Dynamic problems of rate-and-state friction in viscoelasticity

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Experimental background

Figure: System response to jump in velocity (after steady-state sliding)
Rate-and-state friction

Widely used law

\[ \mu(V, \theta) = \mu_\ast + a \log \frac{V}{V_\ast} + b \log \frac{\theta V_\ast}{L}, \quad \dot{\theta}(\theta, V) = \begin{cases} 1 - \frac{\theta V}{L} \\ -\frac{\theta V}{L} \log \frac{\theta V}{L} \end{cases} \]

ageing law

slip law

Transformation: \( \alpha = \log(\theta V_\ast/L) \)

\[ \mu(V, \alpha) = \mu_\ast + a \log \frac{V}{V_\ast} + b \alpha, \quad \dot{\alpha}(\alpha, V) = \begin{cases} \frac{V_\ast e^{-\alpha} - V}{L} \\ -\frac{V}{L} (\log \frac{V}{V_\ast} + \alpha) \end{cases} \]

General setting

- \( \mu \) is monotone in \( V \) for fixed \( \alpha \)
- \( \mu \) is Lipschitz with respect to \( \alpha \) (but not \( \theta \))
- (unlike \( \theta \)), \( \alpha \) follows a gradient flow for fixed \( V \).
- (ideally): \( \dot{\alpha} \) is Lipschitz with respect to \( V \).
A typical continuum mechanical problem

\[ \begin{align*}
\sigma(u) &= B\varepsilon(u) + A\varepsilon(\dot{u}) & \text{in } \Omega & \text{(linear viscoelasticity)} \\
\text{div } \sigma(u) + b &= \rho\ddot{u} & \text{in } \Omega & \text{(momentum balance)} \\
\dot{u}_n &= 0 & \text{on } \Gamma_C & \text{(bilateral contact)}^1 \\
\sigma_t &= -\lambda \dot{u}, \quad \lambda = \frac{|\sigma_t|}{|\dot{u}|} = \frac{|s_n|\mu(|\dot{u}|, \alpha)}{|\dot{u}|} & \text{on } \Gamma_C & \text{with } \lambda = 0 \text{ for } \dot{u} = 0 \\
\dot{\alpha} &= \dot{\alpha}(|\dot{u}|, \alpha) & \text{on } \Gamma_{N,D} & \text{(family of ODEs)} \\
\end{align*} \]

with \( s_n \approx \sigma_n \), constant in time\(^1\).

\(^1\)Inherited from the rate-and-state friction model

With prescribed \( u(0), \dot{u}(0), \) and \( \alpha(0) \).
Weak formulation

We get

\[
\int_\Omega \rho \ddot{u} (v - \dot{u}) + \int_\Omega \mathcal{B} \varepsilon(\dot{u}) : \varepsilon(v - \dot{u}) + \int_\Omega \mathcal{A} \varepsilon(u) : \varepsilon(v - \dot{u}) + \int_{\Gamma_C} \phi(v, \alpha) \geq \int_{\Gamma_C} \phi(\dot{u}, \alpha) + \ell(v - \dot{u})
\]

for every \( v \in \mathcal{H} \) with

\[
\mathcal{H} = \{ v \in H^1(\Omega)^d : v = 0 \text{ on } \Gamma_D, \; v_n = 0 \text{ on } \Gamma_C \}
\]

or briefly

\[
0 \in M \ddot{u} + C \dot{u} + A u + \partial\Phi( \cdot, \alpha)(\dot{u}) - \ell \subset \mathcal{H}^*
\]

and

\[
\dot{\alpha} = \dot{\alpha}(|\dot{u}|, \alpha) \; \text{ a.e. on } \Gamma_C
\]
Time discretisation

Turn

\[ \begin{align*}
0 \in M\ddot{u} + C\dot{u} + Au + \partial\Phi(\cdot, \alpha)(\dot{u}) - \ell, \\
\dot{\alpha} = \dot{\alpha}(|\dot{u}|, \alpha)
\end{align*} \]

into

\[ \begin{align*}
0 \in M\ddot{u}_n + C\dot{u}_n + Au_n + \partial\Phi(\cdot, \alpha_n)(\dot{u}_n) - \ell_n, \\
\dot{\alpha} = \dot{\alpha}(|\dot{u}_n|, \alpha)
\end{align*} \]

and then (using a time discretisation scheme/solving the ODEs)

\[ \begin{align*}
0 \in (M_n + C + A_n)\dot{u}_n + \partial\Phi(\cdot, \alpha_n)(\dot{u}_n) - \tilde{\ell}_n, \\
\alpha_n = \Psi|\dot{u}_n|(\alpha_{n-1})
\end{align*} \]

\[ \rightsquigarrow \text{A coupling of} \]

1. a convex minimisation problem
2. a family of ordinary differential equations (one-dimensional gradient flows)
The big picture

\[ |\gamma(v)| \in L^2(\Gamma_C) \]

\( v \in \mathcal{H} \) \hspace{1cm} \( \alpha \in L^2(\Gamma_C) \)

1. Lipschitz, compact
2. continuous, sublinear growth
3. Lipschitz

\[ T : \mathcal{H} \rightarrow \mathcal{H} \]

\( (1) \) trace map + norm
\( (2) \) solve ODEs
\( (3) \) convex minimisation

- Q: Does \( T \) have a fixed point?
  A: Yes, by Banach’s/Schauder’s fixed point theorem
- Q: Is it unique?
  A: Yes/Maybe (depending on the law)
- Q: Does \( T^n v \) always converge to a fixed point?
  A: Yes/Maybe (depending on the law)
Application: a simplified subduction zone

The lower plate moves at a prescribed velocity while the right end of the wedge is held fixed.
Numerical stability: Number of fixed point iterations

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Recurrence time and rupture width are well reproduced. Peak slip is off by a factor of approximately 6. The error thus lies within an order of magnitude.
Further reading