Abstracts

Arithmetic properties of rigid local systems

Hélène Esnault
(joint work with Michael Groechenig)

1. Introduction

The aim of the lecture, which relies on [3], is to generalize to the quasi-projective non-projective case the theorem described in the introduction and Theorem 5.4 of [2] stating that irreducible complex rigid local systems, while restricted to a p-adic formal scheme with good reduction, for p large, underlie the structure of a Fontaine-Lafaille module and define p-adic local systems.

2. Assumption and preliminary geometric facts

Assumption 1 (Throughout)**.** X smooth quasi-projective $/\mathbb{C}$; all *irreducible* complex local systems $\mathbb{L}$ of rank r with *unipotent monodromies* at ∞ are *strongly cohomologically rigid* i.e. $H^1(X, \mathcal{E}nd(\mathbb{L})) = 0$. (*Verified for Shimura varieties of real rank* ≥ 2 *by Margulis superrigidity.*)

Facts 2 (Preliminary geometry)**. 1)** For $j : X \hookrightarrow \bar{X}$ a good compactification, has

$$H^1(\bar{X}, j_{!*}\mathcal{E}nd(\mathbb{L})) \xrightarrow{\text{inj}} H^1(X, \mathcal{E}nd(\mathbb{L}))$$

so (*strongly cohomologically rigid*) $\Longrightarrow$ (*cohomologically rigid*) (fixing unipotent conjugacy classes at ∞ and the determinant) $\Longrightarrow$ (*rigid*) (fixing unipotent conjugacy classes at ∞ and the determinant).

2) E Deligne's extension of $(\mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}^{\text{an}}, 1 \otimes d)$ with $E \to \Omega^1_{\bar{X}}(\log \infty) \otimes_{\mathcal{O}_{\bar{X}}} E$ with nilpotent residues. So

$$H^1(\bar{X}, \Omega^\bullet(\log \infty) \otimes \mathcal{E}nd(E, \nabla)) = H^1(X, \mathcal{E}nd(\mathbb{L})) = 0.$$

Atiyah class computation + Hodge theory ([4, Appendix B])$\Longrightarrow$

$$0 = c_i(E) \in H^{2i}(\bar{X}, \mathbb{Q}),\ i \geq 1.$$

3) (E, ∇) necessarily semi-stable as saturated sub $(E', \nabla') \subset (E, \nabla)$ has also nilpotent residues, so $(E', \nabla') \subset (E, \nabla)$ locally split outside of a codimension 2 subset $\Sigma \subset \infty \subset \bar{X}$ in $\bar{X}$, and for $j : \bar{X} \setminus \Sigma \hookrightarrow \bar{X}$, has $(E', \nabla') = j_*j^*(E', \nabla') \subset (E, \nabla) = j_*j^*(E, \nabla)$. Thus (E', ∇') is Deligne's extension as it is determined outside of codimension 2. Thus $0 = c_i(E') \in H^{2i}(\bar{X}, \mathbb{Q}),\ i \geq 1$.

4) Langer moduli ([9, Theorem 1.1]) $M_{dR}(r),\ M_{Dol}(r)$ of *stable* log-objects to the Hilbert polynom $P(E) = P(\oplus_1^r \mathcal{O}_{\bar{X}})$ are defined over some S smooth $/\mathbb{Z}$, $M_{dR}(r)_S\ \ M_{Dol}(r)_S \to S$ flat and $X_S, \bar{X}_S \to S$ relative NCD and base change for $H^1(\bar{X}_S, \Omega^\bullet(\log \infty) \otimes \mathcal{E}nd(E_S, \nabla_S))$.

5) The characteristic polynomials of the residues $E_S \to \Omega^1(\log \infty_S) \otimes E_S$ at ∞_S are regular functions on $M_{dR}(r)_S$, so the nilpotent residues condition defines closed subs $M^\circ_{dR}(X)(r)_S \subset M_{dR}(r)_S$, which after shrinking are flat $/S$. $M^\circ_{dR}(X)(r)_S(\mathbb{C})$ consists precisely of the Deligne's extensions of the underlying (strongly cohomologically rigid) *irreducible* local systems. So it is 0-dimensional, say of cardinality N. By taking an étale cover of S, we may assume that $M^\circ_{dR}(X)(r)_S$ consists of N-sections.

6)

Theorem 3 ([1], Theorem 1.1)**.** *All* $\mathbb{L}$ *are integral. (Finiteness of* $\{\mathbb{L}\}$ *implies all* $\mathbb{L}$ *defined over one* $\mathcal{O}_L$*,* L *number field).*

7) Mochizuki ([10, Theorem 10.5]): any (E, ∇) with nilpotent residues deforms real analytically to a polarized $\mathbb{C}$-VHS, so rigidity implies all the (E, ∇) underlie a polarized $\mathbb{C}$-VHS. So with 6)

Claim 4. Assumption 1 $\Longrightarrow$ all $\mathbb{L}$ underlie a polarized $\bar{\mathbb{Z}}$-VHS.

8) Boundedness of possible Hodge filtrations.

Definition 5 (Good model)**.** $S/\mathbb{Z}$ smooth, condition 5), plus: all (E, ∇, Fil) defined over S, Fil locally split $/S$. So $(gr^{Fil}(E), KS)$ stable Higgs, locally free $/S$ nilpotent. We assume also for any $\mathrm{Spec}(W(\mathbb{F}_q)) \to S$, char $\mathbb{F}_q > 2r + 2$.

Theorem 6 ([3], Theorem A.4)**.** *Assumption 1* $\Longrightarrow$ *on a good model, for any* $\mathrm{Spec}(W) \to S$*, with* $W = W(\mathbb{F}_q)$*, the formal connection* $(\hat{E}_W, \hat{\nabla}_W)/\widehat{\bar{X}_W}$ *carries the structure of a locally free Fontaine-Lafaille module.*

(Standard definition of a Fontaine-Lafaille module right now irrelevant as we shall work with an equivalent definition).

3. Sketch of Proof of Theorem 6

s closed point of $\mathrm{Spec}(W)$. Has $M^\circ_{dR}(r)_s =: (dR)^\circ_s$ consisting of N s-points, and $(Dol)^\circ_s$ defined as the *set* of *stable* rank r log Higgs bundles (V, θ) with the residues of θ being nilpotent and Hilbert polynomial $P(V) = P(\oplus_1^r \mathcal{O}_{\bar{X}})$.

Claim 7. C^{-1} (Ogus-Vologodksy [11, Theorem 2.8]) : $(Dol)^\circ_s \to (dR)^\circ_s$ and is injective (in particular $(Dol)^\circ_s$ is finite).

Proof. C^{-1} defined for $p > r + 1$ plus lift to W_2, preserves stability, total Chern classes and nilpotency of the residues at ∞. □

Claim 8. $H^1(\bar{X}_s, \mathcal{E}nd(C^{-1}(V, \theta))) = H^1(\bar{X}_s, \mathcal{E}nd(V, \theta)) = 0$.

Proof. C^{-1} defined for $2r$ rank objects, preserves cohomology, $LHS = 0$ by 4). □

By Claim 7, $|(Dol)^\circ_s| = M \le N = |(dR)^\circ_s|$. Let $M' \le M$ be the number of objects in $(Dol)^\circ_s$ of the shape $(V, \theta) = (gr^{Fil}(E), KS)$.

Corollary 9. *Given* $(V, \theta) \in (Dol)^\circ_s$ *there is at most one possible possible* (E, ∇, Fil) *with* $(V, \theta) = (gr^{Fil}(E), KS)$.

Proof. Given Fil on (E,∇), then Rees $(\oplus_{i\in\mathbb{Z}}(Fil^iE)t^{-i},\nabla_t)$ on $X[t,t^{-1}]$ has fibre $(gr^{Fil}E,KS)$ at $t=0$ and (E,∇) at $t=\infty$. Deformation of $(gr^{Fil}E,KS)$ from $\mathbb{F}_q[t]/(t^n)$ to $\mathbb{F}_q[t]/(t^{n+1})$ is computed by $H^1(\bar{X}_s,\mathcal{E}nd(V,\theta))=0$. As any (E,∇) is endowed with at least one Fil (by 7) and good model), has $N\le M'$. □

Corollary 10. *1) $M'=M=N$; C is bijective; the p-curvature of any $(E,\nabla)\in(dR)^\circ_s$ is nilpotent;*
2) any $(E,\nabla)\in(dR)^\circ_s$ carries precisely one Fil;
3) $gr:(dR)^\circ_s\to(Dol)^\circ_s$, $(E,\nabla)\mapsto(gr^{Fil}E,KS)$ is well defined and bijective.

Proof. Ad 1): $N=M'\le M\le N$ (first inequality from Corollay 9, last inequality by Claim 7). Thus $M'=M=N$. C^{-1} sends nilpotent Higgs to nilpotent p-curvature dR.

Ad 2): any $(E,\nabla)\in(dR)^\circ_s$ carries at least one Fil by the good model and more would imply $M>N$ by Corollary 9. 3) follows. □

Corollary 11. *$\sigma:=C^{-1}\circ gr$ is a permutation of $(dR)^\circ_s$ and has finite order $f|N!$.*

Definition 12. The chain

$$(E_0,\nabla_0,Fil_0,\phi_0:C^{-1}(gr^{Fil_0}E_0,KS)\cong(E_1,\nabla_1),$$
$$E_1,\nabla_1,Fil_1,\dots,E_{f-1},\nabla_{f-1},Fil_{f-1},\phi_{f-1}:C^{-1}(gr^{Fil_{f-1}}E_{f-1},KS)\cong(E_0,\nabla_0))$$

is called a f-periodic Higgs-de Rham flow. ([7], [8]).

Proposition 13. *1) The f-periodic Higgs-de Rham flow lifts to $\widehat{\bar{X}_W}$ in what is still a f-'periodic Higgs-de Rham flow' over $\widehat{\bar{X}_W}$.*

2) The operator σ becomes the Frobenius on the isocrystals $(\hat{E}_W,\hat{\nabla}_W)_K$.

Here $W=W(\mathbb{F}_q)$, $K=\mathrm{Frac}(W)$ and recall that the p-curvatures of the mod p-reduction are nilpotent so we have isocrystals with a Frobenius structure.

Proof. The (E,∇) in $(dR)^\circ_s$ lift by definition to $\widehat{\bar{X}_W}$ together with their Hodge filtration. So gr is defined on $\widehat{\bar{X}_W}$ yielding some $(\hat{V}_W,\hat{\theta}_W)$ so (V_K,θ_K) which in addition are stable. C^{-1} is defined on $\widehat{\bar{X}_W}$ by Ogus-Vologodsky. As the lift $(\hat{E}_W,\hat{\nabla}_W)$ is uniquely determined by its reduction to $s\Longrightarrow f$-periodicity.

Remark 14. Claim 8 $\Longrightarrow$ (semi-continuity of coherent cohomology)

$$H^1(\bar{X}_K,\mathcal{E}nd(V_K,\theta_K))=0\Longrightarrow(V_K,\theta_K)\in M^\circ_{Dol}(r)_S(K).$$

If we define $M^\circ_{Dol}(X)(r)_S\subset M_{dR}(r)_S$, so $(V,\theta)\in M^\circ_{Dol}(\mathbb{C})$ if the residues at ∞ are nilpotent and $H^1(\bar{X},\mathcal{E}nd(V,\theta))=0$, and we assume by étally shrinking S in the good model that in addition $M^\circ_{Dol}(X)(r)_S(S)$ consists of different (finitely many) S-sections, we see that in fact we have N such and they all come from the Higgs-de Rham flow. If we had a log-Simpson correspondence at the level of moduli $/\mathbb{C}$, we would know this and could shorten the argument.

□

Proposition 15. *Lan-Sheng-Zuo, Lan-Sheng-Yang-Zuo: 1) Fully faithful functor: (f-periodic Higgs-de Rham flow with nilpotent residues level $\leq p-1$) $\to$ (log-Fontaine-Lafaille modules with $Frob^f$-structure, with nilpotent residues level $\leq p-1$).*

2) Generalization of Fontaine-Lafaille-Faltings [5] *Theorem 2.6* and p.43 i) applied to $\oplus_{i=0}^{f-1} Frob^i(object)$: fully faithful functor (log-Fontaine-Lafaille modules with $Frob^f$-structure, with nilpotent residues, level $\leq p-1$) $\to$ (crystalline local systems on X_K with values in $GL_r(\mathbb{Z}_{p^f})$).*

Remark 16. Crystalline here is defined by Fontaine on K, Faltings on 'small' opens defined by a ring R, étale over the Tate algebra of $\mathbb{G}_m^d$ over W, then on their rings by admissibility of a $B_{crys}(R)$, then gluing. Generalization by Tan-Tong, and just a few days ago Du-Lu-Moon-Shimizu. *All those concepts restricted to* $\mathrm{Spec}(W) \to X_W$ *yield the same definition, which is Fontaine's one.* Matti Würthen told us that he can construct directly a prismatic F-crystal in the sense of Bhatt-Scholze out of a Fontaine-Lafaille module. Granted this in the log-version, one could enhance a bit Theorem 6 to

Theorem 17 (?)**.** *Assumption 1 $\Longrightarrow$ on a good model, for any* $\mathrm{Spec}(W) \to S$*, the formal connection $(\hat{E}_W, \hat{\nabla}_W)/(\widehat{\bar{X}_W} \setminus \infty_W)$ carries the structure of a prismatic F-crystal.*

4. Étale theorem

Has $\mathcal{O}(S) \subset \mathbb{C}$ and choose $W \subset \mathbb{C}$ for $\mathrm{Spec}(W) \to S$ as in Theorem 6. This defines $\bar{K} \subset \mathbb{C}$ with (Grothendieck) $\pi_1(X_\mathbb{C}) \xrightarrow{\cong} \pi_1(X_{\bar{K}})$. By Theorem 3, under Assumption 1, each $\mathbb{L}$ is integral. A p-place $\mathfrak{p}$ of $\mathcal{O}_L$, $p = \mathrm{char}(\mathbb{F}_q)$, defines $\mathbb{L}_\mathfrak{p}$ on $X_\mathbb{C}$. Keep the same letter $\mathbb{L}_\mathfrak{p}$ for the $\bar{\mathcal{O}_{L_\mathfrak{p}}}$ local system on $X_{\bar{K}}$.

Theorem 18 ([3], Theorem A.21)**.** *Assumption 1 $\Longrightarrow \mathbb{L}_\mathfrak{p}$ defined on $X_{\bar{K}}$ descends to a crystalline local system on X_K with values in $GL_r(\mathbb{Z}_{p^f})$ for some $f|N!$.*

Proof. By the compatibitliy of Faltings p-adic Simpson correspondence on $\widehat{\bar{X}_W}$ [6, Theorem 5] with his Fontaine-Lafaille functor [5] *loc.cit*, calling π_i, $i = 1, \ldots, N$ the $GL_r(\mathbb{Z}_{p^f})$ local systems on X_K defined Proposition 15, where f is a l.c.m. of the periods, which divides $N!$, the $\pi_i/X_{\bar{K}}$ correspond to the Higgs bundles $(V,\theta)_{\bar{K}}$ which are stable, so in particular $\pi_i/X_{\bar{K}}$ is irreducible. Likewise

$$\dim_{\mathbb{C}_p} \mathrm{Hom}(X_{\mathbb{C}_p}, \pi_1, \pi_2) \leq \dim_{\mathbb{C}_p} \mathrm{Hom}(X_{\mathbb{C}_p}, (V_1,\theta_1), (V_2,\theta_2)) = 0$$

as Faltings functor from small p-adic local systems to Higgs bundles is faithful. So the number of isomorphism classes of $\pi_i/X_{\bar{K}}$ is the same as the one of π_i/X_K which is N. But by the complex Riemann-Hilbert correspondence, there are precisely N isomorphism classes of $\mathbb{L}_\mathfrak{p}$.

□

Remark 19. Theorem 18 via Remark 16 is the way Pila-Ananth Shankar-Tsimerman use our work for Shimura varieties of real rank ≥ 2 in their proof of the André-Oort conjecture for those [12, Theorem 1.2].

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