M-ideals of compact operators into ℓ_p

Kamil John[1] and Dirk Werner

Abstract. We show for $2 \leq p < \infty$ and subspaces X of quotients of L_p with a 1-unconditional finite-dimensional Schauder decomposition that $K(X, \ell_p)$ is an M-ideal in $L(X, \ell_p)$.

1. Introduction

A closed subspace J of a Banach space X is called an M-ideal if the dual space X^* decomposes into an ℓ_1-direct sum $X^* = J^\perp \oplus_1 V$, where $J^\perp = \{x^* \in X^*\colon x^*|_J = 0\}$ is the annihilator of J and V is some closed subspace of X^*. This notion is due to Alfsen and Effros [1], and it is studied in detail in [4].

It has long been known that the space of compact operators $K(\ell_p)$ is an M-ideal in the space of bounded operators $L(\ell_p)$ for $1 < p < \infty$ whereas this property fails for $L_p = L_p[0,1]$ unless $p = 2$; cf. Section VI.4 in [4]. More recently, it was shown in [6] that $K(L_p, \ell_p)$ is an M-ideal if $1 < p \leq 2$, and it is not an M-ideal if $p > 2$.

In this paper we wish to examine the M-ideal character of $K(X, \ell_p)$ for subspaces X of quotients of L_p and $2 \leq p < \infty$. Our idea is to exploit the fact that those X have Rademacher cotype p with constant 1. This leads to the result mentioned in the abstract.

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2. Results

Here is our main result.

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Theorem 2.1 *Let $1 < p < \infty$ and suppose that the Banach space X admits a sequence of operators $K_n \in K(X)$ satisfying*

(a) $K_n x \to x$ *for all* $x \in X$,

(b) $K_n^* x^* \to x^*$ *for all* $x^* \in X^*$,

(c) $\|Id_X - 2K_n\| \to 1$.

Then $K(X, \ell_p)$ is an M-ideal in $L(X, \ell_p)$ if

$$\limsup_n (\|x\|^p + \|x_n\|^p)^{1/p} \le \limsup_n \left(\frac{\|x + x_n\|^p + \|x - x_n\|^p}{2} \right)^{1/p} \tag{2.1}$$

for all $x, x_n \in X$ such that $x_n \to 0$ weakly.

Proof. Let $T\colon X \to \ell_p$ be a contraction. We shall show that T has property (M), i.e.,

$$\limsup_n \|y + Tx_n\| \le \limsup_n \|x + x_n\|$$

whenever $x \in X$, $y \in \ell_p$, $\|y\| \le \|x\|$, and $x_n \to 0$ weakly in X. This implies our claim by [6, Th. 6.3].

In fact, we have

$$\begin{aligned} \limsup_n \|y + Tx_n\| &= \limsup_n (\|y\|^p + \|Tx_n\|^p)^{1/p} \\ &\le \limsup_n (\|x\|^p + \|x_n\|^p)^{1/p} \\ &\le \limsup_n \left(\frac{\|x + x_n\|^p + \|x - x_n\|^p}{2} \right)^{1/p}; \end{aligned}$$

so it is enough to show that

$$\limsup_n \|x + x_n\| = \limsup_n \|x - x_n\|. \tag{2.2}$$

Let $\varepsilon > 0$. Pick $m \in \mathbb{N}$ so that

$$\|K_m x - x\| \le \varepsilon, \qquad \|Id - 2K_m\| \le 1 + \varepsilon.$$

Then pick $n_0 \in \mathbb{N}$ so that

$$\|K_m x_n\| \le \varepsilon \qquad \forall n \ge n_0;$$

this is possible since $x_n \to 0$ weakly and K_m is compact. We now have for $n \ge n_0$

$$\begin{aligned} (1 + \varepsilon)\|x_n + x\| &\ge \|(Id - 2K_m)(x_n + x)\| \\ &= \|x_n - x - 2K_m x_n + 2x - 2K_m x\| \\ &\ge \|x_n - x\| - 2\varepsilon - 2\varepsilon \end{aligned}$$

so that

$$\limsup_n \|x_n + x\| \geq \limsup_n \|x_n - x\|,$$

and by symmetry equality holds. □

We note that (2.1) is not a necessary condition, for essentially trivial reasons: e.g., if $p < 2$ and $X = \ell_2$, then every operator from X to ℓ_p is compact and, therefore, $K(X, \ell_p)$ is an M-ideal, but (2.1) fails.

As the proof shows, one can as well consider all the Banach spaces sharing the property

$$\limsup_n \|y + y_n\| \leq \limsup_n (\|y\|^p + \|y_n\|^p)^{1/p}$$

whenever $y_n \to 0$ weakly, e.g., ℓ_q or the Lorentz spaces $d(w, q)$ for $p \leq q < \infty$. So our theorem is closely related to [10, Th. 3] and [11, Prop. 4.2]. Actually, we needed assumptions (a)–(c) only to ensure (2.2), a condition that could be called property (wM) in accordance with Lima's property (wM^*) [7].

Now we wish to give more concrete examples where Theorem 2.1 applies. There is a natural class of Banach spaces in which inequality (2.1) is valid. Recall that a Banach space X has Rademacher type p with constant C if for all finite families $\{x_1, \ldots, x_n\} \subset X$, with $r_1, r_2, \ldots$ denoting the Rademacher functions,

$$\left(\int_0^1 \left\|\sum_{k=1}^n r_k(t) x_k\right\|^p dt\right)^{1/p} \leq C\left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p};$$

it has Rademacher cotype p with constant C if

$$\left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p} \leq C\left(\int_0^1 \left\|\sum_{k=1}^n r_k(t) x_k\right\|^p dt\right)^{1/p}$$

instead. Thus we see that the inequality (2.1) is always satisfied when X has Rademacher cotype p with constant 1, which is the case if X is a subspace of a quotient of L_p for $2 \leq p < \infty$. As for assumptions (a)–(c) from Theorem 2.1, these conditions are obviously fulfilled if X has a shrinking 1-unconditional finite-dimensional Schauder decomposition or merely the shrinking unconditional metric compact approximation property of [2] and [3]. Let us mention that the "shrinking" character of these properties holds, by a well-known convex combinations argument (cf. [4, Lemma VI.4.9]), for reflexive spaces automatically. These observations yield the next corollary.

Corollary 2.2 *Let X be a subspace of a quotient of L_p, $2 \le p < \infty$, and let X have a 1-unconditional finite-dimensional Schauder decomposition or merely the unconditional metric compact approximation property. Then $K(X, \ell_p)$ is an M-ideal in $L(X, \ell_p)$.*

More explicitly, we note that for instance ℓ_p, $\ell_p \oplus_p \ell_r$ and $\ell_p(\ell_r)$, where $2 \le r \le p < \infty$, satisfy these assumptions; but for these spaces the result of Corollary 2.2 has already been known from [11] or [4, p. 327]. Yet there are other examples. In fact, Li [8] has exhibited spaces of Λ-spectral functions $L^p_\Lambda(\mathbb{T})$ for certain $\Lambda \subset \mathbb{Z}$ that enjoy the unconditional metric compact approximation property. Moreover, since for $2 \le q \le p < \infty$ the space L_q is isometric to a quotient of L_p, one can substitute q for p in the above list of examples.

Another way to see that (2.1) holds for L_p, $2 \le p < \infty$, is to observe that (2.1) follows immediately from Clarkson's inequality in L_p, that is

$$\|f\|^p + \|g\|^p \le \frac{\|f+g\|^p + \|f-g\|^p}{2}$$

for $p \ge 2$. Now, Clarkson's inequalities are valid in the Schatten classes as well [9]. Therefore we obtain a noncommutative version of the previous corollary. (Actually, this argument is not that different, because the Clarkson inequality entails the desired cotype property.)

Corollary 2.3 *Let X be a subspace of a quotient of the Schatten class c_p, $2 \le p < \infty$, and let X have a 1-unconditional finite-dimensional Schauder decomposition or merely the unconditional metric compact approximation property. Then $K(X, \ell_p)$ is an M-ideal in $L(X, \ell_p)$.*

There is a dual version of Theorem 2.1 which we state for completeness.

Theorem 2.4 *Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Suppose that the Banach space Y admits a sequence of operators $K_n \in K(Y)$ satisfying*
(a) *$K_n y \to y$ for all $y \in Y$,*
(b) *$K_n^* y^* \to y^*$ for all $y^* \in Y^*$,*
(c) *$\|Id_Y - 2K_n\| \to 1$.*
Then $K(\ell_p, Y)$ is an M-ideal in $L(\ell_p, Y)$ if

$$\limsup_n (\|y^*\|^{p'} + \|y_n^*\|^{p'})^{1/p'} \le \limsup_n \left(\frac{\|y^* + y_n^*\|^{p'} + \|y^* - y_n^*\|^{p'}}{2} \right)^{1/p'} \qquad (2.3)$$

for all $y^*, y_n^* \in Y^*$ such that $y_n^* \to 0$ weak*.

The proof of Theorem 2.4 can be accomplished along the same lines as above using property (M^*) of a contraction (cf. [6, p. 171]) instead.

Again, inequality (2.3) is always satisfied when Y^* has Rademacher cotype p' with constant 1, which is the case if Y has Rademacher type p with constant 1. The latter holds if Y is a subspace of a quotient of L_p or c_p for $1 < p \leq 2$.

3. Concluding remarks

The conditions (2.1) and (2.3) can be understood as averaging conditions. In an earlier draft of this manuscript we used these conditions to establish what we call p-averaged versions of the properties (M) and (M^*) of contractions T, that is

$$\limsup_n \|y + Tx_n\| \leq \begin{cases} \limsup_n \left(\dfrac{\|x + x_n\|^p + \|x - x_n\|^p}{2} \right)^{1/p} & \text{for } p < \infty \\ \limsup_n \max(\|x + x_n\|, \|x - x_n\|) & \text{for } p = \infty \end{cases}$$

whenever $x \in X$, $y \in Y$ with $\|y\| \leq \|x\|$ and $x_n \to 0$ weakly in X; respectively,

$$\limsup_n \|x^* + T^* y_n^*\| \leq \begin{cases} \limsup_n \left(\dfrac{\|y^* + y_n^*\|^p + \|y^* - y_n^*\|^p}{2} \right)^{1/p} & \text{for } p < \infty \\ \limsup_n \max(\|y^* + y_n^*\|, \|y^* - y_n^*\|) & \text{for } p = \infty. \end{cases}$$

for all $x^* \in X^*$, $y^* \in Y^*$ such that $\|x^*\| \leq \|y^*\|$ and for all weak* null sequences $(y_n^*) \subset Y^*$. (As a matter of fact, (2.3) implies the p'-averaged property (M^*) for a contraction T: $\ell_p \to Y$.) Using techniques from [6] (which in turn depend on those from [5]) one can prove the following results.

Proposition 3.1 *Let $1 \leq p \leq \infty$ and suppose that the Banach space X admits a sequence of operators $K_n \in K(X)$ satisfying*

(a) *$K_n x \to x$ for all $x \in X$,*

(b) *$K_n^* x^* \to x^*$ for all $x^* \in X^*$,*

(c) *$\|Id_X - 2K_n\| \to 1$.*

Let Y be a Banach space. Then $K(X,Y)$ is an M-ideal in $L(X,Y)$ if and only if every contraction T: $X \to Y$ has p-averaged (M).

Proposition 3.2 *Let $1 \leq p \leq \infty$ and suppose that the Banach space Y admits a sequence of operators $K_n \in K(Y)$ satisfying*

(a) $K_n y \to y$ *for all* $y \in Y$,
(b) $K_n^* y^* \to y^*$ *for all* $y^* \in Y^*$,
(c) $\|Id_Y - 2K_n\| \to 1$.

Let X *be a Banach space. Then* $K(X,Y)$ *is an* M*-ideal in* $L(X,Y)$ *if and only if every contraction* $T\colon X \to Y$ *has* p*-averaged* (M^*).

It is well known (cf. [4, Th. I.2.2]) that a closed subspace J of a Banach space X is an M-ideal in X if and only if the following 3-ball property holds: For all $y_1, y_2, y_3 \in B_J$, all $x \in B_X$ and all $\varepsilon > 0$ there is $y \in J$ such that $\|x + y_i - y\| \le 1 + \varepsilon$ for $i = 1, 2, 3$. (Here B_X denotes the closed unit ball of X.) Upon replacing the number 3 by some $n \in \mathbb{N}$ we obtain the n-ball property, which is equivalent to the 3-ball property provided $n \ge 3$. One may "average" this condition as well and obtain the following characterisation of M-ideals by means of an averaged 3-ball property.

Proposition 3.3 *A closed subspace* J *of a Banach space* X *is an* M*-ideal in* X *if and only if*

(A) *For all* $y_1, y_2, y_3 \in B_J$, $x \in B_X$ *and* $\varepsilon > 0$ *there is* $y \in J$ *such that*

$$\|x + y_i - y\| + \|x - y_i - y\| \le 2(1 + \varepsilon) \quad \textit{for } i = 1, 2, 3.$$

holds.

Proof. Evidently the 6-ball property implies (A). Conversely, suppose (A). In order to show that J is an M-ideal in X we will verify the ordinary 3-ball property (see above). Now an inspection of the proof of [4, Theorem I.2.2] shows that one may additionally assume that $\operatorname{dist}(x, J) \ge 1 - \varepsilon$, in which case (A) implies that

$$\|x + y_i - y\| \le 2(1 + \varepsilon) - \|x - y_i - y\| \le 1 + 3\varepsilon, \qquad i = 1, 2, 3,$$

and we are done. □

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Mathematical Institute, Czech Academy of Sciences, Žitná 25,
CZ-11 567 Prague 1, Czech Republic; e-mail: kjohn@earn.cvut.cz

I. Mathematisches Institut, Freie Universität Berlin, Arnimallee 2–6,
D-14 195 Berlin, Germany; e-mail: werner@math.fu-berlin.de