# REFERENCE MEASURES AND THE FINE TOPOLOGY

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ABSTRACT. It is proved that a positive kernel on a Polish space $X$ has a reference measure if and only if the associated fine topology is not discrete on any compact perfect subset of $X$.

In [2] and [5], remarkable results concerning the existence of a reference measure for a positive kernel on a Polish space were established. In this note we intend to refine these results using tools from functional analysis, more precisely from the geometry of Banach spaces.

As in [2] we consider a positive kernel $V$ on $\mathscr{B}(X)$, the space of Borel functions on a Polish space $X$; i.e., $V$ is a mapping of the form

$$(Vf)(y) = \int_X f(x)\, d\mu_y(x) \tag{1}$$

with $(\mu_y)_{y\in X}$ a family of positive measures such that $y \mapsto \mu_y(A)$ is Borel for every Borel set $A$. Also $(\mu_y)$ is referred to as a positive kernel. We further assume that $V$ is *proper*, that is, there exists a Borel function $h > 0$ such that $(Vh)(y) < \infty$ for all $y$, and we suppose that the *dominance principle* holds: If $f, g \in \mathscr{B}(X)$ are nonnegative and $(Vf)(y) \geq (Vg)(y)$ on $\{g > 0\}$, then $(Vf)(y) \geq (Vg)(y)$ everywhere.

Let

$$\mathscr{S} = \Bigl\{\sup_n Vf_n \colon f_n \in \mathscr{B}(X),\ f_n \geq 0,\ (Vf_n) \text{ increases}\Bigr\}$$

be the cone of *excessive functions* associated to $V$. We suppose that $u \wedge v \in \mathscr{S}$ whenever $u, v \in \mathscr{S}$ and that $\mathscr{S}$ contains the positive constant functions. The *fine topology* on $X$ is the initial topology for the family $\mathscr{S}$, hence the coarsest topology that makes each $u \in \mathscr{S}$ continuous.

We now have the following theorem.

**Theorem 1.** *Under the above assumptions, the following statements are equivalent:*

(a) *There is a reference measure for $V$; i.e., a measure $m$ such that all the $\mu_y$ from* (1) *are absolutely continuous with respect to $m$.*

(b) *The fine topology satisfies the countable chain condition, meaning that each family of nonvoid pairwise disjoint open sets is at most countable.*

(c) *There is no compact perfect (for the original topology) subset $K \subset X$ on which the fine topology is discrete.*

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The implication (a) ⇒ (b) was proved in [2] as was the converse implication (b) ⇒ (a). The new contribution here is the stronger implication (c) ⇒ (a); note that (b) ⇒ (c) is obvious since a compact perfect set must be uncountable.

Our main tool in proving that (c) implies (a) is a refinement of a theorem of Rosenthal [4] which states that a bounded linear operator $T\colon \mathscr{C}(X) \to \mathscr{C}(X)$ on the Banach space of continuous functions on a compact metric space $X$ either fixes a copy of $\mathscr{C}(X)$, or its adjoint has a separable range. That $T$ fixes a copy of $\mathscr{C}(X)$ means that there is some closed subspace $E \subset \mathscr{C}(X)$ which is isomorphic to $\mathscr{C}(X)$ such that $T$, considered as an operator from $E$ to $T(E)$, is an isomorphism. Should this fail to hold, then, by Rosenthal's theorem, the adjoint operator $T^*\colon \mathscr{C}(X)^* \to \mathscr{C}(X)^*$ has a separable range. By the Riesz representation theorem $\mathscr{C}(X)^*$ can be identified with the space of all finite measures $\mathscr{M}(X)$. Suppose $\nu_1, \nu_2, \dots$ is a sequence of measures dense in the unit ball of $T^*(\mathscr{M}(X))$. Then $m := \sum_{n=1}^\infty 2^{-n}|\nu_n|$ is a positive measure for which $T^*\mu \ll m$ for each $\mu$. Writing $d(T^*\delta_y)/dm = k(\,.\,, y)$ we obtain that

$$(Tf)(y) = (T^*\delta_y)(f) = \int_X f(x)k(x,y)\,dm(x);$$

hence $T$ is an integral operator and $m$ is a reference measure for $T$. This explains the relevance of Rosenthal's theorem in the present context.

Actually, we need the following variant of a refinement of this theorem [6]. Denote by $\mathscr{B}_b(X)$ the sup-normed Banach space of all bounded Borel functions on $X$. $\chi_K$ stands for the indicator of a set $K$ as well as for the multiplication operator with that function.

**Proposition 2.** *Let $X$ and $Y$ be Polish spaces and $T$ be a bounded linear operator from $\mathscr{B}_b(X)$ to $\mathscr{B}_b(Y)$ given by the formula*

$$(Tf)(y) = \int_X f\,d\mu_y$$

*for some positive kernel $(\mu_y)$. Then either there exists a compact perfect set $K \subset Y$ such that $\chi_K T\colon \mathscr{B}_b(X) \to \mathscr{B}_b(K)$ is surjective, or $T$ has a reference measure.*

*Proof.* First of all, we may assume that $X$ is a compact metric space, since a Polish space is homeomorphic to a dense $G_\delta$-subset of some compact metric space [3, Sect. 4C], say $\tilde X$, and we regard the $\mu_y$ as measures on this larger space by simply setting $\mu(A) = 0$ for $A \subset \tilde X \setminus X$. As in [6] we consider the oscillation

$$\omega(y, Y') = \inf_{\delta>0} \sup_{z_1,z_2} \|\mu_{z_1} - \mu_{z_2}\|_{\mathscr{M}(X)}$$

on a subset $Y' \subset Y$, where the supremum is taken over all $z_1$ and $z_2$ in a relative $\delta$-neighbourhood of $y$ in $Y'$. As opposed to the situation in [6], here $y \mapsto \mu_y$ need not be continuous for the weak* topology of $\mathscr{M}(X)$. Now the following lemma, whose proof can be found for instance in [3, Sect. 13], helps.

**Lemma 3.** *Let $P$ be a Polish space.*

(a) *If $f\colon P \to Z$ is a Borel mapping into a second countable topological space $Z$, then there is a finer topology on $P$ making $P$ a Polish space and having the same Borel sets as the original topology such that $f$ is continuous for the new topology.*
(b) *If $A \subset P$ is a Borel set, then there is a finer topology on $P$ making $P$ a Polish space and having the same Borel sets as the original topology such that $A$ is clopen; hence $A$ is a Polish space itself for the new topology.*

Lemma 3 allows us to assume that, after modifying the topology of $Y$ without spoiling its Borel structure, $y \mapsto \mu_y$ is weak* continuous. Now the arguments of [6, Th. 1] apply to show that there is a diffuse positive measure $\lambda$ on $Y$ such that

$$\omega(y, Y) = 2\|\mu_y^s\| \quad \lambda\text{-a.e.},$$

where $\mu_y^s$ is the singular part of $\mu_y$ with respect to the measure $\nu\colon A \mapsto \int_Y \mu_y(A)\, d\lambda(y)$.

If $T$ does not have a reference measure, then (see p. 175 of [6]) for some open set $Y' \subset Y$ we have

$$\alpha := \inf_{y \in Y'} \omega(y, Y) > 0.$$

Also, (ibid., Lemma 2 and p. 175) there exists an uncountable set $D \subset Y'$ such that $\mu_y^s$ and $\mu_z^s$ are pairwise singular for all $y, z \in D$. A theorem of Burgess and Mauldin ([1, Th. 4], see also [6, p. 176]) then provides us with a perfect compact set $K \subset Y'$, a closed set $C \subset X$ with $\nu(C) = 0$ and a Borel mapping $\rho\colon X \to K$ such that

$$\mu_y^s\big(X \setminus (C \cap \rho^{-1}(y))\big) = 0 \qquad \forall y \in K;$$

i.e., $\mu_y^s$ is supported by $\{x \in C\colon \rho(x) = y\}$. (Note that the old and the new topology coincide on the compact set $K$.)

Let us now consider the operator $J\colon \mathscr{B}_b(K) \to \mathscr{B}_b(X)$ given by $Jg = \chi_C \cdot (g \circ \rho)$. We then have for $y \in K$

$$\begin{aligned}\big((TJ)(g)\big)(y) &= \int_C g(\rho(x))\, d\mu_y(x) = \int_C g(\rho(x))\, d\mu_y^s(x) \\ &= \int_C g(y)\, d\mu_y(x) = g(y)\mu_y^s(C \cap \{\rho = y\}) = g(y)\|\mu_y^s\|.\end{aligned}$$

Since $\|\mu_y^s\| \geq \alpha/2 > 0$ on $Y'$, this proves that $\chi_K T$ maps $\mathscr{B}_b(X)$ onto $\mathscr{B}_b(K)$. □

*Proof of Theorem 1*, (c) $\Rightarrow$ (a): Assume that $V$ has no reference measure. Then there exist Borel sets $A$ and $B$ such that $h \cdot \chi_A$, $\chi_B \cdot Vh$, $1/(h \cdot \chi_A)$ and $1/(\chi_B \cdot Vh)$ are bounded and $\bar{V} := \chi_B V \chi_A$ has no reference measure, either. Indeed, $A = \{1/n \leq h \leq n\}$ and $B = \{1/n \leq Vh \leq n\}$ will do for large enough $n$. By Lemma 3, we may pretend that $A$ and $B$ are actually Polish spaces.

Now $\bar{V}$ is a bounded linear operator from $\mathscr{B}_b(A)$ to $\mathscr{B}_b(B)$. From Proposition 2 we infer that for some perfect compact $K \subset B$, the operator $\chi_K \bar{V}$ maps $\mathscr{B}_b(A)$ onto $\mathscr{B}_b(K)$. On the other hand, if $f \in \mathscr{B}_b(A)$, then $Vf \in \mathscr{S}$ and $\chi_K \cdot \bar{V} f = Vf$ on $K$, hence every bounded Borel function on $K$ is of the form $u|_K$ for some $u \in \mathscr{S}$ and thus continuous for the fine topology. In particular, the indicator functions $\chi_{\{x_0\}}$, $x_0 \in K$, are finely continuous, and $K$ is discrete in the fine topology. □

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