# SORTING BY SYMMETRY:

# WALLPAPER AND TESSELLATIONS
## – PART I –
## TRANSLATIONS AND ROTATIONS



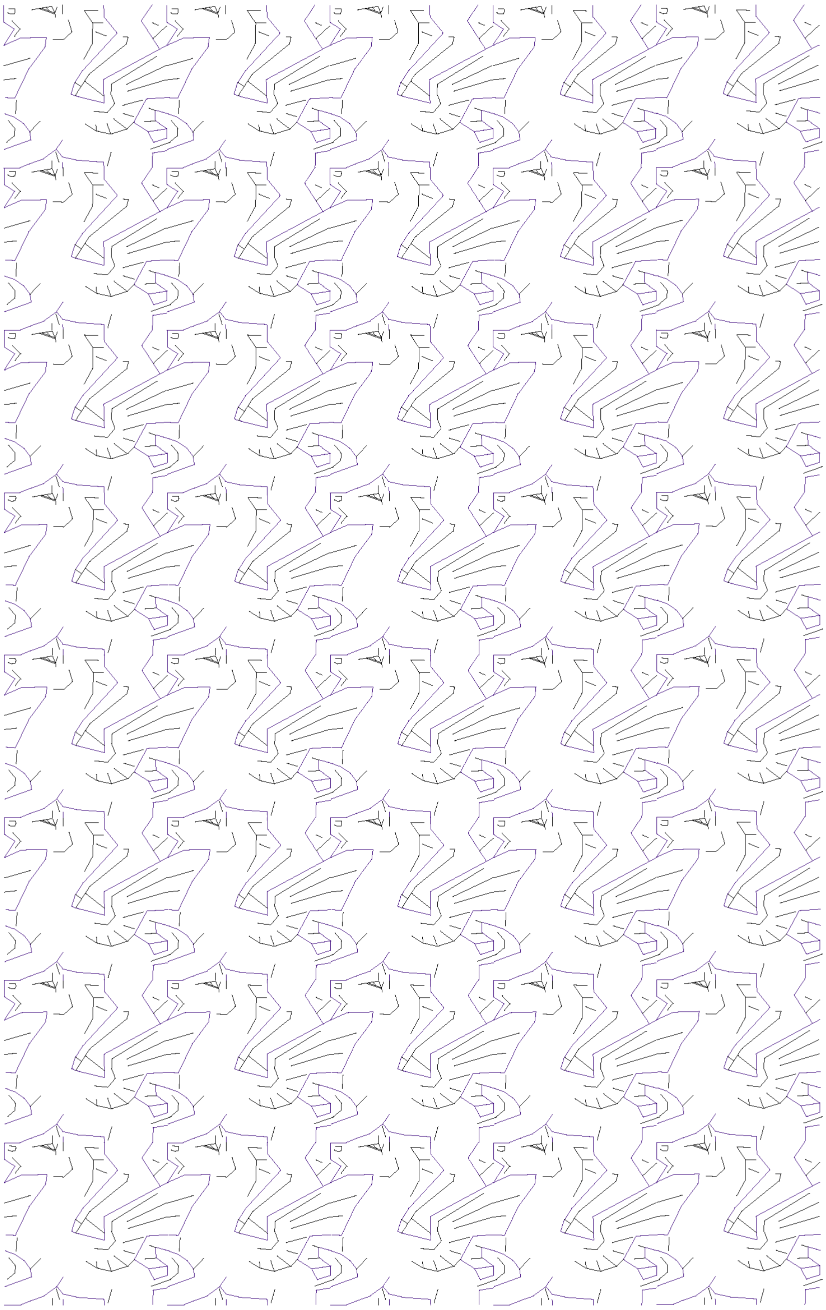
**Introduction**

This booklet is a sequel to *Sorting by symmetry, Patterns with a Centre* (ATM 2005) and *Sorting by Symmetry, Patterns along a line,* freely available at [www.atm.org.uk/resources/friezes.html](www.atm.org.uk/resources/friezes.html). As with the earlier booklets you will need tracing paper for the suggested activities. Many of the exercises and investigations here will need hard copy, that is, a print-out on paper of the diagrams. In *Patterns with a Centre,* reflections and rotations with a common fixed point were explored, and were combined to form cyclic and dihedral patterns. Some basic ideas about symmetry which were developed in *Patterns with a Centre* will be used repeatedly here. *Symmetry* was defined in Section **9** of that booklet and *symmetry group* in section **10** (all the symmetries of a pattern). In *Patterns along a line*, translations were introduced to describe infinite patterns. In order to show that the friezes we examined cover all the possibilities with a shortest translation, we showed in sections **13** and **14** of that book that every symmetry of the plane may be expressed as a reflection or as a product of two or three reflections. Moreover in section **13** we saw that every symmetry of the plane is uniquely determined by a triangle and its image under that symmetry. In *Patterns along a line* all the translations were in the same direction. The style of all three booklets, with each section shaped by **Observing**, **Making**, and **Sharpening** was described in the **Introduction** to *Sorting by Symmetry*, *Patterns with a Centre.*

Wallpaper is made in rolls. One roll of wallpaper shows a frieze pattern. But when two sheets of wallpaper are be laid side by side in such a way that the join matches the pattern and is imperceptible, many sheets may be joined to make the pattern cover an unlimited area. Wallpaper patterns always allow translations in two (or more) directions. This means that there is neither a point nor a line which is fixed by all the symmetries of a wallpaper pattern. As with a frieze, however, there is always a shortest translation (≠ **I**), so that, for example, an empty plane does not count as a wallpaper pattern. In this booklet we will explore firstly what a group of translations of a wallpaper pattern may be like, (**P1**), and then how such a pattern may have rotations (**P2, P4, P3** and **P6**). Just why these are the only possible rotation symmetries of wallpaper will be explored in the last section of this booklet.

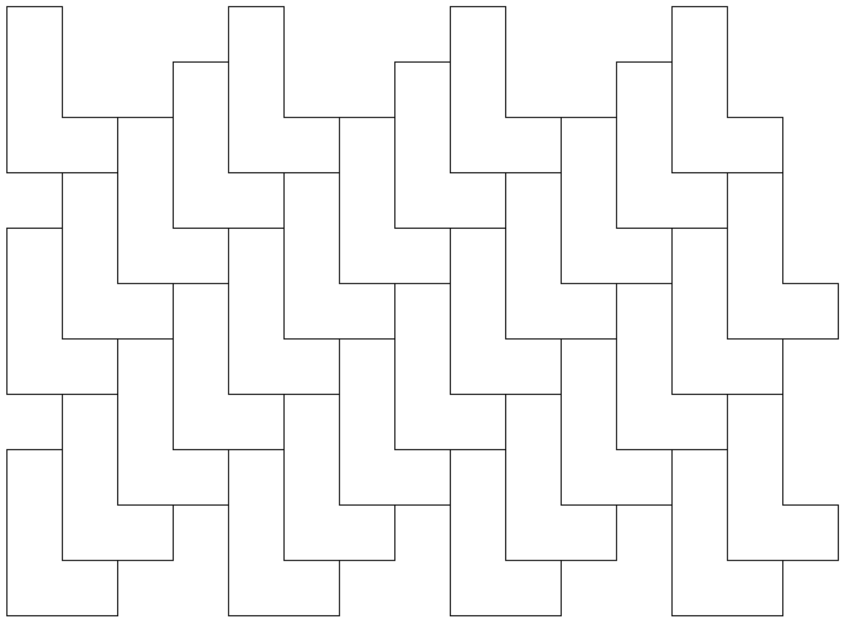
# 1. P1 Translations alone

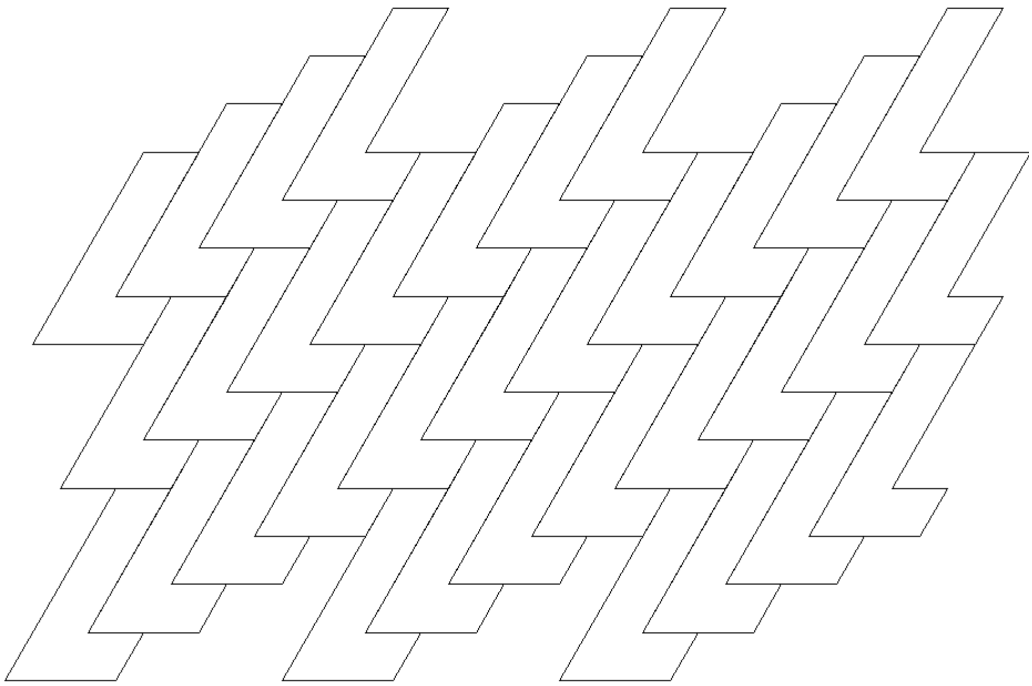
***Observing***



(i) Choose an easily recognisable point on the pattern of flying horses. Mark every copy of that point that you can see. After marking several of these points you may notice that they lie on a pattern of straight lines, and you can use that pattern to help you mark the remaining points. The pattern of points is called a lattice. This pattern of flying horses was drawn by John Sharp after M.C.Escher.

(ii) If you copied one of the horses onto a piece of tracing paper, what movement of the tracing paper will match the traced horse with another on the original tessellation? If this symmetry were copied over the whole plane, would each horse be matched to another horse? What would happen to the lattice of points that you have marked under this movement of the tracing paper?





(iii) Try observations (i) and (ii) on the other tessellations provided. Describe the lattice of points which arises in each case. You should have found a square lattice, a rectangular lattice and a parallelogram lattice.

(iv) What kind of symmetries do each of these tessellations have if they are imagined to extend indefinitely? Each of the lattices which you have constructed is known as the *orbit* of a chosen point under the group of translations – a **translation orbit**.

(v) Examine some patterned wallpaper or wrapping paper. Find the parallelogram lattice which shows how the replication by translations has been effected.

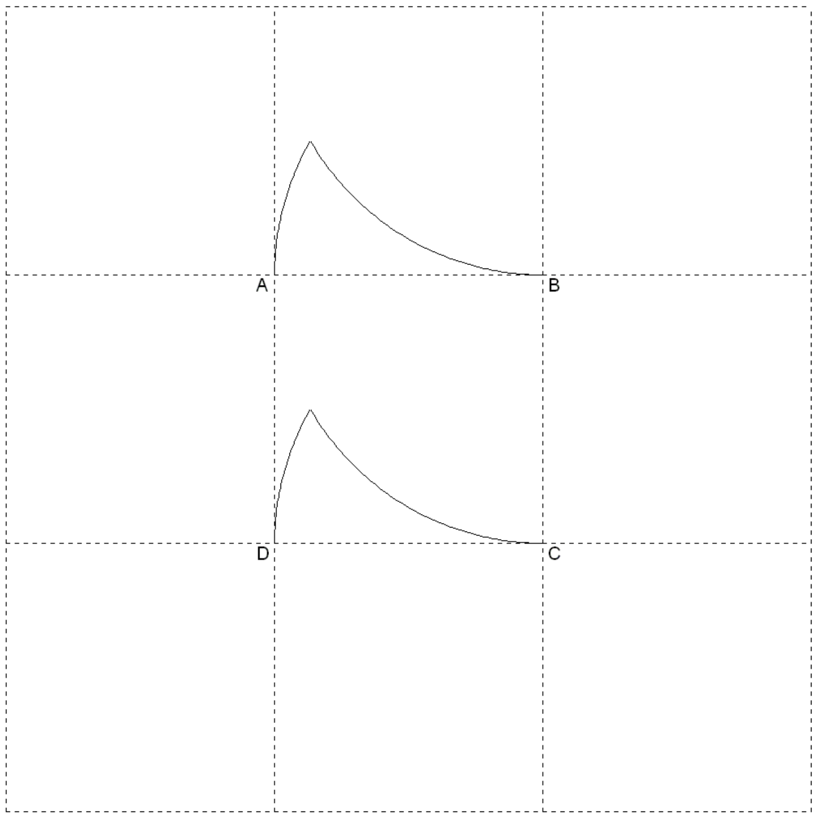
*Making*

(i) Go back to the tessellation of horses and its square lattice. If you made a square tile by joining four of the lattice points which were as close to each other as possible, would that square tile reproduce the whole pattern if it was translated in the same way as the horses were translated?

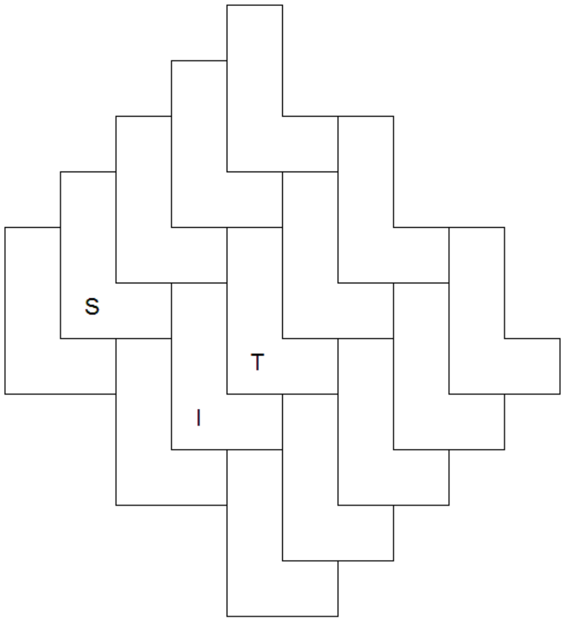
(ii) Look at how the squares of the lattice dissect a horse. And look too at how the tessellation of horses dissects one square of the lattice. Do the pieces match?

(iii) Can you, by joining lattice points, make other tiles that would also cover the whole plane under the same group of translations? Some squared paper, preferably with quite large squares, should help with this.

(iv) Start with a square card with vertices *ABCD*. Take some scissors and make a continuous cut that starts at *D* and ends at *C*. Take the piece you have cut off and attach it to the square tile along the side *AB*. Could you use your newly made card as a template for a wallpaper pattern? This is one of the devices that M.C. Escher used so cleverly. This kind of cutting and pasting is explored in chapter 3 of Ranucci and Teeters, *Creating Escher-type Drawings*, Creative Publications, 1977.

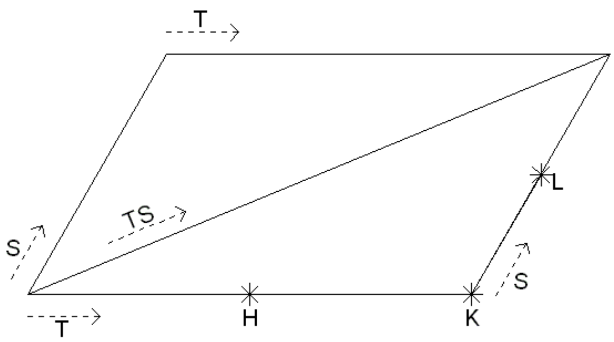


(v) In the diagram below the translation **T** carries the ell labelled **I** to the ell labelled **T**. Also the translation **S** carries the ell labelled **I** to the ell labelled **S**. Mark the remaining ells in the diagram with combinations of **T** and which give the translations which carry the ell labelled **I** to those ells. Notice that **TS = ST**, so there is some ambiguity about the labels.



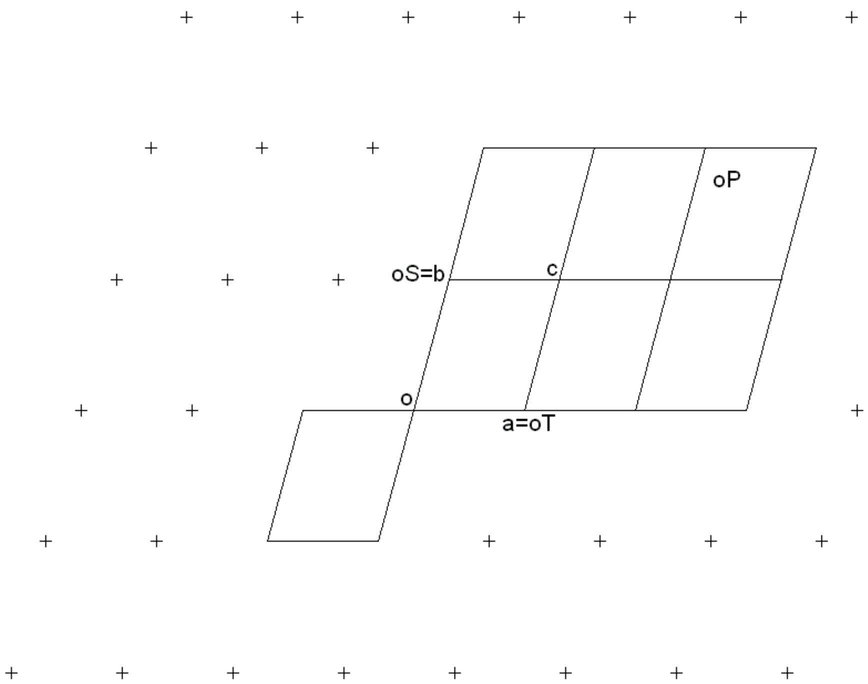

***Sharpening***

(i) **The product of two translations.** The lattices which can be formed by translations from wallpaper are always those from squares, rectangles, rhombuses or parallelograms. This is because of the way translations combine. Try to predict the result of combining two translations. In *Patterns along a line*, **section 5(c)** we saw that a translation was always expressible as the product of two half-turns, with the distance between the centres of the half-turns being half the length of the translation, and the line joining the half-turn centres being in the direction of the translation. This still leaves a lot of choice for the half-turns, **H** and **K**, say, once the translation **T** has been given. For a given translation **T** = **HK** (product of half turns), **H** may be any half-turn of the plane. If **S** is another translation we can express **S** as a product of half-turns taking **K** as the first. If the other half-turn is **L**, so that **S** = **KL**, then **TS** = (**HK**)(**KL**) = **H**(**KK**)**L** = **HL** = a translation. The diagram makes it clear how if **T** takes *A* to *B* and **S** takes *B* to *C*, then **TS** takes *A* to *C*, and a parallelogram gives a good illustration of this. Check that **TS** = **ST**. That is, it makes no difference which translation is done first. Notice that the argument still works if the two translations are in the same direction.



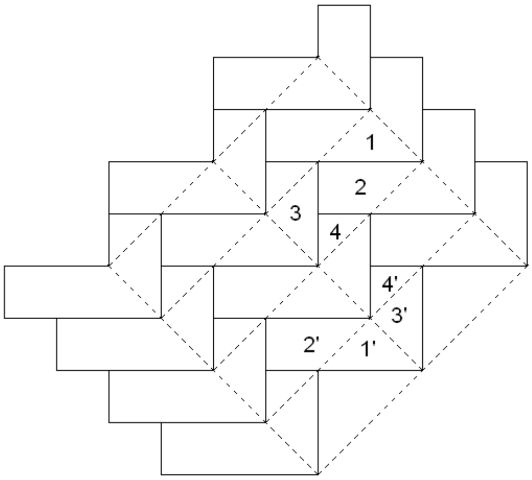
So when two translations are combined, the result is a translation.

(ii) **The translation group is generated by two translations, that is, every translation of the pattern can be made by combinations of two particular translations.** Go back to one of your lattices from ***Observing*** **(i)** or **(iii)**. Label one of the points of the lattice $o$. Then choose a lattice point as near to $o$ as possible and call it $a$. Let **T** be the translation taking $o$ to $a$. Now choose another lattice point as near to $o$ as possible, but not along the line $oa$, and call it $b$. Let **S** be the translation taking $o$ to $b$. Note $ob \geq oa$. Now let the translation **TS** take $o$ to $c$. Then from (i), $oacb$ is a parallelogram. It looks as if we can get from $o$ to any lattice point by taking multiples of **T** and multiples of **S**, but we need to be sure that these are *all* the translations of the tessellation. Just suppose there was another translation **P** (for perhaps) then **P** would take $o$ to a non-lattice point, which necessarily lies within, or on the side of a parallelogram which is the image of $oacb$ under, say, $\mathbf{T}^m\mathbf{S}^n$. This means there is a translation $\mathbf{P}\mathbf{T}^{-m}\mathbf{S}^{-n}$ which must take $o$ to a point in $oacb$. But if this point is within, or on the side of triangle $oab$ the translation is shorter than **T** or **S** which is impossible, and if the point is within triangle $abc$, then the translation $\mathbf{P}\mathbf{T}^{-m-1}\mathbf{S}^{-n-1}$ is shorter than **T** or **S** which is again impossible. So **P** definitely does not exist.




(iii) **Any two tiles which cover the plane by the same translation group have the same area.** In ***Making*** **(ii)** you found that the way the squares dissect a horse, and the way the horses dissect a square may be matched, piece by piece by translations. This proves that the horse and the square have the same area. If you had two arbitrary tiles $t$ and $s$, then the pieces may be matched since each translation **T** takes $t\mathbf{T}^{-1} \cap s$ to $t \cap s\mathbf{T}$. Now suppose that when all the translations of the pattern are applied to $t$, the whole plane is exactly covered without overlaps, and if the same group of translations is applied to $s$, again the plane is covered without overlaps, then the dissections like $t \cap s\mathbf{T}$ add up to $t$, and the dissections like $t\mathbf{T}^{-1} \cap s$ add up to $s$. So the two tiles have the same area since they can be dissected into congruent pieces. So, for example, any two parallelograms which cover the plane by the same translations must have the same area.

The diagram illustrates this for the tessellation of ells and a rectangular tessellation.




A parallelogram of least area which covers the plane by translations of the pattern, without overlaps, is called a ***fundamental parallelogram*** of the pattern

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***P1 Summary:*** *The translation orbit of a point in a wallpaper pattern is a parallelogram lattice.*

*The Conway signature is* ◦ indicating a "wonder-ring".

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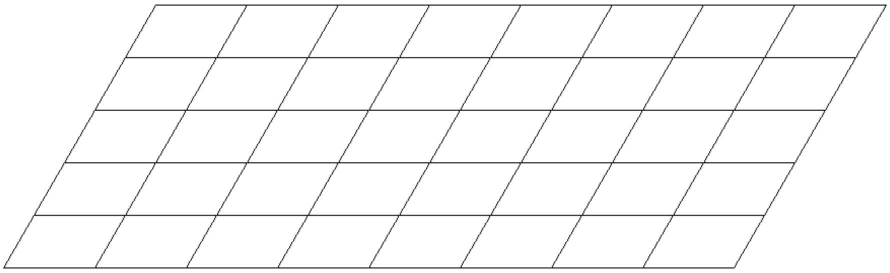
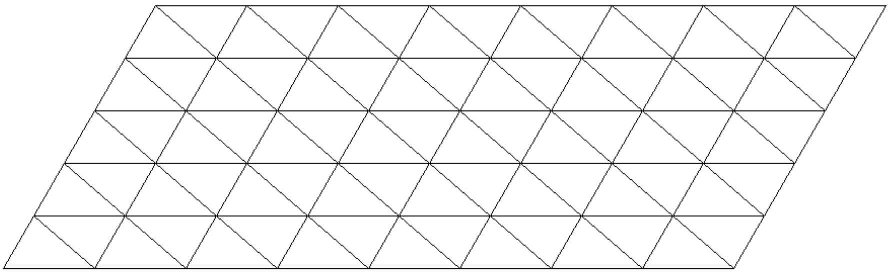
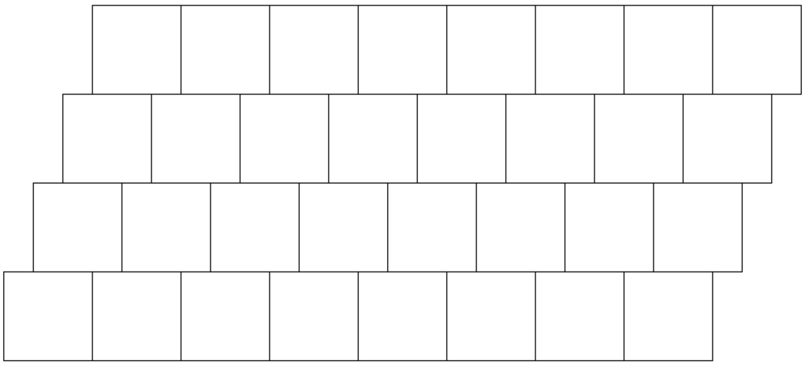
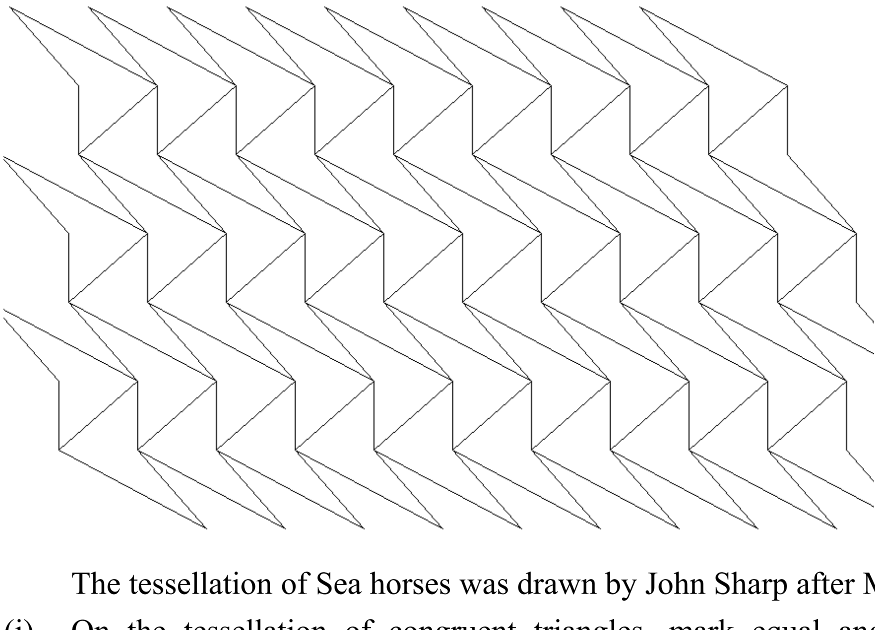
## 2. P2 Translations and half-turns

***Observing***









The tessellation of Sea horses was drawn by John Sharp after M.C.Escher.

(i) On the tessellation of congruent triangles, mark equal angles. Notice alternate angles, corresponding angles and the angle sum of a triangle.

(ii) Find a fundamental parallelogram for each of these patterns.

(iii) Make tracings and locate half-turn centres, by fixing a potential centre, on the original and on the tracing paper, and rotating the tracing paper. We will call such a centre a ***2-centre*** of the pattern. **2** × 180 = 360.

When you think you have found them all, it is a good trick to look half way between two that you have found. Sometimes that reveals another one. We will see why this is a good trick in ***Sharpening*** (below).

(iv) Mark the translation orbit of one of the 2-centres

(v) How many translation orbits of 2-centres are there?

(vi) Is there a relationship between the area of a tile and the area of a fundamental parallelogram? (May be different for different tiles.) You can use the method of **P1** ***Sharpening*** **(iii)** to help with this.

(vii) If you only use the translation orbit of a 2-centre for the vertices, what shapes of fundamental parallelogram are possible? Use the second pattern above to convince yourself of the variety.

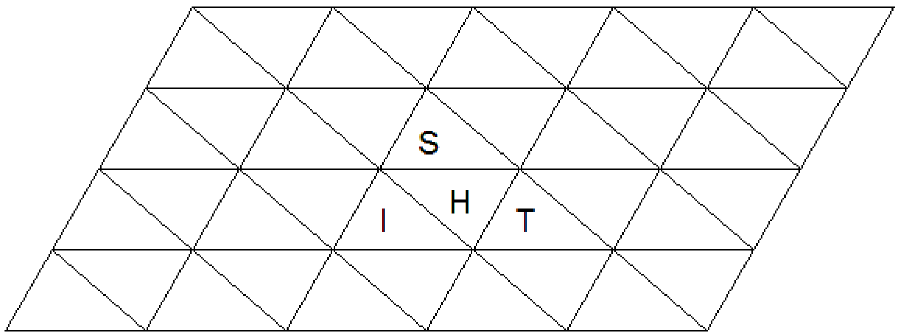
***Making***

(i) Can you use any parallelogram, any triangle, or any quadrilateral to make a tessellation with translations and half turns?

(ii) For the **P2** tessellations shown, can you always make a tile which has half the area of a fundamental parallelogram?

(iii) If you wanted to use the method of **P1** ***Making*** **(iv)** to make a tessellation with half turns, how would you have to modify your scissor cuts?

(iv) In the diagram below the translation **T** carries the triangle labelled **I** to the triangle labelled **T**. The translation **S** carries the triangle labelled **I** to the triangle labelled **S**. Also the half-turn **H** carries the triangle labelled **I** to the triangle labelled **H**. Mark the remaining triangles in the diagram with combinations of **T**, **S** and **H** which give the translations and half turns which carry the triangles labelled **I** to those triangles. Notice that **TS** = **ST**, $\mathbf{H}^2 = \mathbf{I}$, **HT** = $\mathbf{T}^{-1}\mathbf{H}$ and **HS** = $\mathbf{S}^{-1}\mathbf{H}$ so there is some ambiguity about the labels.




(v) Could you do a similar labelling for the arrowhead tessellation above?

***Sharpening***

(i) Quite independently of tessellations, check that the product of two (different) half turns is a translation. (This is discussed in *Patterns along a line*, section **5**.) How does the length of the translation compare with the distance between the centres of the half-turns?

(ii) If **H** and **K** are half turns and **HK** = **T** (a translation), what is **HT**?

(iii) A fundamental parallelogram for a wallpaper pattern is one that when translated by the translations of the pattern covers the whole plane without overlaps of area. If a fundamental parallelogram has the centre of a half-turn at each vertex, where must you find the centres of other half turns in or on the sides of the same fundamental parallelogram?

(iv) If **T** is a shortest translation of a wallpaper pattern, **S** is a shortest translation not in the direction of **T,** and **H** is a half-turn of the pattern, explain why no two of the centres of the half-turns **H**, **HT**, **HS** and **HTS** may lie in the same translation orbit.

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***P2 Summary:*** *The shape of a fundamental parallelogram is not determined by the existence of 2-centres. There are four translation orbits of 2-centres.*

*The Conway signature is 2222.*

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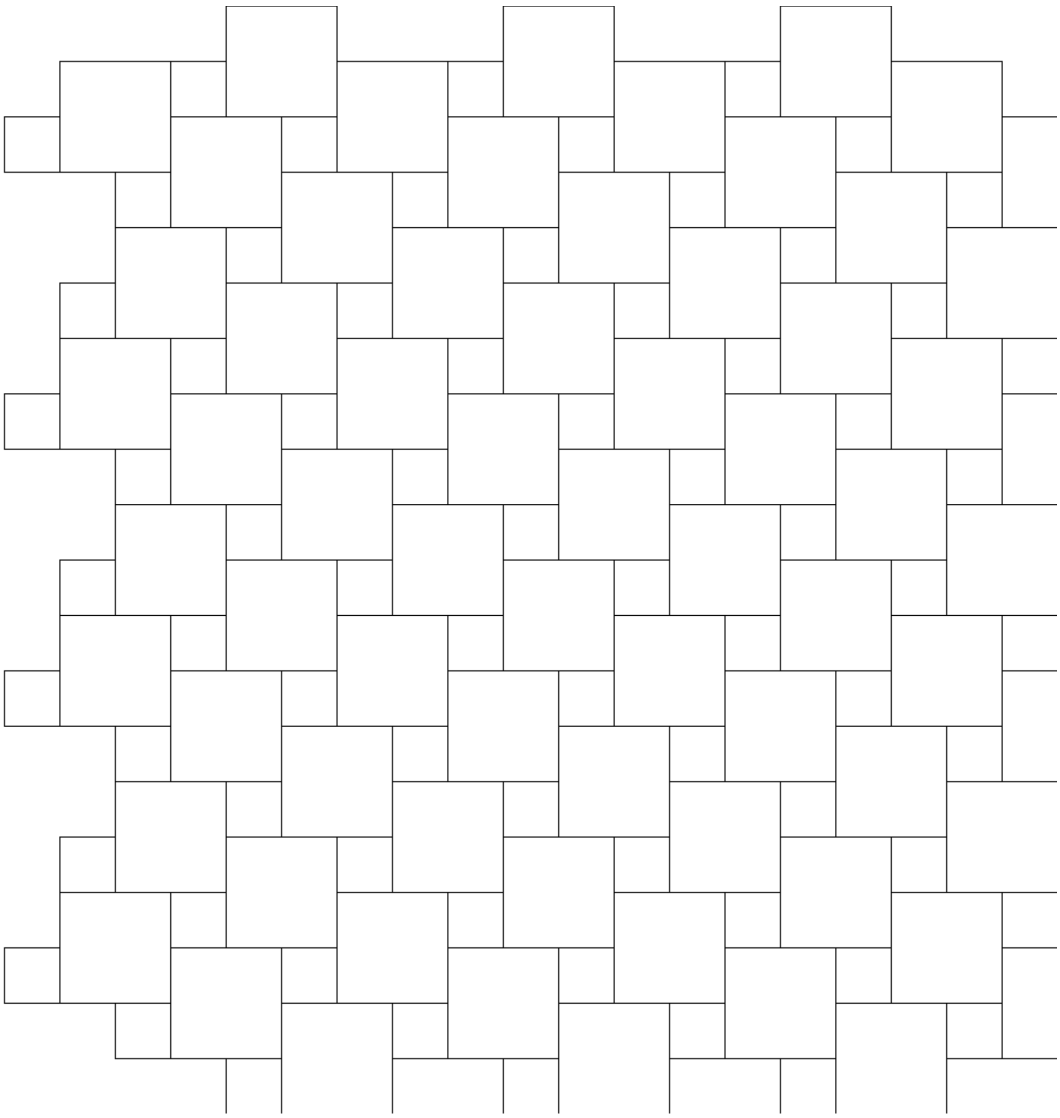
## 3. P4 Translations and quarter-turns

***Observing***

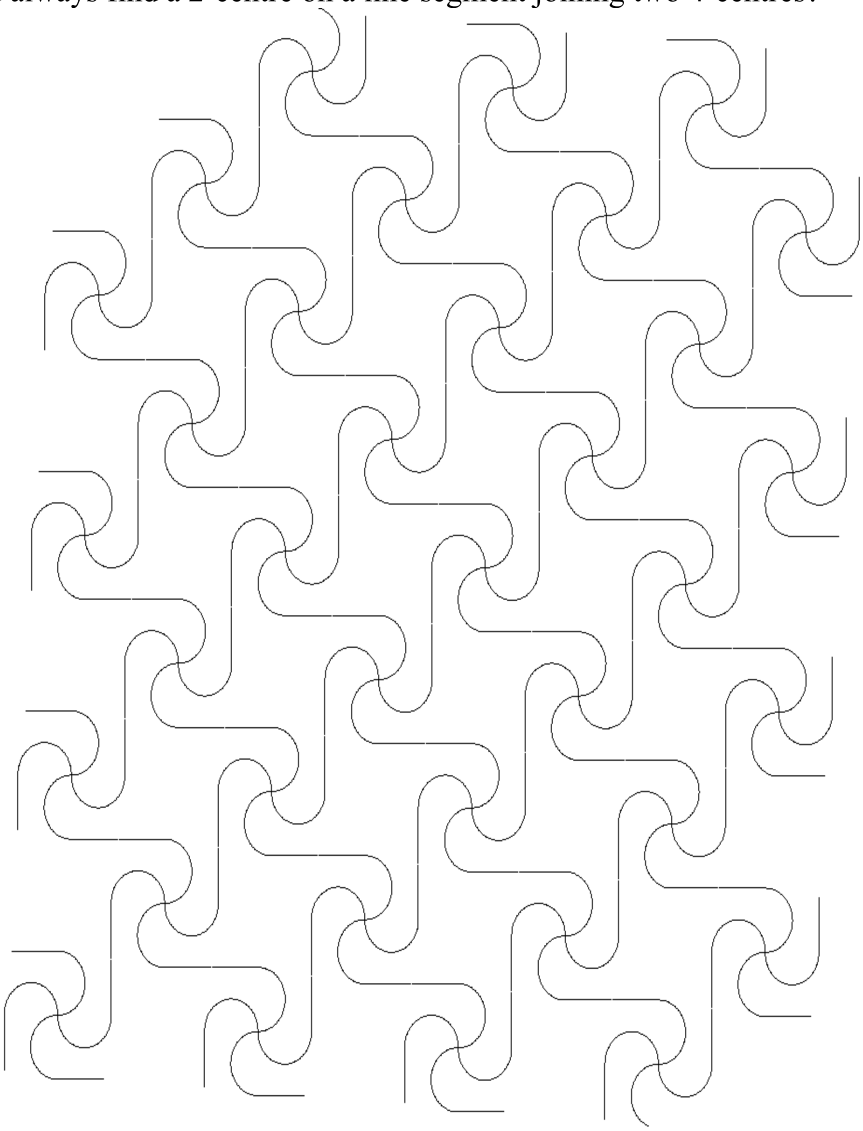


(i) Copy part of the two square tessellation onto tracing paper. Locate the points of one translation orbit. How would you describe the orbit?

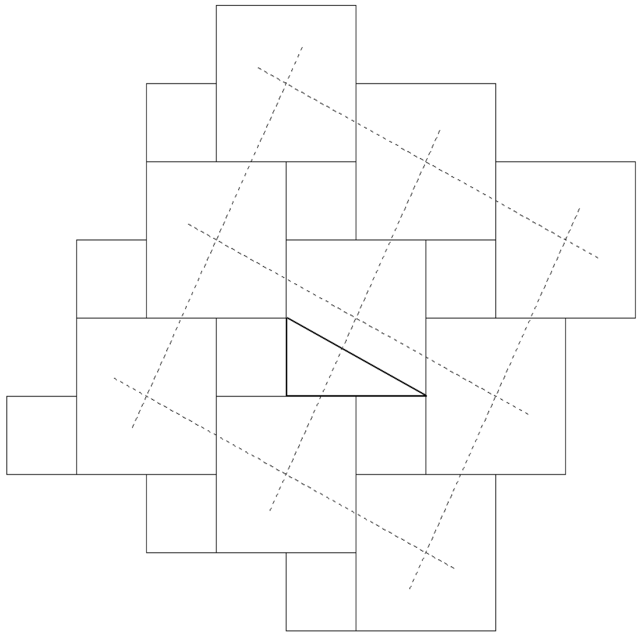
(ii) Are there centres of rotational symmetry through 90° for this pattern? Such centres are called ***4-centres*** for the pattern. **4** × 90 = 360. Locate as many as you can? Do they all lie in one translation orbit?

(iii) Are there 2-centres for this pattern? How many translation orbits of 2-centres are there?

(iv) Can you always find a 2-centre on a line segment joining two 4-centres?



(v) Repeat the exercises (i), (ii), (iii) and (iv) for this curvy cross tessellation.



(vi) In the two square tessellation here, lines have been drawn joining the centres of the large squares, giving, with dotted lines, a square tessellation covering the whole pattern. Using the method of **P1 *Sharpening* (iii),** compare the area of a large dotted square with that of two different smaller squares. A triangle has been marked with bold lines in this figure. Check that the length of the hypotenuse of that triangle is equal to the side of a large square, by a translation.

***Making***

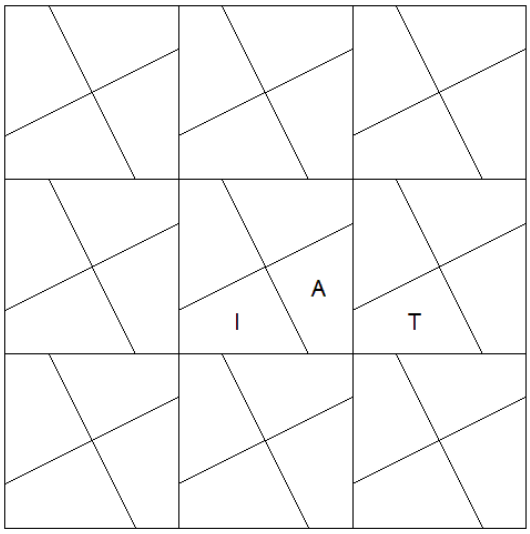
(i) Obtain a collection of identical square tiles. The ATM MAT tiles will do very well. Attempt to replicate the two square tessellation with these tiles in which one of the squares appears as a hole. Are there still 4-centres? How many translation orbits of 4-centres? Are there still 2-centres? How many translation orbits of 2-centres are there? If you change the length of the contact between two tiles, you can change the proportions of the square tile and the square hole to one another. Does this change the number of 4-centres?

(ii) In the two square tessellation, draw isosceles right angled triangles with 4-centres at each end of the hypotenuse. What is distinctive about the location of the right angles?

(iii) Copy a square which is a fundamental parallelogram for the tessellation of two squares, with 4-centres at the vertices. Mark other 4-centres and 2-centres inside or on the edges of this square. Does this drawing also fit a fundamental parallelogram for the curvy cross tessellation?

(iv) Make a Greek cross from five congruent square tiles stuck together. Make a tessellation from Greek crosses. Locate the 4-centres of this tessellation and find a fundamental parallelogram which is a square. Compare the area of this square with that of the Greek cross.

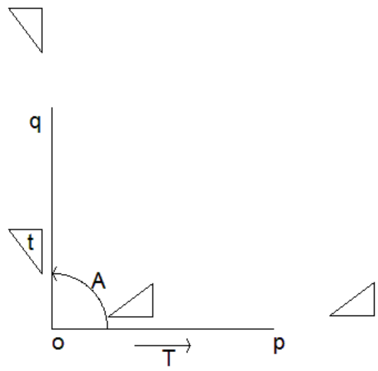
(v) In the diagram below the translation **T** carries the tile labelled **I** to the tile labelled **T**. The quarter-turn **A** carries the tile labelled **I** to the tile labelled **A**. Mark the remaining tiles in the diagram with combinations of **T** and **A** which give the translations, quarter-turns and half-turns which carry the tile labelled **I** to those tiles. Notice that $\mathbf{A}^4 = \mathbf{I}$ and $\mathbf{A}^2\mathbf{T} = \mathbf{T}^{-1}\mathbf{A}^2$ so there is some ambiguity about the labels. [Each ambiguity is a theorem about symmetries.]




Find the centres of the quarter-turns **AT** and **TA**.
The main point of this exercise is to convince yourself that every symmetry of the whole pattern may be expressed as some combination of **A**s and **T**s.

***Sharpening***

(i) If a pattern has a translation and a rotation, you might expect to be able to use the rotation to make a translation in a new direction. To explore this we suppose that a pattern has a rotation **A** with centre *o*, and a translation **T** which carries *o* to *p* (say). We further suppose that **A** carries *op* to *oq*.




We look carefully at the result of first doing $\mathbf{A}^{-1}$, then doing **T** and finally doing **A**.

$$\begin{matrix} & \mathbf{A}^{-1} & & \mathbf{T} & & \mathbf{A} & \\ o & \rightarrow & o & \rightarrow & p & \rightarrow & q \end{matrix}$$

So $\mathbf{A}^{-1}\mathbf{TA}$ takes *o* to *q*.
Now look at what $\mathbf{A}^{-1}\mathbf{TA}$ does to the little triangle marked "*t*".
Mark the triangles *u*, *v*, and *w* in the figure according to the following maps.

$$\begin{matrix} & \mathbf{A}^{-1} & & \mathbf{T} & & \mathbf{A} & \\ t & \rightarrow & u & \rightarrow & v & \rightarrow & w. \end{matrix}$$

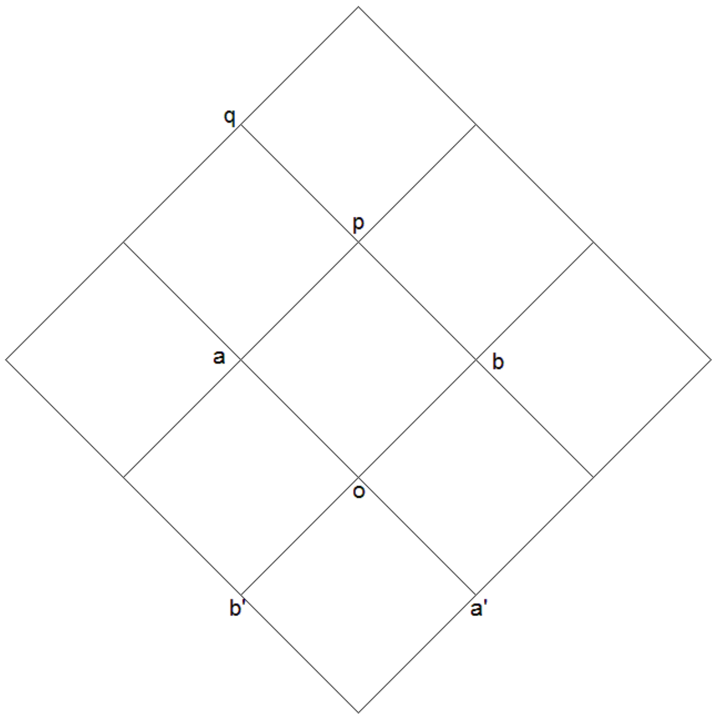
So $\mathbf{A}^{-1}\mathbf{TA}$ takes the triangle *t* to the triangle *w*. And since a symmetry is uniquely determined by a triangle and its image, we can say that the symmetry $\mathbf{A}^{-1}\mathbf{TA}$ is a translation through the same distance as **T**, but rotated through the angle of **A**.
[$\mathbf{A}^{-1}\mathbf{TA}$ is called the *conjugate* of **T** by **A**.]

(ii) If a wallpaper pattern has a shortest translation **T**, and a rotation **A** (not just through 180° or 360°), can you say why **T** and $\mathbf{A}^{-1}\mathbf{TA}$ between them must generate all the translations of the pattern? (If you are not sure, check with **P1 *Sharpening* (ii)**.) If **A** is a rotation through 90°, what is the shape of a translation orbit?

(iii) (Optional) Let **A** be an anti-clockwise rotation through 90° about a centre $a$ and let **B** be an anti-clockwise rotation through 90° about a centre $b$. Work out the product **AB** using the diagram below by determining the intermediate point in the sequences:

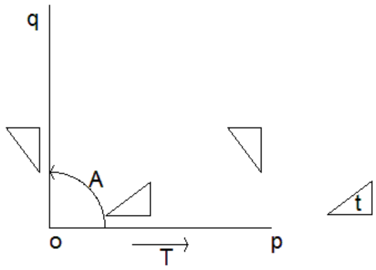
$$o \xrightarrow{\mathbf{A}} ? \xrightarrow{\mathbf{B}} o$$
$$a \xrightarrow{\mathbf{A}} ? \xrightarrow{\mathbf{B}} a'$$
$$b \xrightarrow{\mathbf{A}} ? \xrightarrow{\mathbf{B}} b'$$

and deduce the nature of the product **AB**. So a pattern with two 4-centres must have a 2-centre.




(iv) (Optional) You may like to work out the transformation **AT**.

(v) If a pattern has a translation and a rotation, you might expect to be able to use the translation to make a rotation about a new centre. To explore this we suppose that a pattern has a rotation **A** with centre $o$, and a translation **T** which carries $o$ to $p$ (say). We further suppose that **A** carries $op$ to $oq$.




We look carefully at the result of first doing $\mathbf{T}^{-1}$, then doing **A** and finally doing **T**.

$$\begin{array}{c} \quad \mathbf{T}^{-1} \quad \mathbf{A} \quad \mathbf{T} \\ p \rightarrow o \rightarrow o \rightarrow p \end{array}$$

So $\mathbf{T}^{-1}\mathbf{AT}$ takes $p$ to $p$.

Now look at what $\mathbf{T}^{-1}\mathbf{AT}$ does to the little triangle marked "$t$".

Mark the triangles $u$, $v$, and $w$ in the figure according to the following maps.

$$\begin{array}{c} \quad \mathbf{T}^{-1} \quad \mathbf{A} \quad \mathbf{T} \\ t \rightarrow u \rightarrow v \rightarrow w. \end{array}$$

So $\mathbf{T}^{-1}\mathbf{AT}$ takes the triangle $t$ to the triangle $w$. And since a symmetry is uniquely determined by a triangle and its image, we can say that the symmetry $\mathbf{T}^{-1}\mathbf{AT}$ is a rotation through the same angle as **A**, but with centre at $p$. [$\mathbf{T}^{-1}\mathbf{AT}$ is called the *conjugate* of **A** by **T**.] Since the translation **T** carries the centre of **A** to the centre of $\mathbf{T}^{-1}\mathbf{AT}$, it follows that every point in the translation orbit of a 4-centre is again a 4-centre.

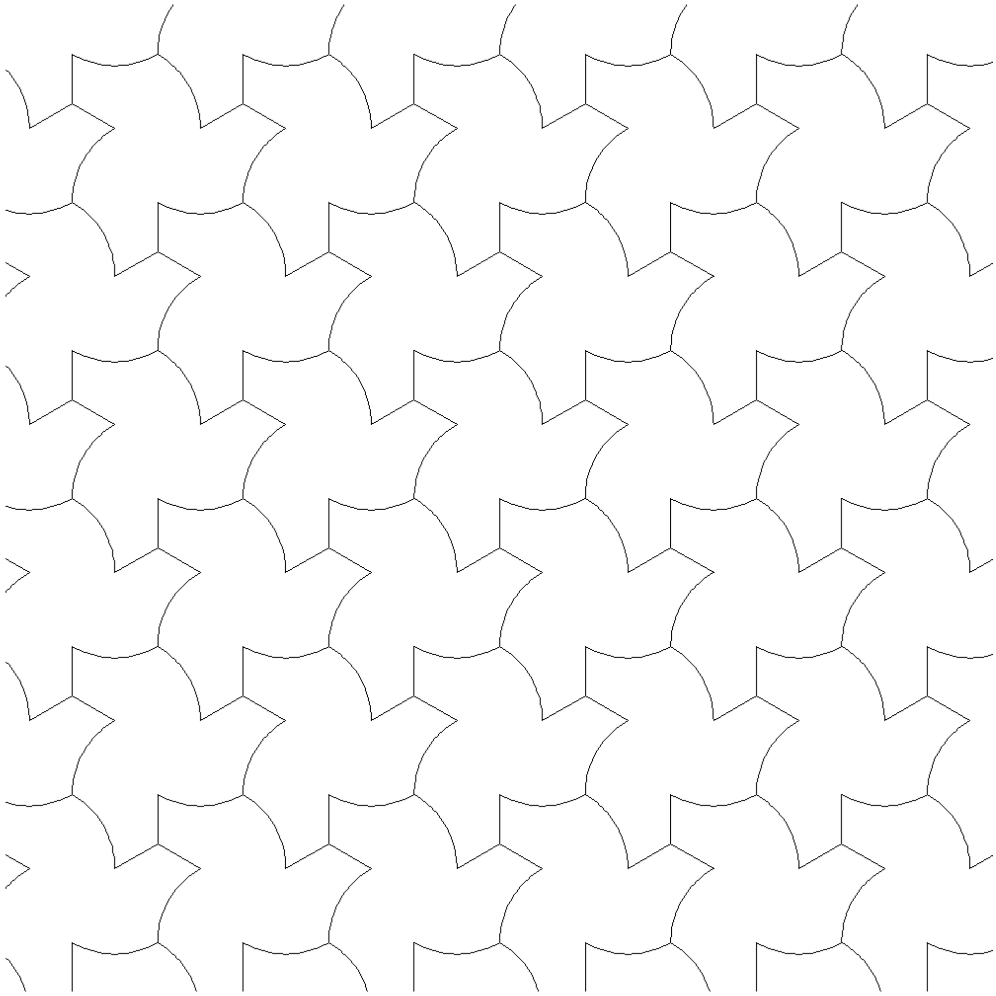
(vi) If **T** is a shortest translation of a wallpaper pattern and **A** is a quarter-turn of the same pattern, explain why the centres of the quarter-turns **A** and **AT** cannot lie in the same translation orbit.

***P4 Summary:*** *The translation orbit of a point in a wallpaper pattern with a rotation of 90° is a square lattice. There are two translation orbits of 4-centres and two translation orbits of 2-centres.*

*The Conway signature is* 442. *The 2-centres all belong to the same orbit in the full group.*

## 4. P3 Translations and 120° turns

***Observing***

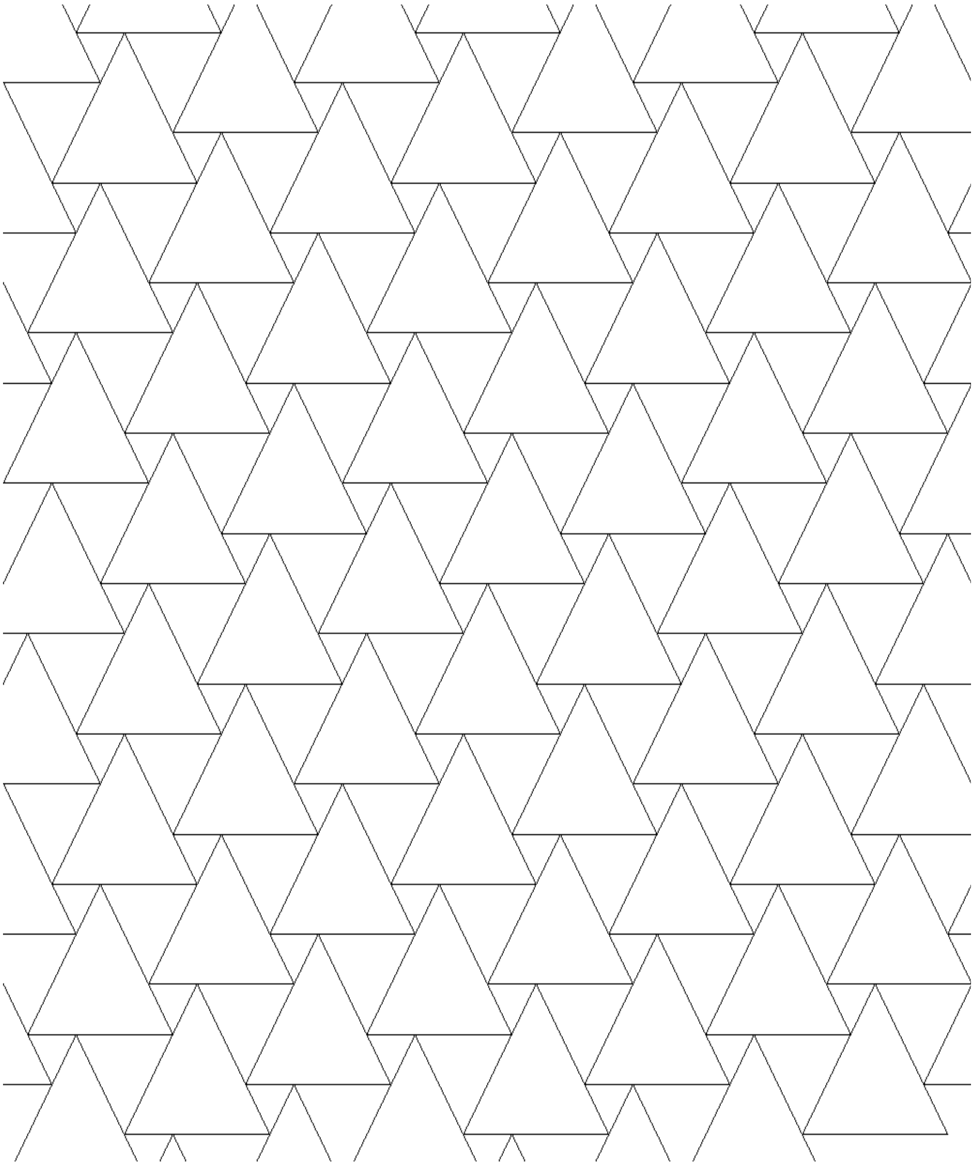


(i) Look at the pattern of fat legs running. Make a tracing of at least four of the tiles which are close to each other. Now use your tracing to locate the centre of a rotation of the pattern through 120°. Use your tracing to find copies of that centre under translations of the pattern. Mark all the points of the translation orbit of that centre. The centre is called a ***3-centre*** of the pattern. **3** × 120 = 360.

(ii) Join two of the 3-centres from the translation orbit that you have drawn, as near to each other as may be, and extend the line as far as you can. What do you notice? If two 3-centres you used were *A* and *B*, look for another 3-centre in this orbit as near to *A* as possible, but not along the line *AB*, and call it *C*. Complete the parallelogram *ABCD*. Is *D* a 3-centre ? Describe the shape and angles of *ABCD*. If *ABCD* is copied by the translations of the pattern, do its images cover the whole plane?

(iii) Look for another 3-centre which is not in the translation orbit you have marked. Mark the points of this new translation orbit differently (say with a different colour). Is the translation orbit for this centre like the translation orbit that you have already drawn in (i); does it have the same shape? Is there a member of this orbit inside the parallelogram *ABCD*? If so, where exactly does it lie?

(iv) Look for yet another 3-centre, not belonging to either of the two translation orbits you have marked. If you can find such a one, mark the points of its translation orbit differently from the orbits in (i) and (ii).



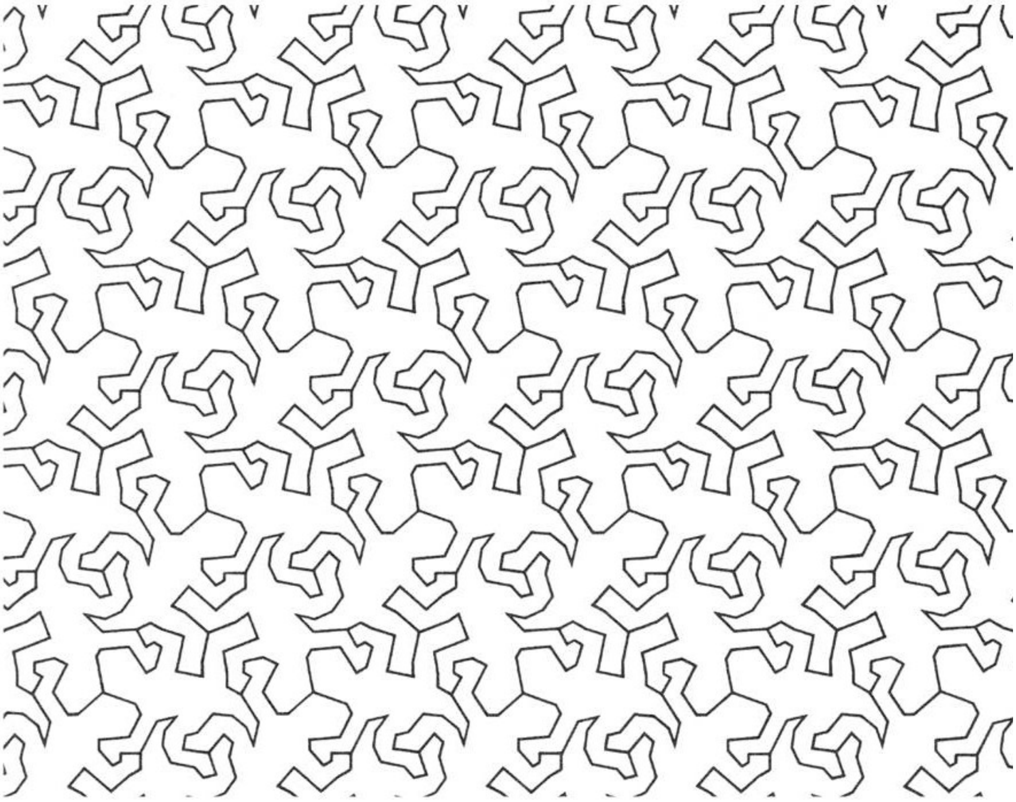
(v) Look at the tessellation of three triangles. Look for 3-centres and then orbits of 3-centres.
If you make a fundamental parallelogram from the 3-centres of one translation orbit, where do members of the other orbits lie inside that parallelogram?

***Making***

(i) Compare the area of a tile with the area of a fundamental parallelogram: for example the fat legs with the parallelogram *ABCD*. You can use the method of **P1 *Sharpening* (iii)**.

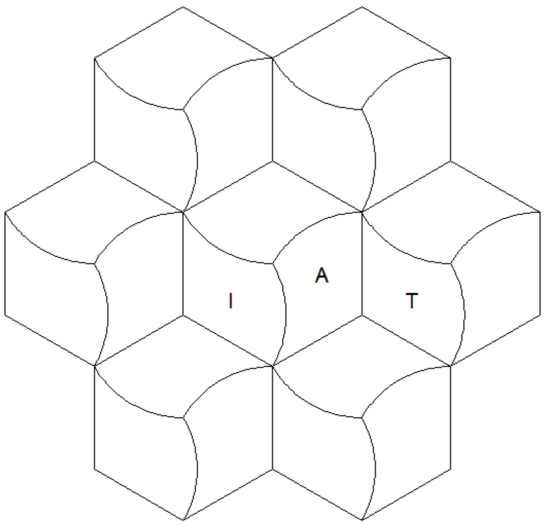
(ii) Compare the combined area of the three triangles with that of a fundamental parallelogram for that pattern.

(iii) Obtain a collection of congruent equilateral triangles. The ATM MAT tiles will do very well. Try replicating the 3 triangle tessellation with the tiles (two of the triangles will appear as holes). In the printed version above, the ratio of the sides was 1 : 2 : 3. Adjust the contact points of the large triangles to change that ratio. Are there still 3-centres? Choose a fundamental parallelogram of 3-centres. How are the other 3-centres located in the parallelogram?



(iv) Find 3-centres in the tessellation of lizards. How does the area of a lizard compare with the area of a fundamental parallelogram?

(v) In the diagram below the translation **T** carries the tile labelled **I** to the tile labelled **T**. The 120° rotation **A** carries the tile labelled **I** to the tile labelled **A**. Mark the remaining tiles in the diagram with combinations of **T** and **A** which give the translations, 120° and 240° rotations which carry the tile labelled **I** to those tiles. Notice that $\mathbf{A}^3 = \mathbf{I}$ and $(\mathbf{AT})^2 = \mathbf{T}^{-1}\mathbf{A}^{-1}$ so there is some ambiguity about the labels.




Check that $\mathbf{A}^{-1}\mathbf{TA}$ is a translation through the same length as **T**, and that the centre of the rotation $\mathbf{T}^{-1}\mathbf{AT}$ is the image of the centre of **A** under the translation **T**. Find the centres of the rotations **AT** and **TAT**.

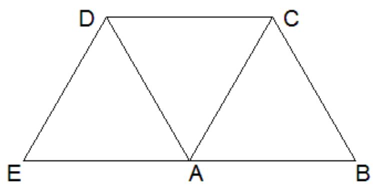
***Sharpening***

(i) If **A** is a 120° anti-clockwise rotation and **T** a shortest translation of a wallpaper pattern, then $\mathbf{A}^{-1}\mathbf{TA}$ is a translation through the same distance as **T**, but in a direction inclined at 120° to that of **T**. The argument of **P4 *Sharpening* (i)** holds even when the angle of rotation of **A** changes from 90° to 120°. So these two translations generate all the translations of the pattern, by **P1 *Sharpening* (ii)**.

(ii) Decide what the shape of the fundamental parallelogram with shortest sides must be.

(iii) If **A** is a 120° rotation and **T** a translation, then $\mathbf{T}^{-1}\mathbf{AT}$ is a rotation through the same angle as **A,** and the distance between the centres of **A** and $\mathbf{T}^{-1}\mathbf{AT}$ is the length of the translation **T.** The argument of **P4 *Sharpening* (iv)** holds even if the angle of **A** changes from 90° to 120°. Thus all the rotations with centres at the points of the translation orbit of the centre of **A** are generated by **A** and **T**.

(iv) If *ABCD* is a fundamental parallelogram with vertices in the translation orbit of the centre of **A**, and with the shortest possible side lengths, the rotation **A** has centre *A* and **A** carries *B* to *D*, and the length of the translation **T** is *AB*. Verify that the transformation **AT** is a rotation through an angle 120° with centre at the centroid of *ABC*, by filling in the question marks in the detail of the transformations below.




$$A \xrightarrow{\mathbf{A}} ? \xrightarrow{\mathbf{T}} B$$

$$B \xrightarrow{\mathbf{A}} ? \xrightarrow{\mathbf{T}} C$$

$$C \xrightarrow{\mathbf{A}} ? \xrightarrow{\mathbf{T}} A$$

So the transformation **AT** rotates the vertices of the triangle *ABC* through 120°.

(v) Deduce that $\mathbf{CT}^{-1}$ is a rotation through an angle 120° with centre at the centroid of *ACD*, where **C** is an anti-clockwise rotation with centre *C* through an angle 120°.

(vi) How do you know that the centres of the 120° turns, **A**, **AT** and $\mathbf{CT}^{-1}$ all lie in different translation orbits?

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***P3 Summary:*** *The translation orbit of a point in a wallpaper pattern with a rotation of 120° is a 60° rhombic lattice. There are three translation orbits of 3-centres.*

*The Conway signature is* 333.

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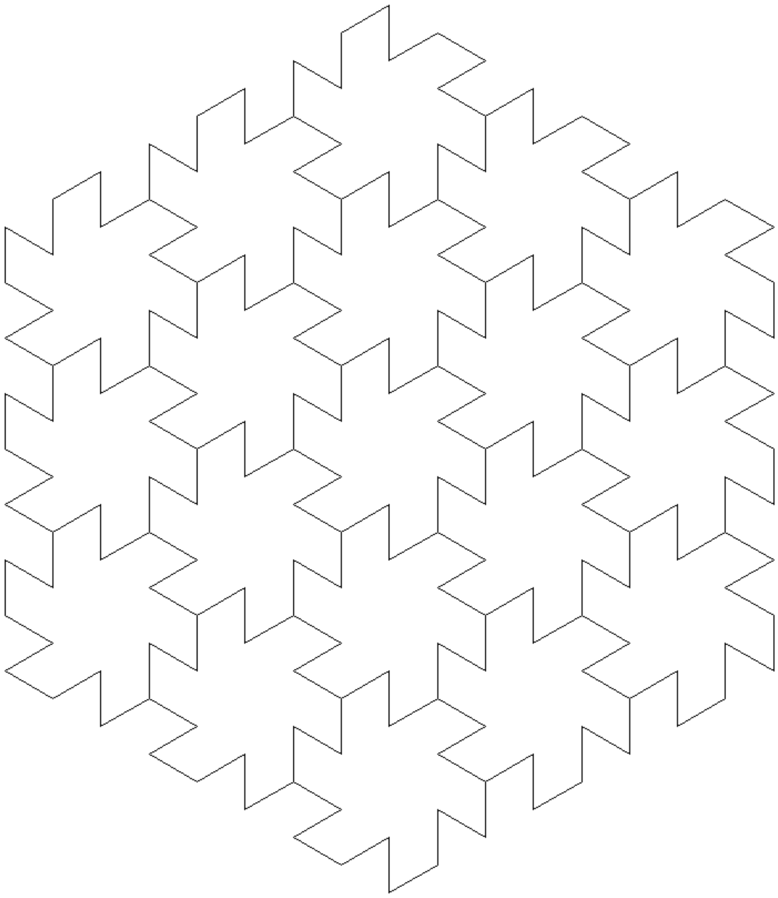
## 5. P6 Translations and 60° turns

***Observing***



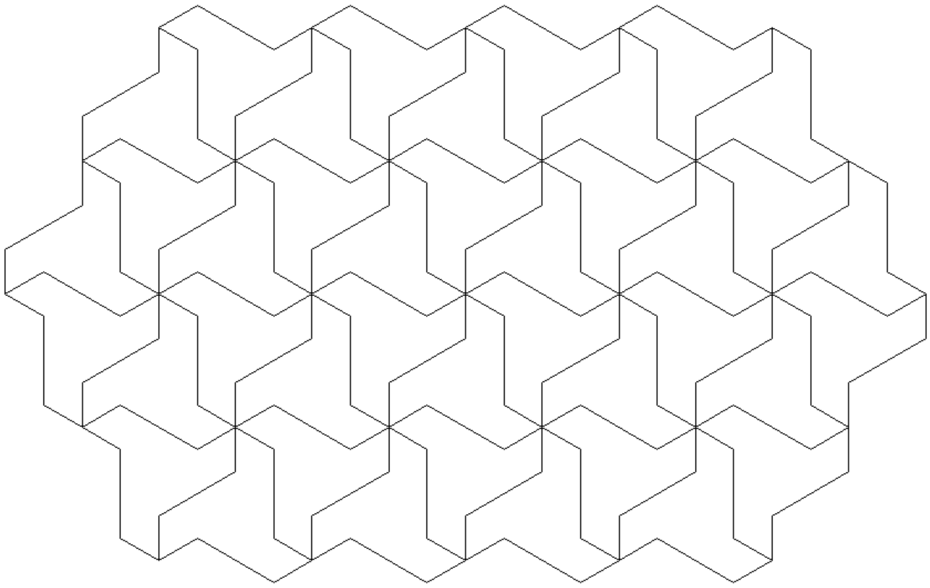
(i) Look at the pattern of cog wheels. Make a tracing of at least four of the wheels which are close to each other. Now use your tracing to locate the centre of a rotation of the pattern through 60°. Use your tracing to find copies of that centre under translations of the pattern. Mark all the points of the translation orbit of that centre. The centre is called a ***6-centre*** of the pattern. **6** $\times$ 60 = 360.

(ii) Join two of the 6-centres from the translation orbit that you have drawn, as near to each other as may be, and extend the line as far as you can. What do you notice? If two 6-centres you used were $A$ and $B$, look for another 6-centre in this orbit as near to $A$ as possible, but not along the line $AB$, and call it $C$. Complete the parallelogram $ABCD$. Is $D$ a 6-centre? Describe the shape and angles of $ABCD$. If $ABCD$ is copied by the translations of the pattern, do its images cover the whole plane?

(iii) Are all the 6-centres in the same translation orbit? Look for 3-centres which are not 6-centres. Where are the 3-centres located within the parallelogram $ABCD$? Is this where you might have expected from what we found with **P3**?

(iv) Look for 2-centres which are not 6-centres. What is the shape of the smallest triangle you can find with a 2-centre, a 3-centre and a 6-centre as vertices?

(v) Decide what the shape of a fundamental parallelogram with shortest sides must be?



(vi) Try observations (i) – (v) on this new tessellation.

***Making***

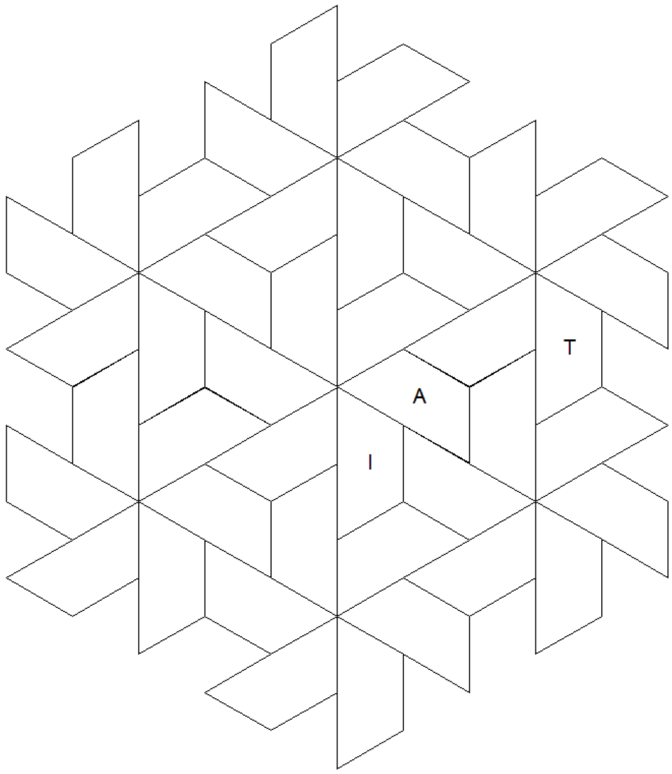
(i) How does the area of a fundamental parallelogram compare with the area of a cog wheel in the pattern of ***Observing*** **(i)**?

(ii) How do the areas of the bent triangles in ***Observing*** **(vi)** compare with that of a fundamental parallelogram?

(iii) Obtain some congruent regular hexagon tiles such as ATM MATs. Make a pattern from them with adjacent hexagons sharing half an edge, with small triangular spaces between the hexagons. Does this pattern have the same symmetry as the **P6** patterns we have investigated?

(iv) Use your congruent regular hexagons as in (iii) and congruent equilateral triangle tiles with the same length of side to make a **P6** pattern with triangles surrounding each hexagon. Again ATM MATs are suitable here. The overall pattern should not have any reflection symmetry.

(v) In the diagram below the translation **T** carries the tile labelled **I** to the tile labelled **T**. The 60° rotation **A** carries the tile labelled **I** to the tile labelled **A**. Mark the remaining tiles in the diagram with combinations of **T** and **A** which give the translations, the 60° and 120° rotations and half-turns which carry the tile labelled **I** to those tiles. Notice that $\mathbf{A}^6 = \mathbf{I}$ and $\mathbf{A}^3\mathbf{T} = \mathbf{T}^{-1}\mathbf{A}^3$ so there is some ambiguity about the labels.




Check that $\mathbf{A}^{-1}\mathbf{T}\mathbf{A}$ is a translation. How does its length compare with that of **T**? Check that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is a 60° rotation. Is its centre the image of the centre of **A** translated by **T**? What is the transformation **AT**?

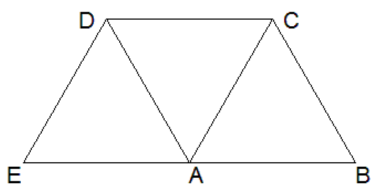
***Sharpening***

(i) If **A** is a 60° anti-clockwise rotation and **T** a shortest translation of a wallpaper pattern, then $\mathbf{A}^{-1}\mathbf{T}\mathbf{A}$ is a translation through the same distance as **T**, but in a direction inclined at 60° to that of **T**. The argument of **P4 *Sharpening* (i)** holds even when the angle of rotation of **A** changes from 90° to 60°. So these two translations generate all the translations of the pattern, by **P1 *Sharpening* (ii)**.

(ii) Decide what the shape of the fundamental parallelogram with shortest sides must be.

(iii) If **A** is a 60° rotation and **T** a translation, then $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is a rotation through the same angle as **A,** and the distance between the centres of **A** and $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is the length of the translation **T.** The argument of **P4 *Sharpening* (iv)** holds even if the angle of **A** changes from 90° to 60°. Thus all the rotations with centres at the points of the translation orbit of the centre of **A** are generated by **A** and **T**.

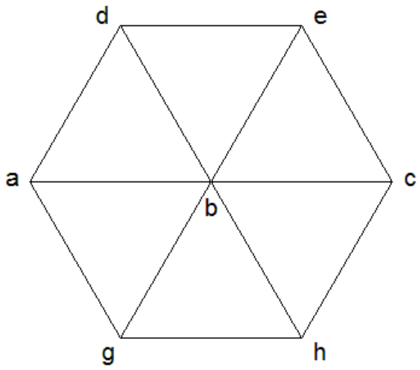
(iv) If *ABCD* is a fundamental parallelogram with vertices in the translation orbit of the centre of **A** (a rotation through 60°), and with the shortest possible side lengths, the rotation **A** has centre *A* and **A** carries *B* to *C*, and the length of the translation **T** is *AB*. Verify that the transformation **AT** is a rotation through an angle 60° with centre at *C*, by filling in the question marks in the detail of the transformations below.



$$D \xrightarrow{\mathbf{A}} ? \xrightarrow{\mathbf{T}} A$$
$$A \xrightarrow{\mathbf{A}} ? \xrightarrow{\mathbf{T}} B$$
$$C \xrightarrow{\mathbf{A}} ? \xrightarrow{\mathbf{T}} C$$

(v) Deduce that $\mathbf{A}^2\mathbf{T}$ is a rotation through an angle 120° with centre at the centroid of $ABC$. What is $\mathbf{A}^3\mathbf{T}$?

(vi) If **A** and **B** are any two anti-clockwise rotations through 60° of the same wallpaper pattern and we can find a translation of the pattern carrying the centre of one to the centre of the other, we will have shown that all the 6-centres of a wallpaper pattern lie in the same translation orbit.




We explore the transformation $\mathbf{BAB}^{-2}$. In the figure $a$ is the centre of **A**, $b$ is the centre of **B** and a regular hexagon has been constructed with centre at $b$, and $a$ as one vertex. Show that $\mathbf{BAB}^{-2}$ carries $a$ to $b$, $d$ to $e$ and $g$ to $h$, and is a translation because of its effect on the triangle $adg$.

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***P6 Summary:*** *A translation orbit of a point in a wallpaper pattern with a rotation of 60° is a 60° rhombic lattice. There is one translation orbit of 6-centres. There are two translation orbits of 3-centres and three translation orbits of 2-centres.*

*The Conway signature is* 632.

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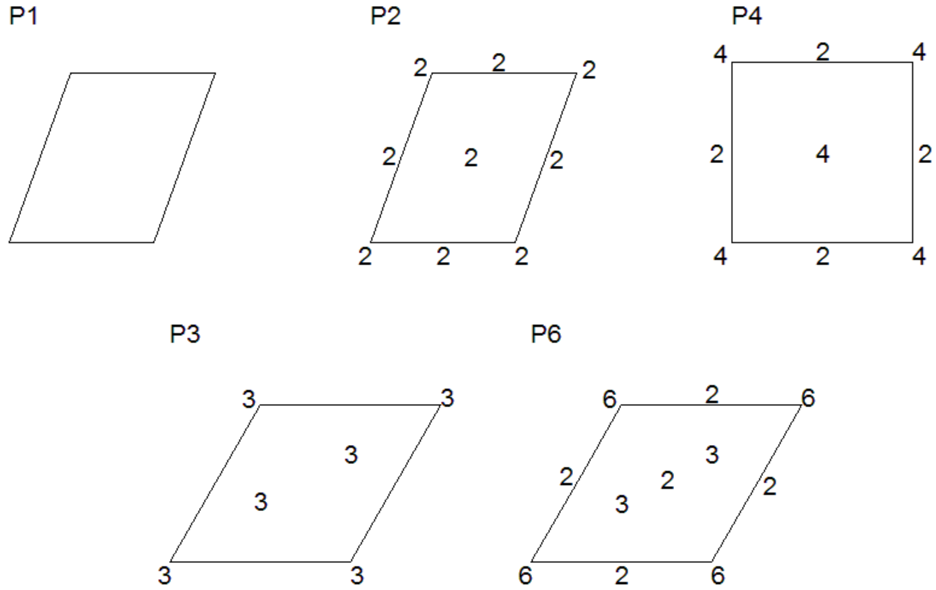
## 6. The Crystallographic restriction

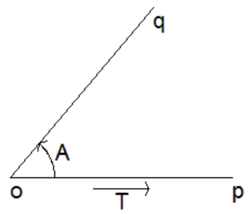
### Fundamental parallelograms




The observing part of this section is the previous five sections. We have examined wallpaper patterns all of which have translations, but the only rotations we have used are through multiples of the angles 60°, 90°, 120° and 180°. Are these really the only possibilities?

If you try making patterns with another angle of rotation, you will find that your group of translations disappears. So why?

(i) Let us suppose we have a wallpaper pattern with a rotation symmetry **A** through an angle less than 60°, and we let **T** be a shortest translation of the pattern.
In the figure, $o$ is the centre of **A**, **T** carries $o$ to $p$, and **A** carries $p$ to $q$.




We found in **P4 *Sharpening* (i)**, that $\mathbf{A}^{-1}\mathbf{TA}$ was a translation carrying $o$ to $q$.

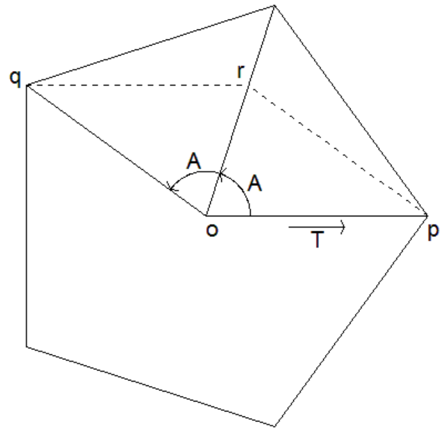
$$p \xrightarrow{\mathbf{T}^{-1}} o \xrightarrow{\mathbf{A}^{-1}\mathbf{TA}} q$$

So $\mathbf{T}^{-1}\mathbf{A}^{-1}\mathbf{TA}$ carries $p$ to $q$. But both $\mathbf{T}^{-1}$ and $\mathbf{A}^{-1}\mathbf{TA}$ are translations, so their product is a translation by **P1 *Sharpening* (i)**. Now the triangle $opq$ is isosceles, and the angle at $o$ is less than 60°, so $pq < op = oq$. So $pq$ is impossibly short for a translation.
So we cannot have a rotation through less than 60° in a wallpaper pattern.

**Exercise**. By repeating a rotation through 61° six times, show that a wallpaper pattern may not have a rotation through 61°. Can you forbid some other angles like this? What about 70°?

(ii) Might a wallpaper pattern have a 5-centre?

We suppose that we have a wallpaper pattern with a rotation **A** through 72°, and a shortest translation **T**. We let $o$ be the centre of **A**, we let **T** carry $o$ to $p$ and we let $\mathbf{A}^2$ carry $p$ to $q$.




As in **P4 *Sharpening* (i)**, $\mathbf{A}^{-2}\mathbf{T}\mathbf{A}^2$ is a translation with the same length as **T** which carries $o$ to $q$. We also let **T** carry $q$ to $r$.

$$\begin{array}{ccccc} & \mathbf{A}^{-2}\mathbf{T}\mathbf{A}^{2} & & \mathbf{T} & \\ o & \rightarrow & q & \rightarrow & r. \end{array}$$

Now both $\mathbf{A}^{-2}\mathbf{T}\mathbf{A}^2$ and **T** are translations, so their product $\mathbf{A}^{-2}\mathbf{T}\mathbf{A}^2\mathbf{T}$ is a translation, by **P1 *Sharpening* (i)**, carrying $o$ to $r$. But $oqr$ is an isosceles triangle and the angle at $q$ is 36°, so $or < oq = op$. So this translation is impossibly short.

**Exercise**. Choose any angle between 60° and 180° other than 90° or 120° and show that it cannot be the angle of a rotational symmetry of a wallpaper pattern.

***Summary:*** *Rotation symmetries of a wallpaper pattern may only be through angles of 60°, 90°, 120°, 180°, 240°, 270° or 300°.*

As a final comment on translations and rotations in wallpaper patterns we note that since both translations and rotations preserve orientation, that is, map clockwise to clockwise, repeating a mixture of them any number of times can only result in translations and rotations, and never reflections or glide-reflections which always convert clockwise into anti-clockwise. In any group of plane symmetries, the translations and rotations, together with the identity, form a subgroup.