Minimum Dual Diameter Triangulations*

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Abstract

Let $\mathcal{P}$ be a simple planar polygon with n vertices. We would like to find a triangulation $\mathrm{MDT}(\mathcal{P})$ of $\mathcal{P}$ that minimizes the diameter of the dual tree. We show that $\mathrm{MDT}(\mathcal{P})$ can be constructed in $O(n^3 \log n)$ time. If $\mathcal{P}$ is convex, we show that the dual of any MDT has diameter $2 \cdot \lceil \log_2(n/3) \rceil$ or $2 \cdot \lceil \log_2(n/3) \rceil - 1$, depending on the value of n. We also investigate the relation between $\mathrm{MDT}(\mathcal{P})$ and the number of ears in $\mathcal{P}$. When $\mathcal{P}$ is convex, we give a construction for MDTs that maximize the number of ears among all triangulations. However, if $\mathcal{P}$ is not convex, we show that triangulations maximizing the number of ears may have diameter quadratic in the diameter of an MDT. Finally, we consider point sets instead of polygons and show that for this case the diameter of the dual graph of an MDT is $O(\log n)$.

1 Introduction

Given a simple planar polygon $\mathcal{P}$ with n vertices, we are interested in finding a triangulation $\mathrm{MDT}(\mathcal{P})$ of $\mathcal{P}$ such that the diameter of the dual tree of the triangulation[1] is minimized. We call this diameter the *dual diameter (of the triangulation)*. Note that $\mathrm{MDT}(\mathcal{P})$ may not be unique.

Shermer [7] considers *thin* and *bushy* triangulations of simple polygons, i.e., triangulations that minimize, resp. maximize, the number of ears. Shermer mentions that bushy triangulations are useful for finding paths in the dual tree, as is needed, e.g., in geodesic algorithms. In that setting, however, the running time is not actually determined by the number of ears, but by the dual diameter of the triangulation. Thus, bushy triangulations would only be useful for geodesic problems if maximizing the number of ears minimized the dual diameter. While this holds for convex polygons, we show that in general there exist polygons for which no MDT maximizes the number of ears. Moreover, forcing a single ear into a triangulation might already nearly double the dual diameter, and the dual diameter of any bushy triangulation may be quadratic in the optimum.

The dual diameter also plays a role in the study of *edge flips*: an edge flip in a triangulation of a convex polygon corresponds to a rotation in the dual tree. For convex polygons, Hurtado, Noy, and Urrutia [2, 8] show that a triangulation with dual diameter k can be transformed into a fan triangulation using at most k parallel flips (i.e., two edges not sharing a triangle may be flipped simultaneously). They also obtain a triangulation with logarithmic dual diameter by recursively cutting off a linear number of ears.

While we focus on the dual graph of a triangulation, optimizing the distance in the primal has also been considered. Kozma [4] addresses the problem of finding a triangulation where the total link distance for all pairs of vertices is minimized. For simple polygons, he gives a sophisticated $O(n^{11})$ time dynamic programming algorithm. Further, he shows that the problem is strongly NP-complete for general point sets and arbitrary edge weights.

2 The Number of Ears of $\mathrm{MDT}(\mathcal{P})$

Since the dual graph has maximal degree 3, the so-called Moore bound implies that the dual diameter is at least $\log_2(\frac{n+2}{3})$ (see, e.g., [5]). For convex polygons, we can get the exact optimum dual diameter.

Proposition 1 *For a convex polygon $\mathcal{P}$ with n vertices, $\mathrm{MDT}(\mathcal{P})$ has dual diameter $2 \cdot \lceil \log_2(n/3) \rceil - 1$ if $n \in \{3 \cdot 2^{m-1} + 1, \ldots, 3 \cdot \frac{4}{3} \cdot 2^{m-1}\}$, and $2 \cdot \lceil \log_2(n/3) \rceil$ if $n \in \{4 \cdot 2^{m-1} + 1, \ldots, 3 \cdot 2^m\}$, for some $m \geq 1$.*

Proof. The dual graph of any triangulation of $\mathcal{P}$ is a tree with $n-3$ vertices and maximum degree 3. Furthermore, every tree with $n-3$ vertices and maximum degree 3 is dual to some triangulation of $\mathcal{P}$.

We consider special triangulations whose dual trees have the following structure: (1) all but at most one

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[1] The dual graph T^* of a triangulation T is the graph with a vertex for each (empty) triangle of T and an edge between two triangles iff they share an edge in T. If T has no interior vertices then T^* is a tree.

inner vertex has degree three, (2) there is a central vertex v (the one corresponding to the central triangle in the triangulation) such that any two paths from v to two leaves differ by at most one in length, and (3) for at most one edge incident to v, paths from v to leaves have different lengths. See Figure 1 for an illustration.

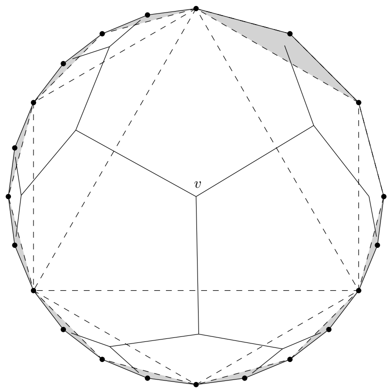


Figure 1: MDTs for convex polygons.

For the upper bound it can be shown (e.g., by case distinction) that the dual diameter of these triangulations is exactly $2 \cdot \lceil \log_2(n/3) \rceil - 1$ or $2 \cdot \lceil \log_2(n/3) \rceil$, depending on the value of n.

For the lower bound, assume there is a tree T with $n-3$ vertices and maximum degree 3 that has diameter k strictly smaller than in the proposition. Consider a longest path π in T and a vertex v on π for which the distances to the endpoints of π differ by at most one. By adding vertices, T can be turned into a tree with structure as in the construction of the upper bound (with v as central vertex) and diameter k, a contradiction. □

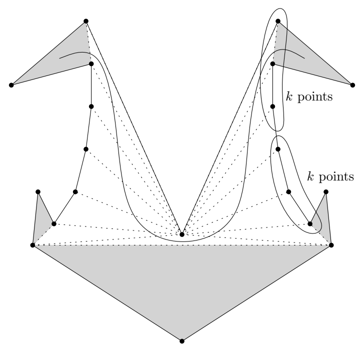
As the dual graph of a triangulation of any simple polygon or point set in the plane also has maximum vertex degree three, we obtain the following corollary from Proposition 1.

Corollary 2 *For any simple polygon $\mathcal{P}$ with n vertices, the dual diameter of* MDT($\mathcal{P}$) *is at least* $2 \cdot \lceil \log_2(n/3) \rceil - 1$ *if* $n \in \{3 \cdot 2^{m-1}+1, \ldots, 4 \cdot 2^{m-1}\}$, *and* $2 \cdot \lceil \log_2(n/3) \rceil$ *if* $n \in \{4 \cdot 2^{m-1}+1, \ldots, 3 \cdot 2^m\}$, *for some* $m \geq 1$.

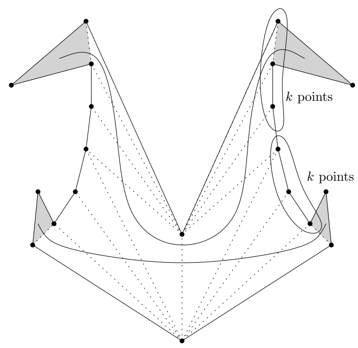
Proposition 1 also implies that for convex $\mathcal{P}$ there exists an MDT that maximizes the number of ears among all triangulations of $\mathcal{P}$. Next, we show that this does not hold for general simple polygons. Hence, any greedy approach that tries to construct MDTs by maximizing the number of ears is bound to fail.

Lemma 3 *There exist simple polygons in which no* MDT *maximizes the number of leaves.*

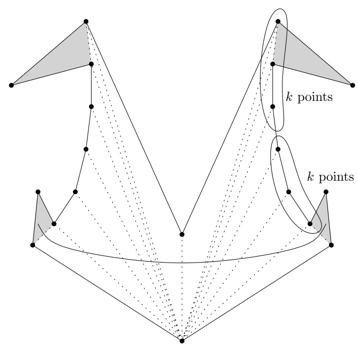
Proof. Let $k \geq 1$ and consider the polygon with $n = 4k+8$ vertices in Figure 2. The triangulation in Figure 2(a) is the only triangulation with 5 ears, and it has dual diameter $4k+2$. However, as depicted in Figure 2(b), omitting the large ear at the bottom allows a triangulation with 4 ears and dual diameter $2k+3$. This shows that forcing even one additional ear might nearly double the dual diameter. □



(a) 5 ears, dual diameter $4k+2$.



(b) 4 ears, dual diameter $2k+3$



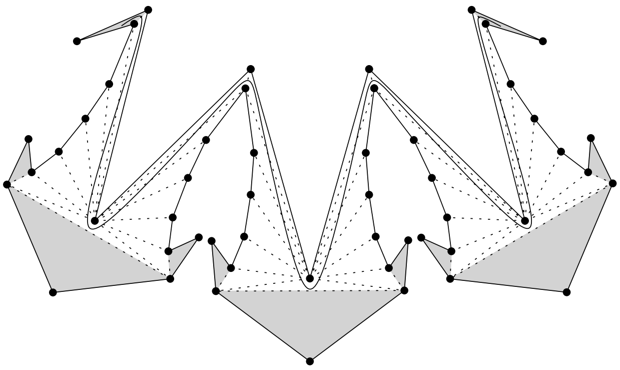
(c) 4 ears, dual diameter $4k+3$

Figure 2: Three triangulations of a polygon with $n = 4k+8$ vertices ($k = 3$) and paths that define their dual diameters. The ears are shaded.

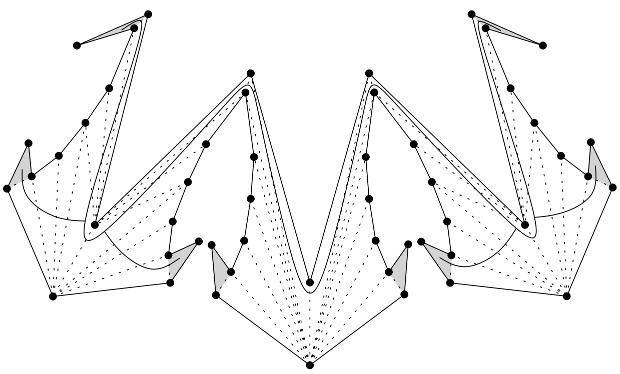
Theorem 4 *There exist simple polygons where triangulations maximizing the number of ears have a dual diameter that is quadratic in the optimum.*

Proof. Consider the polygon $\mathcal{P}'$ from Lemma 3, and let $\mathcal{P}$ be obtained by concatenating c copies of $\mathcal{P}'$ as in Figure 3. $\mathcal{P}$ has $n = c(4k+4)+4$ vertices. Using the triangulation from Figure 2(a) for each copy, we get a triangulation with the maximum number $3c+2$ of ears and dual diameter $c(4k+1)+1$ (the curved line in Figure 3 indicates a longest path). On the other hand, using the triangulation from Figure 2(b) for the leftmost and rightmost part of the polygon and the one from Figure 2(c) for all intermediate parts yields a triangulation with dual diameter $4c+4k-3$ that has only $2c+2$ ears.

For $c = k$, we obtain $c, k = \Theta(\sqrt{n})$. Thus, the dual diameter for the triangulation with maximum number of ears is $\Theta(n)$, while the optimal dual diameter is $O(\sqrt{n})$. □



(a) $3c+2$ ears, dual diameter $4ck+5$.



(b) $2c+2$ ears, dual diameter $4c+4k-3$.

Figure 3: Two triangulations of a polygon with $n = c(4k+4)+4$ vertices ($c = k = 3$) and corresponding longest paths. The ears are shaded.

3 Triangulating a Simple Polygon

We now consider the algorithmic question of computing an MDT of a simple polygon $\mathcal{P}$ with n vertices. Let $v_1, \dots v_n$ be the vertices of $\mathcal{P}$ in clockwise order. The segment $v_i v_j$ is a *diagonal* of $\mathcal{P}$ if it lies completely in $\mathcal{P}$ but is not part of the boundary of $\mathcal{P}$. For a diagonal $v_i v_j$, $i < j$, we define $\mathcal{P}_{i,j}$ as the polygon with vertices $v_i, v_{i+1}, \dots, v_{j-1}, v_j$ (see Figure 4). Observe that $\mathcal{P}_{i,j}$ is a simple polygon contained in $\mathcal{P}$. If

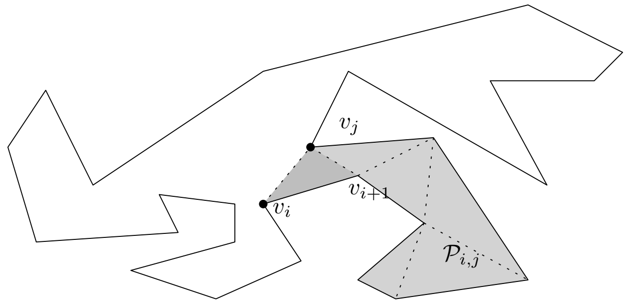


Figure 4: Any triangulation of $\mathcal{P}_{i,j}$ (gray) has exactly one triangle adjacent to $v_i v_j$ (dark gray).

$v_i v_j$ is not a diagonal, we set $\mathcal{P}_{i,j} = \emptyset$.

Theorem 5 *For any simple polygon $\mathcal{P}$ with n vertices, we can compute an* MDT *in $O(n^3 \log n)$ time.*

Proof. We use the classic dynamic programming approach [3], with an additional twist to account for the nonlocal nature of the objective function. Let $v_i v_j$ be a diagonal. Any triangulation Δ of $\mathcal{P}_{i,j}$ has exactly one triangle t adjacent to $v_i v_j$ (see Figure 4). Let $f(\Delta)$ be the length of the longest path in the dual of Δ that has one endpoint in t.

For $d > 0$ and $i, j = 1, \dots, n$, with $i < j$, let $\mathcal{T}_d(i,j)$ be the set of all triangulations of $\mathcal{P}_{i,j}$ with dual diameter at most d (we set $\mathcal{T}_d(i,j) = \emptyset$ if $v_i v_j$ is not a diagonal). We define $M_d[i,j] = \min_{\Delta \in \mathcal{T}_d(i,j)} f(\Delta)+1$, if $\mathcal{T}_d(i,j) \neq \emptyset$, or $M_d[i,j] = \infty$, otherwise. Intuitively speaking, we aim for a triangulation minimizing the distance from $v_i v_j$ to all other triangles of $\mathcal{P}_{i,j}$ while keeping the dual diameter below d. Let $\mathcal{V}(i,j)$ be all vertices v_l of $\mathcal{P}_{i,j}$ such that the triangle $v_i v_j v_l$ lies inside $\mathcal{P}_{i,j}$. We claim that $M_d[i,j]$ obeys the following recursion:

$$M_d[i,j] = \begin{cases} 0, & \text{if } i+1 = j \\ \infty, & \text{if } v_i v_j \text{ is not a diagonal} \\ \min_{v_l \in \mathcal{V}(i,j)} I_d[i,j,l], & \text{otherwise} \end{cases}$$

where

$$I_d[i,j,l] = \begin{cases} \infty, & \text{if } M_d[i,l] + M_d[l,j] > d \\ \max\{M_d[i,l], M_d[l,j]\}+1, & \text{otherwise} \end{cases}$$

We minimize over all possible triangles t in $\mathcal{P}_{i,j}$ incident to $v_i v_j$. For each t, the longest path to $v_i v_j$ is the longer of the paths to the other edges of t plus t itself. If t joins two longest paths of total length more than d, there is no valid solution with t. Thus, we can decide in $O(n^3)$ time whether there is a triangulation with dual diameter at most d, i.e., if $M_d[1,n] \neq \infty$. Since the dual diameter is at most $n-3$, a binary search gives an $O(n^3 \log n)$ time algorithm. □

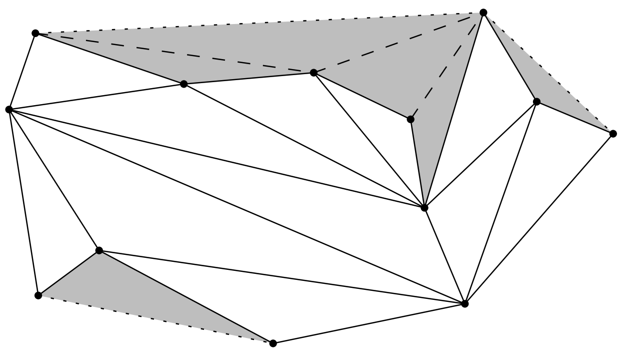


Figure 5: An embedding of a triangulation with logarithmic dual diameter on a point set. The pockets are shown gray, their triangulation (completing the triangulation of S) is dashed.

4 Triangulating a Point Set

We are now given a set S of n points in the plane, and we need to find a triangulation of S whose dual graph has small diameter. The dual graph will be a graph in which no vertex has degree more than three. Thus, it is easy to see that the $\Omega(\log n)$ lower bound for simple polygons also extends for this case. It turns out that this bound can always be achieved.

Theorem 6 *Given a planar n-point set S, we can compute in $O(n \log n)$ time a triangulation of S with dual diameter $\Theta(\log n)$.*

Proof. Let $\mathcal{P}$ be a convex polygon with n vertices and Δ' a triangulation of $\mathcal{P}$ with dual diameter $\Theta(\log n)$ (e.g., the triangulation from Proposition 1). The triangulation Δ' is an outerplanar graph. Any outerplanar graph of n vertices has a plane straight-line embedding on any given n-point set [6]. Furthermore, such an embedding can be found in $O(nd)$ time and $O(n)$ space, where d is the dual diameter of the graph [1].

Let Δ_S be the embedding of Δ' on S. In general, Δ_S does not triangulate S (see Figure 5). Consider the convex hull of Δ_S (which equals the convex hull of S). The untriangulated *pockets* are simple polygons. We triangulate each pocket arbitrarily to obtain a triangulation Δ of S. We claim that the dual diameter of Δ is $O(\log n)$.

Lemma 7 *The dual distance from any triangle in a pocket to any triangle in Δ_S is $O(\log n)$.*

Proof. Let Q be a pocket, and Δ_Q a triangulation of Q. Since Q is a simple polygon, the dual Δ_Q^* is a tree with maximum degree 3. A triangle t of Δ_Q not incident to the boundary of Δ_S either has degree 3 in Δ_Q^*, or it is the unique triangle in Δ_Q that shares an edge with the convex hull of S. Perform a breadth-first-search in Δ_Q^* starting from t, and let k be the maximum number of consecutive layers from the root of the BFS-tree that do not contain a triangle incident to the boundary. By the above observation, for each layer but one the size doubles, so $k = O(\log n)$. □

Given Lemma 7 and the fact that Δ_S has dual diameter $O(\log n)$, Theorem 6 is now immediate. □

5 Conclusion

The proof of the lower bound (Corollary 2) is essentially based on fundamental properties of graphs (i.e., bounded degree) rather than geometric properties. Since the bound is tight when for the convex position case, it cannot be tightened in general. However, we wonder if, by using geometric tools, one can construct a bound that depends on the number of reflex vertices (or interior points for the case of sets of points). Another natural open problem is to extend our algorithm for computing MDT of simple polygons to general polygonal domains, or even sets of points.

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