Density of Arithmetic Representations of Function Fields

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Background and Motivation

X is a smooth connected projective variety of dimension d;
defined over an algebraically closed field k;
$\mathcal{F}$ is a *semi-simple* ℓ-adic (i.e. $\bar{\mathbb{Q}}_\ell$-valued) local system, ℓ prime to $\mathrm{char} k$
(more generally a shifted semi-simple perverse sheaf but forget this generality for the lecture)

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Conjecture

Hard Lefschetz holds: if $\eta \in H^2(X, \mathbb{Q}_\ell)$ *is the Chern class of an ample line bundle, then the cup-product map* $\cup^i \eta : H^{d-i}(X, \mathcal{F}) \to H^{d+i}(X, \mathcal{F})$ *should be an isomorphism for all* $i \in \mathbb{N}$.

Hodge Theory and Weights

Why? $k = \mathbb{C}$: Harmonic theory $\Rightarrow$ Hard Lefschetz. For $\mathcal{F} = \mathbb{C}$: Hodge's proof, in general semi-simplicity $\Leftrightarrow$ existence of an harmonic metric (Simpson).

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Char. $k = p > 0$, may assume $k = \bar{\mathbb{F}}_p$, so $X = X_0 \otimes_{k_0} k$ where $k_0 = \mathbb{F}_q$. Pure weights on $H^j(X, \mathcal{F})$, different for different $j \Rightarrow$ (ultimately) Hard Lefschetz (a central theorem in Weil II).

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If $\mathcal{F}$ is punctually pure, Deligne's theory yields pure weights on $H^j(X, \mathcal{F})$, different for different j.
If $\mathcal{F}$ is defined over finite extension $k_0 \subset k_0' \subset k$, i.e. $\mathcal{F}$ is *arithmetic*, the Langlands correspondence of Drinfeld-Lafforgue implies that $\mathcal{F}$ is punctually pure.

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However, if $\mathcal{F}$ has rank 1, we could develop a strategy, inspired by Drinfeld's proof [de Jong conjecture $\Rightarrow$ Kashiwara conjecture], resting on the following theorem.

Rank 1

G geometric fundamental group;
$G^{\mathrm{ab},\ell}$ its pro$-\ell$ abelianization, assumed to be torsionfree;
$\mathbb{F}$ finite field of char. ℓ, $W(\mathbb{F})$ be the ring of Witt vectors; $W(\mathbb{F}) \hookrightarrow \bar{\mathbb{Q}}_\ell$.

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Then: $\mathrm{Spf}(W(\mathbb{F})[[G^{\mathrm{ab},\ell}]])$ is a formal torus;
$W(\mathbb{F})[[G^{\mathrm{ab},\ell}]] \otimes_{W(\mathbb{F})} \bar{\mathbb{Q}}_\ell$ is noetherian and Jacobson;
$\mathcal{S} := \mathrm{Spm}(W(\mathbb{F})[[G^{\mathrm{ab},\ell}]])(\bar{\mathbb{Q}}_\ell)$ noetherian topological space;
$\mathcal{S} =$set of iso. classes of rank 1 $\mathcal{F}$ with trivial residual representation;
Frobenius $\Phi \in \mathrm{Gal}(k/k_0')$ acts on $G^{\mathrm{ab},\ell}$, thus on $\mathcal{S}$;
arithmetic points: $= \cup_{n \in \mathbb{N}_{>0}} \mathcal{S}^{\Phi^n} =$ (class field theory) torsion points.

The Theorem in Rank 1 (E-K)

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Theorem $\Rightarrow$ Hard-Lefschetz.

$Z^0 = \{\text{bad points}\} \subset \mathcal{S}$ is Zariski constructible ($_{\text{main point}}$), Φ-invariant; Theorem $\Rightarrow$ if Z =Zariski closure $\neq \emptyset$ then Z^0 contains arithmetic points; impossible by Deligne's Hard Lefschetz.

□

Higher rank: the set $\mathcal{S}_{\bar\rho}$ generalizing $\mathcal{S}$

Set Up: X_0 smooth geometrically irreducible over $k_0 = \mathbb{F}_q \hookrightarrow k = \bar{\mathbb{F}}_p$;
$\mathbb{F}$ finite field of char. $\ell \neq p$;
semi-simple residual representation $\bar\rho : G \to GL_r(\mathbb{F})$.
Define $\mathcal{S}_{\bar\rho}$ = set of iso. classes of rank r $\mathcal{F}$ with ($_{\text{semi-simple}}$) residual representation $\bar\rho$.

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Remark

In rank 1, $\bar\rho$ was chosen to be trivial, but if $\bar\rho$ is any character $G \to GL_1(\mathbb{F}) = \mathbb{F}^\times$, then $\mathcal{S}_{\bar\rho}$ is isomorphic to $\mathcal{S}$ by translation with the Teichmüller lift of $\bar\rho$, so the theory is the same. In higher rank however there is no Teichmüller lift and the $\mathcal{S}_{\bar\rho}$ for various $\bar\rho$ are different.

Pseudo-deformations

Taylor-Wiles-...-Chenevier: $\mathcal{C}$ be the category of complete local $W(\mathbb{F})$-algebras $(A, \mathfrak{m}_A)$ such that $W(\mathbb{F}) \to A/\mathfrak{m}_A$ identifies $\mathbb{F}$ with the residue field of A;
The *functor of pseudo-deformations* $\mathrm{PD}_{\bar{\rho}} : \mathcal{C} \to \mathrm{Sets}$ of $\bar{\rho}$ assigns to A the set of continuous r-dimensional A-valued determinants $D : A[G] \to A$ such that $D \otimes_A \mathbb{F} \colon \mathbb{F}[G] \to \mathbb{F}$ is the $\mathbb{F}$-valued determinant induced by $\bar{\rho}$.

Pseudo-deformations II

• A Determinant $D : A[G] \to A$ is simply a compatible collection of $D_B : B[G] \to B$, where B is an A-algebra;
• uniquely determined by the r-coefficients of the (monic) 'characteristic polynomial' $\mathrm{char}_g = D_{A[t]}(t-g)$, $g \in G$;
• any continuous $\rho : G \to GL_r(\mathcal{O}), \mathcal{O}$ finite extension of $W(\mathbb{F})$ in $\mathcal{C}$, with (semi-simple) residual representation isomorphic to $\bar{\rho}$ defines $D(\rho)$ determined by $D(\rho)_{A[t]}(t-g) = \det(t-\rho(g))$, $g \in G$;
• if $r! \in W(\mathbb{F})^\times$, then standard definition: D determined by $D|_G$ and (a) invariance by conjugation (b) $\sum_{\sigma\in\Sigma_{r+1}}(-1)^{\mathrm{sign}\sigma}D(g_{\sigma(1)}\cdots g_{\sigma(r+1)}) = 0$;

$\mathcal{S}_{\bar\rho}$ as a noetherian topological space

Chenevier: $\mathrm{PD}_{\bar\rho}$ is representable by $(\mathcal{C} \ni R^P_{\bar\rho}, D^{R^P_{\bar\rho}} : R^P_{\bar\rho}[G] \to R^P_{\bar\rho})$, $R^P_{\bar\rho}$ noetherian algebra topologically spanned by the coefficients of the char_{g_i}, $g_i \in G$. If $\bar\rho$ is absolutely irreducible, then $R^P_{\bar\rho}$ is Mazur's deformation space of $\bar\rho$ and $D^{R^P_{\bar\rho}}$ is the determinant determined by the characteristic polynomial of the universal representation.

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We deduce: (i) $R^P_{\bar\rho} \otimes_{W(\mathbb{F})} \bar{\mathbb{Q}}_\ell$ noetherian and Jacobson;
(ii) $\mathcal{S}_{\bar\rho} = \mathrm{Spm}(R^P_{\bar\rho} \otimes_{W(\mathbb{F})} \bar{\mathbb{Q}}_\ell)$ noetherian topological space;
(iii) there are closed embeddings: $\mathrm{char}_{\underline{g}} : \mathcal{S}_{\bar\rho} \hookrightarrow (\mathbb{A}^{rm})^{\hat{}}_{\underline{p}}$ where $\underline{p} = (p_1, \ldots, p_m)$ are the characteristic polynomials of $\bar\rho$ on well chosen $g_1, \ldots g_m \in G$.

The arithmetic points of $\mathcal{S}_{\bar\rho}$

(arithmetic) Frobenius $\Phi \in \mathrm{Gal}(k/k_0)$ lifts to $\pi_1^{\text{ét}}(X_0)$;
thus acts by conjugation on G (modulo conjugation by G).
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This defines the arithmetic points $\mathcal{A} = \cup_{n\in\mathbb{N}_{>0}} \mathcal{S}_{\bar\rho}^{\Phi^n} \subset \mathcal{S}_{\bar\rho}$.
So an arithmetic point is a semi-simple ℓ-adic local system with (semi-simple) residue local system $\bar\rho$ which is defined over $X_0 \otimes_{k_0} k_0'$ where $k_0 \subset k_0' \subset k$ is a finite extension.

The conjecture (E-K)

Conjecture

Strong form: A Zariski closed subset $Z \subset \mathcal{S}_{\bar{\rho}}$ invariant under Φ^n for some integer $n > 0$ is the Zariski closure of its arithmetic points $\mathcal{A} \cap Z \subset Z$.
Weak form: $\mathcal{S}_{\bar{\rho}}$ is the Zariski closure of its arithmetic points $\mathcal{A}$.

The theorems (E-K)

Theorem

1) Strong form true for $r = 1$, X_0 projective (mentioned in the motivation) or more generally when Φ is pure on $H^1(X, \mathbb{Q}_\ell)$ of weight $\neq 0$.
2) Weak form true for X_0 curve, $\bar{\rho}$ absolutely irreducible and $\ell > 2$.
3) Strong form true for $X_0 = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $r = 2$ and $\bar{\rho}$ tame.

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3) Strong form true for $X_0 = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $r = 2$ *and* $\bar{\rho}$ *tame.*

Proposition (as a further motivation)

1) Strong form on $X_0 = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $r \geq 2$ *and* $\bar{\rho}$ *tame* $\Rightarrow$ *strong form in any dimension and rank.*
2) Strong form in any dimension and rank $\Rightarrow$ *Hard Lefschetz.*

Comments on the theorems

1) Thm 1) is on the arXiv (last fall). Geometric proof, uses weights and Class Field Theory.
2) Thm 2) (weak form on curves) proved using the Langlands program over function fields, de Jong's method and de Jong's conjecture, proven by Gaitsgory for $\ell > 2$ using the geometric Langlands program.
3) Thm 3) (strong form in rank 2 on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ for a tame $\bar{\rho}$) has a geometric proof, using weights, and ultimately Thm 1). It does not use the Langlands program.

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Remark

Our proof of Thm 3) yields also a geometric proof of de Jong's conjecture in rank 2 on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ for a tame $\bar{\rho}$ and any ℓ, thus without Langlands program.

Leitfaden of Proof of Thm 3)

1-st step: Grothendieck's specialization's theory: the tame quotient G^t of G is a topological quotient of the profinite completion $\hat{F}_2$ of the free group in 2 letters; it is topologically spanned by g_0, g_1 the images (after choosing étale paths from 0, resp. 1 to the base point of G) of the generators of the tame local inertia at 0, resp. 1. Similarly one has g_∞ with, for appropriate choices, the relation $g_0 \cdot g_1 \cdot g_\infty = 1$. Set $\underline{g} = (g_0, g_1, g_\infty)$.

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2nd step: Elementary invariant theory: for $p_a = \mathrm{char}(\bar{\rho}(g_a))$, $a = 0, 1, \infty$, one has $\mathrm{char}_{\underline{g}} : \mathcal{S}_{\bar{\rho}} \to (\mathbb{A}^{2 \cdot 3 = 6})\hat{}_p$ closed embedding.

Leitfaden of Proof of Thm 3) II

3rd step: Arithmetic Frobenius Φ: $\Phi(g_a)$ conjugate to g_a^q, thus can make $\mathrm{char}_{\underline{g}}$ equivariant. It reduces the strong form of the conjecture to $(\mathbb{A}^{2\cdot\overline{3}=6})^{\hat{}}_p$.

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4th and last step: $\pi : (\mathbb{G}_m^{2\cdot 3})\hat{} \to (\mathbb{A}^{2\cdot 3=6})\hat{}_p$ formal torus which separates the roots of the 3 degree 2 polynomials, translated by the Teichmüller lifts of those, on which Φ acts by raising to the q-th power. There one applies Thm 1). □

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Comment: Still for $\bar{\rho}$ tame, which in view of Prop 1) is the most important case, our proof shows that it is enough to find suitable words in the g_a which embed $\mathcal{S}_{\bar{\rho}}$ and on which we control the eigenvalues of Φ. Even for $r = 3$ this is difficult.

Leitfaden of Proof of Prop 1)

1st step: Reduction to X_0 a curve: 'Lefschetz theory à la Wiesend' but for pseudo-characters yields a closed embedding of $\mathcal{S}_{\bar{\rho}}$ on X to $\mathcal{S}_{\bar{\rho}}$ on a good curve.

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2nd step: Reduction to X_0 a curve and $\bar{\rho}$ trivial: taking the Galois cover associated to $\bar{\rho}$, one has to show that the density property is true downstairs if it is upstairs.

3nd step: Choose a tame 'Belyi' map: $h : \bar{X}_0 \to \mathbb{P}^1$, where $X_0 \hookrightarrow \bar{X}_0$ is the normal compactification (possible by Saïdi-Sugiyama-Yasuda), so $h_* \bar{\rho}$, (which has higher rank), is tame. One has to show that the density property is true upstairs if it is downstairs. (Annoyance: no induction of pseudo-representations documented in the literature, so one has to do it by hand). □