section introduces the *shortest distance map* as an auxiliary tool for comparing a smooth surface to a polyhedral surface nearby. Considering this map for comparing two surfaces has been common practice [10, 22, 23].

Definition 2. Let $M \subset \mathbb{R}^3$ be a closed subset. The *medial axis* of M is the set of those points in $\mathbb{R}^3$ which do not have a unique closest neighbor in M. The *reach* of M is the distance of M to its medial axis.

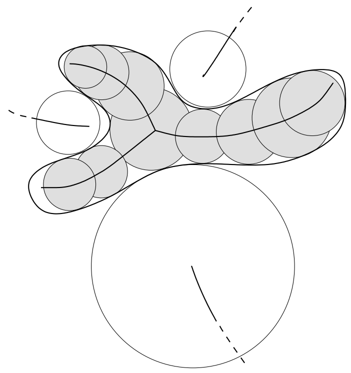

Figure 2: Illustration of several branches of the medial axis of a smooth shape.

If $M \subset \mathbb{R}^3$ is a smooth embedded surface (possibly with boundary) then its medial axis corresponds to the locus of centers of spheres touching M in at least two points without intersecting M (see Figure 2). In this case the reach of M is the infimum over the radii of such spheres. The reach of a smooth surface M is hence bounded above by the radii of osculating spheres:

$$\operatorname{reach}(M) \leq \inf_{p \in M} \frac{1}{|\kappa|_{max}(p)}, \tag{3}$$

where $|\kappa|_{max}(p)$ denotes the maximum absolute value of the normal curvatures at $p \in M$. Note that a compact and smoothly embedded surface M always has positive reach. For a general treatment of sets of *positive reach* we refer to the classical text of Federer [13].

Definition 3 (normal graph)**.** A polyhedral surface M_h is a *normal graph* over a smooth surface M if its distance to M is *strictly less* than the reach of M, and the map $\Phi : M \to M_h$ which takes $p \in M$ to the intersection point $\Phi(p) \in M_h$ of the normal line through p with the polyhedral surface M_h is a *bijection onto the image* up to, and including, the (possibly non-empty) boundary, ∂M.[4]

The *shortest distance map* Φ splits into a tangential and a normal component:

$$\Phi(p) = I(p) + \phi(p) \cdot N(p), \tag{4}$$

where $I : M \to \mathbb{R}^3$ denotes the embedding, N is the oriented normal of M, and ϕ is the scalar-valued (signed) distance function.

[4]Specifically, this implies that we require the image of M under Φ to be contained in M_h, i.e. $\Phi(M) \subset M_h$. However, if $\partial M \neq \emptyset$, we do not require that $\Phi(M) = M_h$.

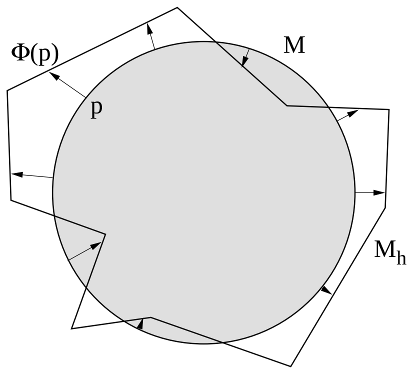

Figure 3: M_h is a normal graph over M. The map Φ takes $p \in M$ to the intersection of the normal line through p with the polyhedral surface M_h. The inverse Φ^{-1} thus realizes the shortest distance from $\Phi(p) \in M_h$ to M.

2.3 The metric distortion tensor

The shortest distance map Φ allows to pull back the polyhedral metric g_{M_h} on M_h to an (almost everywhere defined) metric g_A on the smooth surface M.

Definition 4 (metric distortion tensor)**.** There exists a symmetric positive definite 2×2 matrix field $A(p)$, $p \in M$, uniquely defined M-almost everywhere, such that

$$g_A(X, Y) = g(A(X), Y) = g_{M_h}(\,\mathrm{d}_M\Phi(X),\, \mathrm{d}_M\Phi(Y)) \quad \text{a.e.} \tag{5}$$

for all vector fields X, Y on the smooth surface (M, g).

The metric distortion tensor A is smooth on the pre-image of the interior of triangles of M_h. The next theorem shows that A only depends on the distance between M and M_h, the angles between their normals, and the curvatures of the smooth surface M. A similar result can be found in [23].

Theorem 1 (geometric splitting of metric distortion tensor)**.** *Let M_h be a polyhedral surface which is a normal graph over an embedded smooth surface M. Let N denote the normal field of M, and let N_h denote the pullback under Φ of the normal field of M_h. Then the metric distortion tensor A satisfies*

$$A = P \circ Q^{-1} \circ P \quad \textit{a.e.}, \tag{6}$$

a decomposition into symmetric positive definite 2×2 matrix fields P and Q on M which can be diagonalized (possibly in different ON-frames) such that

$$P = \begin{pmatrix} 1 - \phi \cdot \kappa_1 & 0 \\ 0 & 1 - \phi \cdot \kappa_2 \end{pmatrix}, \tag{7}$$

$$Q = \begin{pmatrix} \langle N, N_h \rangle^2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{8}$$

where κ_1 and κ_2 denote the principal curvatures of the smooth manifold M and ϕ is as in equation (4).

Remark 1. The matrix field P is pointwise a.e. *positive definite* by the assumption that M_h is in the reach of M since inequality (3) implies $1 - \phi \cdot \kappa_i > 0$.

Proof of Theorem 1. It suffices to consider a single triangle T of M_h. Denote by $\Psi = \Phi^{-1} : M_h \to M$ the inverse of the shortest distance map. For any map $f : M \to \mathbb{R}$ let $f_T = f \circ \Psi_{|T} : T \to \mathbb{R}$ denote its pullback to T. Then

$$\Psi_{|T} = I_{M_h} - \phi_T \cdot N_T.$$

Here I_{M_h} is the embedding of M_h into $\mathbb{R}^3$. Note that N_T stands for the pullback of the normal field N of M to the triangle T, rather than for the normal field to T. Let d denote the outer differential on T. Differentiating Ψ and using that $\mathrm{d}N_T = \mathrm{d}_M N \circ \mathrm{d}\Psi = -\mathbf{S} \circ \mathrm{d}\Psi$ yields

$$\mathrm{d}\Psi = (Id - \phi \cdot \mathbf{S})^{-1} \circ (Id - N_T \cdot \mathrm{d}\phi_T) : \quad \mathrm{T}T \to \mathrm{T}M,$$

where $\mathbf{S} = -\,\mathrm{d}_M N$ is the Weingarten operator on M. Setting

$$\begin{aligned} P &:= (Id - \phi \cdot \mathbf{S}) : \quad \mathrm{T}M \to \mathrm{T}M, \\ \tilde{Q} &:= (Id - N_T \cdot \mathrm{d}\phi_T) : \quad \mathrm{T}T \to \mathrm{T}M, \end{aligned}$$

we obtain $\mathrm{d}\Psi = P^{-1} \circ \tilde{Q}$ and hence $\mathrm{d}_M \Phi = \tilde{Q}^{-1} \circ P$. For each $p \in M$ we define a symmetric positive definite operator Q on $\mathrm{T}_p M$ by

$$\langle Q^{-1}(X), Y \rangle_{\mathbb{R}^3} = \langle \tilde{Q}^{-1}(X), \tilde{Q}^{-1}(Y) \rangle_{\mathbb{R}^3}.$$

The definition of the metric distortion tensor A and the symmetry of P yield

$$\begin{aligned} \langle A(X), Y \rangle_{\mathbb{R}^3} &= \langle\, \mathrm{d}_M \Phi(X),\, \mathrm{d}_M \Phi(Y) \rangle_{\mathbb{R}^3} \\ &= \langle P Q^{-1} P(X), Y \rangle_{\mathbb{R}^3}, \end{aligned}$$

and a straightforward calculation delivers that P and Q can be diagonalized as claimed. ☐

Corollary 1 (area distortion)**.** *Under the assumptions of Theorem 1, the volume elements of M and M_h satisfy*

$$\frac{\mathrm{d}vol_{M_h}}{\mathrm{d}vol_M} = (\det A)^{1/2} = \frac{1 + \phi^2\,\mathrm{K} - \phi\,\mathrm{H}}{\langle N, N_h \rangle} \quad a.e., \tag{9}$$

where K *denotes the Gauss curvature, and* H *denotes the scalar mean curvature of M.*

Proof. Equation (9) follows immediately from the explicit representation of the distortion tensor A in Theorem 1, and by using that $\mathrm{K} = \kappa_1 \cdot \kappa_2$ as well as $\mathrm{H} = \kappa_1 + \kappa_2$. ☐

Corollary 2 (length distortion)**.** *The infinitesimal distortion of length satisfies*

$$\min_i (1 - \phi \cdot \kappa_i) \le \frac{\mathrm{d}l_{M_h}}{\mathrm{d}l_M} \le \frac{max_i(1 - \phi \cdot \kappa_i)}{\langle N, N_h \rangle} \quad a.e. \tag{10}$$

Proof. This follows from bounding the smallest and largest eigenvalue of A. ☐

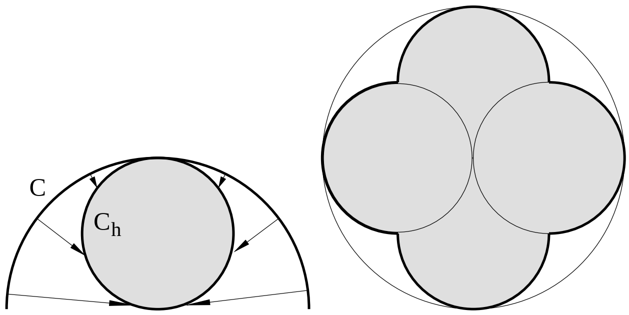

Figure 4: The shortest distance map may induce isometry between non-congruent shapes. Left: the shortest distance map of the half unit circle C induces an isometry to a circle C_h of radius 1/2. Right: by patching together pieces of the left picture one obtains an isometry between the unit circle and what could be called *a dented circle*.

Remark 2 (dented circle). Even if the metric distortion induced by the shortest distance map equals the identity (so that the surfaces are isometric) the surfaces need not be congruent. We give an example of this fact for planar curves; the extension to surfaces is obtained by considering cylinders with cross-sections equal to these planar curves. Consider the half unit circle $C = \{(\cos t, \sin t) : t \in [0, \pi]\}$. Any normal graph C_h over C can be written as

$$C_h = \left\{((1 - \phi(t)) \cdot \cos t, (1 - \phi(t)) \cdot \sin t) \in \mathbb{R}^2 : t \in [0, \pi]\right\},$$

where ϕ is the (signed) distance from C to C_h along the unit circle's (inward) normal N. Setting

$$\phi(t) := 1 - \sin t,$$

one readily checks that C_h becomes a circle of radius 1/2 with center $(0, 1/2)$, compare Figure 4. The inner product between the normals N of C and N_h of C_h is given by

$$\langle N, N_h \rangle = \sin t = 1 - \phi(t). \tag{11}$$

Let $\kappa = 1$ denote the curvature of C. As a special case of (6), the metric distortion between the two planar curves C and C_h with respect to the shortest distance map Φ is given by

$$a = \frac{1 - \phi \cdot \kappa}{\langle N, N_h \rangle} = \frac{1 - \phi}{\langle N, N_h \rangle} = 1. \tag{12}$$

Hence, in this case, the metric distortion is the identity, although the shapes of C and C_h are clearly not congruent.

3 Convergence

Under the assumption of convergence of a sequence $\{M_n\}$ of polyhedral surfaces to a smooth surface M in Hausdorff distance, we show that the following conditions are equivalent: (i) convergence of normals, (ii) convergence of metric distortion tensors, (iii) convergence of area, and (iv) convergence of the Laplace-Beltrami operators. The proof is based on translating properties (ii)-(iv) into corresponding properties of the metric distortion tensor A. We will throughout use the shortest distance map Φ to pull several objects of interest back from the polyhedral surface M_h to the *smooth compact embedded* surface M. Before treating convergence, we set up the relevant terminology. Throughout we write $\|\cdot\|_\infty$ as shorthand for $\|\cdot\|_{\mathcal{L}^\infty(M)}$.

3.1 Terminology

Hausdorff distance. Let $M_1, M_2 \subset \mathbb{R}^3$ be two non empty sets. Then the *Hausdorff distance* between M_1 and M_2 is defined as

$$\mathrm{d}_H(M_1, M_2) = \inf \{\varepsilon > 0 \,|\, M_1 \subset U_\varepsilon(M_2) \,\text{and}\, M_2 \subset U_\varepsilon(M_1)\},$$

where $U_\varepsilon(M) = \left\{p \in \mathbb{R}^3 \,|\, \exists q \in M : \, d_{\mathbb{R}^3}(p,q) < \varepsilon\right\}$.

Totally normal convergence. Let $\{M_n\}$ be a sequence of normal graphs over M. For each n let $N_n = N_{M_n} \circ \Phi_n$ be the pullback of the normal field of M_n to M. The sequence $\{M_n\}$ is said to converge *normally* to M if $\|N_n - N\|_\infty \to 0$. It converges *totally normally* if additionally $d_H(M_n, M) \to 0$.

Convergence of metric tensors. Each polyhedral surface M_n in the approximating sequence induces a metric g_{A_n} on the limit surface M determined by the respective distortion tensor A_n. Let $\|A_n\|_\infty = \operatorname{ess\,sup}_{p\in M} \|A_n(p)\|_{op}$. *Convergence of metric tensors* is defined as $\|A_n - Id\|_\infty \to 0$.

Sobolev norms and spaces. Let M_h be a normal graph over M, so that M_h induces the polyhedral metric g_A on M. In addition to the standard $\mathcal{L}^2$-norm on the smooth reference surface M, the metric g_A yields another norm on $\mathcal{L}^2(M)$. These norms are given by

$$\|u\|^2_{\mathcal{L}^2} = \int_M u^2 \,\mathrm{d}vol, \tag{13}$$

$$\|u\|^2_{\mathcal{L}^2_A} = \int_M u^2 (\det A)^{1/2} \,\mathrm{d}vol, \tag{14}$$

respectively, where $\mathrm{d}vol$ is the volume form on M induced by the Riemannian metric g. Similarly, let $\mathcal{H}^1_0(M) \subset \mathcal{L}^2(M)$ be the space of weakly differentiable functions u on M which either vanish along the (non empty) boundary of M or for which $\int_M u \,\mathrm{d}vol = 0$ if M has no boundary. The space $\mathcal{H}^1_0(M)$ can be equipped with the two norms[5],

$$\|u\|^2_{\mathcal{H}^1_0} = \int_M g(\nabla u, \nabla u) \,\mathrm{d}vol, \tag{15}$$

$$\|u\|^2_{\mathcal{H}^1_{0,A}} = \int_M g(A^{-1}\nabla u, \nabla u)(\det A)^{1/2} \,\mathrm{d}vol, \tag{16}$$

[5]Definition (16) is justified by the equality $\nabla_A = A^{-1}\nabla$.

where ∇ denotes the gradient on M induced by the metric g. Compactness of M implies that (13), (14) and (15), (16) induce equivalent (but not equal) norms. To distinguish these spaces as normed spaces, we shall write $\mathcal{L}^2(M)$, $\mathcal{L}^2_A(M)$, $\mathcal{H}^1_0(M)$, and $\mathcal{H}^1_{0,A}(M)$, respectively. We often drop the argument M.

Remark 3. Before moving on, we issue two warnings. The first warning concerns the case of empty boundary, $\partial M = \emptyset$. In this case the spaces $\mathcal{H}^1_0$ and $\mathcal{H}^1_{0,A}$ can be identified but are strictly speaking not equal since $u \in \mathcal{H}^1_{0,A}$ implies $\int_M u(\det A)^{1/2}\,\mathrm{d}vol = 0$ but not necessarily $\int_M u\,\mathrm{d}vol = 0$. Nonetheless, we are in the sequel silently going to identify $u \in \mathcal{H}^1_{0,A}$ with

$$[u] = u - \frac{1}{|M|}\int_M u\,\mathrm{d}vol,$$

which certainly lies in $\mathcal{H}^1_0$. This identification is justified since the norms defined by (15) and (16) vanish on constants.

The second warning concerns the case of non-empty boundary, $\partial M \neq \emptyset$. Here $\mathcal{H}^1_0$ and $\mathcal{H}^1_{0,A}$ *are* equal as sets (but equipped with different norms). However, in this case the image of M under the shortest distance map Φ is contained in, but may not necessarily be equal to, M_h (compare Definition 3). In particular, $\Phi(\partial M)$ may not be equal to ∂M_h—so that $\mathcal{H}^1_{0,A}(M)$ may not be equal to $\mathcal{H}^1_0(M_h)$. Still, we can identify $\mathcal{H}^1_{0,A}(M)$ with a subspace of $\mathcal{H}^1_0(M_h)$ since $u \in \mathcal{H}^1_{0,A}(M)$ implies $u \circ \Phi^{-1} \in \mathcal{H}^1_0(M_h)$, and,

$$\|u\|_{\mathcal{H}^1_{0,A}(M)} = \|u \circ \Phi^{-1}\|_{\mathcal{H}^1_0(M_h)},$$

which holds by construction.

Laplace-Beltrami operators. The metrics g and g_A both induce a Laplace-Beltrami operator on M. The *weak form* of these operators is given by

$$\langle \Delta u | v\rangle = -\int_M g(\nabla u, \nabla v)\,\mathrm{d}vol, \tag{17}$$

$$\langle \Delta_A u | v\rangle = -\int_M g(A^{-1}\nabla u, \nabla v)(\det A)^{1/2}\,\mathrm{d}vol, \tag{18}$$

respectively, where $\langle\cdot|\cdot\rangle$ denotes the pairing between $\mathcal{H}^1_0(M)$ and its dual $\mathcal{H}^{-1}(M)$. Both Δu and $\Delta_A u$ are elements of $\mathcal{H}^{-1}(M)$ and act on $\mathcal{H}^1_0(M)$ as bounded linear functionals[6]. Convergence of these operators is understood in the operator norm (denoted by $\|\cdot\|_{\mathrm{op}}$) of linear bounded maps between the spaces $\mathcal{H}^1_0(M)$ and $\mathcal{H}^{-1}(M)$.

3.2 Equivalent conditions for convergence

Theorem 2 (equivalent conditions for convergence)**.** *Let $M \subset \mathbb{R}^3$ be a compact embedded smooth surface, and let $\{M_n\}$ be a sequence of polyhedral surfaces which are normal graphs over M and which converge to M in Hausdorff distance. Then the following conditions are equivalent:*

[6]If $u \in C^\infty(M)$, then certainly $\Delta u \in C^\infty(M)$ as well, but $\Delta_A u$ need not even be in $\mathcal{L}^2(M)$ since the metric distortion tensor A is usually discontinuous; in fact, the distributional components of $\Delta_A u$ (located at the pre-image of the edges of M_h) must not be neglected.

i $\|N_n - N\|_\infty \to 0$ *(normal convergence).*

ii $\|A_n - Id\|_\infty \to 0$ *(metric convergence).*

iii $\|\,\mathrm{d}vol_n - \mathrm{d}vol\|_\infty \to 0$ *(convergence of area).*

iv $\|\Delta_n - \Delta\|_{\mathrm{op}} \to 0$ *(convergence of Laplace-Beltrami operators).*

Proof. The proof is based on translating conditions (ii), (iii) and (iv) into corresponding properties of the metric distortion tensors A_n: convergence of metric tensors by definition means $\|A_n - Id\|_\infty \to 0$, convergence of area measure is equivalent to $\|\det A_n\|_\infty \to 1$, and Lemma 1 provides conditions for convergence of Laplace-Beltrami operators. Each single of these conditions can now be shown to be equivalent to the convergence of normals. To see this, let $A_n = P_n \circ Q_n^{-1} \circ P_n$ as in Theorem 1, and let $\bar{A}_n = (\det A_n)^{1/2} A_n^{-1}$. We claim that

$$\begin{aligned}\|A_n - Id\|_\infty \to 0 \iff \|\det A_n\|_\infty \to 1 \iff \|\bar{A}_n - Id\|_\infty \to 0 \\ \iff \|\mathrm{tr}(\bar{A}_n - Id)\|_\infty \to 0\end{aligned}$$

are all equivalent conditions to normal convergence. Indeed, by assumption the surfaces converge in Hausdorff distance, so that $\|P_n - Id\|_\infty \to 0$, and from the diagonalization

$$Q_n = \begin{pmatrix} \langle N, N_n\rangle^2 & 0 \\ 0 & 1 \end{pmatrix},$$

one obtains that the above algebraic expressions involving A_n converge if and only if $\langle N, N_n\rangle \to 1$ in $\mathcal{L}^\infty$ - which is normal convergence. To complete the proof of the theorem, it remains to show Lemma 1. ☐

Remark 4. The prerequisite in Theorem 2 that Hausdorff distance must tend to zero cannot be dropped: in Remark 2 we described an example where the metric tensors converge (are equal) but the surfaces themselves do not.

Lemma 1 (convergence of Laplace-Beltrami operators)**.** *Let $M_h \subset \mathbb{R}^3$ be an embedded compact polyhedral surface which is a normal graph over a smooth embedded surface M. Let A be the metric distortion tensor and $\bar{A} := (\det A)^{1/2} A^{-1}$. Then*

$$\frac{1}{2}\|\mathrm{tr}(\bar{A} - Id)\|_\infty \le \|\Delta_A - \Delta\|_{op} \le \|\bar{A} - Id\|_\infty. \tag{19}$$

Proof. The upper bound is a straightforward application of definitions (17), (18), and Hölder's inequality. To prove the lower bound, let $K \subset M$ be the preimage under the shortest distance map Φ of the 1-skeleton of M_h (its edges and vertices). Then K is a measure zero set. For an arbitrary (but fixed) $p \in M \setminus K$ we will construct a family of functions $\{f_\varepsilon\} \subset \mathcal{H}_0^1(M)$ such that

$$\lim_{\varepsilon \to 0} \frac{|\langle(\Delta_A - \Delta) f_\varepsilon | f_\varepsilon\rangle|}{\|f_\varepsilon\|^2_{\mathcal{H}_0^1}} = \frac{1}{2}\mathrm{tr}(\bar{A} - Id)(p). \tag{20}$$

This will prove the lower bound since it implies

$$\|\Delta_A - \Delta\|_{\mathrm{op}} \ge \frac{1}{2} \sup_{p \in M \setminus K} \mathrm{tr}(\bar{A} - Id)(p).$$

To construct such a family, let $D_\varepsilon(p) \subset M\backslash K$ be a small ε-disk around p, and define in polar coordinates (r, φ) (induced by the exponential map $\exp_p(r, \varphi) : \mathrm{T}_pM \to M$)

$$f_\varepsilon(r, \varphi) = \begin{cases} \varepsilon - r & \text{for} \quad r < \varepsilon \\ 0 & \text{else.} \end{cases}$$

Then $f_\varepsilon \in \mathcal{H}_0^1$ (if M has empty boundary take $f_\varepsilon - \frac{1}{|M|}\int f_\varepsilon$). By the Gauss lemma, $\exp_p$ is a radial isometry so that $g(\nabla f_\varepsilon, \nabla f_\varepsilon) = 1$ on $D_\varepsilon(p) \setminus \{p\}$. By construction, $\nabla f_\varepsilon = 0$ on $M \setminus D_\varepsilon(p)$. It follows that

$$\|f_\varepsilon\|^2_{\mathcal{H}_0^1} = \int_M g(\nabla f_\varepsilon, \nabla f_\varepsilon)\, \mathrm{d}vol = |D_\varepsilon(p)|.$$

Moreover,

$$\langle(\Delta_A - \Delta)f_\varepsilon | f_\varepsilon\rangle = -\int_M g((\bar{A} - Id)\nabla f_\varepsilon, \nabla f_\varepsilon)\, \mathrm{d}vol,$$

so that (20) is equivalent to

$$\lim_{\varepsilon \to 0} \frac{1}{|D_\varepsilon(p)|} \int_{D_\varepsilon(p)} g(\bar{A}\nabla f_\varepsilon, \nabla f_\varepsilon)\, \mathrm{d}vol = \frac{1}{2}\mathrm{tr}(\bar{A})(p). \tag{21}$$

In a first step we are going to prove (21) for the case of constant metric and constant $\bar{A}$. In a second step the general case will be deduced.

1. Step. Let $\mathrm{d}vol_p$ denote the volume form on the tangent space T_pM induced by g_p, and let ∂_r denote the unit radial vector field on T_pM. The coefficients of g_p in polar coordinates are given by

$$(g_p)_{12} = 0, \quad (g_p)_{11} = 1, \quad (g_p)_{22} = r^2.$$

The matrix $\bar{A}_p := \bar{A}(p)$ acts as a linear map from T_pM to itself with eigenvalues λ and $1/\lambda$ (since $\det \bar{A}_p = 1$). On the disk of radius ε, $B_\varepsilon(0) \subset \mathrm{T}_pM$, we have

$$\begin{aligned} \int_{B_\varepsilon(0)} g_p(\bar{A}_p\partial_r, \partial_r)\, \mathrm{d}vol_p &= \int_0^\varepsilon \int_0^{2\pi} (\lambda \cos^2\varphi + \frac{1}{\lambda}\sin^2\varphi) r\, \mathrm{d}r\, \mathrm{d}\varphi \\ &= \frac{1}{2}(\lambda + \frac{1}{\lambda}) \cdot |B_\varepsilon(0)| \\ &= \frac{1}{2}\mathrm{tr}\bar{A}_p \cdot |B_\varepsilon(0)|, \end{aligned}$$

proving (21) for the case of the constant $g = g_p$ and constant $\bar{A} = \bar{A}_p$.

2. Step. To complete the proof, we show that for $\varepsilon \to 0$ one has

$$\frac{1}{|D_\varepsilon(p)|} \int_{D_\varepsilon(p)} g(\bar{A}\nabla f_\varepsilon, \nabla f_\varepsilon)\, \mathrm{d}vol \longrightarrow \frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(0)} g_p(\bar{A}_p\partial_r, \partial_r)\, \mathrm{d}vol_p.$$

Define a 2-form ω_0 on $B_\varepsilon(0) \subset \mathrm{T}_p(M)$ and a 2-form ω_1 on $D_\varepsilon(p)$ by

$$\begin{aligned} \omega_0 &= g_p(\bar{A}_p\partial_r, \partial_r)\, \mathrm{d}vol_p \\ \omega_1 &= g(\bar{A}\nabla f_\varepsilon, \nabla f_\varepsilon)\, \mathrm{d}vol. \end{aligned}$$

Let $\bar{A}^*$ be the pullback of $\bar{A}$, let ω_1^* denote the pullback of ω_1, and let $\mathrm{d}vol^*$ denote the pullback of the volume form $\mathrm{d}vol$ from $D_\varepsilon(p)$ to $B_\varepsilon(0)$. Since $\exp_p$ is a radial isometry (so that $\mathrm{d}\exp_p(\partial_r) = \nabla f_\varepsilon$), it follows that

$$\omega_1^* = g_p(\bar{A}^*\partial_r, \partial_r)\,\mathrm{d}vol^*.$$

From this, and since $\bar{A}$ and the metric are continuous on $D_\varepsilon(p)$, we obtain

$$\|\omega_1^* - \omega_0\|_{\infty,B_\varepsilon(0)} \longrightarrow 0 \quad \text{and} \quad \frac{|B_\varepsilon(0)|}{|D_\varepsilon(p)|} \longrightarrow 1.$$

Hence

$$\begin{aligned}
&\left| \frac{1}{|D_\varepsilon(p)|} \int_{D_\varepsilon(p)} \omega_1 - \frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(0)} \omega_0 \right| \\
&= \left| \frac{1}{|D_\varepsilon(p)|} \int_{B_\varepsilon(0)} \omega_1^* - \frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(0)} \omega_0 \right| \\
&\leq \left| \frac{1}{|D_\varepsilon(p)|} - \frac{1}{|B_\varepsilon(0))|} \right| \int_{B_\varepsilon(0)} |\omega_1^*| + \frac{1}{|B_\varepsilon(0)|} \int_{B_\varepsilon(0)} |\omega_1^* - \omega_0| \\
&\leq \left| \frac{|B_\varepsilon(0)|}{|D_\varepsilon(p)|} - 1 \right| \|\omega_1^*\|_{\infty,B_\varepsilon(0)} + \|\omega_1^* - \omega_0\|_{\infty,B_\varepsilon(0)} \longrightarrow 0,
\end{aligned}$$

proving our claim. □

4 Applications of normal convergence

In this section the general convergence result of Theorem 2 is applied to show convergence of minimizing geodesics, convergence of solutions to the Dirichlet problem in $\mathcal{H}_0^1$, as well as convergence of mean curvature vectors in $\mathcal{H}^{-1}$. Based on the splitting of the metric distortion tensor from Theorem 1, estimates are made *explicit* in terms of pointwise distance, deviation of normals, curvature properties of the smooth limit surface M, and (where appropriate) shapes of the triangles of the polyhedral surface M_h[7].

4.1 Convergence of geodesics

Definition 5 (minimizing geodesic)**.** A minimizing geodesic in a metric space (V, d) is a continuous curve $\gamma : [a, b] \to V$ such that $d(\gamma(t), \gamma(t')) = |t' - t|$ for all t and t' in the interval $[a, b]$.

The Hopf-Rinow theorem for metrically complete length spaces [17] asserts that any two points can be connected by a minimizing geodesic. This ensures that the infimum over all curves which was used in (1) to define the distance between two points is actually attained as a minimum.

[7]We do not provide estimates in terms of mesh size, such as $\mathcal{O}(h)$ (except in Theorem 5). However, it is not hard to obtain such estimates for inscribed meshes (i.e., meshes whose vertices lie on the smooth limit surface) based on Theorem 1 and the triangle shapes of the approximating polyhedra.

Theorem 3 (convergence of geodesics)**.** *Let $\{M_n\}$ be a sequence of polyhedral surfaces converging totally normally to a smooth surface M. Let $p, q \in M$, and let γ_n be a minimizing geodesic connecting $\Phi_n(p)$ to $\Phi_n(q)$ on M_n. Then each accumulation point of $\{\gamma_n\}$ in the compact-open topology on $C^0(\mathbb{R}, \mathbb{R}^3)$ is a minimizing geodesic on M. The set of such accumulation points is not empty. In particular, there exists a minimizing geodesic γ on M and a subsequence of minimizing geodesics $\{\gamma_{n_i}\}$ on M_{n_i} such that $\gamma_{n_i} \to \gamma$ uniformly.*

Proof. We consider all objects to be defined on the smooth reference surface M by using the pull-backs via Φ_n. In particular, we will (by abuse of notation) refer to γ_n as the minimizing g_n-geodesic between p and q on M. Let A_n denote the metric distortion tensor corresponding to g_n, and let

$$\underline{c}_n := \|A_n^{-1}\|_\infty^{-1/2} \quad \text{and} \quad \overline{c}_n := \|A_n\|_\infty^{1/2}.$$

If β is a Lipschitz curve on M, then the g_n-length $l_n(\beta)$ and the g-length $l(\beta)$ are related by

$$\underline{c}_n \cdot l(\beta) \leq l_n(\beta) \leq \overline{c}_n \cdot l(\beta).$$

The geodesic distance between the points p and q on M equals the infimum over the lengths of all Lipschitz curves connecting these points. The last inequality therefore implies

$$\underline{c}_n \cdot d(p, q) \leq d_n(p, q) \leq \overline{c}_n \cdot d(p, q).$$

Hence, if γ_n is a minimizing geodesic connecting p and q in the g_n-metric, then

$$\begin{aligned} \underline{c}_n \cdot d(p, q) \leq d_n(p, q) = l_n(\gamma_n) \leq \overline{c}_n \cdot l(\gamma_n), \\ \overline{c}_n \cdot d(p, q) \geq d_n(p, q) = l_n(\gamma_n) \geq \underline{c}_n \cdot l(\gamma_n). \end{aligned}$$

This implies

$$\frac{\underline{c}_n}{\overline{c}_n} \cdot d(p, q) \leq l(\gamma_n) \leq \frac{\overline{c}_n}{\underline{c}_n} d(p, q).$$

By the assumption of totally normal convergence we have $\underline{c}_n \to 1$ and $\overline{c}_n \to 1$, so that

$$l(\gamma_n) \to d(p, q). \tag{22}$$

Now, assume γ is an accumulation point of $\{\gamma_n\}$. Since the length functional $l : C^0(\mathbb{R}, \mathbb{R}^3) \to \mathbb{R}$ is lower semi-continuous, (22) implies

$$l(\gamma) \leq \liminf l(\gamma_n) = d(p, q).$$

Hence γ is indeed a minimizing geodesic connecting p to q. It remains to show that the set of such accumulation points is not empty. Note that

$$d(\gamma_n(t), \gamma_n(t')) \leq \frac{1}{\underline{c}_n} \cdot d_n(\gamma_n(t), \gamma_n(t')) = \frac{1}{\underline{c}_n} \cdot |t - t'|,$$

for each t, t' in the domain of γ_n. Hence the family $\{\gamma_n\}$ is equicontinuous. Since $|t - t'|$ is bounded by $\sup_n \operatorname{diam}(M_n) \leq \sup_n \overline{c}_n \cdot \operatorname{diam}(M)$, it follows from the Arzelà-Ascoli theorem that there is an accumulation point in the compact-open topology on $C^0(\mathbb{R}, \mathbb{R}^3)$. □

4.2 Convergence of the Dirichlet problem

Let $\{M_n\}$ be a sequence of polyhedral surfaces (possibly with boundary) which converge to the compact smooth surface M totally normally. We prove that the solutions u_n to the respective Dirichlet problems $\Delta_n u_n = f$ on M_n converge to the solution u of the Dirichlet problem $\Delta u = f$ in $\mathcal{H}_0^1(M)$ (Theorem 4). Furthermore we treat convergence of the corresponding finite element discretizations based on an argument by Dziuk [10] (Theorem 5).

Let the polyhedral surface M_h be a normal graph over the compact smooth surface M, so that M_h induces a polyhedral metric g_A on M. We consider the variational formulation of the Dirichlet problem instead of its classical formulation. As before, we use the shortest distance map to define our objects on the smooth surface M. Let $f \in \mathcal{L}^2$, so that $f \in \mathcal{L}_A^2$. The *Dirichlet problem* is to find the solution $u \in \mathcal{H}_0^1$, respectively $u_A \in \mathcal{H}_{0,A}^1$, such that

$$-\langle \Delta u | v \rangle = (f, v)_{\mathcal{L}^2} \quad \forall\, v \in \mathcal{H}_0^1, \tag{23}$$

$$-\langle \Delta_A u_A | v \rangle = (f, v)_{\mathcal{L}_A^2} \quad \forall\, v \in \mathcal{H}_{0,A}^1. \tag{24}$$

We let

$$E : \mathcal{H}_0^1(M) \hookrightarrow \mathcal{L}^2(M)$$

be the natural compact embedding, and let

$$C_E := \sup_{v \in \mathcal{H}_0^1(M)} \frac{\|E(v)\|_{\mathcal{L}^2(M)}}{\|v\|_{\mathcal{H}_0^1(M)}} \tag{25}$$

denote the operator norm of E. Since we use the inner product $(\nabla \cdot, \nabla \cdot)_{\mathcal{L}^2(M)}$ on $\mathcal{H}_0^1(M)$, it follows that C_E is the Poincaré constant of M.

Theorem 4 (consistency error of Dirichlet problem)**.** *The solutions to the Dirichlet problems satisfy*

$$\|u - u_A\|_{\mathcal{H}_0^1} \le C_E \cdot \Big((C_A - 1) + c_A \cdot C_A \left\| 1 - (\det A)^{1/2} \right\|_\infty \Big) \cdot \|f\|_{\mathcal{L}^2}.$$

Here $C_A := \|(\det A)^{1/2} A^{-1}\|_\infty$*, and* $c_A = 1$ *for the case* $\partial M \neq \emptyset$*, whereas* $c_A = 1 + \|\det A^{1/2}\|_\infty \|\det A^{-1/2}\|_\infty$ *for the case* $\partial M = \emptyset$*.*

Proof. We only detail the proof for the case $\partial M \neq \emptyset$. In this case the spaces $\mathcal{H}_0^1$ and $\mathcal{H}_{0,A}^1$ are equal as sets, and by equations (17) and (18), the Dirichlet problems take the form

$$\int_M g(\nabla u, \nabla v)\, \mathrm{d}vol = \int_M f v \, \mathrm{d}vol, \tag{26}$$

$$\int_M g(\bar{A} \nabla u_A, \nabla v)\, \mathrm{d}vol = \int_M f v (\det A)^{1/2} \, \mathrm{d}vol, \tag{27}$$

which has to hold for all $v \in \mathcal{H}_0^1$. As before, we let $\bar{A} := (\det A)^{1/2} A^{-1}$. Subtracting (27) from (26) and dividing by $\|\nabla v\|_{\mathcal{L}^2}$ gives

$$\frac{\big(\nabla u - \bar{A} \nabla u_A, \nabla v\big)_{\mathcal{L}^2}}{\|\nabla v\|_{\mathcal{L}^2}} \le C_E \|1 - (\det A)^{1/2}\|_\infty \|f\|_{\mathcal{L}^2}. \tag{28}$$

Writing

$$\nabla u - \nabla u_A = \bar{A}^{-1}(\nabla u - \bar{A}\nabla u_A) + (Id - \bar{A}^{-1})\nabla u,$$

and using (28) gives

$$\begin{aligned}\|u - u_A\|_{\mathcal{H}_0^1} &= \sup_{v \in \mathcal{H}_0^1} \frac{(\nabla u - \nabla u_A, \nabla v)_{\mathcal{L}^2}}{\|\nabla v\|_{\mathcal{L}^2}} \\ &\leq C_E \|\bar{A}^{-1}\|_\infty \|1 - (\det A)^{1/2}\|_\infty \|f\|_{\mathcal{L}^2} + \|Id - \bar{A}^{-1}\|_\infty \|u\|_{\mathcal{H}_0^1}.\end{aligned}$$

By (26) we have $\|u\|_{\mathcal{H}_0^1} \leq C_E \|f\|_{\mathcal{L}^2}$. The final estimate for the case of non-empty boundary, $\partial M \neq \emptyset$, follows from the fact that the 2×2 matrix field $\bar{A}$ has pointwise positive eigenvalues and $\det \bar{A} = 1$ so that $C_A = \|\bar{A}\|_\infty = \|\bar{A}^{-1}\|_\infty$, and

$$\|Id - \bar{A}^{-1}\|_\infty = \|\bar{A}\|_\infty - 1 = C_A - 1.$$

The case $\partial M = \emptyset$ is similar; a slight technical difficulty is due to the fact that in this case $v \in \mathcal{H}_0^1$ does not imply $v \in \mathcal{H}_{0,A}^1$, see Remark 3. □

Corollary 3 (convergence of Dirichlet problem)**.** *Let $f \in \mathcal{L}^2(M)$. If the sequence of polyhedral surfaces $\{M_n\}$ converges totally normally to the smooth surface M, then the solutions to the Dirichlet problems (24) on M_n converge in $\mathcal{H}_0^1(M)$ to the solution of the Dirichlet problem (23) on M.*

Proof. Let $C_A := \|(\det A)^{1/2} A^{-1}\|_\infty$ as in Theorem 4. Using Theorem 1 one readily checks that totally normal convergence implies $C_A \to 1$ as well as $\|1 - (\det A)^{1/2}\|_\infty \to 0$, so that Theorem 4 guarantees convergence in $\mathcal{H}_0^1(M)$. □

4.2.1 Discretization of the Dirichlet problem: the cotan formula

Since M_h is comprised of flat triangles, it is natural to approximate the solution u_A to the Dirichlet problem (24) by a piecewise linear finite element solution u_h. To obtain u_h, recall that a *Galerkin scheme* is defined by restricting the space of test functions as well as the space of solutions of the Dirichlet problem to the same *finite-dimensional* subspace $S_{h,0}$. We quickly review how to compute u_h explicitly if $S_{h,0}$ is spanned by linear Lagrange basis functions.

Definition 6 (linear Lagrange basis)**.** For vertices $p, q \in M_h$ define

$$\phi_p(q) := \begin{cases} 1 & \text{for} \quad q = p \\ 0 & \text{for} \quad q \neq p, \end{cases}$$

and extended ϕ_p to all of M_h by linear interpolation on triangles. The finite element space $S_{h,0}$ is spanned by the basis $\{\phi_p \,|\, p \in M_h \setminus \partial M_h\}$ corresponding to inner vertices.

Every $u_h \in S_{h,0}$ can be written as $u_h = \sum_q u_h^q \phi_q$ with coefficients u_h^q. Let

$$\Delta_{pq} := -\int_{M_h} g_{M_h}(\nabla_{M_h}\phi_p, \nabla_{M_h}\phi_q)\, \mathrm{d}vol_{M_h} \quad \text{and} \quad b_p := \int_{M_h} f \cdot \phi_p \, \mathrm{d}vol_{M_h}.$$

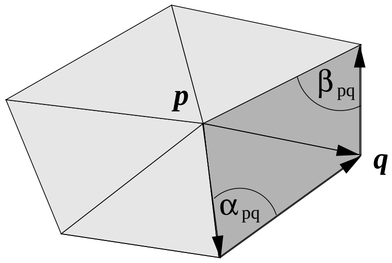

Figure 5: Only the angles α_{pq} and β_{pq} enter into the expression for Δ_{pq}.

Then the Dirichlet problem becomes a *finite linear problem*: find the vector (u_h^q) satisfying

$$\sum_q \Delta_{pq} u_h^q = b_p. \tag{29}$$

One readily verifies the *cotan representation* of Δ_{pq}:

Lemma 2 (cotan formula)**.** *The non-zero entries of the discrete Laplace-Beltrami operator, Δ_{pq}, on a polyhedral surface M_h are given by*

$$\Delta_{pq} = \frac{1}{2}(\cot \alpha_{pq} + \cot \beta_{pq}) \quad \textit{and} \quad \Delta_{pp} = -\sum_{q \sim p} \Delta_{pq}, \tag{30}$$

if p and q share an edge, and where α_{pq} and β_{pq} denote the angles opposite to edge (pq) in the two triangles adjacent to (pq), see Figure 5. The notation $q \sim p$ stands for all vertices q sharing an edge with p.

The cotan representation was introduced by Pinkall and Polthier in [25] as a discretization of the Laplace-Beltrami operator on polyhedral surfaces embedded in $\mathbb{R}^3$ in the context of discrete minimal surfaces. For the case of planar triangulated domains this formula goes back at least to Duffin [9].

Assumptions. Let $u_h \in S_{h,0}$ denote the piecewise linear solution to the finite Dirichlet problem (29) on M_h, and let u be the solution to the Dirichlet problem (23) on the smooth surface M. In order to prove that $u_h \to u$, we need two technical assumptions. First, we assume $u \in \mathcal{H}^2(M)$, and that u satisfies

$$|u|_{\mathcal{H}^2(M)} \leq c\|f\|_{\mathcal{L}^2(M)}.$$

This a priori estimate depends on typical regularity assumptions: it holds if $f \in \mathcal{L}^2(M)$ and $\partial M = \emptyset$ (by classical regularity theory, see [15]), but depends on boundary properties of M if $\partial M \neq \emptyset$. Second, we assume that $u_h \in \mathcal{H}_0^1(M)$. This can be achieved, for example, if f supported sufficiently far away from the boundary, ∂M.

Theorem 5 (convergence of cotan representation)**.** *Let the polyhedral surface M_h be a normal graph over the smooth surface M such that the above assumptions are satisfied. Let $\eta > 0$ be a positive number such that under the shortest distance map*

$$\|\angle(N, N_h)\|_\infty \leq \eta \cdot h \quad \textit{and} \quad \|\phi\|_\infty \leq \eta \cdot h. \tag{31}$$

Then there exists a number $C > 0$ which is a function of M, η, and the aspect ratio[8] *of the triangles of M_h (but not of h) such that*

$$\|u - u_h\|_{\mathcal{L}^2} + h\|u - u_h\|_{\mathcal{H}^1_0} \leq Ch^2\|f\|_{\mathcal{L}^2} \quad \text{if } \partial M \neq \emptyset,$$
$$\|u - u_h\|_{\mathcal{L}^2/\mathbb{R}} + h\|u - u_h\|_{\mathcal{H}^1_0} \leq Ch^2\|f\|_{\mathcal{L}^2} \quad \text{if } \partial M = \emptyset.$$

Here $\|u\|_{\mathcal{L}^2/\mathbb{R}}$ is the $\mathcal{L}^2$-norm of the unique representative of $[u] \in \mathcal{L}^2/\mathbb{R}$ having zero mean.

We omit the proof of the last theorem; it is based on the same arguments as those of Dziuk [10] who treats meshes all of whose vertices lie on the limit surface M, called *inscribed meshes.* For inscribed meshes the constant η in (31) only depends on M and the aspect ratio of the triangles of M_h.

4.3 Convergence of Mean Curvature

In this section we show that mean curvature vectors converges in the sense of *distributions* or *functionals*, that is, as elements of the Sobolev space $\mathcal{H}^{-1}(M)$ (Theorem 6). This result is used to show: if a sequence of discrete minimal surfaces converges totally normally to a smooth surface then the limit surface is minimal in the classical sense (Theorem 7). Finally, we give a counterexample to the convergence of mean curvature vectors in $\mathcal{L}^2$.

Let $I : M \to \mathbb{R}^3$ and $I_{M_h} : M_h \to \mathbb{R}^3$ denote the embeddings of M and M_h, respectively. Set $I_h = I_{M_h} \circ \Phi : M \to \mathbb{R}^3$. The weak mean curvature vectors are (vector-valued) *functionals*, i.e. elements of $\left(\mathcal{H}^{-1}\right)^3$, given by

$$H = \Delta I \quad \text{and} \quad H_A = \Delta_A I_h, \tag{32}$$

respectively. We define the norm of these vector-valued functionals by

$$\|H\|_{\mathcal{H}^{-1}} = \sup_{0 \neq u \in \mathcal{H}^1_0} \frac{\|\langle H|u\rangle\|_{\mathbb{R}^3}}{\|u\|_{\mathcal{H}^1_0}}.$$

Theorem 6 (approximation of weak mean curvature)**.** *Let M_h be a normal graph over the smooth surface M. Then*

$$\|H - H_A\|_{\mathcal{H}^{-1}} \leq \sqrt{|M|} \cdot (C_A - 1 + C_A\|Id - \mathrm{d}\Phi\|_\infty), \tag{33}$$

where $C_A = \|(\det A)^{1/2}A^{-1}\|_\infty$, $|M|$ is the total area of M, and $\|Id - \mathrm{d}\Phi\|_\infty$ denotes the essential supremum over the pointwise operator norm of the operator $(Id - \mathrm{d}\Phi)(p) : \mathrm{T}_pM \to \mathbb{R}^3$.

Proof. We apply the triangle inequality to

$$H - H_A = \Delta I - \Delta_A I_h = (\Delta I - \Delta_A I) + (\Delta_A I - \Delta_A I_h).$$

For any vector field X on M one has $\langle \nabla I, X\rangle_{\mathbb{R}^3} = X$, and $\langle \nabla I_h, X\rangle_{\mathbb{R}^3} = \mathrm{d}\Phi(X)$ almost everywhere. Applying (17), (18), and Hölder's inequality we obtain

$$\begin{aligned}\|\langle \Delta I - \Delta_A I|u\rangle\|_{\mathbb{R}^3} &= \left\| \int_M (A^{-1}(\det A)^{1/2} - Id)\nabla u \,\mathrm{d}vol \right\|_{\mathbb{R}^3} \\ &\leq \sqrt{|M|} \cdot (C_A - 1) \cdot \|u\|_{\mathcal{H}^1_0},\end{aligned}$$

[8]The aspect ratio of a triangle T is commonly defined as the ratio of the radius of the circle which circumscribes T to the radius of the circle which is inscribed into T.

and similarly

$$\begin{aligned}\|\langle \Delta_A I - \Delta_A I_h | u\rangle\|_{\mathbb{R}^3} &= \left\| \int_M \left\langle (\nabla I - \nabla I_h), A^{-1}(\det A)^{1/2}\nabla u \right\rangle \,\mathrm{d}vol \right\|_{\mathbb{R}^3} \\ &= \left\| \int_M (Id - \mathrm{d}\Phi)(A^{-1}(\det A)^{1/2})\nabla u \,\mathrm{d}vol \right\|_{\mathbb{R}^3} \\ &\leq \sqrt{|M|}\cdot C_A \cdot \|Id - \mathrm{d}\Phi\|_\infty \cdot \|u\|_{\mathcal{H}_0^1},\end{aligned}$$

proving the claim. □

Corollary 4 (convergence of weak mean curvature)**.** *If a sequence of polyhedral surfaces $\{M_n\}$ converges totally normally to the smooth surface M, then the corresponding mean curvature functionals converge in $\mathcal{H}^{-1}(M)$.*

Proof. The assumption of totally normal convergence yields $C_A \to 1$. It remains to show that $\|Id - \mathrm{d}\Phi\|_\infty \to 0$. Consider a single triangle T of M_h. Let $N_T = N \circ \Phi^{-1}$ denote the pullback of the normal field N on M to the triangle T. From the proof of Theorem 1 we know that $\mathrm{d}\Phi = \tilde{Q}^{-1} \circ P$, where P is as in (7), and $\tilde{Q}$ is given by $\tilde{Q}(Y) = Y - N_T \cdot \langle N_T, Y\rangle$. Totally normal convergence implies $P \to Id$ as well as $\tilde{Q} \to Id$, and hence $\mathrm{d}\Phi \to Id$ almost everywhere. □

4.3.1 Discrete minimal surfaces

Following [25] we consider discrete minimal surfaces as stationary points for the area functional within the class of polyhedral surfaces having the same underlying simplicial complex, and the same piecewise linear boundary. In [25], Pinkall and the second author considered a numerical flow for finding discrete minimal surfaces. To date a rich pool of *explicitly computable* discrete minimal surfaces has been discovered [16, 19, 26, 27, 28, 31]. In this section we show that if a sequence of discrete minimal surfaces converges to a smooth surface totally normally, then the smooth limit surface must be a minimal surface in the classical sense. Let $H_{M_h} = \Delta_{M_h} I_{M_h}$ denote M_h's distributional mean curvature vector[9]. A polyhedral surface M_h is *discrete minimal* if

$$\langle H_{M_h} | \phi_p \rangle = \frac{1}{2} \sum_{q \sim p} (q - p)(\cot \alpha_{pq} + \cot \beta_{pq}) = 0 \tag{34}$$

for all $p \in M_h \setminus \partial M_h$ (compare Figure 5 for notation). The notation $q \sim p$ stands for all vertices q connected to p by a single edge. In other words, discrete minimality implies that the distribution $H_{M_h} \in \left(\mathcal{H}^{-1}(M_h)\right)^3$ vanishes if paired with any $u_h \in S_{h,0}$. However, this does not imply that $H_{M_h} = 0$. Discrete minimality is thus a *weaker* condition than $H_{M_h} = 0$. In spite of this we have the following result:

Theorem 7 (convergence of discrete minimal surfaces)**.** *Let $\{M_n\}$ be a sequence of discrete minimal surfaces whose triangles have bounded aspect ratio. If M is smooth and $M_n \to M$ totally normally then M is a minimal surface in the classical sense.*

[9] We need to distinguish between H_{M_h} and H_A since in the case $\partial M \neq \emptyset$, we may have $\mathcal{H}^1_{0,A}(M) \subsetneq \mathcal{H}^1_0(M_h)$, see Remark 3.

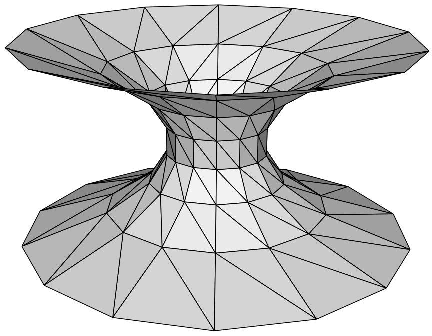

Figure 6: Discrete catenoid as an example of an explicitly computable discrete minimal surface (cf. Polthier and Rossman [28]).

Proof. Let H denote the (smooth) mean curvature vector of the smooth limit surface M. We are going to show that

$$\langle H|u\rangle = \int_M H \cdot u \, \mathrm{d}vol = 0$$

for all $u \in C_0^\infty(M)$ which are supported *away from the boundary*, ∂M. By $S_{n,0}$ we denote the finite element spaces (pulled back under Φ_n) induced by linear Lagrange elements on the meshes M_n. Let u_n be the *orthogonal projection* of u to $S_{n,0} \cap \mathcal{H}_0^1(M)$ with respect to the $\mathcal{H}^1$-inner product $(\nabla u, \nabla v)_{\mathcal{L}^2(M)}$. We have

$$\|\langle H|u\rangle\|_{\mathbb{R}^3} \le \|\langle H|u - u_n\rangle\|_{\mathbb{R}^3} + \|\langle H|u_n\rangle\|_{\mathbb{R}^3} . \tag{35}$$

We are going to show that the right hand side of (35) tends to zero as the mesh size of M_n goes to zero. Since H is smooth, it follows that

$$\begin{aligned}\|\langle H|u - u_n\rangle\|_{\mathbb{R}^3} &= \left\| \int_M H \cdot (u - u_n) \, \mathrm{d}vol \right\|_{\mathbb{R}^3} \\ &\le \|H\|_{\mathcal{L}^2(M)} \, \|u - u_n\|_{\mathcal{L}^2(M)} \\ &\le \|H\|_{\mathcal{L}^2(M)} \cdot C_E \cdot \|u - u_n\|_{\mathcal{H}_0^1(M)} ,\end{aligned}$$

where C_E is the Poincaré constant defined in (25). Since the aspect ratios of the triangles of M_n are assumed to be bounded, and $u \in C_0^\infty(M)$ is smooth and supported away from the boundary, ∂M, it follows that the projections u_n converge to u, i.e. $\|u - u_n\|_{\mathcal{H}_0^1(M)} \to 0$ (the proof is analogous to the planar case; for details see [37]).

To estimate the last term in (35), let $H_{A_n} \in \left(\mathcal{H}^{-1}(M)\right)^3$ denote the distributional mean curvature vector associated with A_n as defined in (32). By assumption, H_{A_n} vanishes on $S_{n,0} \cap \mathcal{H}_0^1$, so that $\langle H_{A_n}|u_n\rangle = 0$. We obtain

$$\begin{aligned}\|\langle H|u_n\rangle\|_{\mathbb{R}^3} &= \|\langle H - H_{A_n}|u_n\rangle\|_{\mathbb{R}^3} \\ &\le \|H - H_{A_n}\|_{\mathcal{H}^{-1}(M)} \cdot \|u_n\|_{\mathcal{H}_0^1(M)} \\ &\le \|H - H_{A_n}\|_{\mathcal{H}^{-1}(M)} \cdot \|u\|_{\mathcal{H}_0^1(M)}.\end{aligned}$$

From Corollary 4 we know that $\|H - H_{A_n}\|_{\mathcal{H}^{-1}(M)} \to 0$. From (35) we obtain $H = 0$, as asserted. □

4.3.2 Failure of convergence of mean curvature in $\mathcal{L}^2$

We have defined weak mean curvature vectors in the sense of distributions, and Corollary 4 shows their convergence in $\mathcal{H}^{-1}$. In this section we define the *discrete mean curvature vector* as the vector-valued *PL-function* associated with the mean curvature functional. We show that these discrete mean curvature vectors in general fail to converge in $\mathcal{L}^2$ (despite totally normal convergence of the meshes). This failure of $\mathcal{L}^2$-convergence comes to no surprise considering the various counterexamples to pointwise convergence of curvatures (see e.g. [20, 38, 39]). Although it has not been disproved that there exists a discrete notion of mean curvature for polyhedra which converges pointwise almost everywhere to the mean curvature of a *general smooth limit surface*, it seems plausible to conjecture that such a notion does not exist (that is, without any additional assumptions such as restrictions on the valence of vertices or a priori knowledge of the normals of the limit surface).

Definition 7 (discrete mean curvature vector)**.** Let $H_{M_h} = \Delta_{M_h} I_{M_h}$ denote the distributional mean curvature vector of the polyhedral surface M_h. The *discrete mean curvature vector* is the vector-valued PL-function $H_h \in (S_{h,0})^3$ defined by

$$(H_h, u_h)_{\mathcal{L}^2(M_h)} = \langle H_{M_h} | u_h \rangle \quad \forall\, u_h \in S_{h,0}.$$

In other words, the discrete mean curvature vector corresponds to the restriction of the full (distributional) mean curvature vector H_{M_h} to the finite element space $S_{h,0}$. Note that this definition is in accordance with our notion of discrete minimality: a polyhedral surface is discrete minimal if and only if $H_h = 0$.

Remark 5. Only because the dimension of S_h is finite, it is possible to associate a discrete function in the above way to the mean curvature functional. There is no infinite-dimensional analogue of this construction.

Example 1 (counterexample to $\mathcal{L}^2$-convergence). Denote by H the smooth mean curvature vector of the smooth surface M, and let $\{H_n\}$ denote the sequence of *discrete mean curvature vectors* associated with the sequence of polyhedral surfaces $\{M_n\}$. We show that in general $\|H_n - H\|_{\mathcal{L}^2}$ does not converge to zero. Consider the cylinder M of height 2π and radius 1. We construct a sequence, $\{M_n\}$, of polyhedral surfaces whose vertices lie on this cylinder and which converges to M totally normally. Let the cylinder be parameterized as

$$x = \cos u, \quad y = \sin u, \quad z = v.$$

Let the vertices of M_n be given by

$$u = \frac{i\pi}{n} \qquad i = 0, ..., 2n-1$$
$$v = \begin{cases} 2j \sin \frac{\pi}{2n} & j = 0, ..., 2n-1 \\ 2\pi & j = 2n \end{cases}$$

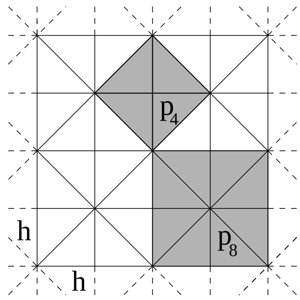

Figure 7: Discrete mean curvature does not converge in $\mathcal{L}^2$ for a $4-8$ tessellation of a regular quad grid, because the ratio between the areas of the stencils of p_4 and p_8 does not converge to 1.

This corresponds (up the uppermost layer) to folding along the vertical lines a regular planar quad-grid of edge length

$$h_n = 2\sin\frac{\pi}{2n}.$$

In other words, all faces of M_n are rectangular (in fact quadratic except for the uppermost layer). It will now depend on the *tessellation pattern* of this quad-grid whether there is $\mathcal{L}^2$-convergence of discrete mean curvature or not. Indeed, consider the regular $4-8$ tessellation scheme depicted in Figure 7. There are two kinds of vertices - those of valence 4 and those of valence 8. Call them p_4 and p_8, respectively. Let ϕ_{p_4} and ϕ_{p_8} denote the corresponding Lagrange basis functions. Using the cotan-formula, it is easy to see that the coefficients of the weak mean curvature satisfy

$$\langle H|\phi_{p_4}\rangle = \langle H|\phi_{p_8}\rangle = -2\left(1-\cos\frac{\pi}{n}\right)\cdot\partial_r,$$

where ∂_r denotes the (radial) outward cylinder normal field. By the symmetry of the problem there exist constants $a_n, b_n \in \mathbb{R}$ such that

$$H_n = \sum_{p_4} a_n\cdot\phi_{p_4}\cdot\partial_r + \sum_{p_8} b_n\cdot\phi_{p_8}\cdot\partial_r + \text{boundary contributions},$$

where the contributions from the boundary include all vertices one layer away from the upper boundary (as symmetry breaks there). Set

$$\lambda_n := -\left(1-\cos\frac{\pi}{n}\right).$$

One verifies that

$$a_n = 12\cdot\frac{\lambda_n}{h_n^2}\cdot\frac{4+\lambda_n}{8-\lambda_n^2} \quad \text{and} \quad b_n = 12\cdot\frac{\lambda_n}{h_n^2}\cdot\frac{\lambda_n}{\lambda_n^2-8}.$$

Since $\lim_{n\to\infty}(\lambda_n/h_n^2) = -1/2$, it follows that

$$\lim_{n\to\infty} a_n = -3 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0,$$

so that asymptotically *only the vertices of valence* 4 *but not those of valence* 8 *contribute to discrete mean curvature*,

$$H_n \sim -3 \sum_{p_4} \phi_{p_4} \cdot \partial_r + \text{boundary contributions.}$$

Hence, H_n is a family of PL-functions oscillating between -3 (at the vertices of valence 4) and 0 (at the vertices of valence 8) with ever growing frequencies. Such a family does not converge in $\mathcal{L}^2$.

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