CHAPTER I

Basic theory of M-ideals

I.1 Fundamental properties

In this book we shall be concerned with decompositions of Banach spaces by means of projections satisfying certain norm conditions. The essential notions are contained in the following definition. We denote the annihilator of a subspace J of a Banach space X by $J^\perp = \{x^* \in X^* \mid x^*(y) = 0 \ \forall y \in J\}$.

Definition 1.1 *Let X be a real or complex Banach space.*

(a) *A linear projection P is called an M-*projection *if*

$$\|x\| = \max\{\|Px\|, \|x - Px\|\} \qquad \text{for all } x \in X.$$

*A linear projection P is called an L-*projection *if*

$$\|x\| = \|Px\| + \|x - Px\| \qquad \text{for all } x \in X.$$

(b) *A closed subspace $J \subset X$ is called an M-*summand *if it is the range of an M-projection. A closed subspace $J \subset X$ is called an L-*summand *if it is the range of an L-projection.*

(c) *A closed subspace $J \subset X$ is called an M-*ideal *if $J^\perp$ is an L-summand in X^*.*

Some comments on this definition are in order. First of all, every Banach space X contains the trivial M-summands $\{0\}$ and X. All the other M-summands will be called nontrivial. (Sometimes only trivial M-summands exist as will presently be shown.) The same remark applies to L-summands and M-ideals.

There is an obvious duality between L- and M-projections:

- P is an L-projection on X iff P^* is an M-projection on X^*.
- P is an M-projection on X iff P^* is an L-projection on X^*.

This remark yields the following characterisation of M-projections which will be useful in the sequel:

A projection $P \in L(X)$ is an M-projection if and only if

$$\|Px_1 + (Id - P)x_2\| \leq \max\{\|x_1\|, \|x_2\|\} \quad \textit{for all } x_1, x_2 \in X. \tag{$*$}$$

In fact, $(*)$ means that the operator

$$(x_1, x_2) \mapsto Px_1 + (Id - P)x_2$$

from $X \oplus_\infty X$ to X is contractive whence its adjoint

$$x^* \mapsto (P^*x^*, (Id - P)^*x^*)$$

from X^* to $X^* \oplus_1 X^*$ is contractive. ($X \oplus_p Y$ denotes the direct sum of two Banach spaces, equipped with the ℓ^p-norm.) This means that P^* is an L-projection, and P must be an M-projection.

Turning to (b) let us note that there is only one M-projection P with $J = \operatorname{ran}(P)$ $(= \ker(Id - P))$ if J is an M-summand and only one L-projection P with $J = \operatorname{ran}(P)$ $(= \ker(Id - P))$ if J is an L-summand (cf. Proposition 1.2 below which contains a stronger statement). Consequently, there is a uniquely determined closed subspace $\widehat{J}$ such that

$$X = J \oplus_\infty \widehat{J}$$

resp.

$$X = J \oplus_1 \widehat{J}.$$

Then $\widehat{J}$ is called the complementary M- (resp. L-)summand. The duality of L- and M-projections may now be expressed as

- $X = J \oplus_\infty \widehat{J}$ iff $X^* = J^\perp \oplus_1 \widehat{J}^\perp$,
- $X = J \oplus_1 \widehat{J}$ iff $X^* = J^\perp \oplus_\infty \widehat{J}^\perp$.

It follows that M-summands are M-ideals and that the M-ideal J is an M-summand if and only if the L-summand complementary to $J^\perp$ is weak* closed. Let us note that the fact that J and $\widehat{J}$ are complementary L-summands in X means geometrically that B_X, the closed unit ball of X, is the convex hull of B_J and $B_{\widehat{J}}$.

As regards (c) the reader might wonder why we didn't introduce the notion of an "L-ideal", meaning a subspace whose annihilator is an M-summand. The reason is that such an "L-ideal" is automatically an L-summand (see Theorem 1.9 below). Note, however, that the expression "L-ideal" has occasionally appeared in the literature as a synonym of L-summand, e.g. in [11].

Proposition 1.2

(a) *If P is an M-projection on X and Q is a contractive projection on X satisfying* $\operatorname{ran}(P) = \operatorname{ran}(Q)$*, then $P = Q$.*

(b) *If P is an L-projection on X and Q is a contractive projection on X satisfying* $\ker(P) = \ker(Q)$*, then $P = Q$.*

PROOF: We first prove (b). The decisive lever for our argument is that, for an L-summand J in X, there is for a given $x \in X$ one and only one best approximant y_0 in J, that is

$$\|x - y_0\| = \inf_{y \in J} \|x - y\|,$$

namely the image of x under the L-projection onto J. (In the language of approximation theory, L-summands are Chebyshev subspaces.) We apply this remark with $J = \ker(P)$. For $x \in X$ we have $x - Px \in \ker(P) = \ker(Q)$, hence

$$\begin{aligned} \|x - (x - Qx)\| &= \|Qx\| \\ &= \|Q(x - (x - Px))\| \\ &\leq \|Q\| \cdot \|Px\| \\ &\leq \|x - (x - Px)\|. \end{aligned}$$

This means that $x - Qx \in \ker(Q) = \ker(P)$ is at least as good an approximant to x in J as $x - Px$ which is the best one. From the uniqueness of the best approximant one deduces $Qx = Px$, thus $P = Q$, as claimed.

(a) follows from (b) since $\ker(P^*) = \operatorname{ran}(P)^\perp = \ker(Q^*)$. ◻

Corollary 1.3 *If an M-ideal is the range of a contractive projection Q, then it is an M-summand.*

PROOF: $\ker(Q^*) = J^\perp$. Thus, the L-projection with kernel $J^\perp$ is Q^* and hence weak* continuous. ◻

We now discuss some examples of M-ideals and M-summands.

Example 1.4(a) Let S be a locally compact Hausdorff space. Then $J \subset C_0(S)$ is an M-ideal if and only if there is a closed subset D of S such that

$$J = J_D := \{x \in C_0(S) \mid x(s) = 0 \quad \text{for all } s \in D\}.$$

It is an M-summand if and only if D is clopen (= closed and open).

PROOF: Obviously, $\mu \mapsto \chi_D \mu$ is the L-projection from $C_0(S)^* = M(S)$ onto $J_D^\perp$ so that J_D is an M-ideal if D is closed, and J_D is an M-summand if D is clopen. Suppose now that $J \subset C_0(S)$ is an M-ideal. In order to find the set D, we make use of the following elementary lemma which we state for future reference. The set of extreme points of a convex set C is denoted by $\operatorname{ex} C$.

Lemma 1.5 *For $Z = J_1 \oplus_1 J_2$ we have (using the convention $\operatorname{ex} B_{\{0\}} = \emptyset$)*

$$\operatorname{ex} B_Z = \operatorname{ex} B_{J_1} \cup \operatorname{ex} B_{J_2}.$$

We shall apply Lemma 1.5 here with $Z = M(S)$, $J_1 = J^\perp$ and $J_2 =$ the complementary L-summand. Let

$$D = \{s \in S \mid \delta_s \in J^\perp\}.$$

Then D is closed, and by construction $J \subset J_D$. The inclusion $J_D \subset J$ follows from the Hahn-Banach and Krein-Milman theorems: By virtue of these results we only have to show $\operatorname{ex} B_{J^\perp} \subset J_D^\perp$, and this is true by Lemma 1.5 and by definition of D.
Finally, if $J = J_D$ is an M-summand, then the complementary M-summand is of the same form, $J_{\widehat{D}}$ say, consequently $D \cup \widehat{D} = S$ and D is clopen. □

In particular, c_0 is an M-ideal in $\ell^\infty = C(\beta\mathbb{N})$ (which could also be proved directly); the additional feature is that ℓ^∞ is the bidual of c_0. We shall study Banach spaces which are M-ideals in their biduals in detail in Chapter III.

Example 1.4(b) Let A be the disk algebra, that is the complex Banach space of continuous functions on the closed unit disk which are analytic in the open unit disk. It will be convenient to consider A (via boundary values) as a subspace of $C(\mathbb{T})$, where $\mathbb{T}$ is the unit circle. We claim that J is a nontrivial M-ideal in A if and only if there is a closed subset $D \neq \emptyset$ of $\mathbb{T}$ with linear Lebesgue measure 0 such that[1]

$$J = J_D \cap A = \{x \in A \mid x(t) = 0 \quad \text{for all } t \in D\}.$$

Proof: Note first that $J_D \cap A = \{0\}$ if D is a subset of $\mathbb{T}$ having positive linear measure, cf. e.g. [544, Theorem 17.18]. To see that an M-ideal J has the form $J_D \cap A$ one proceeds exactly as in Example 1.4(a); one only has to recall that

$$\operatorname{ex} B_{A^*} = \{\lambda \cdot \delta_t|_A \mid |\lambda| = 1,\ t \in \mathbb{T}\}. \qquad (*)$$

[This amounts to saying that every $t \in \mathbb{T}$ is in the Choquet boundary of A, and a proof of this fact is contained in [239, p. 54ff.]; for an explicit statement see e.g. [586, p. 29]. Since this example will have some importance in the sequel, we would like to sketch a direct argument: The right hand side of $(*)$ is weak* closed and norming by the maximum modulus principle, therefore the Krein-Milman theorem (or rather its converse) implies "$\subset$". On the other hand, $\operatorname{ex} B_{A^*} \neq \emptyset$. So let us assume $p_0 := \lambda_0 \cdot \delta_{t_0}|_A \in \operatorname{ex} B_{A^*}$. (Here we use the fact that for $p \in \operatorname{ex} B_{X^*}$ and $X \subset Y$, p has an extension to some $q \in \operatorname{ex} B_{Y^*}$.) For $|\lambda| = 1,\ t \in \mathbb{T}$

$$(\Phi x)(s) := \frac{\lambda}{\lambda_0} \cdot x\Big(\frac{t}{t_0} s\Big)$$

defines an isometric isomorphism on A, hence $\lambda \cdot \delta_t|_A = \Phi^*(p_0) \in \operatorname{ex} B_{A^*}$.]
Now let $D \neq \emptyset$ be a closed subset of $\mathbb{T}$ of linear measure 0. We wish to find an L-projection from A^* onto $(J_D \cap A)^\perp$. A functional $p \in A^*$ may be represented as $p = \mu|_A$ for some measure $\mu \in M(\mathbb{T}) = C(\mathbb{T})^*$. Let $q = (\chi_D \mu)|_A$. Then the mapping $P : p \mapsto q$ is well-defined: If $p = \nu|_A$ is another representation, then $\nu - \mu$ annihilates A. The F. and M. Riesz theorem (cf. e.g. [544, Theorem 17.13]) implies that $\nu - \mu$ is absolutely continuous with respect to Lebesgue measure so that $\chi_D \mu = \chi_D \nu$ if D has measure 0. It

[1] Throughout, J_D will have the same meaning as in Example 1.4(a).

is easy to check that P is the required L-projection. Finally, the nontriviality follows from a theorem of Fatou [316, p. 80] which also shows that different D give rise to different M-ideals. $\square$

We shall present a characterisation of M-ideals in a general function algebra in Theorem V.4.2. As regards this example, see also the abstract version of our approach in Corollary 1.19.

Example 1.4(c) Let K be a compact convex set in a Hausdorff locally convex topological vector space. As usual, $A(K)$ denotes the space of real-valued affine continuous functions on K. Let us recall the definition of a split face of K ([7, p. 133], [9]). A face F of K is called a split face if there is another face F' such that every $k \in K \backslash (F \cup F')$ has a *unique* representation

$$k = \lambda k_1 + (1 - \lambda) k_2 \quad \text{with } k_1 \in F,\ k_2 \in F',\ 0 < \lambda < 1.$$

It is known that every closed face of a simplex is a split face [7, p. 144].
Then J is an M-ideal in $A(K)$ if and only if there exists a closed split face F of K such that

$$J = J_F \cap A(K) = \{x \in A(K) \mid x(k) = 0 \quad \text{for all } k \in F\}.$$

The proof of this fact can be given along the lines of (b), the crucial step being the measure theoretic characterisation of closed split faces (see [7, Th. II.6.12]) which replaces the use of the F. and M. Riesz theorem.

Example 1.4(c) was one of the forerunners of the general M-ideal theory (cf. [9]). Another forerunner of the general theory is contained in the next example.

Example 1.4(d) In a C^*-algebra the M-ideals coincide with the closed two-sided ideals.

We shall present a proof of this fact in Theorem V.4.4. For the time being let us notice that in particular $K(H)$ is an M-ideal in $L(H)$ (where H denotes a Hilbert space) which was first shown by Dixmier [164] back in 1950. An independent proof of Dixmier's result will be given in Chapter VI. There we shall study the class of Banach spaces X for which $K(X)$ is an M-ideal in $L(X)$. Let us indicate at this point that this class contains all the spaces ℓ^p for $1 < p < \infty$ and c_0 as well as certain of their subspaces and quotient spaces (cf. Example VI.4.1, Corollary VI.4.20). Moreover, $K(X)$ is never an M-summand in $L(X)$ unless $\dim X < \infty$ (Proposition VI.4.3).

Example 1.4(e) Although there are certain similarities between algebraic ideals and M-ideals, there is one striking difference: Unlike the case of algebraic ideals the intersection of (infinitely many) M-ideals need not be an M-ideal. An example to this effect will be given in II.5.5. Also let us advertise now the description of M-ideals in G-spaces (Proposition II.5.2) and Theorem II.5.4 which are related to this phenomenon.

Here are some examples of L-summands.

Example 1.6(a) Consider $X = L^1(\mu)$. We assume that μ is localizable (e.g. σ-finite) so that $L^1(\mu)^* \cong L^\infty(\mu)$ in a natural fashion. Then the L-projections on $L^1(\mu)$ coincide with the characteristic projections $P_A(f) = \chi_A f$ for measurable sets A.

PROOF: Trivially, P_A is an L-projection. Conversely, for a given L-projection P the adjoint P^* is an M-projection on $L^\infty(\mu)$. Now $L^\infty(\mu)$ is a commutative unital C^*-algebra, and as such it may be represented as $C(K)$ by the Gelfand-Naimark theorem. (This representation is also possible in the real case.) Since the Gelfand transform is multiplicative, the idempotent elements in $L^\infty(\mu)$ (i.e., the measurable characteristic functions) correspond exactly to the idempotent elements in $C(K)$ (i.e., the continuous characteristic functions). Now Example 1.4(a) tells us that M-projections are characteristic projections. Thus the same is true for $L^\infty(\mu)$ so that P^*, and hence P, is a characteristic projection. $\square$

The description of the L-projections on $L^1(\mu)$ in terms of the measure space is a bit more involved if μ is arbitrary; we refer to [66, p. 58]. However, every space $L^1(\mu)$ is order isometric to a space $L^1(m)$ where m is (even strictly) localizable [559, p. 114]. Therefore, our initial restriction concerning μ is not a severe one, since we are dealing with Banach space properties of L^1 rather than properties of the underlying measure space.
It is worthwhile adopting the point of view of Banach lattice theory in this example. Using the notions of Banach lattice theory we have established the fact that the L-projections on an L^1-space (or an (AL)-space if one prefers) coincide with the band projections and that the L-summands coincide with the (projection) bands (which are the same as the order ideals in L^1); cf. [559, p. 113 and passim] for these matters. The advantage of this description is that it avoids explicit reference to the underlying measure space.
Incidentally, we have shown that the M-projections on $L^1(\mu)^*$ are weak* continuous. This is true for every dual Banach space, see Theorem 1.9 below.

Example 1.6(b) The Lebesgue decomposition $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sing}}$ with respect to a given probability measure furnishes another example of an L-decomposition. Note that neither of the L-summands in

$$C[0,1]^* = M[0,1] = L^1[0,1] \oplus_1 M_{\mathrm{sing}}[0,1]$$

is weak* closed. (In fact, both of them are weak* dense.)

Example 1.6(c) In Chapter IV we shall study Banach spaces which are L-summands in their biduals. Prominent examples will be the L^1-spaces as well as their "noncommutative" counterparts, i.e., the preduals of von Neumann algebras.

The preceding examples have shown a variety of situations where M-ideals or L-summands arise in a natural way. Sometimes, however, it will also be of interest to have a criterion at hand in order to show that a given Banach space does not contain any nontrivial M-ideal or L-summand.

Proposition 1.7 *If X is smooth or strictly convex, then X contains no nontrivial M-ideal and no nontrivial L-summand.*

Proof: Since points of norm one which are contained in a nontrivial L-summand never have a unique supporting hyperplane, we conclude that a smooth space cannot contain a nontrivial L-summand. On the other hand, if X is smooth and $x^* \in S_{X^*}$ attains its norm, then x^* is an extreme functional. Now the Bishop-Phelps theorem (see Theorem VI.1.9) yields that $\operatorname{ex} B_{X^*}$ is norm dense in S_{X^*}. By Lemma 1.5, X cannot contain a nontrivial M-ideal either. (Actually, we have shown the (by virtue of Proposition V.4.6) stronger statement that all the L-summands in the dual of a smooth space are trivial. However, we shall eventually encounter the dual of a smooth space which contains nontrivial M-ideals, namely L^∞/H^∞ (Remark IV.1.17).)
As for the case of strictly convex spaces, all the L-summands are trivial by Lemma 1.5, and the triviality of the M-ideals is a consequence of Corollary II.1.5 below. $\square$

Proposition 1.7 applies in particular to the L^p-spaces for $1 < p < \infty$.
The following theorem describes a dichotomy concerning the existence of L-summands and M-ideals.

Theorem 1.8 *A complex Banach space or a real Banach space which is not isometric to $\ell^\infty(2)$ $(= (\mathbb{R}^2, \|\,.\,\|_\infty))$ cannot contain nontrivial M-ideals and nontrivial L-summands simultaneously.*

Proof: Suppose that the real Banach space X admits nontrivial decompositions

$$\begin{aligned} X &= J \oplus_\infty \widehat{J} \\ &= Y \oplus_1 \widehat{Y}. \end{aligned}$$

Our aim is to show that under this assumption Y must be one-dimensional. Once this is achieved we conclude by symmetry that $\widehat{Y}$ must be one-dimensional as well so that X is isometric to $\ell^1(2)$ $(\cong \ell^\infty(2))$. If X is a complex Banach space, then the above reasoning applies to the underlying real space and yields that X is $\mathbb{R}$-isometric to $\ell^\infty_{\mathbb{R}}(2)$ which of course is impossible.
Let us now present the details of the proof that $\dim(Y) = 1$. We denote by P the M-projection onto J and by π the L-projection onto Y. We first claim:

$$J \cap Y = \{0\} \tag{1}$$

Assume to the contrary that there exists some $u \in J \cap Y$ with $\|u\| = 1$. Let $x \in \widehat{J}$, $\|x\| = 1$. We shall show $x \in Y$: Since $\|u \pm x\| = 1$ we have

$$\begin{aligned} 2 &= \|u + x\| + \|u - x\| \\ &= \|u + \pi(x)\| + \|x - \pi(x)\| + \|u - \pi(x)\| + \|\pi(x) - x\| \\ &\geq 2 \cdot \|u\| + 2 \cdot \|x - \pi(x)\| \end{aligned}$$

so that

$$x = \pi(x) \in Y.$$

This shows $\widehat{J} \subset Y$ and in particular $\widehat{J} \cap Y \neq \{0\}$. Thus we may repeat the same argument with the duo $\widehat{J}$ & Y to obtain $J \subset Y$ as well. Consequently $Y = X$ in contrast to our assumption that the L-decomposition is nontrivial. Therefore (1) holds.

To show that Y is one-dimensional we again argue by contradiction. Suppose there exists a two-dimensional subspace Y_0 of Y. By (1), $P_{|Y_0}$ is injective so that $J_0 := P(Y_0)$ is two-dimensional, too. By Mazur's theorem (e.g. [317, p. 171]) (or since every convex function on $\mathbb{R}$ has a point of differentiability) J_0 contains a smooth point z, i.e., $\|z\| = 1$ and

$$\ell(x) := \lim_{h\to 0} \frac{1}{h}\Big(\|z + hx\| - 1\Big)$$

exists for all $x \in J_0$. Since J_0 is supposed to be two-dimensional, we can find some $x \in J_0,\ \|x\| = 1$, such that

$$\lim_{h\to 0} \frac{1}{h}\Big(\|z + hx\| - 1\Big) = 0. \tag{2}$$

Let us write $z = \dfrac{Py}{\|Py\|}$ for some $y \in Y_0,\ \|y\| = 1$. We next claim:

$$\lim_{h\to 0} \frac{1}{h}\Big(\|y + hx\| - 1\Big) = 0 \tag{3}$$

To prove this we note

$$\|y + hx\| = \max\{\|Py + hx\|, \|y - Py\|\}$$

since P is an M-projection and $x \in J$. Also note

$$1 = \|y\| = \max\{\|Py\|, \|y - Py\|\}.$$

Thus, if $\|Py\| < 1$, we have for sufficiently small h

$$\|y + hx\| = \|y - Py\| = 1,$$

and (3) follows immediately. If $\|Py\| = 1$ (hence $z = Py$) and $\|y - Py\| < 1$, we have for sufficiently small h

$$\|y + hx\| = \|Py + hx\| = \|z + hx\|,$$

and (3) follows from (2). It is left to consider the case where $1 = \|Py\| = \|y - Py\|$. In this case we have

$$\|y + hx\| = \max\{\|z + hx\|, 1\},$$

and again (3) follows from (2).

We can now conclude the proof as follows. For $h > 0$ we have

$$\|y + hx\| = \|y + h\pi(x)\| + h\|x - \pi(x)\|,$$

$$\|y - hx\| = \|y - h\pi(x)\| + h\|x - \pi(x)\|.$$

Observing

$$\|y + h\pi(x)\| + \|y - h\pi(x)\| \ge 2 \cdot \|y\| = 2$$

we obtain from the above equations

$$\frac{1}{h}\Big(\|y + hx\| - 1\Big) + \frac{1}{h}\Big(\|y - hx\| - 1\Big) \ge 2 \cdot \|x - \pi(x)\|,$$

which shows $x = \pi(x) \in Y$ by virtue of (3). Since $0 \neq x \in J$ by choice of x we have arrived at a contradiction to (1). Therefore, Y must be one-dimensional. $\square$

The real space $\ell^\infty(2)$ has to be excluded because it contains the nontrivial M-summand $\{(s,t) \mid s = 0\}$ and the nontrivial L-summand $\{(s,t) \mid s + t = 0\}$.
Theorem 1.8 implies that an L^1-space does not contain any nontrivial M-ideal whereas a $C(K)$-space does not contain any nontrivial L-summand unless the scalars are real and the space is two-dimensional.

Next we are going to collect several results on M-ideals and L-summands which will frequently be used.

Theorem 1.9 *An M-summand in a dual space X^* is weak* closed and hence of the form $J^\perp$ for some L-summand J in X. Consequently, an M-projection on a dual space is weak* continuous.*

PROOF: Given a decomposition $X^* = J_1 \oplus_\infty J_2$ let us assume to the contrary that J_1 is not weak* closed. In this case there exists a net (x_i^*) in the unit ball of J_1 which converges in the weak* sense to some $x^* \in J_2$, $x^* \neq 0$. (Here the Krein-Smulian theorem enters.) Then $y_i^* = x_i^* + x^*/\|x^*\|$ defines a net in the unit ball of X^* whose weak* limit has norm $1 + \|x^*\| > 1$: a contradiction.
It is easily checked that a projection whose range and kernel are weak* closed is weak* continuous so that the remaining assertion follows. $\square$

REMARKS: (a) Theorem 1.9 can be thought of as a "localized" version of Grothendieck's result that L^1 is the only predual of L^∞ [281].

(b) We will eventually prove in Corollary II.3.6 that a weak* closed M-ideal in a dual space is an M-summand and hence the annihilator of an L-summand.

(c) In Proposition V.4.6 we present an example of a space X without nontrivial M-ideals such that X^* has infinitely many L-summands; see also Example IV.1.8.

Theorem 1.10

(a) *Two L- (resp. M-)projections commute.*

(b) *The set $\mathbb{P}_L(X)$ of all L-projections on X forms a complete Boolean algebra under the operations*

$$P \wedge Q = PQ, \;\; P \vee Q = P + Q - PQ, \;\; P^c = Id - P.$$

(c) *The set $\mathbb{P}_M(X)$ of all M-projections on X forms a (generally not complete) Boolean algebra. However, for a dual space X^* $\mathbb{P}_M(X^*)$ is isomorphic to $\mathbb{P}_L(X)$ and hence complete.*

PROOF: (a) Let P and Q be L-projections. Then

$$\begin{aligned}
\|Qx\| &= \|PQx\| + \|(Id - P)Qx\| \\
&= \|QPQx\| + \|(Id - Q)PQx\| + \|Q(Qx - PQx)\| + \|(Id - Q)(Qx - PQx)\| \\
&= \|QPQx\| + 2 \cdot \|PQx - QPQx\| + \|Qx - QPQx\| \\
&\geq \|Qx\| + 2 \cdot \|PQx - QPQx\|
\end{aligned}$$

so that $PQ = QPQ$. Likewise we obtain $P(Id - Q) = (Id - Q)P(Id - Q)$ which is equivalent to $QP = QPQ$, hence the result.

(b) PQ is a projection by (a), and it is an L-projection since

$$\begin{aligned} \|x\| &= \|Qx\| + \|(Id - Q)x\| \\ &= \|PQx\| + \|Qx - PQx\| + \|x - Qx\| \\ &\geq \|PQx\| + \|x - PQx\| \\ &\geq \|x\|, \end{aligned}$$

and so is $P + Q - PQ = Id - (Id - P)(Id - Q)$. It is routine to verify that $\mathbb{P}_L$ has the structure of a Boolean algebra.
Let $\mathbb{M}$ be a family of L-projections. To prove that $\mathbb{P}_L$ is complete we have to show that the supremum of $\mathbb{M}$ (for the induced order: $P \leq Q$ if and only if $PQ = P$) exists. Upon replacing $\mathbb{M}$ by its family of finite suprema one may assume without loss of generality that $\mathbb{M}$ is upward directed. Since

$$\begin{aligned} \|Qx\| &= \|PQx\| + \|Qx - PQx\| \\ &= \|Px\| + \|Qx - Px\| \\ &\geq \|Px\| \end{aligned}$$

for $P \leq Q$ and since $\{\|Px\| \mid P \in \mathbb{M}\}$ is bounded we see that $(Px)_{P\in\mathbb{M}}$ is a Cauchy net for every x: If $\|P_0x\| \geq \sup_{P\in\mathbb{M}} \|Px\| - \varepsilon$, then $\|Q_1x - Q_2x\| \leq 2\varepsilon$ for $Q_1, Q_2 \geq P_0$. It is not hard to verify that its pointwise limit $Sx = \lim_{P\in\mathbb{M}} Px$ is an L-projection, and that $Q \geq S$ if and only if $Q \geq P$ for all $P \in \mathbb{M}$ which means that S is the supremum of $\mathbb{M}$.

(c) $\mathbb{P}_M$ is a Boolean algebra by the duality of L- and M-projections. That $\mathbb{P}_M$ need not be complete can be seen from Example 1.4(a): For $X = c = C(\alpha\mathbb{N})$ the family of M-projections onto the M-summands J_D, $D = \{2\}, \{2,4\}, \{2,4,6\}, \ldots$, has no infimum. For dual spaces combine Theorem 1.9 and (b). □

Proposition 1.11 *Let J_i $(i \in I)$, J_1, J_2 be M-ideals in X.*

(a) $\overline{\text{lin}} \bigcup_i J_i$ *is an M-ideal in X. (Equivalently, the intersection of weak* closed L-summands is a weak* closed L-summand.)*
(b) $J_1 \cap J_2$ *is an M-ideal in X.*
(c) $J_1 + J_2$ *is closed and hence an M-ideal in X. Moreover*

$$J_1/(J_1 \cap J_2) \cong (J_1 + J_2)/J_2.$$

PROOF: (a) and (b) are consequences of Theorem 1.10(b) (cf. [51, p. 37] for details). (c) is a special case of Proposition 1.16 below. □

We again stress that in general an arbitrary intersection of M-ideals need not be an M-ideal; see II.5.5 for a counterexample.
The following proposition is very important, though its proof is immediate from the definition of an M-ideal. It states that M-ideals are "Hahn-Banach smooth".

Proposition 1.12 *Let J be an M-ideal in X. Then every $y^* \in J^*$ has a unique norm preserving extension to a functional $x^* \in X^*$.*

PROOF: By assumption, $J^\perp$ is an L-summand so that there is a decomposition

$$X^* = J^\perp \oplus_1 J^\#. \tag{$*$}$$

But $J^\#$ can explicitly be described, since there are canonical isometric isomorphisms

$$J^* \cong X^*/J^\perp \cong J^\#$$

so that

$$J^\# = \{x^* \in X^* \mid \|x^*\| = \|x^*|_J\|\}.$$

The result now follows. □

Remark 1.13 Proposition 1.12 enables us to view J^* as a subspace of X^*. Accordingly we shall replace $(*)$ above by

$$X^* = J^\perp \oplus_1 J^* \tag{$**$}$$

in the sequel. Thus, it makes sense to consider the (generally non-Hausdorff) topology $\sigma(X, J^*)$, which we shall occasionally meet. (In the special case where J is an M-ideal in J^{**} (see Chapter III) this is nothing but the weak* topology on J^{**}.)
The Hahn-Banach theorem implies that B_J is $\sigma(X, J^*)$-dense in B_X. (Note that this holds even in the case when $\sigma(X, J^*)$ is not Hausdorff.) This simple observation has important consequences when applied to operator spaces (Proposition VI.4.10).

Next we wish to consider the question whether an M-ideal in a Banach space induces an M-ideal in a subspace or a quotient space. We first collect some general facts in this direction.

Lemma 1.14 *Let J and Y be closed subspaces in a Banach space X.*

(a) *$J+Y$ is closed in X if and only if $J^\perp+Y^\perp$ is closed in X^* if and only if $J^\perp+Y^\perp$ is weak* closed in X^*. In this case $J^\perp + Y^\perp = (J \cap Y)^\perp$, and $(J+Y)/J$ is isomorphic to $Y/(J \cap Y)$, $(J^\perp + Y^\perp)/Y^\perp$ is isomorphic to $J^\perp/(J^\perp \cap Y^\perp)$.*

(b) *Suppose $J^\perp$ is the range of a projection P such that $P(Y^\perp) \subset Y^\perp$. Then the assertions of* (a) *hold. If P is contractive we even have*

$$\begin{aligned} (J+Y)/J &\cong Y/(J \cap Y), \\ (J^\perp + Y^\perp)/Y^\perp &\cong J^\perp/(J^\perp \cap Y^\perp). \end{aligned}$$

If $Id - P$ is contractive we have

$$(J+Y)/Y \cong J/(J \cap Y).$$

PROOF: (a) To prove the asserted equivalence apply the closed range theorem to the operator $y \mapsto y + J$ from Y to X/J (cf. [521] for details). The sum $J^\perp + Y^\perp$ is always weak* dense in $(J \cap Y)^\perp$ by the Hahn-Banach theorem so that the next assertion obtains. Finally, the operator

$$T : Y/(J \cap Y) \longrightarrow (J+Y)/J, \quad T(y + (J \cap Y)) = y + J$$

is a well-defined bijective contractive operator between Banach spaces, hence an isomorphism. Similarly,

$$S : J^\perp/(J^\perp \cap Y^\perp) \longrightarrow (J^\perp + Y^\perp)/Y^\perp, \quad S(z^* + (J^\perp \cap Y^\perp)) = z^* + Y^\perp$$

is an isomorphism.

(b) Let us prove that $J^\perp + Y^\perp$ is closed. Suppose

$$J^\perp + Y^\perp \ni z_n^* + \xi_n^* \to x^* \in X^*$$

as $n \to \infty$. Then

$$\xi_n^* - P\xi_n^* = (Id - P)(z_n^* + \xi_n^*) \to x^* - Px^*$$

which is seen to lie in $Y^\perp$ by assumption on P. Hence

$$x^* = Px^* + (x^* - Px^*) \in J^\perp + Y^\perp.$$

It is left to prove that T^{-1} and S^{-1} are contractive. The latter follows from the estimate (valid for $z^* \in J^\perp$)

$$\begin{aligned}
&\inf\{\|z^* - \xi^*\| \mid \xi^* \in J^\perp \cap Y^\perp\} \\
&\qquad = \inf\{\|Pz^* - P\xi^*\| \mid \xi^* \in X^*,\ P\xi^* \in Y^\perp\} \\
&\qquad \le \inf\{\|z^* - \xi^*\| \mid \xi^* \in X^*,\ P\xi^* \in Y^\perp\} \\
&\qquad \le \inf\{\|z^* - \xi^*\| \mid \xi^* \in Y^\perp\}
\end{aligned}$$

(where we used $\|P\| \le 1$ and $P(Y^\perp) \subset Y^\perp$); for the former observe that

$$\begin{aligned}
\|y + J \cap Y\| &= d(y, J \cap Y) \\
&= \sup\{|\langle x^*, y\rangle| \mid \|x^*\| \le 1,\ x^* \in (J \cap Y)^\perp = J^\perp + Y^\perp\}, \\
\|y + J\| &= d(y, J) \\
&= \sup\{|\langle x^*, y\rangle| \mid \|x^*\| \le 1,\ x^* \in J^\perp\}.
\end{aligned}$$

Now for $x^* = z^* + \xi^* \in J^\perp + Y^\perp$ we have $x^* - Px^* = \xi^* - P\xi^* \in Y^\perp$, hence $\langle x^*, y\rangle = \langle Px^*, y\rangle$. Since $\|Px^*\| \le \|x^*\|$ the result follows.
The remaining isometry is established in the same way. □

Lemma 1.15 *Let P be a projection on X and let $Y \subset X$ be a closed subspace. We suppose $P(Y) \subset Y$ so that*

$$P_{|Y} : Y \longrightarrow Y, \quad y \mapsto Py$$

and

$$P/Y : X/Y \longrightarrow X/Y, \quad x + Y \mapsto Px + Y$$

are well-defined projections. We have

$$\begin{aligned}
\operatorname{ran}(P_{|Y}) &= \operatorname{ran}(P) \cap Y, \\
\operatorname{ran}(P/Y) &= (\operatorname{ran}(P) + Y)/Y,
\end{aligned}$$

and

$$\mathrm{ran}(P/Y) \cong \mathrm{ran}(P)/(\mathrm{ran}(P) \cap Y)$$

if P is contractive. Moreover, $P_{|Y}$ and P/Y are L- (resp. M-) projections if P is.

Proof: All the assertions concerning $P_{|Y}$ are obvious. Note that $P^*(Y^\perp) \subset Y^\perp$ and thus $(P/Y)^* = P^*{}_{|Y^\perp}$. Taking into account the duality of L- and M-projections and Lemma 1.14 we infer the assertions concerning P/Y. □

Proposition 1.16 *Suppose J is an M-ideal in X with corresponding L-projection P from X^* onto $J^\perp$. Suppose in addition that Y is a closed subspace of X such that*

(a) $$P(Y^\perp) \subset Y^\perp.$$

Then $J \cap Y$ is an M-ideal in Y, $J+Y$ is closed, and $(J+Y)/Y$ is isometric with $J/(J \cap Y)$ and an M-ideal in X/Y.
Moreover, (a) *is equivalent to either of the conditions*

(b) $J \cap Y$ *is $\sigma(X, J^*)$-dense in Y,*

(b_1) $B_{J \cap Y}$ *is $\sigma(X, J^*)$-dense in B_Y.*

Proof: If one identifies Y^* with $X^*/Y^\perp$ then $P/Y^\perp$ defines an L-projection from Y^* onto the annihilator of $J \cap Y$ in Y^* by Lemma 1.15 so that $J \cap Y$ is an M-ideal in Y. To show that $(J+Y)/Y$ is an M-ideal in X/Y employ the L-projection $P_{|Y^\perp}$. The other assertions are special cases of Lemma 1.14. Now that $J \cap Y$ is an M-ideal in Y under assumption (a), we see from Remark 1.13 that (b_1) holds, and (b_1) trivially implies (b). Next assume (b). For $\xi^* \in Y^\perp$ we have $\xi^* - P\xi^* \in \ker(P) = J^*$. Now $\xi^* \in Y^\perp \subset (J \cap Y)^\perp$ and $P\xi^* \in J^\perp \subset (J \cap Y)^\perp$ imply that $\xi^* - P\xi^*$ annihilates $\overline{J \cap Y}^{\sigma(X, J^*)}$ so that by (b) $\xi^* - P\xi^* \in Y^\perp$. Hence (a) follows. □

Remarks: (a) The reader should have no difficulty in recovering Example 1.4(b) from 1.4(a) using condition (a) of Proposition 1.16. This is the abstract argument behind our reasoning in Example 1.4(b), and it will be expressively mentioned in Corollary 1.19 below.

(b) We remark that (a) is not necessary to ensure that $J \cap Y$ is an M-ideal. For example, let $Y = \ell^\infty(2)$, $K = B_{Y^*} (= \{(s,t) \in \mathbb{K}^2 \mid |s| + |t| \le 1\})$ and $X = C(K)$ so that $Y \subset X$ in a canonical fashion. Let $U = \{(0,t) \mid t \in \mathbb{K}\}$ and

$$D = \{y^* \in K \mid y^*{}_{|U} = 0\} = \{(s,0) \mid |s| \le 1\}.$$

Then U is the intersection of the M-ideal J_D with Y and also itself an M-ideal in Y (in fact, an M-summand). Nevertheless, assumption (a) in Proposition 1.16 is violated: If $\mu \in M(K)$ is the discrete measure

$$\mu = \delta_{(1/2,1/2)} + \delta_{(1/2,-1/2)} + \delta_{(-1,0)}$$

then $\mu \in Y^\perp$, but

$$P(\mu) = \chi_D \mu = \delta_{(-1,0)} \notin Y^\perp.$$

For a positive result involving maximal measures we refer to Theorem 2.4.

(c) Condition (a) in 1.16 is fulfilled if Y, too, is an M-ideal, since L-projections commute (1.10(a)). This provides a proof of Proposition 1.11(c).

Proposition 1.17

(a) *If J_1 and J_2 are M-ideals in X, then the canonical images of J_1 and J_2 in $(J_1 + J_2)/(J_1 \cap J_2)$ are complementary M-summands.*

(b) *Let J be an M-ideal in X.*

- *$Y \subset J$ is an M-ideal in X if and only if it is an M-ideal in J.*
- *$Y \subset X/J$ is an M-ideal if and only if it is the image of an M-ideal in X under the quotient map.*
- *If $Y \subset J$, then J/Y is an M-ideal in X/Y.*
- *If $J \subset Y \subset X$, then J is an M-ideal in Y.*

PROOF: These assertions are easily verified; for details cf. [51, p. 39f.]. □

We recall from the fundamental Example 1.4(a) that the M-ideals in $C(K)$ coincide with the ideals J_D. The following proposition says that the J_D are in fact the ancestors of all M-ideals, since every Banach space is a subspace of some $C(K)$.

Proposition 1.18 *Let X be a closed subspace of $C(K)$, and let J be an M-ideal in X. Then there is a closed subset D of K such that $J = J_D \cap X$.*

PROOF: Let $D = \{k \in K \mid \delta_k|_X \in J^\perp\}$. Then D is a closed set, and $J \subset J_D \cap X$ by construction. If the inclusion were proper, we could separate a certain $x_0 \in J_D \cap X$ from J by a functional $p \in J^\perp$. We may even assume $p \in \operatorname{ex} B_{J^\perp}$ by the Krein-Milman theorem and thus (Lemma 1.5) $p \in \operatorname{ex} B_{X^*}$. Such a p is of the form $p(x) = \lambda \cdot x(k)$ for some $k \in K$, $|\lambda| = 1$. Since $p \in J^\perp$ we must have $k \in D$ and hence $x_0(k) = 0$. On the other hand $p(x_0) \neq 0$ since p is a separating functional: a contradiction. □

We turn to the converse of this proposition, which, of course, will generally not hold. The following is a special case of Proposition 1.16.

Corollary 1.19 *Let X be a closed subspace of $C(K)$. Then $J_D \cap X$ is an M-ideal in X if the L-projection $\mu \mapsto \chi_D \mu$ from $C(K)^*$ onto $J_D^\perp$ leaves $X^\perp$, the annihilator of X in $C(K)^*$, invariant.*

Let us formulate an important example where this occurs. Suppose X is a closed subspace of $C(K)$ and suppose $D \subset K$ is closed. We let $X|_D$ be the space of all restrictions $\{x|_D \mid x \in X\}$. Following [444] one says that $(X|_D, X)$ has the *bounded extension property* if there exists a constant C such that, given $\xi \in X|_D$, $\varepsilon > 0$ and an open set $U \supset D$, there is some $x \in X$ such that

$$\begin{aligned} x|_D &= \xi, \\ \|x\| &\leq C \cdot \|\xi\|, \\ \|x(k)\| &\leq \varepsilon \qquad \text{for } k \notin U. \end{aligned}$$

(Note that $X|_D$ is closed under this assumption.) For example, if $X \subset C(K)$ is a subalgebra and $D \subset K$ is a subset of the form $f^{-1}(\{1\})$ for some $f \in B_X$ (a "peak set"), then the pair $(X|_D, X)$ has the bounded extension property: Let $\xi \in X|_D$ and $g \in X$ such that $g|_D = \xi$. First of all we remark that replacing f by $(1+f)/2$ permits us to assume that in addition $|f(k)| < 1$ if and only if $k \notin D$. Then $g \cdot f^n$ meets the requirements of the above definition if n is large enough. (The argument easily extends to intersections of peak sets, the so-called p-sets.)
We now have:

Proposition 1.20 *If $(X|_D, X)$ has the bounded extension property, then $J_D \cap X$ is an M-ideal in X.*

Instead of proving this proposition now with the help of Corollary 1.19 we prefer to provide a proof using the methods of the next section, see p. 24. In fact, not only could one prove that under the assumption of the bounded extension property the condition of Corollary 1.19, namely $\chi_D \mu \in X^\perp$ for all $\mu \in X^\perp$, is fulfilled; it is even true that the two conditions are equivalent. This follows from [238]. Let us add that Hirsberg [311] has obtained necessary and sufficient conditions on a closed subset D to ensure that $J_D \cap X$ is an M-ideal in X, where $X \subset C(K)$ is assumed to be a closed subspace separating points and containing the constants.

As a counterpart to the preceding results we now discuss the L-structure of subspaces of L^1-spaces. (In contrast to the situation presented in Proposition 1.18 and Corollary 1.19, not every Banach space is a subspace of some L^1-space.) Recall from Example 1.6(a) that the L-projections on L^1 coincide with the band projections.

Proposition 1.21 *Let X be a closed subspace of some L^1-space and let $P : X \longrightarrow X$ be an L-projection. Then P can be extended to an L-projection on L^1. More precisely, if P_B denotes the L-projection from L^1 onto the band B generated by $P(X)$, then X is an invariant subspace of P_B and $P_B|_X = P$.*

Proof: We shall use the slightly unusual notation ($A \subset L^1$ a given subset)

$$A^{\mathrm{s}} = \{y \in L^1 \mid |x| \wedge |y| = 0 \quad \text{for all } x \in A\}.$$

(Here s stands for singular. The usual notation $A^\perp$ collides with our symbol for the annihilator of A.) Then ([558, p. 210]) $B = P(X)^{\mathrm{ss}}$ and $B^{\mathrm{s}} = P(X)^{\mathrm{s}}$. We first claim:

(a) For $x \in X$ there is a measurable set E (depending on x) such that $Px = \chi_E \cdot x$.

In fact, for $f, g \in L^1$ with $\|f \pm g\| = \|f\| + \|g\|$ we have $|f| \wedge |g| = 0$, since $|f(\omega) \pm g(\omega)| = |f(\omega)| + |g(\omega)|$ almost everywhere which implies $f(\omega) \cdot g(\omega) = 0$ almost everywhere. In particular $|Px| \wedge |x - Px| = 0$, and our claim obtains with $E = \{Px \neq 0\}$.
We now show

(b) $$(Id - P)(X) = X \cap B^{\mathrm{s}},$$

(c) $$P(X) = X \cap B,$$

which will prove Proposition 1.21.

ad (b): For $x \in X \cap B^{\rm s}$ we have $|x| \wedge |Px| = 0$. But $Px = \chi_E \cdot x$ for a certain E by (a) so that $Px = 0$ and $x \in (Id - P)(X)$. Conversely, let $Px = 0$. We wish to prove $|x| \wedge |y| = 0$ for all $y \in P(X)$. Now choose E such that $P(x+y) = \chi_E \cdot (x+y)$. It follows $y = P(x+y) = \chi_E \cdot (x+y)$ and $\chi_{\complement E} \cdot y = \chi_E \cdot x$. This implies $|x| \wedge |y| = 0$.

ad (c): "$\subset$" is trivial. Let $x \in X \cap B$. Since $Px \in B$ we have $x - Px \in B$. On the other hand $x - Px \in B^{\rm s}$ by (b). Thus $x = Px \in P(X)$. $\square$

Corollary 1.22 *Let X and Y be closed subspaces of $C(K)$ with $Y \subset X$. Let B be the band in $M(K)$ generated by $X^\perp$, the annihilator of X in $M(K)$, and let P_B be the band projection onto B. Then X/Y is an M-ideal in $C(K)/Y$ if and only if*

$$P_B(Y^\perp) \subset Y^\perp \tag{1}$$

$$B \cap Y^\perp = X^\perp. \tag{2}$$

PROOF: By standard duality, X/Y is an M-ideal if and only if there is an L-projection from $Y^\perp$ onto $X^\perp$. Now, if (1) and (2) hold, the restriction of P_B to $Y^\perp$ is such a projection. The converse follows from Proposition 1.21. $\square$

The final part of this section is devoted to the possible dependence of results on the choice of the scalar field. Up to now we have not encountered the necessity of treating real and complex spaces separately. This is not accidental.

Proposition 1.23 *Let X be a complex Banach space, and denote by $X_{\mathbb{R}}$ the same space, considered as a real Banach space. If P is an L- (resp. M-) projection on $X_{\mathbb{R}}$, then P is complex linear and thus an L- (resp. M-) projection on X. Consequently, X and $X_{\mathbb{R}}$ have the same L-summands, M-summands and M-ideals.*

PROOF: Let P be an $\mathbb{R}$-linear L-projection. It is enough to prove

$$z \in \operatorname{ran}(P) \implies iz \in \operatorname{ran}(P) \tag{$*$}$$

since then $iPx \in \operatorname{ran}(P)$ and, by symmetry, $i(Id - P)x \in \ker(P)$ for every $x \in X$ which shows

$$iPx = P(iPx) = P(ix - i(Id-P)x) = P(ix).$$

For the proof of $(*)$ let $Qx = i^{-1}P(ix)$. This is an $\mathbb{R}$-linear L-projection because multiplication by i is an isometry. Consequently, P and Q commute (1.10(a)). Now we have for $z \in \operatorname{ran}(P)$

$$z - Qz = Pz - QPz = P(z - Qz) \in \operatorname{ran}(P)$$

and

$$i(z - Qz) = iz - P(iz) \in \ker(P)$$

so that

$$\begin{aligned} \sqrt{2} \cdot \|z - Qz\| &= \|(z - Qz) + i(z - Qz)\| \\ &= \|z - Qz\| + \|z - Qz\| \\ &= 2 \cdot \|z - Qz\|. \end{aligned}$$

Hence $z = Qz$, i.e. $iz = P(iz) \in \operatorname{ran}(P)$. $\square$

There are, however, several phenomena particular to complex spaces. We now discuss one of them. Suppose X is a complex Banach space and $T \in L(X)$. We let

$$\Pi(X) = \{(x^*, x) \in B_{X^*} \times B_X \mid \langle x^*, x\rangle = 1\}.$$

Recall that T is called *hermitian* if

$$V(T) := \{\langle x^*, Tx\rangle \mid (x^*, x) \in \Pi(X)\} \subset \mathbb{R}.$$

The set $V(T)$ is called the spatial numerical range of T. Equivalently, T is hermitian if and only if $\|\exp(itT)\| = 1$ for all $t \in \mathbb{R}$. We refer to [84] for a proof of this fact and related information. As an application, one easily shows that L- and M-projections are hermitian.

We wish to prove that M-ideals are invariant subspaces of hermitian operators on complex Banach spaces. This will be obtained as a corollary to the following technical proposition, which will be used again in Lemma V.6.7.

Proposition 1.24 *Suppose X is a complex Banach space, and let $J \subset X$ be an M-ideal. Moreover, let $T \in L(X)$ be an operator whose numerical range $V(T)$ is contained in the strip $R(\varepsilon) := \mathbb{R} \times [-\varepsilon, \varepsilon]i$ of the complex plane. Then for $x \in J$*

$$d(Tx, J) \leq \varepsilon \cdot \|x\|.$$

Proof: Let us assume $\|x\| = 1$. Since

$$d(Tx, J) = \|Tx + J\| = \sup\{|\langle x^*, Tx\rangle| \mid x^* \in B_{J^\perp}\}$$

we have to show

$$|\langle x^*, Tx\rangle| \leq \varepsilon \quad \text{for all } x^* \in S_{J^\perp}.$$

So let $x^* \in S_{J^\perp}$. Taking into account the canonical duality $(J^\perp)^* = (J^*)^\perp$ (recall Remark 1.13), we may choose $x^{**} \in B_{(J^*)^\perp}$ such that $\langle x^{**}, x^*\rangle = 1$, i.e. $(x^{**}, x^*) \in \Pi(X^*)$. Then

$$\langle \lambda x^*, x + \overline{\lambda} x^{**}\rangle = 1$$

and

$$\|x + \overline{\lambda} x^{**}\| = \max\{\|x\|, \|x^{**}\|\} = 1$$

if $|\lambda| = 1$. Consequently $(x + \overline{\lambda} x^{**}, \lambda x^*) \in \Pi(X^*)$ so that

$$\lambda\langle x^*, Tx\rangle + \langle x^*, T^{**}x^{**}\rangle \in V(T^*) \overset{(*)}{\subset} \overline{V(T)} \subset R(\varepsilon)$$

if $|\lambda| = 1$ (cf. [86, p. 11] for the inclusion marked $(*)$). But this implies $|\langle x^*, Tx\rangle| \leq \varepsilon$, as requested. $\square$

Letting $\varepsilon \to 0$ in this proposition we obtain:

Corollary 1.25 *A hermitian operator on a complex Banach space leaves M-ideals invariant.*

I.2 Characterisation theorems

The definition of an M-ideal was given in terms of projections on the dual space. The first aim of the present section is to derive an equivalent condition for J to be an M-ideal which avoids mentioning the dual space. We start with an easy lemma.

Lemma 2.1 *Suppose J is an M-summand in X, and suppose $(B(x_i, r_i))_{i \in I}$ is a family of closed balls satisfying*

$$B(x_i, r_i) \cap J \neq \emptyset \quad \textit{for all } i \in I \tag{1}$$

and

$$\bigcap_i B(x_i, r_i) \neq \emptyset. \tag{2}$$

Then

$$\bigcap_i B(x_i, r_i) \cap J \neq \emptyset.$$

PROOF: Let P be the M-projection onto J, and let $x \in \bigcap_i B(x_i, r_i)$. We claim $Px \in \bigcap_i B(x_i, r_i)$.
In fact, if $y_i \in B(x_i, r_i) \cap J$ then

$$\begin{aligned} r_i &\geq \|x_i - y_i\| \\ &= \|(Px_i - y_i) + (x_i - Px_i)\| \\ &= \max\{\|Px_i - y_i\|, \ \|x_i - Px_i\|\} \\ &\geq \|x_i - Px_i\| \end{aligned}$$

so that

$$\|x_i - Px\| = \max\{\|Px_i - Px\|, \ \|x_i - Px_i\|\} \leq r_i. \qquad \square$$

It will be shown in Proposition II.3.4 that the intersection condition of Lemma 2.1 actually characterises M-summands. Let us now characterise M-ideals by means of an intersection condition which turns out to be a powerful tool for detecting M-ideals.

Theorem 2.2 *For a closed subspace J of a Banach space X, the following assertions are equivalent:*

(i) *J is an M-ideal in X.*

(ii) (The n-ball property)
For all $n \in \mathbb{N}$ and all families $(B(x_i, r_i))_{i=1,\dots,n}$ of n closed balls satisfying

$$B(x_i, r_i) \cap J \neq \emptyset \quad \textit{for all } i = 1, \dots, n \tag{1}$$

and

$$\bigcap_{i=1}^{n} B(x_i, r_i) \neq \emptyset \tag{2}$$

the conclusion

$$\bigcap_{i=1}^{n} B(x_i, r_i + \varepsilon) \cap J \neq \emptyset \quad \textit{for all } \varepsilon > 0$$

obtains.

(iii) *Same as* (ii) *with* $n = 3$.

(iv) (The [restricted] 3-ball property)
For all $y_1, y_2, y_3 \in B_J$, *all* $x \in B_X$ *and all* $\varepsilon > 0$ *there is* $y \in J$ *satisfying*

$$\|x + y_i - y\| \leq 1 + \varepsilon \qquad (i = 1, 2, 3).$$

(v) (The strict n-ball property)
For all $n \in \mathbb{N}$ *and all families* $(B(x_i, r_i))_{i=1,\dots,n}$ *of* n *closed balls satisfying*

$$B(x_i, r_i) \cap J \neq \emptyset \quad \textit{for all } i = 1, \dots, n \tag{1}$$

and

$$\operatorname{int} \bigcap_{i=1}^{n} B(x_i, r_i) \neq \emptyset \tag{2'}$$

the conclusion

$$\bigcap_{i=1}^{n} B(x_i, r_i) \cap J \neq \emptyset$$

obtains.

Proof: (i) $\Rightarrow$ (ii): We consider X as a subspace of X^{**} and

$$B_{X^{**}}(x_i, r_i) = \{x^{**} \in X^{**} \mid \|x^{**} - x_i\| \leq r_i\}.$$

Now $J^{\perp\perp}$ is an M-summand in X^{**}, hence by Lemma 2.1 we find some

$$x_0^{**} \in \bigcap_i B_{X^{**}}(x_i, r_i) \cap J^{\perp\perp}.$$

If (ii) were false, we could even assume that, for some $\varepsilon > 0$, $D := \bigcap_i B(x_i, r_i + \varepsilon)$ and J have positive distance and hence can be separated strictly. Thus, there is $x_0^* \in J^\perp$ with $\operatorname{Re} x_0^*|_D \geq 1$. Let $E := \operatorname{lin}\{x_0^{**}, x_1, \dots, x_n\} \subset X^{**}$ and $\delta = \min \varepsilon / r_i$.
The principle of local reflexivity assures the existence of an operator $T \in L(E, X)$ satisfying

- $\|T\| \leq 1 + \delta$
- $Tx_i = x_i \qquad (i = 1, \dots, n)$
- $\langle Tx_0^{**}, x_0^* \rangle = \langle x_0^{**}, x_0^* \rangle$.

It follows that

$$\|Tx_0^{**} - x_i\| \ \leq \ \|T\| \cdot \|x_0^{**} - x_i\| \ \leq \ (1+\delta)r_i \ \leq \ r_i + \varepsilon,$$

i.e.

$$Tx_0^{**} \in D,$$

thus

$$1 \ \leq \ \mathrm{Re}\,\langle Tx_0^{**}, x_0^* \rangle = \mathrm{Re}\,\langle x_0^{**}, x_0^* \rangle \ = \ 0,$$

a contradiction. (Variants of this proof will be discussed in the Notes and Remarks.)

(ii) $\Rightarrow$ (iii): This is more than obvious.

(iii) $\Rightarrow$ (iv): (iv) is just the special case $x_i = x + y_i,\ r_i = 1$.

(iv) $\Rightarrow$ (i): We put

$$J^\# = \{x^* \in X^* \mid \ \|x^*\| = \|x^*|_J\|\}.$$

Step 1: Each $x^* \in X^*$ can be written as

$$x^* = x_1^* + x_2^*$$

with $x_1^* \in J^\perp,\ x_2^* \in J^\#$.

[Proof: Let x_2^* be a Hahn-Banach extension of $x^*|_J$ and put $x_1^* = x^* - x_2^*$.]

Step 2: $\|x_1^* + x_2^*\| = \|x_1^*\| + \|x_2^*\|$ for all $x_1^* \in J^\perp,\ x_2^* \in J^\#$.

[Proof: Given $\varepsilon > 0$, choose $x \in B_X$ and $z \in B_J$ such that $\langle x_1^*, x\rangle$ and $\langle x_2^*, z\rangle$ are real and

$$\begin{aligned} \langle x_1^*, x\rangle &\geq \|x_1^*\| - \varepsilon, \\ \langle x_2^*, z\rangle &\geq \|x_2^*\| - \varepsilon. \end{aligned}$$

An application of (iv) with $y_1 = y_2 = z,\ y_3 = -z$ yields

$$\|x \pm z - y\| \leq 1 + \varepsilon$$

for some $y \in J$. Hence

$$\begin{aligned} &(1+\varepsilon)(\|x_1^* + x_2^*\| + \|x_1^* - x_2^*\|) \\ &\qquad \geq \ |\langle x_1^* + x_2^*, x + z - y\rangle + \langle x_1^* - x_2^*, x - z - y\rangle| \\ &\qquad = \ 2\ |\langle x_1^*, x\rangle + \langle x_2^*, z\rangle| \\ &\qquad \geq \ 2\|x_1^*\| + 2\|x_2^*\| - 4\varepsilon \\ &\qquad \geq \ \|x_1^*\| + \|x_2^*\| + \|x_1^* - x_2^*\| - 4\varepsilon \end{aligned}$$

from which Step 2 follows.]

Step 3: The decomposition in Step 1 is unique.

[Proof: If $x_1^* + x_2^* = y_1^* + y_2^*$, then $x_2^* = (y_1^* - x_1^*) + y_2^* \in J^\perp + J^\#$. Since $x_2^*|_J = y_2^*|_J$, Step 2 shows $\|y_1^* - x_1^*\| = 0$.]

Thus, $P : x^* \mapsto x_1^*$ is a well-defined idempotent map onto $J^\perp$ which fulfills the norm condition of L-projections. To finish the proof we must show the linearity of P which easily follows from the

Step 4: $J^\#$ is a linear subspace of X^*.

[Proof: Obviously $J^\#$ is a cone. Now let $x^*, y^* \in J^\#$. Then we have a unique decomposition

$$x^* + y^* = x_1^* + x_2^* \in J^\perp + J^\#.$$

We wish to show $x_1^* = 0$. To this end, let $x \in B_X$. Given $\varepsilon > 0$, choose $y_0, y_1, y_2 \in B_J$ satisfying

$$\begin{array}{rcrcl} \mathbb{R} & \ni & \langle x^*, y_0 \rangle & \geq & \|x^*\| - \varepsilon \\ \mathbb{R} & \ni & \langle y^*, y_1 \rangle & \geq & \|y^*\| - \varepsilon \\ \mathbb{R} & \ni & -\langle x_2^*, y_2 \rangle & \geq & \|x_2^*\| - \varepsilon. \end{array}$$

Use (iv) to obtain $y \in J$ such that

$$\|x + y_i - y\| \leq 1 + \varepsilon \qquad (i = 0, 1, 2).$$

We then have

$$\begin{array}{cl} & (1+\varepsilon)(\|x^*\| + \|y^*\| + \|x_2^*\|) \\ \geq & |\langle x^*, x + y_0 - y \rangle + \langle y^*, x + y_1 - y \rangle - \langle x_2^*, x + y_2 - y \rangle| \\ \geq & \operatorname{Re} \langle x_1^*, x \rangle + \|x^*\| + \|y^*\| + \|x_2^*\| - 3\varepsilon. \end{array}$$

Consequently

$$\operatorname{Re} \langle x_1^*, x \rangle \leq 0 \quad \text{for all } x \in B_X$$

so that $x_1^* = 0$.]

(ii) $\iff$ (v): Since (v) is apparently stronger than (ii), the proof will be completed if we can show the following for a fixed number n:

- If (ii) holds for collections of $n+1$ balls, then (v) holds for collections of n balls.

So let us assume the balls $B(x_i, r_i)$, $i = 1, \ldots, n$, satisfy (1) and (2′). By (2′) there exist $y_0 \in X$ and $\delta > 0$ such that

$$\|y_0 - x_i\| \leq r_i - \delta, \quad i = 1, \ldots, n.$$

Let $r = \min r_i$. Next we wish to construct a sequence $y_1, y_2, \ldots$ in J such that for $k \in \mathbb{N}$

$$\begin{array}{rcll} \|y_k - y_{k-1}\| & \leq & 2^{-k} \cdot 4r, & \\ \|y_k - x_i\| & \leq & r_i + 2^{-k}\delta & \qquad (i = 1, \ldots, n) \end{array}$$

hold. (Note that we are not claiming $y_0 \in J$.) Once this is achieved we see that the limit y of the Cauchy sequence (y_k) belongs to $\bigcap_{i=1}^n B(x_i, r_i) \cap J$.
To construct y_1 consider the balls $B(y_0, 2r-\delta)$ and $B(x_i, r_i)$ for $i = 1, \dots, n$. These $n+1$ balls fulfill the assumptions of (ii) by the definition of y_0, hence there is some

$$y_1 \in B(y_0, 2r) \cap \bigcap_{i=1}^n B(x_i, r_i + \delta/2) \cap J,$$

which is what we are looking for.
Now suppose $y_1, \dots, y_k$ have already been constructed. To find y_{k+1} consider the balls $B(y_k, (2^{-(k+1)} - 2^{-(2k+1)}) \cdot 4r)$ and $B(x_i, r_i + (2^{-(k+1)} - 2^{-(2k+1)})\delta)$ for $i = 1, \dots, n$. Obviously each of them intersects J (note $y_k \in J$ for $k \geq 1$). Let us observe that

$$z_k := 2^{-(k+1)} y_0 + (1 - 2^{-(k+1)}) y_k$$

lies in the intersection of these $n+1$ balls:

$$\begin{aligned}
\|z_k - y_k\| &= 2^{-(k+1)} \|y_0 - y_k\| \\
&\leq 2^{-(k+1)} (\|y_0 - y_1\| + \dots + \|y_{k-1} - y_k\|) \\
&\leq 2^{-(k+1)} \sum_{i=1}^k 2^{-i} \cdot 4r \\
&= (2^{-(k+1)} - 2^{-(2k+1)}) \cdot 4r, \\
\|z_k - x_i\| &\leq 2^{-(k+1)} \|y_0 - x_i\| + (1 - 2^{-(k+1)}) \|y_k - x_i\| \\
&\leq 2^{-(k+1)} (r_i - \delta) + (1 - 2^{-(k+1)})(r_i + 2^{-k}\delta) \\
&= r_i + (2^{-(k+1)} - 2^{-(2k+1)})\delta.
\end{aligned}$$

An application of (ii) yields some

$$y_{k+1} \in B(y_k, 2^{-(k+1)} \cdot 4r) \cap \bigcap_{i=1}^n B(x_i, r_i + 2^{-(k+1)}\delta) \cap J,$$

as requested. □

Remarks 2.3 (a) Two balls do not suffice in the *real* case. For example, in the real space $X = L^1(\mu)$, μ a positive measure, the subspace J of X consisting of those functions whose integral vanishes satisfies (ii) in Theorem 2.2 with $n = 2$, but fails to be an M-ideal if $\dim X > 2$. (We remark that only recently such an example was found in the complex case, see the Notes and Remarks.)
[Proof: To prove that J satisfies the 2-ball property suppose that the balls $B(f_1, r_1)$ and $B(f_2, r_2)$ have nonvoid intersection (i.e., $\|f_1 - f_2\| \leq r_1 + r_2$) and both of them meet J (i.e., $\|g_i - f_i\| \leq r_i$ for some $g_i \in J$). If $B(f_1, r_1 + \varepsilon) \cap B(f_2, r_2 + \varepsilon) \cap J = \emptyset$ for some

$\varepsilon > 0$, we could separate the set $D = \{(x_1, x_2) \mid \|x_i - f_i\| \le r_i\} \subset L^1 \oplus L^1$ strictly from $\{(y, y) \mid y \in J\}$ by means of some functional $(F_1, F_2) \in L^\infty \oplus L^\infty$. Thus we would have

$$F_1 + F_2 \in J^\perp = \operatorname{lin}\{\mathbf{1}\}, \tag{1}$$

and

$$\sup_{(x_1,x_2)\in D} \int (F_1 x_1 + F_2 x_2)\, d\mu < 0. \tag{2}$$

On the other hand, each L^∞-function F may be decomposed as $F = c\mathbf{1} + G$ with $c = (\operatorname{ess\,sup} F + \operatorname{ess\,inf} F)/2$ so that

$$\|F\| = |c| + \|G\|.$$

By (1) above we may then write

$$F_1 = c_1\mathbf{1} + G_1, \qquad \|F_1\| = |c_1| + \|G_1\|,$$

$$F_2 = c_2\mathbf{1} - G_1, \qquad \|F_2\| = |c_2| + \|G_1\|.$$

Hence we obtain

$$\begin{aligned}
&\sup_{(x_1,x_2)\in D} \int (F_1 x_1 + F_2 x_2)\, d\mu \\
&= \sup_{(x_1,x_2)\in D} \int \Big(F_1(x_1 - f_1) + F_1 f_1 + F_2(x_2 - f_2) + F_2 f_2\Big)\, d\mu \\
&= \|F_1\| r_1 + \|F_2\| r_2 + c_1 \int f_1\, d\mu + \int G_1(f_1 - f_2)\, d\mu + c_2 \int f_2\, d\mu \\
&= \|F_1\| r_1 + \|F_2\| r_2 + c_1 \int (f_1 - g_1)\, d\mu + \int G_1(f_1 - f_2)\, d\mu + c_2 \int (f_2 - g_2)\, d\mu \\
&\ge \|F_1\| r_1 + \|F_2\| r_2 - \Big(|c_1| r_1 + \|G_1\|(r_1 + r_2) + |c_2| r_2\Big) \\
&= 0,
\end{aligned}$$

a contradiction to (2).
Clearly, J is not an M-ideal by Theorem 1.8 if $\dim(X) > 2$.]
An examination of the proof of Theorem 2.2 reveals that the 2-ball property yields a nonlinear "L-projection" onto $J^\perp$; in order to establish linearity we needed an intersection property involving three balls. We refer to the Notes and Remarks section for some information on spaces with the 2-ball property.

(b) One can even guarantee $\|y\| \le 1 + \varepsilon$ in (iv). This is a consequence of the 4-ball property applied to the balls $B(x + y_i, 1)$ $(i = 1, 2, 3)$ and B_X.

(c) One may also consider infinite collections of balls in (ii) provided the centres form a relatively compact set and the radii depend (lower semi-) continuously on the centres. (The proof is straightforward.)

(d) $\varepsilon = 0$ is not admissible in (ii). By way of example, let A be the disk algebra, and let

$$J = \{x \in A \mid x(1) = 0\}.$$

By Example 1.4(b) J is an M-ideal in A. Consider now the Möbius transform $w(z) = \frac{i-z}{1-iz}$ and the function $f(z) = w(\sqrt{w(z)})$. Then, by elementary complex analysis, f maps the open unit disk $\mathbb{D}$ onto $\{z \in \mathbb{D} \mid \operatorname{Re} z < 0\}$ and the unit circle $\mathbb{T}$ onto the boundary of that region. Moreover, we have $f \in A$ and $f(1) = 0$, hence $f \in J$; and since $|f(z)| = 1$ on $\{z \in \mathbb{T} \mid \operatorname{Re} z \le 0\}$ we conclude $f \in \operatorname{ex} B_A$ [316, p. 139]. This means $B(\mathbf{1} + f, 1) \cap B(\mathbf{1} - f, 1) = \{\mathbf{1}\}$, and therefore

$$J \cap B(\mathbf{1} + f, 1) \cap B(\mathbf{1} - f, 1) = \emptyset.$$

(It may seem counterintuitive that an M-ideal contains extreme points of the ambient space, since this is impossible for an M-summand.)

(e) The presence of some $\varepsilon > 0$ in (ii) is due to the fact that some openness assumptions are needed in order to apply the Hahn-Banach theorem. This assumption is shifted to condition $(2')$ in (v).

One can see Theorem 2.2 at work at various places of this book, e.g., II.2.3, II.5.2, III.1.4, III.3.4, V.3.2, VI.2.1, VI.4.1, VI.5.3. Let us now give a first application of the 3-ball property in that we provide the still missing

PROOF OF PROPOSITION 1.20:

To begin with we observe that the constant C appearing in the definition of the bounded extension property may be chosen as close to 1 as we wish. To see this let $U \supset D$ be an open set and let $\varepsilon > 0$. Applying the bounded extension property with $U_1 = U$ and ε yields an extension x_1 of a given $\xi \in X|_D$ (w.l.o.g. $\|\xi\| = 1$) such that $\|x_1\| \le C$ and $|x_1| \le \varepsilon$ off U_1. Then we repeat this procedure with $U_2 = \{k \mid |x_1(k)| < 1 + \varepsilon/2\} \cap U_1$ and obtain an extension x_2 such that $\|x_2\| \le C$ and $|x_2| \le \varepsilon$ off U_2. In the third step one applies the bounded extension property with $U_3 = \{k \mid |x_2(k)| < 1 + \varepsilon/2\} \cap U_2$ etc. This yields a sequence of extensions (x_n) with $\|x_n\| \le C$ and $|x_n| \le \varepsilon$ off U_n. Let $x = \frac{1}{N}\sum_{n=1}^{N} x_n$. Obviously we have $x|_D = \xi$ and $|x| \le \varepsilon$ off U. Finally, if $k \in U$, then by construction $|x_n(k)|$ is big (but $\le C$) for at most one n, and $|x_n(k)| \le 1 + \varepsilon/2$ otherwise. It follows

$$|x(k)| \le \frac{1}{N}\Big(C + (N-1)(1+\varepsilon/2)\Big) \le 1 + \varepsilon$$

for sufficiently large N. (As a consequence of this one may remark that $X/(J_D \cap X) \cong X|_D$.)

Now we can prove that $J_D \cap X$ is an M-ideal employing the 3-ball property. Thus, let y_1, y_2, y_3 in the unit ball of $J_D \cap X$, $x \in B_X$ and $\varepsilon > 0$. With the help of the $(1+\varepsilon)$-bounded extension property we may find $\widehat{x} \in X$ such that

$$\widehat{x}|_D = x|_D,$$

$$\|\widehat{x}\| \le (1+\varepsilon)\|x|_D\| \le 1 + \varepsilon,$$

$$|\widehat{x}(k)| \leq \varepsilon \quad \text{if } \max_i |y_i(k)| \geq \varepsilon.$$

Let $y = x - \widehat{x}$ so that $y \in J_D \cap X$. Distinguishing whether or not $\max_i |y_i(k)| \geq \varepsilon$ one can immediately verify that

$$|(x + y_i - y)(k)| = |y_i(k) + \widehat{x}(k)| \leq 1 + 2\varepsilon$$

for all $k \in K$, i.e.

$$\|x + y_i - y\| \leq 1 + 2\varepsilon \qquad (i = 1, 2, 3).$$ □

We now turn to a characterisation of M-ideals with a different, namely measure theoretic flavour. In the following we shall make use of the basic notions of integral representation theory, for the weak* compact convex set B_{X^*}. By a measure on B_{X^*} we shall understand a regular Borel measure on this compact set. A measure μ is called maximal (or boundary measure) if its variation is maximal with respect to Choquet's ordering. A measure μ has a unique barycentre (or resultant), denoted by $r(\mu)$, which is defined by

$$\langle r(\mu), x\rangle = \int_{B_{X^*}} p(x)\, d\mu(p) \quad \text{for all } x \in X.$$

Note $r(\mu) \in B_{X^*}$ for probability measures μ. Conversely, every $x^* \in X^*$ may be represented by a maximal measure μ on B_{X^*} in that $r(\mu) = x^*$. (For a detailed discussion see [7] and [495].)

Let J be a closed subspace of X. For $\mu \in M(B_{X^*})$ we define the restricted measure $\mu|_{J^\perp} \in M(B_{X^*})$ by $\mu|_{J^\perp}(E) = \mu(E \cap J^\perp)$. Note that $\mu|_{J^\perp}$ is maximal if μ is.

Now the characterisation theorem reads as follows:

Theorem 2.4 *$J \subset X$ is an M-ideal if and only if the following requirements are fulfilled:*

(1) *$\|p\| + \|q - p\| = \|q\|$ and $q \in J^\perp$ imply $p \in J^\perp$.*

(2) *If $\mu \in M(B_{X^*})$ is maximal and $r(\mu) = 0$, then also $r(\mu|_{J^\perp}) = 0$.*

PROOF OF THE "IF" PART:
Given $x^* \in X^*$, choose a maximal measure μ with $r(\mu) = x^*$. By (2), $r(\mu|_{J^\perp})$ depends only on x^* and not on the particular choice of μ so that $P(x^*) = r(\mu|_{J^\perp})$ gives rise to a well-defined mapping $P : X^* \to X^*$. By construction P is linear. We wish to show that P is an L-projection onto $J^\perp$. Since we have $\|r(\nu)\| \leq \|\nu\|$ for every measure $\nu \in M(B_{X^*})$ we conclude

$$\begin{aligned}
\|x^*\| &\leq \|Px^*\| + \|x^* - Px^*\| \\
&= \|r(\mu|_{J^\perp})\| + \|r(\mu - \mu|_{J^\perp})\| \\
&\leq |\mu|(B_{X^*} \cap J^\perp) + |\mu|(B_{X^*} \backslash J^\perp) \\
&= \|\mu\|.
\end{aligned}$$

Since one may choose such a μ with $\|\mu\| = \|x^*\|$, one obtains

$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\|.$$

Also, P is a projection by the remark preceding Theorem 2.4, and $\operatorname{ran}(P) \subset J^\perp$ is clear. Now let $q \in J^\perp$ with $\|q\| = 1$, say. Represent q by a maximal probability measure μ. Then there is a net of discrete measures (ν_α) with $r(\nu_\alpha) = q$ and weak*-$\lim \nu_\alpha = \mu$ [7, Prop. I.2.3]. Let

$$\nu_\alpha = \sum \lambda_i \delta_{p_i}$$

be such a measure. Then

$$\|q\| = \Big\| \sum \lambda_i p_i \Big\| \le \sum \lambda_i \|p_i\| \le \sum \lambda_i = 1 = \|q\|$$

so that $\lambda_i = 0$ or $p_i \in J^\perp$ as a consequence of (1). Therefore, the support of μ must be contained in $J^\perp$, and it follows $\mu = \mu|_{J^\perp}$ so that $q = P(q)$. ($\Box$)

The proof of the "only if" part will be given as a consequence of several lemmas which are of independent interest. We fix some notation. In the following we assume that J is an M-ideal in the *real* Banach space X with $P : X^* \to X^*$ denoting the corresponding L-projection onto $J^\perp$. (This assumption is made merely for simplicity of notation. The modifications to be made in the complex case will be sketched at the end of the proof of Theorem 2.4.) Furthermore we shall write

$$K = B_{X^*},\ D = K \cap J^\perp,\ D' = K \cap \ker(P).$$

Given $x \in X$ we define $h_x : K \to \mathbb{R}$ by

$$h_x(x^*) = \begin{cases} x^*(x) & \text{if } x^* \in D \text{ and } x^*(x) \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $h_x = \chi_D \cdot x \vee 0$.
Also, we recall the definition of the upper envelope $\widehat{h_x}$ of h_x :

$$\widehat{h_x}(x^*) = \inf a(x^*)$$

where a runs through the set of those functions $a \ge h_x$ which are affine and weak* continuous on K. Then $\widehat{h_x}$ is concave and weak* upper semicontinuous.

Lemma 2.5

$$\langle P^* x, x^* \rangle = \langle x, P x^* \rangle = \widehat{h_x}(x^*) - \widehat{h_x}(-x^*) \quad \textit{for all } x \in X,\ x^* \in K.$$

(Here the adjoint P^* of P is considered as a map from X to X^{**}.)

Proof: We first note

$$\begin{aligned} \mathcal{S}_x &:= \{(x^*, r) \in K \times \mathbb{R} \mid 0 \le r \le \widehat{h_x}(x^*)\} \\ &= \overline{\mathrm{co}}^{w*}\{(x^*, r) \mid 0 \le r \le h_x(x^*)\} \end{aligned}$$

which results from the Hahn-Banach theorem. ($\mathcal{S}_x$ is the truncated subgraph of $\widehat{h_x}$.) Also

$$\{(x^*, r) \mid 0 \le r \le h_x(x^*)\} = \{(x^*, r) \mid x^* \in D,\ 0 \le r \le x^*(x)\} \cup (K \times \{0\}).$$

Since the latter two sets are weak* compact and convex, so is the convex hull of their union. Therefore, taking into account

$$K = \mathrm{co}\,(D \cup D')$$

we have

$$\mathcal{S}_x \;=\; \mathrm{co}\,(\{(x^*, r) \mid x^* \in D,\ 0 \le r \le x^*(x)\} \cup (D \times \{0\}) \cup (D' \times \{0\})).$$

Now let $x^* \in K$. Then $(x^*, \widehat{h_x}(x^*)) \in \mathcal{S}_x$. Accordingly there is a convex combination

$$x^* = \lambda_1 x_1^* + \lambda_2 x_2^* + \lambda_3 x_3^* \tag{1}$$

with

$$x_1^*, x_2^* \in D,\ x_3^* \in D' \tag{2}$$

$$x_1^*(x) \ge 0 \tag{3}$$

$$\widehat{h_x}(x^*) = \lambda_1 x_1^*(x). \tag{4}$$

By concavity of $\widehat{h_x}$

$$\begin{aligned}\widehat{h_x}(x^*) &\ge \sum \lambda_i \widehat{h_x}(x_i^*) \\ &\ge \sum \lambda_i h_x(x_i^*) \\ &\ge \lambda_1 x_1^*(x) + \lambda_2 x_2^*(x)\end{aligned}$$

so that

$$0 \;\ge\; \lambda_2 x_2^*(x).$$

In case $\lambda_2 = 0$ we could have chosen $x_2^* = 0$, hence we may conclude $x_2^*(x) \le 0$ and thus

$$h_x(x_2^*) = 0, \qquad h_x(-x_2^*) = -x_2^*(x).$$

We employ the concavity of $\widehat{h_x}$ once again to obtain

$$\begin{aligned}\widehat{h_x}(-x^*) &\ge \sum \lambda_i h_x(-x_i^*) \\ &\ge -\lambda_2 x_2^*(x).\end{aligned}$$

Together with (4) this implies

$$\begin{aligned}\widehat{h_x}(x^*) - \widehat{h_x}(-x^*) &\le (\lambda_1 x_1^* + \lambda_2 x_2^*)(x) \\ &= \langle Px^*, x\rangle \qquad \text{(by (1) and (2))}.\end{aligned}$$

Since both sides of this inequality are odd functions of x^*, the inequality is in fact an equality, and the lemma is proved. □

From Lemma 2.5 one concludes immediately that $P^*x|_K$, being a difference of weak* upper semicontinuous functions, is weak* Borel. Even more: the points of continuity of an upper semicontinuous function on a compact space form a dense G_δ-set, cf. e.g. [204, p. 87]. By Baire's theorem this property is shared by linear combinations of such functions, in particular by P^*x restricted to K or any compact subset. A result of Choquet's [7, p. 16] now yields:

Lemma 2.6 *$P^*x|_K$ is a bounded affine weak* Borel function which satisfies the barycentric calculus, i.e.*

$$\langle P^*x, r(\mu)\rangle = \int_K P^*x \; d\mu$$

for all $\mu \in M(K)$.

Of course, Lemma 2.6 is completely trivial if J is an M-summand, since in this case $P^*x \in X$, i.e., P^*x is weak* continuous. Now Lemma 2.6 implies that P^*x is not very discontinuous in the general case, either. In fact, if X is separable then $P^*x|_K$ is of the first Baire class by what we have seen and Baire's classification theorem [157, p. 67]. One should mention that the weak* closedness of $\operatorname{ran}(P)$ is crucial for Lemma 2.6: for example, consider the L-projection P defined on $C[0,1]^*$ by $P(\nu)$ = atomic part of ν. Then $P^*\mathbf{1}$ is of the second Baire class (as a function on the dual unit ball), yet fails the barycentric calculus [7, Ex. I.2.10].

Lemma 2.7 *$r(\chi_D\mu) = P(r(\mu))$ for every maximal measure $\mu \in M(K)$.*

PROOF: With the help of Lemmas 2.5 and 2.6 one obtains

$$\begin{aligned}
\langle P(r(\mu)), x\rangle &= \langle r(\mu), P^*x\rangle \\
&= \int_K P^*x \; d\mu \\
&= \int_K (\widehat{h_x}(x^*) - \widehat{h_x}(-x^*)) \; d\mu(x^*) \\
&= \int_D (h_x(x^*) - h_x(-x^*)) \; d\mu(x^*) \\
&= \int_D x^*(x) \; d\mu(x^*) \\
&= \langle r(\chi_D\mu), x\rangle.
\end{aligned}$$

(In the fourth line we used the fact that a maximal measure is concentrated on $\{h = \widehat{h}\}$ for all upper semicontinuous functions h [7, p. 35].) $\square$

Now it is easy to give the

PROOF OF THE "ONLY IF" PART OF THEOREM 2.4:

(1) is an elementary calculation:

Since

$$\|p\| = \|P(p)\| + \|p - P(p)\|$$

and

$$\|q - p\| = \|q - P(p)\| + \|p - P(p)\|$$

(because $q \in J^\perp$) we have

$$\begin{aligned} \|q\| &= \|p\| + \|q-p\| \\ &= \|P(p)\| + \|q - P(p)\| + 2\|p - P(p)\| \\ &\geq \|q\| + 2\|p - P(p)\|, \end{aligned}$$

hence $p = P(p) \in J^\perp$.

(2) is a consequence of Lemma 2.7:
If $r(\mu) = 0$, then we conclude in the case of real Banach spaces $r(\mu^+) = r(\mu^-)$ $(\mu = \mu^+ - \mu^-$ denoting the Hahn decomposition; note that μ^+ and μ^- are maximal), hence

$$\begin{aligned} r(\chi_D \mu) &= r(\chi_D \mu^+) - r(\chi_D \mu^-) \\ &= P(r(\mu^+)) - P(r(\mu^-)) \\ &= 0. \end{aligned}$$

The proof of the complex case can be reduced to the real case by considering real and imaginary parts. (We remark that the above lemmas remain valid, too; h_x should be defined as $(\mathrm{Re}\,\chi_D x) \vee 0$ in the complex case.) □

We finish this section with a theorem stating that every vector in a Banach space X has a kind of unconditional expansion into a series of elements from a separable M-ideal J, with the convergence being taken with respect to the topology $\sigma(X, J^*)$. (Recall from Remark 1.13 that J^* can naturally be identified with a subspace of X^*.) The proof of that result (Theorem 2.10) relies on the above Lemma 2.5 as well as the following two lemmas which are of independent interest.

Lemma 2.8 *Let H and K be compact Hausdorff spaces and $\rho : K \to H$ a continuous surjection. Suppose $\sigma : H \to \mathbb{R}$ is a function such that $\tau = \sigma \circ \rho$ is a difference of two positive lower semicontinuous functions $\tau = g_1 - g_2$ with the additional property that $g_1(t) + g_2(t) \leq 1$ for all $t \in K$. Then there are positive lower semicontinuous functions $f_1, f_2 : H \to \mathbb{R}$ such that $\sigma = f_1 - f_2$ and $f_1(s) + f_2(s) \leq 1$ for all $s \in H$.*

PROOF: We let

$$f_i(s) := \inf\{g_i(t) |\ \rho(t) = s\} = \min\{g_i(t) |\ \rho(t) = s\}.$$

The infimum defining f_i is actually a minimum, as lower semicontinuous functions on compact sets attain their infimum. Clearly the f_i are well-defined and positive.
To show that f_i ($i = 1$ or 2) is lower semicontinuous we pick $s_0 \in H$ and $\alpha \in \mathbb{R}$ such that $f_i(s_0) > \alpha$. Then, by definition of f_i, $\rho^{-1}(\{s_0\}) \subset V := \{g_i > \alpha\}$, which is an open set by assumption on g_i. Now $\bigcap_W \rho^{-1}(W) \cap \complement V = \emptyset$, where the intersection is taken over all compact neighbourhoods of s_0. Then the compactness of K produces a finite void intersection which in turn yields a compact neighbourhood W_0 of s_0 such that $\rho^{-1}(W_0) \subset V$. This says $g_i(t) > \alpha$ whenever $\rho(t) \in W_0$, and as a result $f_i(s) > \alpha$ whenever $s \in W_0$. Thus the lower semicontinuity is proved.

Next we show that $\sigma = f_1 - f_2$. Given $s \in H$ there are $t_1, t_2 \in \rho^{-1}(\{s\})$ such that $f_i(s) = g_i(t_i)$ and therefore $g_1(t_1) \le g_1(t_2)$, $g_2(t_2) \le g_2(t_1)$. Hence

$$f_1(s) - f_2(s) = g_1(t_1) - g_2(t_2) \le g_1(t_2) - g_2(t_2) = \tau(t_2) = \sigma(s)$$

and likewise $f_1(s) - f_2(s) \ge \sigma(s)$.
Finally, for s and t_i as above,

$$f_1(s) + f_2(s) = g_1(t_1) + g_2(t_2) \le g_1(t_2) + g_2(t_2) \le 1,$$

which completes the proof of Lemma 2.8. □

Lemma 2.9 *Let E be a Banach space, $F \subset E$ a (not necessarily closed) subspace and $x^{**} \in F^{\perp\perp}$ such that there is a sequence (x_n) in E and a constant $C > 0$ satisfying*

$$\begin{aligned} x^{**} &= \text{weak}^*\text{-}\sum_{n=1}^{\infty} x_n, \\ \sup_{|\varepsilon_n| \le 1} \left\| \sum_{n=1}^{N} \varepsilon_n x_n \right\| &\le C\|x^{**}\| \qquad \forall N \in \mathbb{N}. \end{aligned}$$

Then there is, for every $\varepsilon > 0$, a sequence (y_n) in F satisfying

$$\begin{aligned} x^{**} &= \text{weak}^*\text{-}\sum_{n=1}^{\infty} y_n, \\ \sup_{|\varepsilon_n| \le 1} \left\| \sum_{n=1}^{N} \varepsilon_n y_n \right\| &\le (C+\varepsilon)\|x^{**}\| \qquad \forall N \in \mathbb{N}. \end{aligned}$$

PROOF: We may assume $\|x^{**}\| = 1$. To begin with we show that x^{**} is a weak* limit of some sequence (z_m) from F. First of all one concludes from the Hahn-Banach theorem that the distance between F and $\operatorname{co}\{\sum_{n=1}^{m} x_n, \sum_{n=1}^{m+1} x_n, \ldots\}$ vanishes for all $m \in \mathbb{N}$. Hence there are $z_m \in F$ and $\xi_m \in \operatorname{co}\{\sum_{n=1}^{m} x_n, \sum_{n=1}^{m+1} x_n, \ldots\}$ such that $\|z_m - \xi_m\| \to 0$, and we obtain

$$\text{weak}^*\text{-}\lim z_m = \text{weak}^*\text{-}\lim \xi_m = \text{weak}^*\text{-}\sum_{n=1}^{\infty} x_n = x^{**}.$$

Now $v_n = z_n - \sum_{i=1}^{n} x_i$ defines a weakly null sequence so that a sequence of convex combinations tends to 0 strongly. More precisely, we can define a strictly increasing sequence of integers $0 = p_0 < p_1 < \ldots$, a sequence of real numbers $\lambda_n \ge 0$ with $\sum_{i=p_{n-1}+1}^{p_n} \lambda_i = 1$ and $u_n = \sum_{i=p_{n-1}+1}^{p_n} \lambda_i v_i$ to obtain $\|u_n\| \le \varepsilon/2^{n+1}$ for all n. Setting now

$$w_n = \sum_{i=p_{n-1}+1}^{p_n} \lambda_i z_i$$

and

$$y_1 = w_1, \qquad y_{n+1} = w_{n+1} - w_n$$

for $n \geq 1$, we have

$$\begin{aligned} y_1 &= \sum_{j=1}^{p_1} \mu_j^0 x_j + u_1, \\ y_{n+1} &= \sum_{j=p_{n-1}+1}^{p_{n+1}} \mu_j^n x_j + u_{n+1} - u_n \end{aligned}$$

with

$$\begin{aligned} \mu_j^0 &= \sum_{i=j}^{p_1} \lambda_i, \\ \mu_j^n &= \begin{cases} 1 - \displaystyle\sum_{i=j}^{p_n} \lambda_i & \text{if } p_{n-1} + 1 \leq j \leq p_n, \\ \displaystyle\sum_{i=j}^{p_{n+1}} \lambda_i & \text{if } p_n + 1 \leq j \leq p_{n+1}. \end{cases} \end{aligned}$$

Since $0 \leq \mu_j^n \leq 1$ for all j and n, we obtain

$$\begin{aligned} \left\| \sum_{n=1}^{N} \varepsilon_n y_n \right\| &\leq \left\| \sum_{n=1}^{N-1} \sum_{j=p_{n-1}+1}^{p_n} (\varepsilon_n \mu_j^{n-1} + \varepsilon_{n+1} \mu_j^n) x_j + \sum_{j=p_{N-1}+1}^{p_N} \varepsilon_N \mu_j^{N-1} x_j \right\| + \varepsilon \\ &\leq C + \varepsilon \end{aligned}$$

because $\mu_j^{n-1} + \mu_j^n = 1$ for $p_{n-1} + 1 \leq j \leq p_n$ and hence $|\varepsilon_n \mu_j^{n-1} + \varepsilon_{n+1} \mu_j^n| \leq 1$. □

We now state the promised "unconditional expansion".

Theorem 2.10 *Let X be a Banach space, and suppose J is a separable M-ideal in X. Then there is, for each $x \in X$ and each $\varepsilon > 0$, a sequence (y_n) in J such that*

$$x = \sum_{n=1}^{\infty} y_n$$

with respect to the topology $\sigma(X, J^*)$ and

$$\sup_{|\varepsilon_n| \leq 1} \left\| \sum_{n=1}^{N} \varepsilon_n y_n \right\| \leq (1 + \varepsilon) \|x\| \qquad \text{for all } N \in \mathbb{N}.$$

In fact, such y_n can be picked from any given dense subspace of J.

PROOF: It is clearly sufficient to prove this for real Banach spaces and for $x \in B_X$. We will retain the notation $K = B_{X^*}$, $P : X^* \to J^\perp$ the L-projection and h_x from Lemma 2.5. Putting

$$\tau : K \to \mathbb{R}, \quad \tau(x^*) = \langle x^* - Px^*, x \rangle,$$

we obtain from that lemma the representation

$$\tau(x^*) = \left[\frac{1 + x^*(x)}{2} - \widehat{h_x}(x^*)\right] - \left[\frac{1 - x^*(x)}{2} - \widehat{h_x}(-x^*)\right]$$

of τ as a difference of two lower semicontinuous functions $\tau = g_1 - g_2$, where in addition $g_1 \geq 0$, $g_2 \geq 0$ and $g_1 + g_2 \leq \mathbf{1}$ as is easily checked. Now, if $H = B_{J^*}$ (equipped with the topology $\sigma(J^*, J)$) and $\rho : K \to H$ is the restriction map, then there is, by Lemma 2.8, a corresponding representation of the function $\sigma : H \to \mathbb{R}$, $\sigma(y^*) = \langle y^*, x \rangle$ as a difference $\sigma = f_1 - f_2$ of positive lower semicontinuous functions on H whose sum does not exceed 1. (To define the duality between y^* and x we have identified J^* with $\ker(P)$, as in Remark 1.13. Note also $\sigma \circ \rho = \tau$.)

The assumption that J is separable, i.e. H is metrizable and $C(H)$ is separable, permits us to write f_1 as a pointwise converging series of positive continuous functions on H. [In fact, since f_1 is lower semicontinuous and positive, we have

$$f_1(y^*) = \sup\{\varphi(y^*) |\ 0 \leq \varphi \leq f_1 \text{ and } \varphi \in C(H)\}$$

for each $y^* \in H$. If (φ_n) is a uniformly dense sequence in $\{\varphi \in C(H) |\ 0 \leq \varphi \leq f_1\}$ and $\varphi_1 = 0$, then clearly $f_1 = \sup \varphi_n$, and the functions $h_{1,n} = (\varphi_1 \vee \cdots \vee \varphi_{n+1}) - (\varphi_1 \vee \cdots \vee \varphi_n)$ have the required properties.] Likewise $f_2 = \sum_{n=1}^\infty h_{2,n}$ for some positive continuous functions $h_{2,n}$. Hence, using the notation $h_n = h_{1,n} - h_{2,n}$, we obtain a pointwise converging series of continuous functions $\sigma = \sum_{n=1}^\infty h_n$ such that

$$\begin{aligned}
\left\|\sum_{n=1}^N \varepsilon_n h_n\right\| &\leq \sup_{y^*}\left(\left|\sum_{n=1}^N \varepsilon_n h_{1,n}(y^*)\right| + \left|\sum_{n=1}^N \varepsilon_n h_{2,n}(y^*)\right|\right) \\
&\leq \sup_{y^*}\left(\sum_{n=1}^N h_{1,n}(y^*) + \sum_{n=1}^N h_{2,n}(y^*)\right) \\
&\leq \sup_{y^*}\left(f_1(y^*) + f_2(y^*)\right) \\
&\leq 1
\end{aligned}$$

whenever $|\varepsilon_n| \leq 1$ and $N \in \mathbb{N}$, and

$$\langle y^*, x \rangle = \sigma(y^*) = \sum_{n=1}^\infty h_n(y^*)$$

for each $y^* \in H$.

We would like to replace $\sum h_n$ by a series of functions from $A_0(H)$, the space of affine continuous functions on H vanishing at 0. To achieve this we employ Lemma 2.9. Clearly

σ is a bounded affine function on the compact convex set H, $\sigma(0) = 0$ and moreover, by what we have just proved, it is of the first Baire class. Hence σ is continuous on a dense G_δ-set as is every restriction $\sigma|_{H'}$ of σ to a compact subset H'. Choquet's theorem referred to in the paragraph preceding Lemma 2.6 now yields that σ satisfies the barycentric calculus. The upshot of this argument is that $\sigma \in A_0(H)^{\perp\perp} \subset C(H)^{**}$: In fact, since σ is measurable and bounded we have that $\sigma \in C(H)^{**}$ in a natural way. If $\mu \in C(H)^*$ annihilates $A_0(H)$, then the resultant $r(\mu)$ is 0; consequently $\int_H \sigma \, d\mu = \sigma(0) = 0$.
Of course, $A_0(H)$ is canonically isometrically isomorphic with J by the Krein-Smulian theorem; and the dominated convergence theorem yields $\sigma = \sum_{n=1}^{\infty} h_n$ in the topology $\sigma(C(H)^{**}, C(H)^*)$. Therefore an application of Lemma 2.9 with $E = C(H)$ and $F = A_0(H)$ shows how to find such a sequence (y_n) in J. If a dense subspace of J is considered, then we only have to apply Lemma 2.9 with the corresponding dense subspace of $A_0(H)$. □

Remark: Another way to see that $\sigma \in A_0(H)^{\perp\perp}$ in the above proof is to use a result from [457] (see also [157, p. 235]) stating that a bounded affine function which is a pointwise limit of a sequence of continuous functions is even a pointwise limit of a sequence of continuous affine functions.

We shall apply Theorem 2.10 in Theorem III.3.8 and Theorem VI.4.21.

I.3 The centralizer of a Banach space

In this section we shall briefly discuss an algebra of operators on a Banach space X which has close relations to the M-ideal structure of X. In the same way as the M-ideals of a Banach space correspond to the closed ideals of a C^*-algebra, this operator algebra, called the centralizer of X, corresponds to the centre of a C^*-algebra (cf. the revisited Example 3.4(h)). It must be stressed that there are some differences between the real and complex case, here; cf. the Notes and Remarks.
For the most part, proofs of the results presented in this section have already appeared in E. Behrends' monograph [51]. Instead of repeating these arguments we prefer to give due references.
We start with the basic notion.

Definition 3.1 *$T \in L(X)$ is called a* multiplier *if every $p \in \operatorname{ex} B_{X^*}$ is an eigenvector of T^*, with eigenvalue $a_T(p)$ say, i.e.*

$$T^*p = a_T(p)p \qquad \textit{for all} \;\; p \in \operatorname{ex} B_{X^*}. \tag{$*$}$$

The collection of all multipliers is called the multiplier algebra *and denoted by* $\operatorname{Mult}(X)$. $\operatorname{Mult}(X)$ *is said to be trivial if* $\operatorname{Mult}(X) = \mathbb{K} \cdot Id$.

It is an immediate consequence of the Krein-Milman theorem that $\operatorname{Mult}(X)$ is a closed commutative unital subalgebra of $L(X)$.

The following lemma is sometimes quite helpful for deciding whether a given operator belongs to $\mathrm{Mult}(X)$. We let

$$Z_X = \overline{\mathrm{ex}}^{w*}\, B_{X^*}\backslash\{0\}.$$

This weak* locally compact space will turn out to be of importance later; see Theorem II.5.9.

Lemma 3.2 *Suppose that E is a weak* dense subset of Z_X and that $T \in L(X)$. Then $T \in \mathrm{Mult}(X)$ if and only if for all $p \in E$ there is a number $a_T(p)$ such that*

$$T^*p = a_T(p)p.$$

In this case, a_T may be extended to a weak continuous function on Z_X, and each $p \in Z_X$ is an eigenvector of T^* with eigenvalue $a_T(p)$.*

This lemma is a direct consequence of the definition. Next we will collect a few easy-to-prove properties of the functions a_T appearing in $(*)$ which we will understand to be defined on Z_X.

Lemma 3.3 *Let $T \in \mathrm{Mult}(X)$.*

(a) *a_T is bounded and, moreover, $\|a_T\|_\infty = \|T\|$.*
(b) *$a_T(p) = a_T(-p)$ for all $p \in Z_X$.*
(c) *$T \mapsto a_T$ is an algebra homomorphism.*

By these properties, $\mathrm{Mult}(X)$ is isometric to a closed subalgebra of $C^b(Z_X)$, where Z_X is equipped with the weak* topology. Consequently, $\mathrm{Mult}(X)$ is a function algebra.

Examples 3.4

(a) Let S be locally compact. Then $\mathrm{Mult}(C_0(S))$ consists exactly of the multiplication operators

$$M_z : x \mapsto x \cdot z$$

with bounded continuous functions z as can easily be proved [51, p. 55]. Hence

$$\mathrm{Mult}(C_0(S)) \cong C^b(S) \cong C(\beta S)$$

as algebras. If S is compact, then $\mathrm{Mult}(C(S)) \cong C(S)$.

(b) Let $A \subset C(\mathbb{T})$ be the disk algebra. Since the extreme functionals are given by

$$x \mapsto \lambda \cdot x(t), \quad |\lambda| = 1,\ t \in \mathbb{T}$$

(cf. p. 4) we conclude that a multiplier must be of the form M_z as above, for some function z. Since $z = M_z(\mathbf{1}) \in A$,

$$\mathrm{Mult}(A) = \{M_z | z \in A\} \cong A.$$

(c) The same reasoning as in (b) applies to any complex function algebra A. [If $S \in \mathrm{Mult}(A)$, then $Sx(t) = (S\mathbf{1})(t) \cdot x(t)$ for all t in the Choquet boundary of A, hence $S = M_{S\mathbf{1}}$.] This example will be of importance in Chapter V.

(d) Now consider $A_{\mathbb{R}}$, that is the disk algebra as a real Banach space. For $T \in \mathrm{Mult}(A_{\mathbb{R}})$, T^* is subject to having real eigenvalues so that $T = M_z$ for some real-valued analytic function on the unit disk. Hence $\mathrm{Mult}(A_{\mathbb{R}}) = \mathbb{R}\cdot Id$. Consequently, the multiplier algebra depends on the choice of the scalar field.

(e) Every M-projection P belongs to $\mathrm{Mult}(X)$, the eigenvalues of P^* being 0 or 1. This is immediate from Lemma 1.5.

(f) For a compact convex set K we let $A(K)$ be the space of real-valued affine continuous functions on K. Then T is a multiplier on $A(K)$ if and only if T is order bounded. This is proved in [7, II.7.10].

(g) In a sense example (a) comprises the most general case. Given X, consider the locally compact space Z_X as above. Then there is a canonical isometric embedding of X into $C_0(Z_X)$, and by definition $T \in L(X)$ is a multiplier if and only if T is the restriction of a multiplication operator on $C_0(Z_X)$ which leaves X invariant.

We shall need the following lemma in Chapter VI (Lemma VI.2.2).

Lemma 3.5 *Let $T \in \mathrm{Mult}(X)$.*
(a) $\|x+y\| = \max\{\|x\|, \|y\|\}$ *if* $x \in \ker(T)$, $y \in \mathrm{ran}(T)$.
(b) $T(J) \subset J$ *if J is an M-ideal in X.*

Proof: (a) We have $y = Tz$ for some z and $Tx = 0$ so that $0 = p(Tx) = a_T(p)p(x)$ for all $p \in \operatorname{ex} B_{X^*}$ and, as a result, $a_T(p) = 0$ or $p(x) = 0$. It follows

$$\begin{aligned} \|x+y\| &= \sup\{\operatorname{Re} p(x+y) \mid p \in \operatorname{ex} B_{X^*}\} \\ &= \sup\{\operatorname{Re} p(x) + \operatorname{Re} a_T(p)p(z) \mid p \in \operatorname{ex} B_{X^*}\} \\ &= \max\{\|x\|, \|y\|\}. \end{aligned}$$

(b) follows from $T^*(\operatorname{ex} B_{J^\perp}) \subset J^\perp$, the Krein-Milman theorem and the Hahn-Banach theorem. (Recall $\operatorname{ex} B_{J^\perp} \subset \operatorname{ex} B_{X^*}$ from Lemma 1.5.) □

The following theorem characterises multipliers without recourse to the dual space.

Theorem 3.6 *For a real or complex Banach space X and $T \in L(X)$, the following assertions are equivalent:*
(i) $T \in \mathrm{Mult}(X)$.
(ii) *For all $x \in X$, Tx is contained in every closed ball which contains*

$$\{\lambda x \mid \lambda \in \mathbb{K},\ |\lambda| \le \|T\|\}.$$

Proof: [11, p. 151] and [51, p. 57] in the complex case. □

In the real case, $\mathrm{Mult}(X) \cong \{a_T \mid T \in \mathrm{Mult}(X)\}$ is the self-adjoint part of the C^*-subalgebra $\{a_S + ia_T \mid S, T \in \mathrm{Mult}(X)\}$ of $C^b_{\mathbb{C}}(\operatorname{ex} B_{X^*})$ and thus $\mathrm{Mult}(X) \cong C_{\mathbb{R}}(K_X)$ for a (uniquely determined) compact Hausdorff space K_X by the Gelfand-Naimark theorem. In the complex case, $\mathrm{Mult}(X)$ need not be self-adjoint (cf. Example 3.4(b)). This is the motivation for the next definition.

Definition 3.7 *The* centralizer *of X, $Z(X)$, consists of all those $T \in \mathrm{Mult}(X)$ for which there exists $\overline{T} \in \mathrm{Mult}(X)$ such that $a_{\overline{T}}(p) = \overline{a_T(p)}$ for all $p \in \mathrm{ex}\, B_{X^*}$. We let*

$$Z_{\mathbb{R}}(X) = \{T \in Z(X) \mid a_T \ \text{ real valued}\}$$

and

$$Z_{0,1}(X) = \{T \in Z_{\mathbb{R}}(X) \mid 0 \le a_T \le 1\}.$$

$Z(X)$ is called trivial if $Z(X) = \mathbb{K} \cdot Id$.

Obviously, $Z(X) = \mathrm{Mult}(X)$ for real Banach spaces and $Z(X) = Z_{\mathbb{R}}(X) + iZ_{\mathbb{R}}(X)$ in the complex case. By construction $Z(X)$ is a commutative unital C^*-algebra, thus $Z(X) \cong C(K_X)$, with K_X the Gelfand space of this C^*-algebra. One can prove that X has a representation as a Banach space of sections in a bundle with base space K_X such that the section of norms is upper semicontinuous [139], [51, Chap. IV]. We will not pursue this idea, but invite the reader to consult the above mentioned literature and the Notes and Remarks section.

For the sake of easy reference we remark in addition:

Lemma 3.8 *Every $T \in Z_{\mathbb{R}}(X)$ is hermitian.*

PROOF: $\|e^{itT}\| = \|e^{ita_T}\|_\infty = 1$ for all $t \in \mathbb{R}$. □

Examples 3.4 (revisited)

(a) Here we have $Z(C_0(S)) = \mathrm{Mult}(C_0(S)) \cong C^b(S)$.

(b) If $T = M_z \in Z(A)$ has an "adjoint" $\overline{T} = M_{z^*}$ for some $z^* \in A$, then $z^* = \overline{z}$ (the complex conjugate function) so that z is constant. Consequently $Z(A) = \mathbb{C} \cdot Id$.

(c) Here we have $Z(A) = \{x \in A \mid \overline{x} \in A\}$.

(e) Since a_P is real-valued, we have $P \in Z(X)$. In fact, a projection belongs to $Z(X)$ if and only if it is an M-projection. This is a consequence of $(*)$ from p. 2 and Proposition 3.9 below.

(g) If X is canonically embedded in $C_0(Z_X)$, then the operators in $Z(X)$ are exactly the restrictions of multiplication operators $M_h : f \to f \cdot h$ on $C_0(Z_X)$ such that M_h and $M_{\overline{h}}$ leave X invariant.

(h) For a unital C^*-algebra A, $Z(A) = \{M_z \mid z \in \mathrm{centre}(A)\}$. (This will be proved in Theorem V.4.7.)

The following proposition should be compared with $(*)$ on p. 2.

Proposition 3.9 *$T \in L(X)$ belongs to $Z_{0,1}(X)$ if and only if*

$$\|Tx_1 + (Id - T)x_2\| \ \le \ \max\{\|x_1\|, \|x_2\|\}$$

for all $x_1, x_2 \in X$.

PROOF: It is elementary to verify that $T \in Z_{0,1}(X)$ has the announced property. Now suppose that T fulfills the above norm condition. Then we have for $p \in \operatorname{ex} B_{X^*}$ (assuming w.l.o.g. $T^*p \neq 0$ and $T^*p \neq p$)

$$p = \|T^*p\| \frac{T^*p}{\|T^*p\|} + \|p - T^*p\| \frac{p - T^*p}{\|p - T^*p\|}.$$

But this is a convex combination since there are, given $\varepsilon > 0$, $x_1, x_2 \in B_X$ such that

$$\begin{aligned} \|T^*p\| + \|p - T^*p\| - 2\varepsilon &\leq p(Tx_1) + p(x_2 - Tx_2) \\ &\leq \|Tx_1 + (x_2 - Tx_2)\| \\ &\leq 1 \end{aligned}$$

by assumption on T. The extremality of p now yields

$$T^*p = \|T^*p\|\, p. \qquad \square$$

We are now going to discuss the problem of characterising those functions a on $\operatorname{ex} B_{X^*}$ (or Z_X if you prefer) which give rise to operators in $Z(X)$. In view of the (revisited) Example 3.4(g) this is the case if and only if multiplication by a and $\overline{a}$ (defined on $C_0(Z_X)$) leaves X invariant. Eventually we shall state a characterisation of those functions a in terms of a topological condition involving M-ideals (Theorem 3.12).
First, however, we shall present a result of Stone-Weierstraß type which gives a necessary and sufficient condition for $g \in C_0(Z_X)$ to be in X. It will be convenient to take a slightly broader view. So let us fix some notation. Let L be a locally compact space and suppose that X is a closed subspace of $C_0(L)$. Put

$$Z(X, C_0(L)) = \{f \in C^b(L) \mid f \cdot X \subset X \text{ and } \overline{f} \cdot X \subset X\}.$$

We further denote by $\mathcal{F}(X, C_0(L))$ the set of equivalence classes which are obtained from the equivalence relation

$$s \sim t \quad \Longleftrightarrow \quad f(s) = f(t) \quad \forall f \in Z(X, C_0(L))$$

on L. Finally, for a subset $F \subset L$, we let

$$X_{|F} = \{x_{|F} \mid x \in X\}.$$

With this notation, we may state:

Theorem 3.10 *Let X be a closed subspace of $C_0(L)$. Then $g \in C_0(L)$ belongs to X if and only if*

$$g_{|F} \in X_{|F} \qquad \text{for all } F \in \mathcal{F}(X, C_0(L)).$$

PROOF: The "only if" part is trivial. For the "if" part, we start by showing that the support of each measure $\mu \in \operatorname{ex} B_{X^\perp}$ is contained in some $F_0 \in \mathcal{F}(X, C_0(L))$. To this end let $f \in Z(X, C_0(L))$ be given. We wish to show

$$f_{|\operatorname{supp}(\mu)} = \text{const}.$$

Since $\operatorname{Re} f \in Z(X, C_0(L))$, too, we may assume that f is real-valued, and a simple scaling procedure allows us to assume

$$0 < f(s) < 1 \qquad \text{for all} \quad s \in L.$$

We then have

$$\|f\mu\| + \|(1-f)\mu\| = \int_L f\, d|\mu| + \int_L (1-f)\, d|\mu| = \|\mu\| = 1.$$

Hence μ can be represented as a convex combination

$$\mu = \|f\mu\| \frac{f\mu}{\|f\mu\|} + \|(1-f)\mu\| \frac{(1-f)\mu}{\|(1-f)\mu\|}$$

so that by extremality (note that by the very choice of f we have $f\mu \in X^\perp$)

$$\|f\mu\|\, \mu = f\mu,$$

i.e.

$$\|f\mu\| = f \qquad \mu\text{-a.e.},$$

and the claim follows.
Now consider $g \in C_0(L)$ with

$$g|_F \in X|_F \qquad \forall F \in \mathcal{F}(X, C_0(L)).$$

If $g \notin X$, there would be some $\mu \in \operatorname{ex} B_{X^\perp}$ with $\int_L g\, d\mu \neq 0$. However, if $\operatorname{supp}(\mu) \subset F_0$, $F_0 \in \mathcal{F}(X, C_0(L))$, then, since $g|_{F_0} = x|_{F_0}$ for some $x \in X$,

$$\int_L g\, d\mu = \int_{F_0} g\, d\mu = \int_{F_0} x\, d\mu = \int_L x\, d\mu = 0.$$

This contradiction proves $g \in X$, as claimed. $\square$

We remark that for a subalgebra X, $\mathcal{F}(X, C_0(L))$ is the well-known maximal X-antisymmetric decomposition of L, and Theorem 3.10 reduces to Bishop's generalisation of the Stone-Weierstraß theorem. We hasten to add that our proof of Theorem 3.10 is nothing but a minor modification (if any) of the de Branges-Glicksberg proof of Bishop's theorem, cf. [239, p. 60].
An application of Theorem 3.10 will be given in Theorem II.5.9; for others see the Notes and Remarks section.

To indicate the links between the centralizer of X and the M-ideals of X, which will enable us to characterise those functions on $\operatorname{ex} B_{X^*}$ which arise as a_T for some $T \in Z(X)$, we need the notion of the structure topology.

Definition 3.11 *The collection of sets* $\operatorname{ex} B_{X^*} \cap J^\perp$*, where* J *runs through the family of* M*-ideals in* X*, is the collection of closed sets for some topology on* $\operatorname{ex} B_{X^*}$ *(this follows from Proposition 1.11), called the* structure topology.

This topology is never Hausdorff, since p and $-p$ cannot be separated. For this reason we will also consider the quotient space $\operatorname{ex} B_{X^*}/\sim$ (where $p \sim q$ iff p and q are linearly dependent) equipped with the corresponding quotient topology. This topological space will be denoted by E_X.
The richness of the M-ideal structure is reflected by topological qualities of the structure topology. For example, if X has no nontrivial M-ideals, then E_X has no nontrivial structurally open sets. On the other hand, if E_X satisfies the T_1-separation axiom, then $\ker p$ is an M-ideal in X for all $p \in \operatorname{ex} B_{X^*}$ so that X has in fact many M-ideals. Surely, E_X is a T_1-space if X^* is isometric to an L^1-space, but this condition is not necessary, as the example of the disk algebra shows. We will consider certain strengthenings of the T_1-axiom for E_X and will study the Banach spaces with such a structure space in Section II.5.
We can now state the announced connection between M-ideals and the centralizer in the following theorem of Dauns-Hofmann type. These authors obtained a corresponding result in the setting of C^*-algebras [146].

Theorem 3.12 *For a bounded function $a : \operatorname{ex} B_{X^*} \to \mathbb{K}$, the following assertions are equivalent:*

(i) *a is structurally continuous.*
(ii) *$a = a_T$ for some $T \in Z(X)$.*

PROOF: [11, p. 153], [201] or [51, Th. 3.13(ii)], where in addition the complex case is treated. ◻

As a corollary one obtains: If $Z(X)$ is nontrivial, then X contains a nontrivial M-ideal. We shall later encounter Banach spaces which show that the converse is not true, see p. 88 or the discussion following Theorem III.2.3. Another example is the disk algebra; but note that here the multiplier algebra is nontrivial.

Finally, we would like to briefly sketch the dual situation. We give one more definition.

Definition 3.13 *The closed linear span of the set of L-projections on X is called the* Cunningham algebra *of X, denoted by* $\operatorname{Cun}(X)$.

$\operatorname{Cun}(X)$ is a commutative algebra of operators by virtue of Theorem 1.10(a). For instance, by Example 1.6(a) the Cunningham algebra of $L^1(\mu)$ coincides with the algebra of multiplication operators with L^∞-functions.
The connection between Definitions 3.13 and 3.7 is expressed in the following theorem.

Theorem 3.14

(a) *$T \in Z(X)$ if and only if $T^* \in \operatorname{Cun}(X^*)$.*
(b) *$T^* \in Z(X^*)$ if and only if $T \in \operatorname{Cun}(X)$.*
(c) *Every operator $T \in Z(X^*)$ is weak* continuous so that*

$$\operatorname{Cun}(X) \cong Z(X^*) \cong C(K_{X^*})$$

where K_{X^*} is hyperstonean, and

$$Z(X^*) = \overline{\operatorname{lin}}\,\{P \mid P \text{ is an } M\text{-projection on } X^*\}.$$

Moreover, $\operatorname{Mult}(X^*)$ *and $Z(X^*)$ are dual Banach spaces.*

(d) *If X is a complex Banach space or a real Banach space not isometric to $\ell^\infty(2)$, then either $Z(X)$ or* $\mathrm{Cun}(X)$ *is trivial.*

PROOF: (a) [11, p. 151] or [51, Th. 3.12] (real scalars) and [56] (complex scalars).

(b) and (c) Let us first note that $L(X^*)$ is a dual Banach space with canonical predual $X^* \widehat{\otimes}_\pi X$ and that the corresponding weak* topology (henceforth called "the" weak* topology) on $L(X^*)$ coincides with the weak* operator topology on bounded sets, i.e.

$$T_i \xrightarrow{w*} T \quad \Longleftrightarrow \quad \langle T_i x^*, x\rangle \to \langle T x^*, x\rangle \quad \forall x^* \in X^*,\ x \in X$$

if $\sup_i \|T_i\| < \infty$. (Cf. e.g. [158, Chap. VIII].)
Now we show that $\mathrm{Mult}(X^*)$ is weak* closed in $L(X^*)$. Let (T_i) be a bounded net in $\mathrm{Mult}(X^*)$ and $T \in L(X^*)$ such that $T_i \xrightarrow{w*} T$. There is no loss of generality in assuming $\|T\| = \|T_i\| = 1$ throughout. We shall verify the condition of Theorem 3.6(ii). So suppose

$$\|x_0^* - \lambda x^*\| \le r \quad \text{for all} \quad |\lambda| \le 1.$$

Since the T_i are multipliers we obtain

$$\|x_0^* - T_i x^*\| \le r \quad \text{for all} \quad i.$$

Passing to the limit entails by weak* lower semicontinuity of the norm

$$\|x_0^* - T x^*\| \le r$$

so that our assertion follows from Theorem 3.6 and the Krein-Smulian theorem.
Hence, if the scalars are real, $Z(X^*) = \mathrm{Mult}(X^*)$ is a weak* closed subspace of $L(X^*)$, and in the complex case $Z_{\mathbb{R}}(X^*)$, which is $\mathbb{R}$-isometric with $\mathrm{Mult}(X^*_{\mathbb{R}})$, is weak* closed in $L(X^*_{\mathbb{R}})$. Consequently they are isometric to dual Banach spaces. The point of these considerations is that, regardless of the scalar field, $Z(X^*)$ is isometric to a dual $C(K)$-space. By a theorem due to Grothendieck [385, p. 96] we conclude that K_{X^*} is hyperstonean; in particular, the algebra $C(K_{X^*})$ is generated by its idempotents, and the same must be true for $Z(X^*)$. Taking into account that the idempotents in the centralizer are exactly the M-projections (cf. the revisited Example 3.4(e)) we obtain the assertions made in (b) and (c).
For a different line of reasoning, see [51, Th. 5.9] along with [51, Prop. 1.16] and [66, p. 24ff.].

(d) is a consequence of (a) – (c) and Theorem 1.8, applied to X^*. □

Formally, Theorem 3.14 generalises both Theorem 1.8 and Theorem 1.9, but its proof relies on these results.

Corollary 3.15

(a) *If $T \in Z(X)$, then also $T^{**} \in Z(X^{**})$.*

(b) *If X is a complex Banach space, then $T \in Z(X)$ commutes with every hermitian operator.*

PROOF: (a) is an immediate consequence of Theorem 3.14.

(b) Since the adjoint of an hermitian operator is hermitian, we may – by passing to second adjoints – assume from the outset that T is an M-projection (by (a) and 3.14(c)). In this case the assertion to be proved is equivalent with saying that a hermitian operator leaves M-summands invariant, which we already know from Corollary 1.25. $\square$

We remark that the assertions of Corollary 3.15 extend to $T \in \mathrm{Mult}(X)$; see [56] and [68] for a proof.

I.4 Notes and remarks

GENERAL REMARKS. In 1972 the seminal paper "Structure in real Banach spaces" [11] by Erik M. Alfsen and Edward G. Effros appeared. There they initiated a study of general Banach spaces based on the notion of an M-ideal, proceeding in much the same way as in C^*-algebra theory where the notion of a closed two-sided ideal is instrumental. Effros [186] proved in 1963 that the closed left ideals in a C^*-algebra correspond in a one-to-one fashion to the closed faces in the state space, thus paving a way towards a geometric approach to structural questions. (The result was independently obtained by Prosser [509, Th. 5.11], as well.) Then it was proved in a series of articles ([186], [584], [9]) that the closed two-sided ideals are in one-to-one correspondence with the closed split faces of the state space. Similar developments of general ideal theories were encountered in the theory of ordered Banach spaces ([10], [491], [637]), L^1-predual spaces ([189], [214]; cf. Section II.5) and compact convex sets ([9], [187], [188]; cf. Example 1.4(c)). In order to provide a unified treatment, Alfsen and Effros defined the concept of an M-ideal; around the same time the same notion arises in Ando's work [19], however under a different name ("splittable convex set") and for a different purpose.

The key result of part I of the Alfsen-Effros paper is the characterisation of M-ideals by means of an intersection property of balls; more precisely they proved (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (v) of Theorem 2.2 for real Banach spaces. Their proof was anything but simple and used a theorem on "dominated extensions" [11, Th. 5.4], based on Theorem 2.4 which is also due to them [11, Th. 4.5], as a decisive step. (Alfsen adapted this approach to complex spaces in [8].) Simpler proofs, covering the case of complex scalars, too, were devised by Lima [399], who added the useful condition (iv), Behrends [51, Chap. 2D] and Yost [649]. The latter paper also contains the counterexample of Remark 2.3(d); further counterexamples appear in [11, p. 126], [648] and [650]. The example of Remark 2.3(a) is due to Lima [399]; a three-dimensional version can already be found in [11].

The proof of the implication (iv) $\Rightarrow$ (i) in Theorem 2.2 follows [399], and for (ii) $\iff$ (v) we follow [650]. Our proof of the implication (i) $\Rightarrow$ (ii) using the principle of local reflexivity, however natural this approach may be, seems to be new. One could even shortcut the proof if one knew in advance that a local reflexivity operator T can always be found with the additional property that $T(J^{\perp\perp} \cap E) \subset J$ for a given M-ideal J and a finite dimensional subspace $E \subset X^{**}$. As a matter of fact this is the case as was recently shown in [62], see also [69] and [167]. However we decided to present the slightly longer proof, since it employs well known tools only. Lima's idea in [399] to show (i) $\Rightarrow$ (ii) is to use the Hahn-Banach theorem in the space $X \oplus_\infty \cdots \oplus_\infty X$. If (ii) were false one could separate the "diagonal" $\{(y, \ldots, y) \mid y \in J\}$ from $B(x_1, r_1) \times \cdots \times B(x_n, r_n)$, from which Lima derives a contradiction. Yost's approach in [649] uses the $1\frac{1}{2}$-ball property to be discussed in the Notes and Remarks to Chapter II.

We remark that Proposition 1.2 and Corollary 1.3 first appeared in [292]. Theorem 1.8 is due to Behrends whose original proof is quite tedious albeit elementary (cf. [51, p. 24ff.]). We have followed the line of reasoning by Payá and Rodríguez [479]. Theorem 1.9 comes from [142], and Theorem 1.10 from [138]. (For the importance of part (b) of this theorem see below.) Proposition 1.16 was observed in [148] in order to obtain best approximation results in the setting of C^*-algebras. For another account of Proposition 1.20 we refer to [641, Chap. III.D]. Proposition 1.21 and Corollary 1.22 can be found in [240] where a different proof is given. Proposition 1.23 is due to Effros and was published in [311], see [476] for another proof. Corollary 1.25 made its first appearance in [473] and, independently, [476] and [477]; our proof and Proposition 1.24 are taken from [630]. Theorem 2.10 represents a general version of the main result of Godefroy's and Li's paper [264] where the case $X = J^{**}$ is treated and also draws on the related paper [396]. Lemma 2.9 is in essence a classical result due to Pełczyński [484] and can be found for example in [422, p. 32] or [572, p. 446ff.].

Part II of the Alfsen-Effros paper centres around the notions of Section I.3, notably the centralizer, the Cunningham algebra and the structure topology. These notions are patterned after similar concepts in C^*-algebra theory, the analogue of the structure topology being the so-called hull-kernel topology. The main result describes their connection in the Dauns-Hofmann type theorem 3.12. However, as the title of [11] suggests, only real Banach spaces are considered in that paper. The wish to cover complex spaces as well necessitates distinguishing between the centralizer and the multiplier algebra. This was first done by Behrends in [51]. Let us remark that M-ideals in complex Banach spaces were first studied in [311]. With the help of Proposition 1.23 one may conclude that most of the results on M-ideals in real Banach spaces extend – mutatis mutandis – to the complex case. For the multiplier algebra things are not so easy. For example it is not known if a complex Banach space such that $\mathrm{Mult}(X)$ is nontrivial must contain a nontrivial M-ideal. (For $Z(X)$ this follows from 3.12, though a direct proof is possible, too.) Wodinski [638] has shown that this is so if in addition the existence of a nontrivial $T \in \mathrm{Mult}(X)$ such that T^* attains its norm is supposed. Also, it is not hard to show that a strictly convex space has a trivial centralizer. The corresponding result for the multiplier algebra holds, too, but is considerably more difficult to prove [338, Th. 12.7], [638].

Most of the material of Section I.3 appears in [11] and [51], cf. the references given in the text. Proposition 3.9 and Theorem 3.10 come from [627] and [629]. There it is also pointed out that 3.10 implies that a unital C^*-algebra is commutative if and only if its centre separates the weak* closure of its pure states (which represents a special case of Théorème 11.3.1 in [166]) and that a compact convex set K is a Bauer simplex if and only if the order bounded operators on $A(K)$ separate the closure of $\operatorname{ex} K$.

For a detailed discussion of results related to Section I.3 we refer to the monographs [35] and [51]. In addition we mention the papers [53], [56], [161], [162], [339], [452], [453], [454].

Semi M-ideals. We have pointed out in Remark 2.3(a) that the 2-ball property and the 3-ball property are not equivalent, whereas it follows from Theorem 2.2 that the n-ball property and the 3-ball property are in fact equivalent if $n \geq 3$. Let us call a closed subspace a *semi M-ideal* if it satisfies the 2-ball property. (By the way, the 2-ball property and the strict 2-ball property coincide [649].) The arguments of Theorem 2.2, (iv) $\Rightarrow$ (i), and of Remark 2.3(a) yield that J is a semi M-ideal in X if and only if there is a (nonlinear) projection P from X^* onto $J^\perp$ such that

$$\begin{aligned} P(\lambda x^* + Py^*) &= \lambda Px^* + Py^*, \\ \|x^*\| &= \|Px^*\| + \|x^* - Px^*\| \end{aligned}$$

for all $x^*, y^* \in X^*$, $\lambda \in \mathbb{K}$. Such a projection is called a *semi L-projection* and its range a *semi L-summand*. Thus the above result (due to Lima [399]) says that J is a semi M-ideal if and only if $J^\perp$ is a semi L-summand. (Actually, Lima defines a semi M-ideal in terms of semi L-projections, and the characterisation by the 2-ball property is a theorem of his.)

The example given in 2.3(a) seems to be one of the few natural occurrences of (proper) semi M-ideals resp. semi L-summands. It shows that M-ideals and semi L-summands can appear simultaneously, in contrast to the situation described in Theorem 1.8. However, in this case it is necessary that the semi L-summand is one-dimensional and that the ambient space is isometric to an $A(K)$-space. This is proved by Payá and Rodríguez [479]. A lot more examples of semi M-ideals in real Banach spaces which are not M-ideals are constructed in [481] and [412]. As a matter of fact, these authors show that every real Banach space can be represented as a quotient space X/J where J is a semi M-ideal, but not an M-ideal. On the other hand, occasionally a semi M-ideal is automatically forced to be an M-ideal. For example, if X is a semi M-ideal in X^{**} then it is already an M-ideal ([551], [404]). This is due to the fact that a natural contractive linear projection with kernel $J^\perp$ is available. In fact, the proof of Proposition 1.2 extends to show: If P is a semi L-projection and Q is a contractive linear projection with $\operatorname{ran}(P) = \ker(Q)$, then $P = Id - Q$ so that P is linear, too. Applying this to $P =$ the semi L-projection onto the annihilator $X^\perp$ of X in X^{***} and $Q =$ the natural projection from X^{***} onto X^* (see III.1.2) we obtain that P is actually an L-projection. The same holds for the subspace $K(X)$ of compact operators in $L(X)$ provided X has the metric compact approximation property. (The natural linear projection will be described in the Notes and Remarks to Chapter VI.) Also, semi M-ideals in $C(K)$-spaces and, more generally, in C^*-algebras are ideals and hence M-ideals ([399], [439]).

Semi M-ideals share a number of properties with M-ideals; for example, the sum of two semi M-ideals is closed and a semi M-ideal [399], hermitian operators leave semi M-ideals invariant [439], semi M-ideals are Hahn-Banach smooth and proximinal (see Section II.1 for M-ideals) etc. [399]. A striking difference from the theory of M-ideals is that the intersection of two semi M-ideals need not be a semi M-ideal and that there are weak* closed semi M-ideals in a dual space which are not M-summands [399]; for M-ideals this is not possible as will be shown in II.3.6(b).

It is worth mentioning that the annihilator $J^\perp$ of a semi M-ideal J is always the *range* of a contractive linear projection which, however, in general does not enjoy decent norm properties. This follows from Lindenstrauss' work [414, p. 270] (see also [488, p. 62] for a simpler proof) since the semi L-projection onto $J^\perp$ can be shown to be 1-Lipschitz [647]. The most intriguing problem about semi M-ideals is to find examples of semi M-ideals in a complex Banach space which are not M-ideals. Only recently the existence of such examples was established by Behrends [63], thus refuting the long-standing conjecture that complex M-ideals might be characterised by the 2-ball property. (We shall discuss in the Notes and Remarks to Chapter II which results led to this conjecture.) Previously Yost [652] proved the equivalence of the following conjectures:

- There exists a semi M-ideal in some complex Banach space which is not an M-ideal.
- There exists a compact convex set $\Delta \subset \mathbb{C}^2$ such that $\ell(\Delta)$ is a disk for all linear maps $\ell : \mathbb{C}^2 \to \mathbb{C}$ which is not symmetric.

Further reformulations appear in [654] and [412], see also [656]. Using a very delicate and involved construction Behrends produces a set Δ as above and hence shows the existence of proper complex semi M-ideals. Let us remark that the equivalence of the two conjectures holds for real Banach spaces as well (here "disk" has of course to be replaced by "interval"); in this case, however, both of them were already known to be true. (For the former consult Remark 2.3(a), for the latter take Δ to be a triangle.) Another relevant paper on this topic is [64].

Unique Hahn-Banach extensions. It was shown in Proposition 1.12 that a functional which is defined on an M-ideal can be extended to a functional on the whole space preserving its norm in a unique manner. Following [573] and [587] we shall refer to this property of a closed subspace as *Hahn-Banach smoothness*. The study of the phenomenon of unique Hahn-Banach extensions was initiated by Phelps in [494] who proved for example that the subspace J is Hahn-Banach smooth if and only if $J^\perp$ is a Chebyshev subspace of X^* (cf. Section II.1 for the definition of this notion); in Phelps' terminology Hahn-Banach smoothness is called property (U).
Obviously, being an M-ideal is far stronger than being Hahn-Banach smooth. For this reason Hennefeld [304] has proposed a weakening of the notion of an M-ideal in that he requires that there be a projection P on X^* whose kernel is $J^\perp$ such that

$$\|Px^*\| \quad < \quad \|x^*\| \qquad \text{if } x^* \neq Px^* \tag{1}$$

$$\|x^* - Px^*\| \quad \leq \quad \|x^*\|. \tag{2}$$

In this case J is called an *HB-subspace*. It is known that, for all Banach spaces X and for $1 < p < \infty$, $K(X, \ell^p)$ is an *HB*-subspace of $L(X, \ell^p)$ without generally being an M-ideal [461], cf. also [304], [305].
It is easy to see that an *HB*-subspace is in fact Hahn-Banach smooth, but actually inequality (2), which may contain welcome information, has nothing to do with the problem of unique Hahn-Banach extensions. Let us say that a subspace J is *strongly Hahn-Banach smooth* if there is a projection on X^* with kernel $J^\perp$ which satisfies solely (1). In [404] Lima characterises the difference between strongly Hahn-Banach smooth subspaces and Hahn-Banach smooth subspaces by an intersection property akin to the

n-ball property of Theorem 2.2. More precisely, he has proved that a Hahn-Banach smooth subspace J is strongly Hahn-Banach smooth if and only if for $\varepsilon > 0$, all $n \in \mathbb{N}$ (equivalently: for $n = 3$), all $x_1, \ldots, x_n \in J$ the implication

$$\bigcap_{i=1}^{n} B(x_i, r_i) \neq \emptyset \quad \Longrightarrow \quad \bigcap_{i=1}^{n} B(x_i, r_i) \cap J \neq \emptyset$$

holds. (Note that one requires here all the centres to lie in J!) Incidentally, Lima seems to have overlooked inequality (2) since he claims to provide a characterisation of *HB*-subspaces. This was pointed out by E. Oja in [460]. She also presents examples of strongly Hahn-Banach smooth subspaces which are not *HB*-subspaces and of Hahn-Banach smooth subspaces which are not strongly Hahn-Banach smooth. One way of distinguishing between Hahn-Banach smoothness and its strong version is as follows: J is Hahn-Banach smooth if and only if there is only one norm preserving extension function $\Phi : J^* \to X^*$. The subspace J is strongly Hahn-Banach smooth if in addition Φ is linear!

More general norm decompositions. A natural generalisation of the concept of L- and M-projections is the concept of an L^p-*projection*, defined by the norm condition

$$\|x\|^p = \|Px\|^p + \|x - Px\|^p.$$

L^p-summands are defined in the obvious way. This notion is discussed in detail in [66]. The nontrivial fact about these projections is that two L^p-projections commute if $p \neq 2$. This was first proved by Behrends [47], an easier proof can be found in [110], and for complex scalars there is a simple argument in [476]. If we agree to call an M-projection an L^∞-projection, then it is also true that L^p- and L^r-projections cannot exist simultaneously on a given Banach space if $p \neq r$ (except for $\ell^\infty_{\mathbb{R}}(2)$). This is proved in [47] and [110], too. Moreover, the set of all L^p-projections forms a complete Boolean algebra, and L^p-projections on dual spaces are weak* continuous if $p > 1$. Therefore, there is a complete duality between L^p-projections in X and L^q-projections in X^* (with q denoting the index conjugate to p); in particular there is no need for "L^p-ideals". This fact seems to be responsible for the extreme lack of nontrivial examples of L^p-projections apart from the characteristic projections in L^p-spaces, including L^p-spaces of vector valued functions (Bochner L^p-spaces). In fact, one can characterise L^p-spaces by a richness condition on the set of L^p-projections ([138] for $p = 1$, [66, p. 54]). Let us mention one other instance where L^p-projections arise naturally: the 1-complemented subspaces of the Schatten classes c_p for $1 < p < \infty$, $p \neq 2$, were recently characterised as those subspaces which can be decomposed into a sequence of L^p-summands of a special nature (Cartan factors) ([29], [30]). (The boundary cases $p = 1, \infty$ were treated in [28].) However, it follows from the results of [580] that the Schatten class c_p itself does not admit any nontrivial L^p-projections.

L^p-projections also turn out to be of interest in the discussion of ergodic theorems [282] and isometries for Bochner L^p-spaces [277] [278]. Fortunately, it is the lack of L^p-projections of the range space which is of importance for the description of the isometric isomorphisms of Bochner L^p-spaces. For example, Greim describes in [277] the surjective isometries of $L^p(\mu, V)$ if V is separable and has no nontrivial L^p-summands.

Yet more general norm decompositions are discussed in [438] and the subsequent papers [439], [440]. Let F be a norm on $\mathbb{R}^2$ such that

$$\begin{aligned} F(0,1) &= F(1,0) = 1 \\ F(s,t) &= F(|s|,|t|) \qquad \forall (s,t) \in \mathbb{R}^2. \end{aligned}$$

An *F-projection* is defined by the equation

$$\|x\| = F(\|Px\|, \|x - Px\|) \qquad \forall x \in X,$$

and the range of an F-projection is called an F-summand. The dual norm F^* is defined by

$$F^*(a,b) = \sup_{F(s,t)\leq 1} |at + bs|,$$

and an F-ideal of X is a closed subspace J for which $J^\perp$ is an F^*-summand. One can show that the existence of F-ideals which are not F-summands depends decisively on the geometry of the two-dimensional unit ball $B_F = \{(s,t) |\ F(s,t) \leq 1\}$. In fact, if $(0,1) \in \operatorname{ex} B_F$, then every F-ideal is actually an F-summand. [One way to prove this is to show by duality that a G-summand in X^* is weak* closed if G is differentiable at (1,0), i.e., $\lim_{h\to 0}(G(1,h) - 1)/h = 0$. This can be achieved by an argument similar to the one leading to Theorem 1.9.]
The authors of [438] also consider the corresponding "semi" notions, where the additivity of the projections involved is replaced by the weaker quasiadditivity condition

$$P(x + Py) = Px + Py \qquad \forall x, y \in X.$$

At this level of abstraction another dualisation procedure leads to a new concept: If $J^\perp$ is a semi F^*-ideal, then J is called a semi F-idealoid. Luckily, further dualisation has no effect, since J can be shown to be a semi F-ideal if $J^\perp$ is a semi F^*-idealoid. Again, one can prove that the geometry of B_F and of B_{F^*} governs the problem whether several of these notions coincide for a given F. For details we refer to [438].

Boolean algebras of projections. In Theorem 1.10, which is due to Cunningham [138], we proved the completeness of the Boolean algebra $\mathbb{P}_L$ of all L-projections. (Actually, not only is $\mathbb{P}_L$ complete as an abstract Boolean algebra, but the supremum of an upwards filtrating family is its limit in the strong operator topology. This refined version of completeness is known as Bade completeness and of importance in operator theory.) By the Stone representation theorem (cf. e.g. [283, p. 78]) $\mathbb{P}_L$ is isomorphic to the Boolean algebra of clopen sets of some compact Hausdorff space Ω. Since $\mathbb{P}_L$ is complete, Ω is extremally disconnected, meaning that the closure of each open set is open. But even more is true: as Cunningham shows there is a Borel measure m on Ω with the following properties.

- m is regular on sets of finite measure.
- $m(A) = 0$ if A is nowhere dense.
- $m(C) > 0$ if $\emptyset \neq C$ is clopen.
- Every Borel set contains a Borel set of finite measure.

[Idea of construction: Let $C \mapsto E_C$ be the Boolean isomorphism between the clopen sets of Ω and $\mathbb{P}_L$. For $x \in X$ and closed $B \subset \Omega$ let $m_x(B) = \inf\{\|E_C x\| \mid C \supset B \text{ clopen}\}$. One can show that m_x extends to a uniquely determined finite regular Borel measure on Ω. Let $\{m_{x_i} | \ i \in I\}$ be a maximal family of pairwise singular measures of this type (which exists by Zorn's lemma) and put $m(B) = \sum_i m_{x_i}(B)$. (Details can be found in [66, p. 24ff.].)]
An extremally disconnected compact space admitting such a measure is called hyperstonean; equivalently, the order continuous functionals on $C(\Omega)$ ("normal measures") separate the points of $C(\Omega)$. (This is the usual definition to be found e.g. in [385].) It is now possible to represent X as a Banach space of "integrable sections" in a product $\prod_{\omega\in\Omega} X_\omega$ such that $\|x\| = \int_\Omega \|x(\omega)\|_{X_\omega}\, dm(\omega)$. If X is represented in this way the operators in the Cunningham algebra (Definition 3.13) correspond exactly to the multiplication operators $Tx(\cdot) = f(\cdot)x(\cdot)$ for $f \in C(\Omega)$. This representation, called the integral module representation, is proved in [66].
Furthermore one can show that the operators in the Cunningham algebra behave like normal operators on Hilbert space; more precisely they are spectral operators of scalar type in the sense of Dunford and Schwartz [179], see [48], [66].
These theorems extend to the setting of L^p-projections for $1 < p < \infty,\ p \neq 2$ [66]. Also, Evans proves in [206] that a Boolean algebra of F-projections necessarily consists solely of L^p-projections for some p under some natural additional assumption on F.
The case $p = \infty$ differs from the previous ones in that the Boolean algebra $\mathbb{P}_M$ need not be complete. As a matter of fact, there need not exist any nontrivial M-projections on a $C(K)$-space (e.g. $K = [0,1]$) so that such a space cannot be "resolved" in terms of M-projections although it certainly should have a fairly large "M-structure". The idea here is to have the role of the algebra generated by $\mathbb{P}_M$ taken over by the centralizer, and now a representation analogous to the integral module representation is possible. This was shown – up to the terminology – by Cunningham in [139]. More specifically, let K_X be the Gelfand space of the commutative C^*-algebra $Z(X)$. Then X is isometric to a subspace of a product $\prod_{k\in K_X} X_k$ in such a way that the numerical function $k \mapsto \|x(k)\|_{X_k}$ is upper semicontinuous on K_X and $\|x\| = \sup \|x(k)\|$. Moreover, the operators $T \in Z(X)$ correspond exactly to the multiplication operators $Tx(\cdot) = f(\cdot)x(\cdot)$ for $f \in C(K_X)$. In particular, the copy of X in $\prod X_k$ has the structure of a $C(K)$-module. The above representation is called the maximal function module representation in [51, Chap. 4]. The reader should observe that this procedure is reminiscent of von Neumann's reduction theory of operator algebras [165, Chap. II]. We refer to [139], [51] or [244] for exhaustive information, moreover we mention the papers [161], [245], [377] and [562]. A general view of integral and function module representation theorems is given in Evans' paper [207], see also [208].