ON WOLFGANG LUSKY'S PAPER "THE GURARIJ SPACES ARE UNIQUE"

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ABSTRACT. This note surveys Wolfgang Lusky's proof of uniqueness of the Gurariy spaces and mentions further developments.

1. INTRODUCTION

In 1966, V. I. Gurariy [11] defined the notion of a *Banach space of (almost) universal disposition* by a certain extension property; see Definition 2.1. He proved the existence of (separable) such spaces and investigated some of their properties; henceforth, such spaces were called *Gurariy spaces* (alternative spellings: Gurarii, Gurarij, Gurariĭ, ...); we shall reserve this name to separable spaces of this kind. While it is not a daunting task to prove that any two Gurariy spaces are almost isometric in the sense that their Banach-Mazur distance is 1, it remained open to decide whether they are actually isometric. This was asked for instance by J. Lindenstrauss and his collaborators at various junctures ([20, Problem II.4.13], [17]).

The isometry problem was solved in 1976 by a fresh PhD from the (likewise rather freshly established) University of Paderborn, Wolfgang Lusky, in his first-ever published paper (the title says it all)

[L] The Gurarij spaces are unique. *Arch. Math.* 27, 627–635 (1976).

We shall refer to this paper, which is [23] in the bibliography, simply by [L].

The present note aims at surveying the background, Lusky's proof, and the ramifications of this result along with an outlook.

Interestingly, some 30 years later Gurariy and Lusky cooperated intensively on a rather different topic, the Müntz spaces, which has led to their monograph [12].

The notation in this note is standard; B_X stands for the closed unit ball of X and $\operatorname{ex} B_X$ for the set of its extreme points. We are considering only real Banach spaces.

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2. Banach spaces of almost universal disposition

V. I. Gurariy (1935–2005) was a member of the Kharkiv school of Banach spaces led by M. I. Kadets (sometimes spelled Kadec), one of the strongest in Europe which had its heyday from the late 1950ies till the collapse of the Soviet Union that produced a brain-drain in all fields of science. Gurariy himself emigrated to the United States in the early 1990ies. After 2000, the Kharkiv school was basically reduced to V. Kadets and his students. In 2022 the terror regime in Moscow set out to destroy the university of Kharkiv altogether [31], but remembering a slogan from many years back, ¡No pasarán!

Here is the key definition of his paper [11].

Definition 2.1. Let X be a Banach space with the following property.

- For finite-dimensional spaces E and F, isometries $T\colon E \to X$ and $S\colon E \to F$, and for $\varepsilon > 0$, there exists an operator $\widehat{T}\colon F \to X$ satisfying $\widehat{T}S = T$ and
$$(1+\varepsilon)^{-1}\|y\| \le \|\widehat{T}y\| \le (1+\varepsilon)\|y\| \qquad (y \in F)$$
("an ε-isometry").

Then X is called a Banach space of *almost universal disposition.* A separable such space will also be called a *Gurariy space.*

The epithet "almost" in this definition refers to the quantifier "for all $\varepsilon > 0$"; if $\varepsilon = 0$ is permissible above, then the "almost" will be dropped. However, Gurariy proved in [11, Th. 10] that no separable space of universal disposition exists, but see Subsection 6.3 below.

If in the above definition, S is the identical inclusion, i.e., $E \subset F$, then $\widehat{T}$ is an extension of T, which can likewise be considered as the identical inclusion.

To see that the condition of Definition 2.1 is quite restrictive, let us discuss two examples.

Example 2.2. (a) c_0 is not a space of almost universal disposition. Indeed, let $E = \mathbb{R}$, $T\colon E \to c_0$, $T(r) = (r, 0, 0, \dots)$, $F = \ell^2_\infty = \mathbb{R}^2$ with the max-norm, $S\colon E \to F$, $S(r) = (r, r)$. Assume that $\widehat{T}$ has the properties of Definition 2.1, and let $\widehat{T}(-1, 1) = (x_1, x_2, \dots)$. Note that $\widehat{T}(1, 1) = (1, 0, 0, \dots)$ and therefore
$$\widehat{T}(0,1) = \Bigl(\frac{1+x_1}{2}, \frac{x_2}{2}, \dots\Bigr),$$
$$\widehat{T}(1,0) = \Bigl(\frac{1-x_1}{2}, \frac{-x_2}{2}, \dots\Bigr).$$
This shows that $\widehat{T}$ cannot be an ε-isometry for small ε. (If x is a real number close to 1 in modulus, then $\frac{1\pm x}{2}$ cannot both be close to 1.)

(b) $C[0, 1]$ is not a space of almost universal disposition. Indeed, let $E = \mathbb{R}$, $T\colon E \to C[0, 1]$, $T(r) = r\mathbb{1}$ (the constant function), $F = \ell^2_2 = \mathbb{R}^2$ with the ℓ_2-norm, $S\colon E \to F$, $S(r) = (r, 0)$. Assume that $\widehat{T}$ has the properties of Definition 2.1, and let $\widehat{T}(0, 1) = f$. Note that $\widehat{T}(1, 0) = \mathbb{1}$ and therefore

$$\widehat{T}(1,1) = \frac{\mathbb{1} + f}{2},$$

which must have norm $\sqrt{2} = \|(1, 1)\|_2$ up to ε. Since $(1+\varepsilon)^{-1} \le \|f\| \le 1+\varepsilon$, this is impossible for small ε.

These examples indicate that positive results might not be very easy to come by. By a technical inductive argument, Gurariy shows in [11, Th. 2] the following existence theorem.

Theorem 2.3. *There exists a separable Banach space of almost universal disposition.*

As for uniqueness, he proves the following result. To formulate it succinctly, let us recall the *Banach-Mazur distance* between (isomorphic) Banach spaces

$$d(X, Y) = \inf\{\|\Phi\|\|\Phi^{-1}\|\colon\ \Phi\colon X \to Y \text{ is an isomorphism}\}$$

and call two Banach spaces *almost isometric* if their Banach-Mazur distance equals 1.

Now for Theorem 5 of [11].

Theorem 2.4. *Any two separable spaces of almost universal disposition are almost isometric.*

A quick sketch of the proof can also be found in [20, p. 168].

3. The Lazar-Lindenstrauss approach

A key property of the Gurariy spaces (from now on we shall use this terminology) is that they are L_1-preduals. Recall that an *L_1-predual* (a.k.a. a *Lindenstrauss space*) is a Banach space whose dual is isometrically isomorphic to a space $L_1(\mu)$ of integrable functions on some measure space. This class of spaces is the subject of Lindenstrauss's epoch-making memoir [18].

Proposition 3.1. *Every Gurariy space is an L_1-predual.*

In the literature, especially from the previous century, there are only vague indications as to why this is so. Since a recent article [6] admits that this proposition is "not completely evident from the definition" and since it is instrumental for Lusky's proof, I'll sketch a proof. To begin with, we have to recall a characterisation of L_1-preduals from Lindenstrauss's memoir; see [20, Th. 6.1] in conjunction with [20, Lemma 4.2], or [16, §21].

Theorem 3.2. *A Banach space X is an L_1-predual if and only if any four open balls $U(x_i, r_i)$ that intersect pairwise have a nonvoid intersection. It is enough to check this for balls of radius* 1.

Let us verify that a Gurariy space X has this property. So suppose $U(x_1,1),\dots,U(x_4,1)$ are four open balls in X with radius 1 that intersect pairwise, i.e., $\|x_i-x_j\|<2$. Choose $\varepsilon>0$ such that even $\|x_i-x_j\|<2-4\varepsilon$. Let E be the span of $x_1,\dots,x_4$. There are some $N\in\mathbb{N}$ and a linear operator $S_1\colon E\to\ell_\infty^N$ such that

$$\frac{1}{1+\varepsilon}\|S_1x\|_\infty\le\|x\|\le\|S_1x\|_\infty \qquad (x\in E).$$

Let us consider the balls $U_{\ell_\infty^N}(S_1x_i,1-\varepsilon)$ in ℓ_∞^N. They intersect pairwise since

$$\|S_1x_i-S_1x_j\|_\infty\le(1+\varepsilon)\|x_i-x_j\|<(1+\varepsilon)(2-4\varepsilon)<2-2\varepsilon.$$

Being pairwise intersecting balls in ℓ_∞^N, these balls have a point in common. This means that there exists some $z\in\ell_\infty^N$ such that

$$\|z-Sx_i\|_\infty<1-\varepsilon \qquad (i=1,\dots,4).$$

Unfortunately, S_1 is not an isometry and therefore is not eligible for being used in Definition 2.1. However, we can renorm ℓ_∞^N to make it an isometry: note that $B_{\ell_\infty^N}\cap S_1(E)\subset S_1(B_E)$, and we can renorm ℓ_∞^N by letting the new unit ball be the convex hull of $S_1(B_E)$ and $B_{\ell_\infty^N}$. Call this renorming F, and let $S=S_1$ considered as an operator from E to F; this is an isometry. We have

$$\frac{1}{1+\varepsilon}\|y\|_\infty\le\|y\|_F\le\|y\|_\infty \qquad (y\in F)$$

and thus

$$\|z-Sx_i\|_F\le\|z-S_1x_i\|_\infty<1-\varepsilon.$$

Since X is a Gurariy space, there is an ε-isometry $\widehat{T}\colon F\to X$ satisfying $\widehat{T}Sx=x$ for $x\in E$. Let $x_0=\widehat{T}z$; then $x_0\in\bigcap U(x_i,1)$:

$$\|x_0-x_i\|=\|\widehat{T}z-\widehat{T}Sx_i\|\le(1+\varepsilon)\|z-Sx_i\|_F<(1+\varepsilon)(1-\varepsilon)<1.$$

In the more contemporary literature one can find explicit proofs of Proposition 3.1 based on another characterisation of L_1-preduals and a "pushout argument" [9, Th. 2.17], [5, Prop. 6.2.8].

Now let X be a separable L_1-predual. By the results of Michael and Pełczyński [26] and Lazar and Lindenstrauss [17] there is a chain of finite-dimensional subspaces E_n of X such that

(a) $E_1\subset E_2\subset\dots$;
(b) $\dim E_n=n$, and E_n is isometrically isomorphic to ℓ_∞^n,
(c) $\bigcup E_n$ is dense in X.

The inclusion $E_n\subset E_{n+1}$ entails some degree of freedom, namely the choice of an isometry $\psi_n\colon\ell_\infty^n\to\ell_\infty^{n+1}$. To study the structure of these ψ_n, we need the ad-hoc notion of an admissible basis: if $\delta_1,\dots,\delta_n$ denotes the canonical unit vector basis of ℓ_∞^n and $\psi\colon\ell_\infty^n\to\ell_\infty^n$ is an isometry, then $\psi(\delta_1),\dots,\psi(\delta_n)$ is called an *admissible basis* for ℓ_∞^n. Note that ψ takes a vector $(a_1,\dots,a_n)$ to $(\vartheta_1a_{\pi(1)},\dots,\vartheta_na_{\pi(n)})$ for some permutation π and some signs $\vartheta_j=\pm1$.

Thus, an admissible basis is just a permutation of the unit vector basis up to signs, and the isometric image of an admissible basis is again an admissible basis.

Let us return to the isometric embedding $\psi_n\colon \ell_\infty^n \to \ell_\infty^{n+1}$, and let $e_{1,n}, \dots, e_{n,n}$ be an admissible basis for ℓ_∞^n. We can develop the vectors $f_j := \psi_n(e_{j,n})$ into the unit vector basis of ℓ_∞^{n+1}. Since ψ_n is an isometry, there is at least one coordinate i where $|f_j(i)| = 1$. Then, if $k \neq j$, $f_k(i) = 0$: pick a sign λ such that

$$|f_j(i) + \lambda f_k(i)| = |f_j(i)| + |f_k(i)| = 1 + |f_k(i)|$$

and so

$$1 = \|e_{j,n} + \lambda e_{k,n}\| = \|f_j + \lambda f_k\| \geq |f_j(i) + \lambda f_k(i)| = 1 + |f_k(i)|,$$

hence the claim. Since $\|\psi_n\| = 1$, we also have

$$\Big|\sum_{j=1}^n f_j(i)\Big| = \Big|\psi_n\Big(\sum_{j=1}^n e_{j,n}\Big)(i)\Big| \leq \Big\|\sum_{j=1}^n e_{j,n}\Big\| = 1.$$

Therefore, there is an admissible basis $e_{1,n+1}, \dots, e_{n+1,n+1}$ for ℓ_∞^{n+1} such that for some numbers a_{jn}

$$\psi_n(e_{j,n}) = e_{j,n+1} + a_{jn} e_{n+1,n+1} \qquad (j = 1, \dots, n)$$

and

$$\sum_{j=1}^n |a_{jn}| \leq 1.$$

We can rephrase these representations in terms of the E_n as follows.

Proposition 3.3. *There exist admissible bases in each E_n and real numbers a_{jn} such that*

$$e_{j,n} = e_{j,n+1} + a_{jn} e_{n+1,n+1} \qquad (j = 1, \dots, n;\ n = 1, 2, \dots)$$
$$\sum_{j=1}^n |a_{jn}| \leq 1 \qquad (n = 1, 2, \dots).$$

This proposition is due to Lazar and Lindenstrauss [17]. The triangular matrix $(a_{jn})_{j \leq n, n \in \mathbb{N}}$ is called a *representing matrix* for the given L_1-predual X. Conversely does the choice of admissible bases and of an array (a_{jn}) lead to an L_1-predual.

Lazar and Lindenstrauss use this approach to present another proof of the existence of Gurariy spaces. Let $a_n = (a_{1n}, \dots, a_{nn}, 0, 0, \dots)$ be the n^{th} column of a matrix as in Proposition 3.3; then each a_n is in the unit ball of ℓ_1.

Theorem 3.4. *If $\{a_1, a_2, \dots\}$ is dense in the unit ball of ℓ_1, then the corresponding matrix is associated to a Gurariy space.*

It should be noted that the representing matrix A of an L_1-predual X is not uniquely determined, and much work has been done to study the relation of A and X for certain classes of L_1-preduals; see e.g. Lusky's paper [24].

4. Lusky's uniqueness proof

Here is Lusky's uniqueness theorem.

Theorem 4.1. *Any two Gurariy spaces are isometrically isomorphic.*

Let us first remark that almost isometric spaces (cf. Theorem 2.4) need not be isometric. The following is a classical counterexample due to Pełczyński from [28]: Let X and Y be c_0 equipped with the equivalent norms ($x = (x_n)$)

$$\|x\|_X = \|x\|_\infty + \Big(\sum_{n=1}^\infty \frac{|x_n|^2}{2^n}\Big)^{1/2},$$

$$\|x\|_Y = \|x\|_\infty + \Big(\sum_{n=1}^\infty \frac{|x_{n+1}|^2}{2^n}\Big)^{1/2}.$$

The operators $\Phi_n\colon X \to Y$, $x \mapsto (x_n, x_1, \dots, x_{n-1}, x_{n+1}, \dots)$ are isomorphisms satisfying $\lim_n \|\Phi_n\|\|\Phi_n^{-1}\| = 1$ so that X and Y are almost isometric; but X is strictly convex while Y isn't, therefore X and Y are not isometric.

Benyamini [3] has shown that such counterexamples also exist among L_1-preduals.

The proof of Theorem 4.1 consists of a delicate inductive construction of ℓ_∞^n-subspaces and admissible bases. The key problem to be solved here is this.

Problem 4.2. Let X be a Gurariy space and $E \subset F$ be finite-dimensional spaces with $E \cong \ell_\infty^n$ and $F \cong \ell_\infty^{n+1}$. Let $T\colon E \to X$ be an isometry. When does there exist an isometric extension $\widehat{T}\colon F \to X$?

Lusky notes that this is not always the case [L, p. 630], and he gives the following useful criterion in terms of admissible bases. W.l.o.g. suppose that T is the identity. Let $e_1, \dots, e_n$ and $f_1, \dots, f_{n+1}$ be admissible bases for E resp. F such that

$$e_i = f_i + r_i f_{n+1}, \qquad i = 1, \dots, n.$$

Lemma 4.3. *Problem 4.2 has a positive solution if* $\sum_{i=1}^n |r_i| < 1$.

This criterion is a little hidden in the proof of the Corollary [L, p. 630], where the extreme point condition $\operatorname{ex} B_E \cap \operatorname{ex} B_F = \emptyset$ is spelled out to be sufficient; but the heart of the matter is Lemma 4.3.

Now let's take a quick glimpse at the proof of Theorem 4.1. Suppose that X and Y are Gurariy spaces coming with ℓ_∞^n-approximations $\bigcup_n E_n$ and $\bigcup F_n$, respectively. Comparing Proposition 3.3 with Lemma 4.3 one realises that one has to perturb the given admissible bases so that Lemma 4.3 becomes applicable. The details of this process are quite technical [L, pp. 631–633] and lead to sequences of admissible bases. Ultimately one can pass to the limit and obtain admissible bases $\{e_{i,n}\colon i \le n,\ n \ge 1\}$ resp. $\{f_{i,n}\colon i \le n,\ n \ge 1\}$ spanning dense subspaces of X resp. Y, and the operator $e_{i,n} \mapsto f_{i,n}$ acts as a well-defined isometry.

In an addendum to [L], dated January 10, 1976, Lusky applies his methods to Mazur's rotation problem that asks whether a separable transitive space is isometric to a Hilbert space; a Banach space X is called *transitive* if whenever $\|x\| = \|y\| = 1$, there is an isometric automorphism $T\colon X \to X$ mapping x to y, i.e., $Tx = y$. This problem is open to this day, and recent papers on the subject include [4] and [6].

What Lusky proves in his addendum is that the Gurariy space (now that we know it's unique we may use the definite article) is transitive for smooth points. Recall that x_0 is a smooth point of the unit ball B_X if $\|x_0\| = 1$ and there is exactly one $x_0^* \in X^*$ such that $\|x_0^*\| = x_0^*(x_0) = 1$; equivalently, the norm function $x \to \|x\|$ is Gâteaux differentiable at x_0. It is a theorem of Mazur that smooth points are dense in the unit sphere of a separable Banach space.

Theorem 4.4. *Let x and y be smooth points of the unit ball of the Gurariy space G. Then there is an isometric automorphism $T\colon G \to G$ mapping x to y.*

Another result of [L] is a refined version of a theorem originally due to Wojtaszczyk [32] (see also [24]).

Theorem 4.5. *Let X be a separable L_1-predual and G be the Gurariy space. Then there exist an isometry $T\colon X \to G$ and a norm-1 projection $P\colon G \to G$ onto $T(X)$; further $(\mathrm{Id} - P)(G)$ is isometrically isomorphic to G.*

This indicates that the Gurariy space is "maximal" among the separable L_1-predual spaces; in particular it contains $C[0,1]$ and is universal, a fact proved by other means by Gevorkyan in [10].

We close this section by mentioning another proof of Theorem 4.1, due to W. Kubiś and S. Solecki [15]. Their proof avoids the Lazar-Lindenstrauss machinery and just depends on the defining properties of a Gurariy space. They also prove the universality of the Gurariy space from first principles, without relying on the universality of $C[0,1]$. Still another proof is in Kubiś's paper [14] in *Archiv der Mathematik*, which builds on a Banach-Mazur type game.

5. The Poulsen simplex

This note wouldn't be complete without mentioning the cousin of the Gurariy space in the world of compact convex sets, the *Poulsen simplex*. The traditional definition of a (compact) simplex is a compact convex subset S of a Hausdorff locally convex space E such that the cone generated by

$S \times \{1\}$ in $E \oplus \mathbb{R}$ is a lattice cone. Thus, a triangle in the plane is a simplex while a rectangle isn't. For our purposes it is important to note that the space $A(S)$ of affine continuous functions on a compact convex set is an L_1-predual if and only if S is a simplex.

Poulsen [29] had proved the existence of a metrisable simplex, which now bears his name, whose set of extreme points is dense. It is a result due to Lindenstrauss, Olsen, and Sternfeld [19] that such a simplex is uniquely determined up to affine homeomorphism. They write:

> We discovered the uniqueness of the Poulsen simplex after reading Lusky's paper [L] on the uniqueness of the Gurari space. Our proof of the uniqueness uses the same idea which Lusky used in [L].

The role of admissible bases is now played by peaked partitions of unity.

The authors mention a lot of similarities between the Poulsen simplex and the Gurariy space. For example, the counterpart of the defining property of the Poulsen simplex S_P is Lusky's theorem from [L] and [24] that a separable L_1-predual is a Gurariy space G if and only if $\operatorname{ex} B_{G^*}$ is weak* dense in the unit ball B_{G^*}. However, $A(S_P)$ is not the Gurariy space since for example the transitivity property of Theorem 4.4 fails. But, as shown by Lusky [25], one can salvage this by requiring a slightly more stringent assumption on x and y, which are now supposed to be positive: in addition, $\mathbb{1} - x$ and $\mathbb{1} - y$ should be smooth points.

6. Outlook

6.1. **Fraïssé theory.** The Gurariy space is a very homogeneous object, for example [11, Th. 3]: If E and F are finite-dimensional subspaces of the same dimension of a Gurariy space G, then for every $\varepsilon > 0$, every isometric isomorphism from E to F extends to an ε-isometric automorphism of G. In recent years, such homogeneous structures were investigated by methods of model theory known as Fraïssé theory ([8], [2], [13]). Fraïssé theory associates a unique limit to certain substructures. This approach is at least implicit in the Kubiś-Solecki uniqueness proof, and a detailed exposition involving the Gurariy space, the Poulsen simplex and a whole lot more can be found in M. Lupini's paper [22].

6.2. **Noncommutative Gurariy spaces.** T. Oikhberg, in his *Archiv der Mathematik* paper [27], proved the existence and uniqueness of a "noncommutative" Gurariy space, i.e., a Gurariy-like object in the setting of operator spaces à la Effros-Ruan. Again, this can also be viewed from the perspective of Fraïssé theory [21].

6.3. **Nonseparable spaces.** We have already mentioned in Section 2 Gurariy's result that no space of universal disposition can be separable. Since the definition of (almost) universal disposition makes perfect sense beyond the separable case, it was studied in several papers, e.g., [1], [7], [9]. It turns out that there are spaces of almost universal disposition of density character

$\aleph_1$, but the uniqueness breaks down (Th. 3.6 and Th. 3.7 in [9]). Likewise, there are spaces of universal disposition of density $\aleph_1$, and again, uniqueness fails ([1], [7]). Indeed, it should be noted that in these papers also the variant of being of (almost) universal disposition with respect to separable spaces, already considered by Gurariy, is studied: in Definition 2.1 one now allows E and F to be separable rather than finite-dimensional.

6.4. **Banach lattices.** Recently, M. A. Tursi [30] proved the existence of a uniquely determined Gurariy-like Banach lattice. She exploits ideas of Fraïssé theory.

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