At the request of Adrian Langer and Lei Zhang, we detail the proof of the *Full Faithfulness* part of Theorem 3.5. We thank Adrian Langer and Lei Zhang for their interest.

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Recall the statement. Let $U \subset X$ be smooth in X a normal projective variety of dimension d over a characteristic $p > 0$ algebraically closed field k such that

$$\mathrm{codim}_X(X \setminus U) \geq 2.$$

Let $i : C \hookrightarrow X$ be a reduced closed subscheme which is set theoretically the intersection of $(d-1)$ ample divisors. By the Bertini theorem, for each given degree of the ample divisors, there is an open dense of the Hilbert schemes of such complete intersections with the property

$$C \subset U = X \setminus X_{\mathrm{sing}} \tag{1}$$

where X_{sing} is the singular locus. We denote by $\mathrm{Strat}(U)$, resp. $\mathrm{Strat}(C)$ the (tannakian) category of stratified bundles on U, resp. C.

Theorem 0.1. *Assuming* (1)*, the restriction functor*

$$i^* : \mathrm{Strat}(U) \to \mathrm{Strat}(C)$$

is fully faithful.

We now justify Theorem 0.1.

Proof. Theorem 0.1 precisely says that for all $W_i \in \mathrm{Strat}(U),\ i = 1, 2$ the restriction functor

$$\mathrm{Hom}_U(W_1, W_2) \xrightarrow{i^*} \mathrm{Hom}_C(i^*W_1, i^*W_2)$$

is an isomorphism. This is equivalent to

$$H^0_{\mathrm{strat}}(U, W) \xrightarrow{i^*} H^0_{\mathrm{strat}}(C, i^*W)$$

being an isomorphism for any $W \in \mathrm{Strat}(U)$, which in turn is equivalent to the same statement with $W \in \mathrm{Pic}^{\mathrm{strat}}(U)$, the full subcategory of $\mathrm{Strat}(U)$ of rank 1 objects.

Lemma 0.2. $\pi_1(C) \to \pi_1(U)$ *is surjective.*

Proof. Let $f : V \to U$ be a finite étale connected cover. The normalization Y of U in $k(V)$ yields a finite morphism $Y \to X$ extending f (and ramified along $X \setminus U$). The pullback D of C in Y is set theoretically the intersection of ample divisors in a normal projective variety so is connected.

□

The surjective restriction $\mathrm{Cl}(X) \to \mathrm{Pic}(U)(k)$ from the class group of X to the group of k-line bundles on U is an isomorphism as there are no Weil divisor supported in $X \setminus U$. By [Kin13, Proof of 3.16] (see also [BGS11, Proposition 3.2]) there is an exact sequence

$$0 \to A_X(k) \to \mathrm{Cl}(X) \to NS(X) \to 0$$

where A_X is an abelian variety and $NS(X)$ an abelian group of finite type. Its restricts via i^* to

$$0 \to A_C(k) \to \mathrm{Pic}(C)(k) \to NS(C) \to 0$$

where $A_C = \mathrm{Pic}^0(C)_{\mathrm{red}}$ is a connected commutative algebraic group variety over k. As $H^0(U, \mathcal{O}^\times) = H^0(C, \mathcal{O}^\times) = k^\times$ by [Kin13, Cor.3.7] it holds

$$\mathrm{Pic}^{\mathrm{strat}}(U) = \mathrm{lim}_p \mathrm{Pic}(U)(k), \ \mathrm{Pic}^{\mathrm{strat}}(C) = \mathrm{lim}_p \mathrm{Pic}(C)(k)$$

where $_p$ indicates the inverse system over the multiplication by p, and is endowed with the projection

$$\tau : \mathrm{Pic}^{\mathrm{strat}}(U) \to \mathrm{Pic}(U)(k)$$

on the first component. Moreover, the torsion in $\mathrm{Pic}^{\mathrm{strat}}(U)$ maps to the torsion in $\mathrm{lim}_p NS$ which is precisely the finite group $NS[p']$. So for $L \in \mathrm{Pic}^{\mathrm{strat}}(U)$ there is $N \in \mathbb{N}_{\geq 1}$ prime to p with $NL \in A_X(k)$. On the other hand $A_X(k)[p']$, which lies in $\mathrm{Pic}^{\mathrm{strat}}(U)[p']$, is dense in $A_X(k)$ and by Lemma 0.2 $\mathrm{Pic}^{\mathrm{strat}}(U)[p'] \subset \mathrm{Pic}^{\mathrm{strat}}(C)[p']$. Thus

$$\mathrm{Ker}\big(A_X(k) \subset A_C(k)\big) \subset A_X(k)[p] \subset \mathrm{Pic}(U)(k)[p]$$

and is finite. However

$$\tau(\mathrm{Pic}^{\mathrm{strat}}(U)) \cap \mathrm{Pic}(U)(k)[p] = 0.$$

So if $i^*L = 0$ then $NL = 0$. But then again by Lemma 0.2 i^*L is torsion of precisely the same order as L. This shows

$$i^* : \mathrm{Pic}^{\mathrm{strat}}(U) \to \mathrm{Pic}^{\mathrm{strat}}(C)$$

is injective and finishes the proof.

□

References

[Kin13] Kindler, L.: *Evidence for a generalization of Gieseker's conjecture on stratified bundles in positive characteristic*, Doc. Math. **18** (2013), 1215–1242.

[BGS11] Boissière, S., Gabber, O., Serman, O.: *Sur le produit de variétés localement factorielles ou* **Q***-factorielles*, `https://arxiv.org/pdf/1104.1861`.