Freie Universität Berlin

Kurt Gödel - Selected Topics

Intuitionistic Logic versus Classical Logic

Gödel's Interpretation and Conjectures

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VII

AN INTERPRETATION OF THE INTUITIONISTIC SENTENTIAL LOGIC

KURT GÖDEL

ONE can interpret[1] Heyting's sentential logic in terms of the concepts of the usual sentential logic and of the concept 'p is provable' (denoted by Bp), if one assumes for the latter the following axiom system $\mathfrak{S}$:

1. $Bp \rightarrow p$
2. $Bp \rightarrow .B(p \rightarrow q) \rightarrow Bq$
3. $Bp \rightarrow BBp$

In addition, we have to assume the axioms and rules of inference of the usual sentential logic for the concepts $\rightarrow, \sim, \cdot, \vee$, plus a new rule of inference: From A one may infer BA.

Heyting's basic concepts are to be translated in the following way:

$\neg p$	$\sim Bp$
$p \supset q$	$Bp \rightarrow Bq$
$p \vee q$	$Bp \vee Bq$
$p \wedge q$	$p \cdot q$

We could also translate equally well $\neg p$ with $B \sim Bp$, and $p \wedge q$ with $Bp \cdot Bq$. The translation of an arbitrary valid formula of Heyting's system follows from $\mathfrak{S}$, whereas the translation of $p \vee \neg p$ does not follow from $\mathfrak{S}$. In general, no formula of the form $BP \vee BQ$ is provable from $\mathfrak{S}$, unless BP or BQ is provable from $\mathfrak{S}$. Presumably, a formula of Heyting's calculus is valid if and only if its translation is provable from $\mathfrak{S}$.

The system $\mathfrak{S}$ is equivalent to Lewis's system of strict implication, if Bp is translated with Np (cf. p. 15 of this number [i.e. *Ergebnisse*, Vol. 4—Ed.]) and if Lewis's system is completed with Becker's[2] 'Zusatzaxiom' $Np \prec NNp$.

From *Ergebnisse eines mathematischen Kolloquiums*, Vol. 4 (Verlag Franz Deuticke, Vienna, 1933), pp. 39–40; translated here by J. Hintikka and L. Rossi. Printed by permission of Verlag Franz Deuticke and the author.

[1] Kolmogorov (*Mathematische Zeitschrift*, Vol. 35, p. 58) has given a somewhat different interpretation of the intuitionistic sentential logic, though without giving any precise formalism.

[2] 'Zur Logik der Modalitäten', *Jahrbuch für Philosophie und phänomenologische Forschung*, Vol. 11 (1930), p. 497.

It can be pointed out that not all formulas provable from $\mathfrak{S}$ hold for the concept 'provable in a given formal system S'. For example, $B(Bp \rightarrow p)$ never holds for the latter, i.e. it holds for no system which includes arithmetic. For otherwise e.g. $B(0 \neq 0) \rightarrow 0 \neq 0$ and hence also $\sim B(0 \neq 0)$ were provable in S, i.e. the consistency of S were provable in S.

"An Interpretation of the Intuitionistic Sentential Logic."

K. Gödel, 1933

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Structure

- Intuitionism, Intuitionistic Logic & Heyting's Calculus
- Classical Propositional Logic
- Gödel’s Interpretation
- Gödel’s Results

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Intuitionism

Intuitionism is a philosophy of mathematics that was introduced by Luitzen Egbertus Jan Brouwer in 1908.

Intuitionism

Intuitionism is a philosophy of mathematics that was introduced by Luitzen Egbertus Jan Brouwer in 1908.

> "It does not make sense to think of truth or falsity of a mathematical statement independently of our knowledge concerning the statement.
> A statement is *true* if we have proof of it, and *false* if we can show that the assumption that there is a proof for the statement leads to a contradiction."
>
> A. S. Troelstra and D. van Dalen, *Constructivism in Mathematics*, 1988

The truth of a mathematical statement can only be conceived via a mental construction (a *proof* or *verification*) that proves it to be true.

⇛ Intuitionism centers on proof rather than truth.

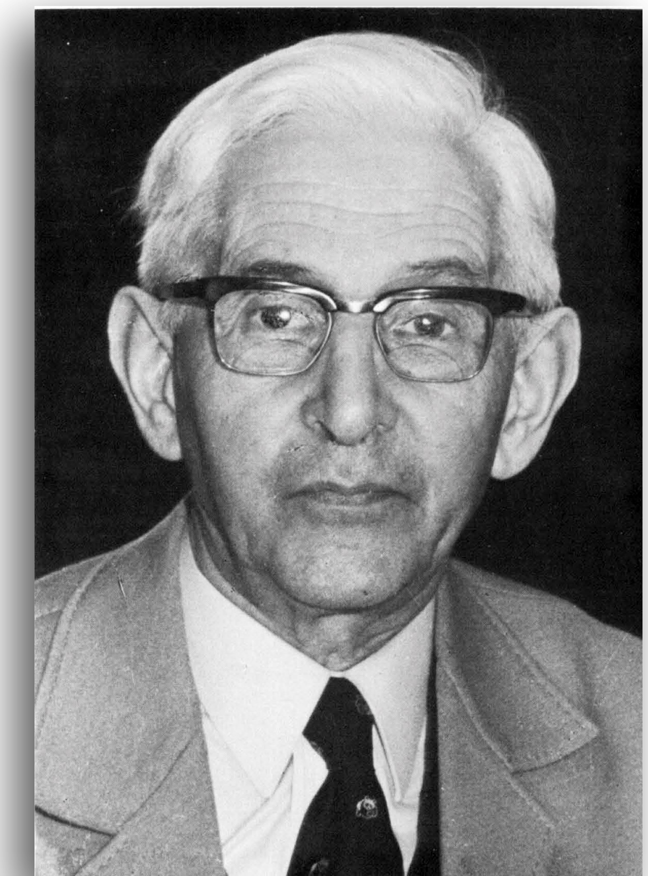
Intuitionistic Propositional Calculus (IPC)

Introduced by Arend Heyting in 1930.

A logical calculus describing rules for the derivation of propositions that are valid from the point of view of intuitionism.

Intuitionistic logic is most easily described as classical logic without the principle of excluded middle.

$$\varphi \vee \neg \varphi$$



Syntax of IPC

Alphabet. Propositional variables ($A, B, C \cdots$), logical connectives $\cdot$, $+$, $\supset$ and constant symbol $\perp$.

The negation of a formula φ , denoted as $\sim \varphi$, is abbreviated by $\varphi \supset \perp$.

Atomic Formulas. Any propositional variable or $\perp$ is an atomic formula.

Formulas. The *(well-formed) formulas* of $\mathbf{IPC}$ are defined inductively as follows:

- Each atomic formula is a *well-formed formula*.
- If φ and ψ are *well-formed formulas*, so are $\varphi \cdot \psi$, $\varphi + \psi$ and $\varphi \supset \psi$.
- Nothing else is a *well-formed formula*.

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Semantics of IPC

The *Brouwer-Heyting-Kolmogorov (BHK) interpretation* states informally what is intended to be a proof of a given formula:

- A proof of $\varphi \cdot \psi$ consists of a proof of φ and a proof of ψ.
- A proof of $\varphi + \psi$ is given by presenting either a proof of φ or a proof of ψ.
- A proof of $\varphi \supset \psi$ is a construction which, given a proof of φ, returns a proof of ψ.
- $\perp$ has no proof.

[- A proof of $\sim \varphi$ is a construction which, given a proof of φ, would return a proof of $\perp$.]

where φ, ψ are formulas in $\mathbf{IPC}$.

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Proof system for IPC

Hilbert-style system for IPC

Axiom schemes where φ, ψ, μ are formulas in **IPC**:

1. $\varphi \supset (\psi \supset \varphi)$
2. $(\varphi \supset (\psi \supset \mu)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \mu))$
3. $(\varphi \cdot \psi) \supset \varphi$
4. $(\varphi \cdot \psi) \supset \psi$
5. $\varphi \supset (\psi \supset (\varphi \cdot \psi))$
6. $\varphi \supset (\varphi + \psi)$
7. $\psi \supset (\varphi + \psi)$
8. $(\varphi \supset \mu) \supset ((\psi \supset \mu) \supset ((\varphi + \psi) \supset \mu))$
9. $\perp \supset \varphi$

Hilbert-style system for IPC

Axiom schemes where φ, ψ, μ are formulas in $\mathbf{IPC}$:

1. $\varphi \supset (\psi \supset \varphi)$

   $\vdots$

7. $\psi \supset (\varphi + \psi)$
8. $(\varphi \supset \mu) \supset ((\psi \supset \mu) \supset ((\varphi + \psi) \supset \mu))$
9. $\perp \supset \varphi$

Inference rules where φ, ψ are formulas in $\mathbf{IPC}$:

1. From φ and $\varphi \supset \psi$, conclude ψ. *(Modus Ponens)*

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Classical Propositional Logic (CPL)

Syntax of CPL

Alphabet. Propositional variables ($A, B, C \cdots$), logical connectives $\wedge$, $\vee$, $\rightarrow$ and constant symbol $\perp$.

The negation of a formula φ , denoted as $\neg\varphi$, is abbreviated by $\varphi \rightarrow \perp$.

Atomic Formulas. Any propositional variable or $\perp$ is an atomic formula.

Formulas. The *(well-formed) formulas* of $\mathbf{CPL}$ are defined inductively as follows:

- Each atomic formula is a *well-formed formula*.
- If φ and ψ are *well-formed formulas*, so are $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\varphi \rightarrow \psi$.
- Nothing else is a *well-formed formula*.

Semantics of CPL

The semantics of $\mathbf{CPL}$ is subject to the usual conditions ("truth tables"):

- $\varphi \wedge \psi$ is *true* if and only if φ is *true* and ψ is *true*.
- $\varphi \vee \psi$ is *true* if and only if φ is *true* or ψ is *true* (or both).
- $\varphi \to \psi$ is *false* if and only if φ is *true* and ψ is *false*.
- $\bot$ is *false*.

[- $\neg\varphi$ is *true* if and only if φ is *false*.]

where φ, ψ are formulas in $\mathbf{CPL}$.

Proof system for CPL

Hilbert-style system for CPL

Axiom schemes where φ, ψ, μ are formulas in **CPL**:

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \mu)) \to ((\varphi \to \psi) \to (\varphi \to \mu))$
3. $(\varphi \wedge \psi) \to \varphi$
4. $(\varphi \wedge \psi) \to \psi$
5. $\varphi \to (\psi \to (\varphi \wedge \psi))$
6. $\varphi \to (\varphi \vee \psi)$
7. $\psi \to (\varphi \vee \psi)$
8. $(\varphi \to \mu) \to ((\psi \to \mu) \to ((\varphi \vee \psi) \to \mu))$
9. $\perp \to \varphi$
10. $\varphi \vee (\varphi \to \perp)$

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Hilbert-style system for CPL

Axiom schemes where φ, ψ, μ are formulas in $\mathbf{CPL}$:

1. $\varphi \to (\psi \to \varphi)$

   $\vdots$

8. $(\varphi \to \mu) \to ((\psi \to \mu) \to ((\varphi \vee \psi) \to \mu))$
9. $\perp \to \varphi$
10. $\varphi \vee (\varphi \to \perp)$

Inference rules where φ, ψ are formulas in $\mathbf{CPL}$:

1. From φ and $\varphi \to \psi$, conclude ψ. *(Modus Ponens)*

Expansion of CPL into system $\mathcal{G}$

Additional concept 'φ is provable' (denoted by $\mathrm{B}\varphi$ with an additional unary operator B).

Additional axiom schemes where φ, ψ are formulas in $\mathcal{G}$:

11. $\mathrm{B}\varphi \to \varphi$
12. $\mathrm{B}\varphi \to (\mathrm{B}(\varphi \to \psi) \to \mathrm{B}\psi)$
13. $\mathrm{B}\varphi \to \mathrm{B}\mathrm{B}\varphi$

Additional inference rules where φ, ψ are formulas in $\mathcal{G}$:

2. From φ conclude $\mathrm{B}\varphi$.

Hilbert-style system for $\mathscr{G}$

Axiom schemes where φ, ψ, μ are formulas in $\mathscr{G}$:

1. $\varphi \to (\psi \to \varphi)$

$\vdots$

10. $\varphi \vee (\varphi \to \perp)$
11. $\mathsf{B}\varphi \to \varphi$
12. $\mathsf{B}\varphi \to (\mathsf{B}(\varphi \to \psi) \to \mathsf{B}\psi)$
13. $\mathsf{B}\varphi \to \mathsf{B}\mathsf{B}\varphi$

Inference rules where φ, ψ are formulas in $\mathscr{G}$:

1. From φ and $\varphi \to \psi$, conclude ψ. *(Modus Ponens)*
2. From φ conclude $\mathsf{B}\varphi$.

Interpretation

Interpretation function $g: \ \mathbf{IPC} \to \mathscr{G}$ is defined as follows:

$$
\begin{aligned}
g(A) &= A \\
g(\varphi \cdot \psi) &= g(\varphi) \ \wedge \ g(\psi) \\
g(\varphi + \psi) &= \mathsf{B}\ g(\varphi) \ \vee \ \mathsf{B}\ g(\psi) \\
g(\varphi \supset \psi) &= \mathsf{B}\ g(\varphi) \ \to \ \mathsf{B}\ g(\psi) \\
g(\ \bot\ ) &= \bot
\end{aligned}
$$

[$g(\sim \varphi) = \neg\, \mathsf{B}\ g(\varphi)$]

where A is a propositional variable and φ, ψ are formulas in $\mathbf{IPC}$.

Variant Interpretation

Interpretation function $g: \ \mathbf{IPC} \rightarrow \mathscr{G}$ is defined as follows:

$$
\begin{aligned}
g(A) &= A \\
g(\varphi \cdot \psi) &= \mathsf{B}\, g(\varphi) \wedge \mathsf{B}\, g(\psi) \\
g(\varphi + \psi) &= \mathsf{B}\, g(\varphi) \vee \mathsf{B}\, g(\psi) \\
g(\varphi \supset \psi) &= \mathsf{B}\, g(\varphi) \rightarrow \mathsf{B}\, g(\psi) \\
g(\perp) &= \perp \\
[\ g(\sim \varphi) &= \mathsf{B} \neg \mathsf{B}\, g(\varphi) \ ]
\end{aligned}
$$

where A is a propositional variable and φ, ψ are formulas in $\mathbf{IPC}$.

Exemplary Interpretation

$$\varphi = A + (A \supset \bot)$$

$$g(\varphi) = ?$$

$$\begin{aligned}
g(A + (A \supset \bot)) &= \mathsf{B}\, g(A) \vee \mathsf{B}\, g(A \supset \bot) \\
&= \mathsf{B}A \vee \mathsf{B}\, g(A \supset \bot) \\
&= \mathsf{B}A \vee \mathsf{B}(\mathsf{B}\, g(A) \to \mathsf{B}\, g(\bot)) \\
&= \mathsf{B}A \vee \mathsf{B}(\mathsf{B}A \to \mathsf{B}\bot)
\end{aligned}$$

Interpretation function $g : \mathbf{IPC} \to \mathscr{G}$:

$$\begin{aligned}
g(A) &= A \\
g(\varphi + \psi) &= \mathsf{B}\, g(\varphi) \vee \mathsf{B}\, g(\psi) \\
g(\varphi \supset \psi) &= \mathsf{B}\, g(\varphi) \to \mathsf{B}\, g(\psi) \\
&\vdots
\end{aligned}$$

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Exemplary Interpretation

$$\varphi = A + (A \supset \bot)$$

$$g(\varphi) = \mathsf{B}A \vee \mathsf{B}(\mathsf{B}A \to \mathsf{B}\bot)$$

$$\begin{aligned} g(A + (A \supset \bot)) &= \mathsf{B}\, g(A) \vee \mathsf{B}\, g(A \supset \bot) \\ &= \mathsf{B}A \vee \mathsf{B}\, g(A \supset \bot) \\ &= \mathsf{B}A \vee \mathsf{B}(\mathsf{B}\, g(A) \to \mathsf{B}\, g(\bot)) \\ &= \mathsf{B}A \vee \mathsf{B}(\mathsf{B}A \to \mathsf{B}\bot) \end{aligned}$$

Interpretation function $g : \mathbf{IPC} \to \mathscr{G}$:

$$\begin{aligned} g(A) &= A \\ g(\varphi + \psi) &= \mathsf{B}\, g(\varphi) \vee \mathsf{B}\, g(\psi) \\ g(\varphi \supset \psi) &= \mathsf{B}\, g(\varphi) \to \mathsf{B}\, g(\psi) \\ &\vdots \end{aligned}$$

Remarks

No formula $\mathsf{B}\varphi \vee \mathsf{B}\psi$ is derivable from $\mathscr{G}$, unless $\mathsf{B}\varphi$ *or* $\mathsf{B}\psi$ is derivable from $\mathscr{G}$.

⇛ The Law of Excluded Middle, $\mathsf{B}\varphi \vee \mathsf{B}(\varphi \rightarrow \perp)$, is not derivable from $\mathscr{G}$.

The operator B should be interpreted as 'provable by any correct means' and must not be interpreted as 'provable in a given formal system' because this would contradict Gödel's second incompleteness theorem.

Gödel's Results

Gödel claims that if a formula is derivable from intuitionistic logic, then its 'translation' is derivable from $\mathscr{G}$, that is:

$$\text{If } \vdash_{\mathbf{IPC}} \varphi \text{ , then } \vdash_{\mathscr{G}} g(\varphi) \text{ .}$$

He conjectures that the converse also holds, and thus we should have:

$$\vdash_{\mathbf{IPC}} \varphi \text{ if, and only if } \vdash_{\mathscr{G}} g(\varphi) \text{ .}$$

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Lewis Modal System $\mathbf{S}_4$

The system $\mathbf{S}_4$ is a modal propositional logic.

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What is Modal Logic?

Modal logic describes the logical relations of modalities as necessities and possibilities.

φ	φ is *true*
$\Box\varphi$	φ is necessarily *true*
$\Diamond\varphi$	φ is possibly *true*

Modal logic extends classical propositional logic to include operators expressing modality, namely $\Box$ for necessity and $\Diamond$ for possibility.

$\neg\Box EarthHasExactlyOneMoon$ The Earth has exactly one moon.

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Lewis Modal System $\mathbf{S}_4$

The system $\mathbf{S}_4$ is a modal propositional logic.

Lewis Modal System $\mathbf{S}_4$

The system $\mathbf{S}_4$ is a modal propositional logic with necessity operator $\Box$.

For $\bot$ and $\to$, $\wedge$, $\vee$ we take the rules and axioms of classical propositional logic as before.

Additional axiom schemes where φ, ψ are formulas in $\mathbf{S}_4$:

$$\Box \varphi \to \varphi$$

$$\Box \varphi \to (\Box (\varphi \to \psi) \to \Box \psi)$$

$$\Box \varphi \to \Box \Box \varphi$$

Additional inference rules where φ, ψ are formulas in $\mathbf{S}_4$:

From φ, conclude $\Box \varphi$. *(Necessity Rule)*

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Lewis Modal System $\mathbf{S}_4$

Additional axiom schemes where φ, ψ are formulas in $\mathbf{S}_4$:

$$\Box \varphi \to \varphi$$

$$\Box \varphi \to (\Box (\varphi \to \psi) \to \Box \psi)$$

$$\Box \varphi \to \Box \Box \varphi$$

Additional inference rules where φ, ψ are formulas in $\mathbf{S}_4$:

From φ, conclude $\Box \varphi$. *(Necessity Rule)*

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System $\mathscr{G}$

Additional axiom schemes where φ, ψ are formulas in $\mathscr{G}$:

$$B\ \varphi \rightarrow \varphi$$

$$B\ \varphi \rightarrow (\ B\ (\varphi \rightarrow \psi) \rightarrow B\ \ \psi)$$

$$B\ \varphi \rightarrow\ B\ B\ \varphi$$

Additional inference rules where φ, ψ are formulas in $\mathscr{G}$:

From φ, conclude $B\ \varphi$.

Relation of $\mathscr{G}$ and $\mathbf{S}_4$

If Bφ is understood as 'φ is necessary' the expanded system $\mathscr{G}$ results as the Lewis modal system $\mathbf{S}_4$, with B written for the necessity operator $\square$.

Hence, Gödel's result shows that there is an embedding of the intuitionistic propositional logic $\mathbf{IPC}$ into the modal logic $\mathbf{S}_4$. Therefore,

$$\vdash_{\mathbf{IPC}} \varphi \text{ if, and only if } \vdash_{\mathbf{S}_4} g(\varphi).$$

Thank you

Literature:

S. Artemov. On Two Models of Provability. In Mathematical Problems from Applied Logics, New Logics for the XXIst Century, II, Dov M. Gabbay (Eds.), International Mathematical Series, Springer, 2006.

K. Gödel. Eine Interpretation des intuitionistischen Aussagenkalküls. Ergebnisse Math. Kolloq. 4, 1933, pp. 39–40; English translation in: Kurt Gödel Collected Works. Vol. 1, S. Feferman et al. (Eds.), Oxford Univ. Press, Oxford, Clarendon Press, New York, 1986, pp. 301–303.

A. S. Troelstra, D. van Dalen. Constructivism in Mathematics: An Introduction. North-Holland Publishing, Amsterdam.

A. Heyting. Die formalen Regeln der intuitionistischen Logik. Sitzungsberichte der preussischen Akademie der Wissenschaften, 1930, pp. 42–71, 158–169.

https://plato.stanford.edu/entries/logic-intuitionistic/. Retrieved February 27, 2019.

A. Akbar Tabatabai. Provability Interpretation of Propositional and Modal Logics. 2017.