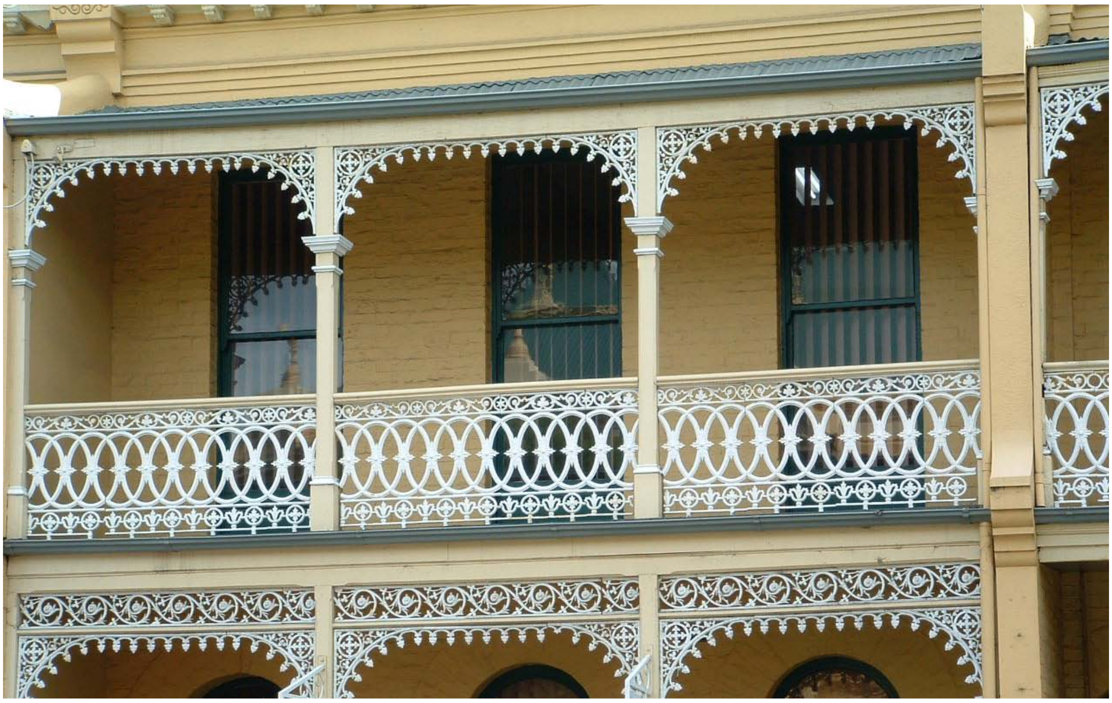
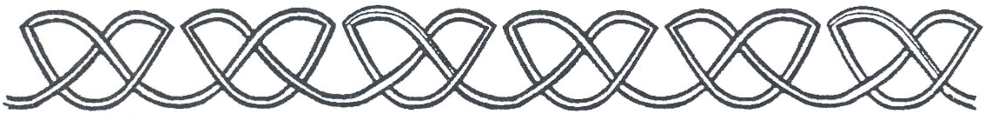
Sorting by Symmetry

Patterns along a Line

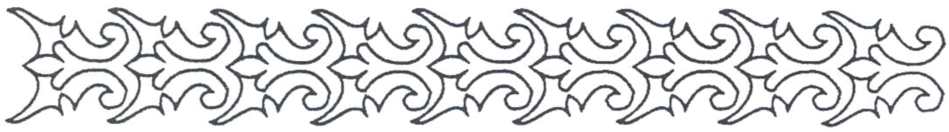


Bob Burn

Celtic Designs



Hungarian Designs



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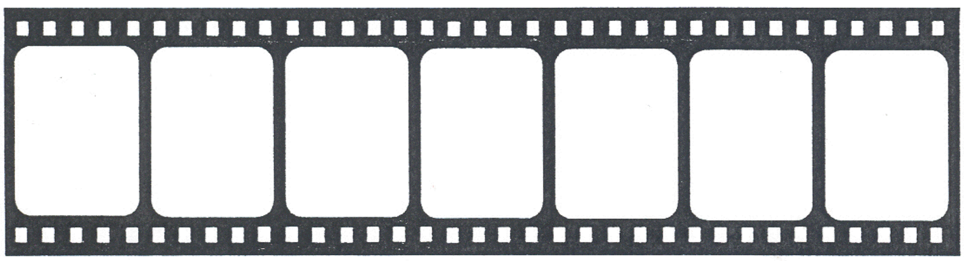

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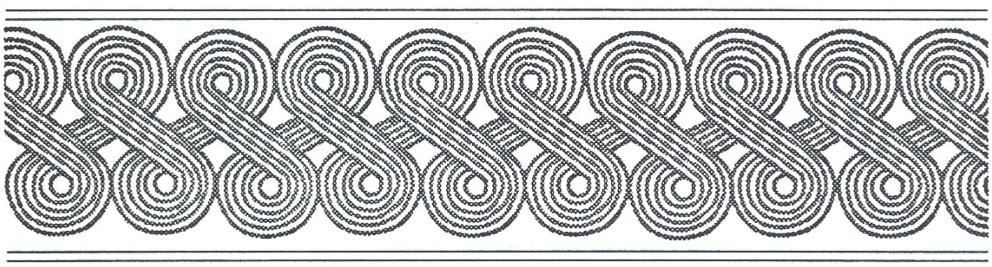

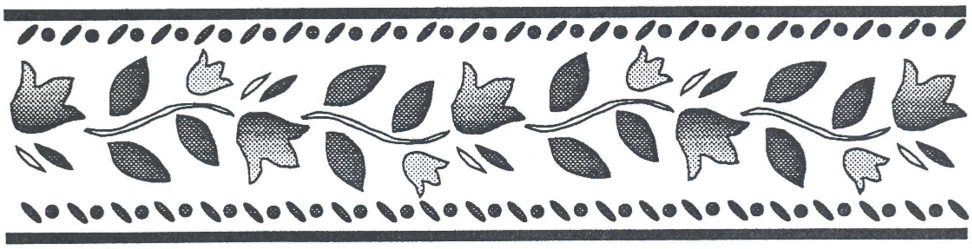

Sorting by Symmetry
Patterns along a Line

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This booklet is a sequel to *Sorting by Symmetry*, *Patterns with a Centre* (ATM, 2005). As with the earlier book, you will need squared paper, tracing paper and hand mirrors for the suggested activities. Access to a dynamic geometry package and to a version of LOGO will also be useful. (MSWLogo may be freely downloaded from the internet.) In *Patterns with a Centre*, reflections and rotations with a common fixed point were explored, and were combined to form cyclic and dihedral patterns. Although the main result in *Patterns with a Centre* is not used in this booklet, some basic ideas about symmetry which were developed in *Patterns with a Centre* will be used repeatedly here. <u>Symmetry</u> was defined in Section **9** of that booklet and <u>symmetry group</u> in Section **10** (all the symmetries of a pattern). The effect of a reflection on orientation was investigated in Section **7** (interchanging clockwise with anti-clockwise). Our understanding of half-turns in this book depends on the result in Section **8** that combining two reflections with intersecting axes results in a rotation through twice the angle between the axes.

In *Patterns with a Centre* all the figures considered were bounded, that is, contained within some circle. Only the two diagrams 13.3 and 13.4 in *Patterns with a Centre* were intended to indicate a process of construction of unlimited size. In *Patterns along a Line* every pattern that we consider is infinite in extent, so only a part of each one can be shown, and the reader has to think “and so on” to understand the patterns.

References to *Centre* **6**, for example, refer to section **6** of the earlier booklet *Sorting by Symmetry*, *Patterns with a Centre*. The style of both booklets, with each section shaped by **Observing**, **Making**, and **Sharpening** was described in the **Introduction** to *Sorting by Symmetry*, *Patterns with a Centre*.

Patterns along a Line is structured as follows.

Sections 1 – 5

A pair of parallel mirrors is used to introduce translation, the symmetry common to all the patterns in this booklet. Then we find how a translation combines with a reflection in a perpendicular axis and also with a half-turn. We study our first three types of frieze pattern and have a glimpse of some more.

Sections 6 – 7

We find how a translation combines with a reflection with axis along the translation, and we identify a new kind of symmetry, the glide-reflection. We study two new types of frieze pattern.

Section 8

By examining the fixed lines of symmetries we can decide which ones may be symmetries of a frieze pattern: translations, glide-reflections, half-turns and reflections in axes along and perpendicular to the translation.

Sections 9 – 10

We find two more ways to combine symmetries to make a frieze group, that is, the symmetry group of a frieze pattern.

Sections 11 – 12

We find how the seven frieze groups are built up and how the seven types are related to one another.

Sections 13 – 14

We see that every symmetry of the plane is either a reflection, a rotation, a translation or a glide-reflection. So our examination of the possible symmetries in the group of a frieze pattern in Section **8** was complete.

1 Parallel mirrors

In which we see how reflections in two parallel mirrors may be combined to make a translation.

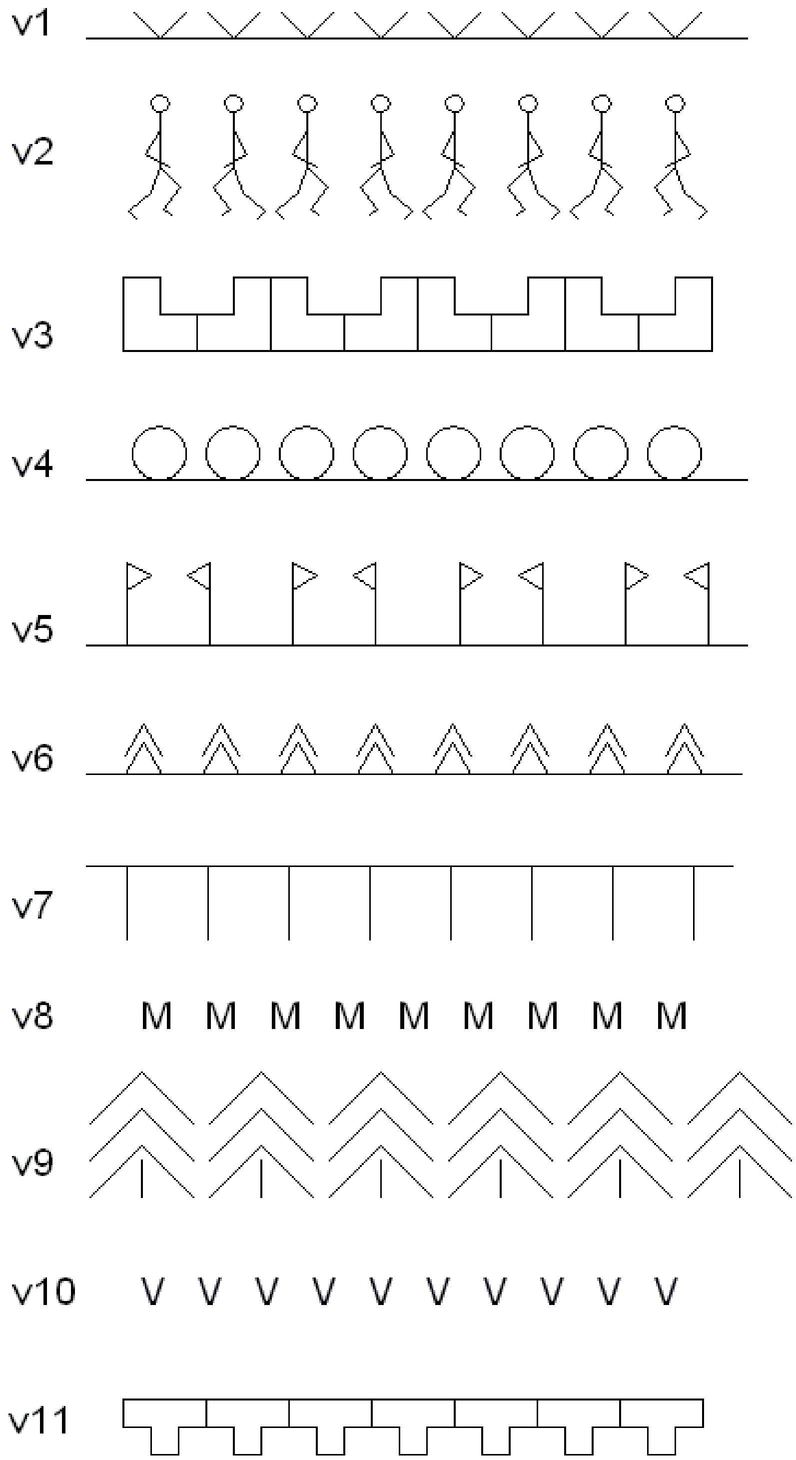

Figure 1.1

(a) Observing

- We will start by getting as close as we can to observing an infinite pattern. Get hold of two hand mirrors. Hold them with reflecting surfaces towards each other, perpendicular to a flat piece of paper and parallel to each other. Place some unsymmetrical object between them and then look at the reflection of the object, the reflection of the reflection, the reflection of the reflection of the reflection, and so on. If you have managed to get the two mirrors parallel, the images of the object will appear along a line. Look carefully. This is your one chance to observe the real thing, the kind of infinite pattern that this booklet is about.
- Cover all but one of the patterns **v1 – v11** in figure 1.1. For the one pattern showing, use a hand mirror to find as many mirror lines for this pattern as you can, presuming that the pattern continues indefinitely to left and right. Figure 1.2 shows the mirror lines of **v10**.
- For each of the patterns **v1 – v11**, mark all the mirror lines on the part of the pattern which is shown, with a pencil, still presuming that the pattern continues indefinitely to left and right. Are all the mirror lines parallel? Are they equally spaced? (The letter **v**, used in naming the patterns in Figure 1.1, is the first letter of 'vertical'.)

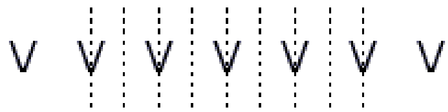

Figure 1.2

(b) Making

- Cut a piece of A4 in half, lengthways. Using one of these long thin strips, fold it to halve the length and press the fold firmly. Fold it to halve the length again, and press the fold firmly. Halve the length again by one more fold, then either using a hole-puncher or a pair of scissors, cut out a piece from one of the folded edges, or punch a couple of holes at different distances from the edge. Unfold the paper. Look at the overall pattern. If it were to be extended to left and right, would it have mirror lines? Would they be parallel? Are the fold lines the mirror lines?
- Fold half a sheet of A4, as above. Then unfold, and refold using the same fold lines but folding the paper like a concertina, so that the folds come alternately left, right, left, right, … . Again punch holes or cut out with scissors, and unfold to examine the pattern. Where are the mirror lines? Some people can make a row of dancers this way.
- Explore the combination of two reflections with parallel axes with dynamic geometry software. Start with a triangle (or any polygon) and two parallel mirrors, **V** and **W**. Show the image of the triangle under **V**, **VW**, **VWV**, **VWVW**,… and under **W**, **WV**, **WVW**, **WVWV**, …

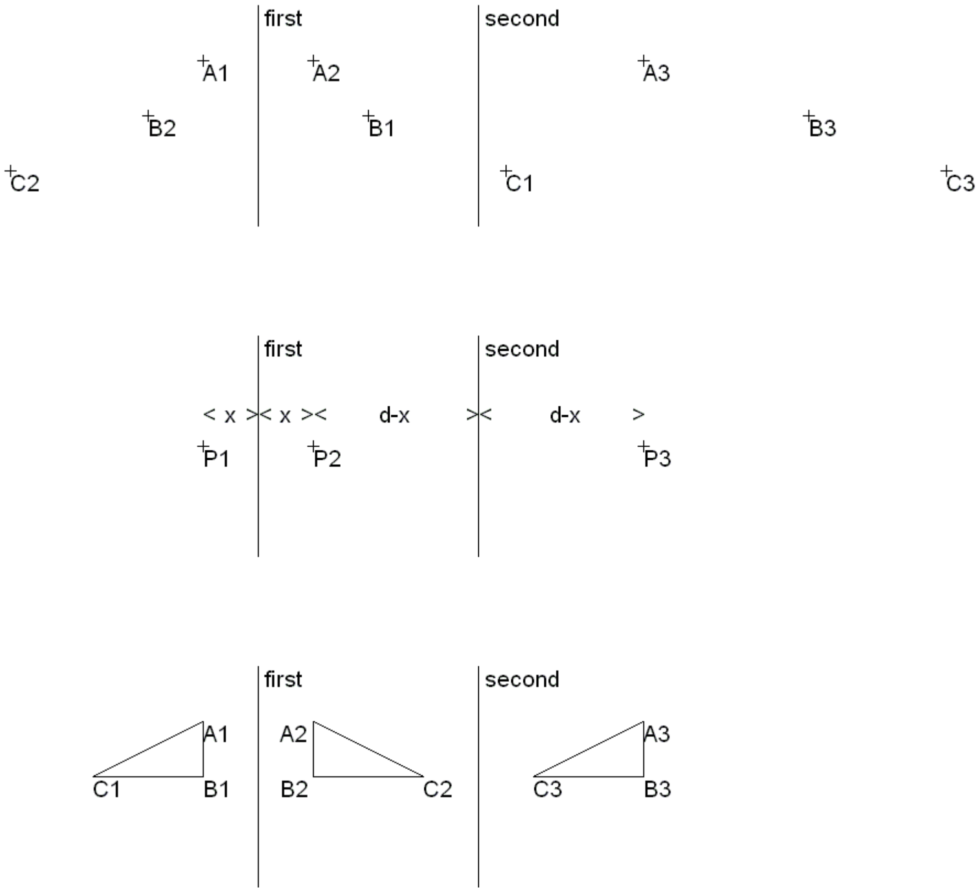

Figure 1.3

(c) Sharpening

We are going to investigate parallel mirrors, or rather, reflections with parallel axes, in a manner similar to the investigation in *Centres* **8(b)**, to find out what we get when reflections in two parallel axes are combined. Get some squared paper and mark two parallel grid lines boldly, naming them **first** and **second** respectively. Don't take them too far apart.

Take the lines **first** and **second** parallel to the short edge of the paper. Choose a square on your paper and label it 1. Find the reflection of 1 in **first** and label it 2. Then find the reflection of 2 in **second** and label it 3. The combination of the reflections in **first** and **second** takes 1 to 3. Now choose a grid point and label it A_1. See Figure 1.3. (It is best to start with one *not* on the lines labelled **first** or **second**, though such points should be considered later, when you have some confidence about what is going on). Find the reflection of A_1 in **first** and label it A_2. Find the reflection of A_2 in **second** and label it A_3. Do A_1, A_2 and A_3 lie on a line? What is the angle between the line A_1A_3 and the parallel lines **first** and **second**? Start again with a different grid point on the same squared paper and label it B_1. Find the reflection of B_1 in **first** and label it B_2. Find the reflection of B_2 in **second** and label it B_3. Continue building up such triples of points until you think you can describe the transformation which carries P_1 (*via* P_2) to P_3. How does the length P_1P_3 compare with the distance between the **first** and **second** axes? This investigation may be done with a dynamic geometry package.

Draw two parallel lines (**first** and **second**) on plain paper. Take the distance between them as d and calculate the distances of P_2 from **first** and from **second**, and the distance of P_3 from **second**, given that P_1 is distant x from **first**. What is the length of P_1P_3? See figure 1.3.

Combining two reflections: When two reflections in parallel axes are combined, the result is a movement perpendicular to the axes (called a **translation**), and through twice the distance between the axes.

2 Translations

In which we look at patterns whose only symmetries are translations.

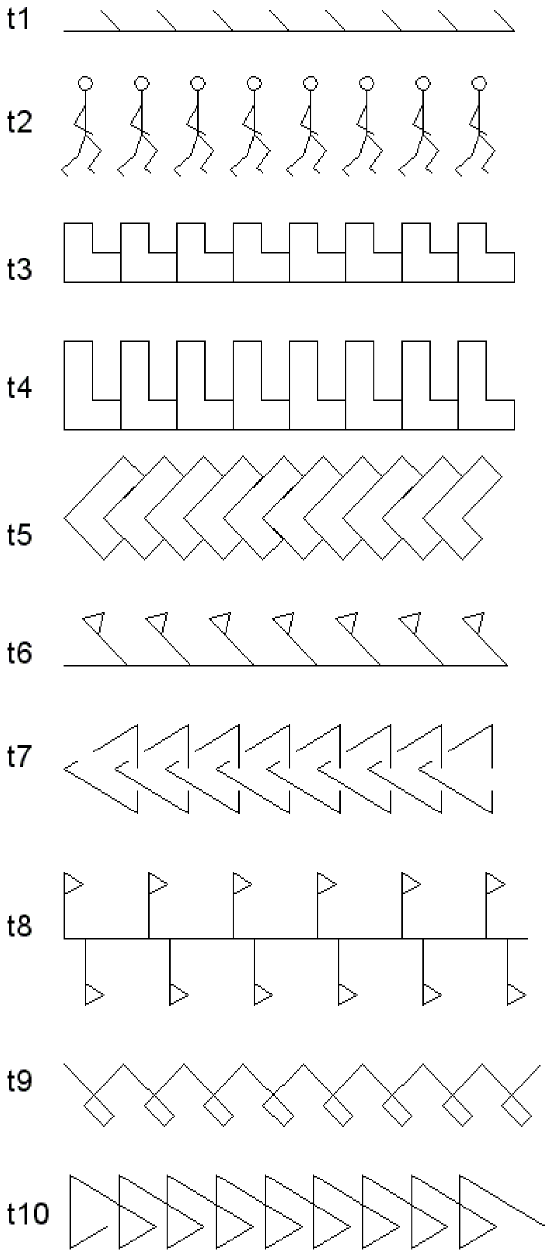

Figure 2.1

(a) Observing

Make tracings of some of the patterns **t1 - t10** in figure 2.1, and, without turning the tracing over, find out how to slide the tracing over the pattern so as to give a perfect match. You need to pretend that the pattern goes on and on in both directions (to left and right).

Is there a minimum length of slide to give a match? Are all the other lengths of slides, that give a match, multiples of this one? (The letter **t**, used in naming the patterns in Figure 2.1 is the first letter of the word 'translation'.)

(b) Making

- Make a (potentially infinite) strip pattern repeating a capital letter which, by itself, has only the identity as symmetry. We have used the letter L at the end of Section **2**.
- You are given a translation **T** taking the point *A* to the point *B*. Use the idea of Section **1(c)** to find two reflections **V** and **W** (*not* symmetries of any of the patterns **t1 – t10**), such that **T** = **VW**. Where does the combination **WV** take the point *B*? What could you say about the translation **WV**? What is the combination (**VW**)(**WV**)? The choice of the letter **V** here and the letter **v** in Section **1** is as the first letter of the word *vertical*, as in Section **4**.
- Use the two reflection construction of Section **1(c)**, to show how a triangle $A_1B_1C_1$ is reflected in **first** to give a triangle $A_2B_2C_2$, and then $A_2B_2C_2$ is reflected in **second** to give $A_3B_3C_3$. Check that if $A_1 \to B_1 \to C_1 \to A_1$ is clockwise, then $A_2 \to B_2 \to C_2 \to A_2$ is anti-clockwise, and $A_3 \to B_3 \to C_3 \to A_3$ is clockwise, as in *Centre* **7**. See figure 1.3.

  What would have happened if $A_1 \to B_1 \to C_1 \to A_1$ had been anti-clockwise?

  So what happens to clockwise and anti-clockwise under a translation?

This investigation may be done with a dynamic geometry package.

(c) Sharpening

The name given to the movement of sliding every point the same distance in the same direction is **translation**. It may seem a bit dull. But it feels more substantial if you think of a train moving on straight railway lines, or a car travelling on a straight piece of road. Look back to the definition of symmetry in *Centre* **9** [if $A \to A'$ and $B \to B'$, then $AB = A'B'$] and decide whether a translation is a symmetry. (You will probably want to think of a translation as having a life of its own. But if you remember that we first made a translation by combining two reflections with parallel axes, that, by itself, guarantees that a translation must be a symmetry.) If a particular pattern has a translation symmetry **T**, use *Centre* **10** [closure] to say why $\mathbf{T}^2$, $\mathbf{T}^3$, and indeed $\mathbf{T}^n$ for any positive integer *n*, must be symmetries of the pattern. Again, use *Centre* **10** [inverses], to say why $\mathbf{T}^{-1}$ must be a symmetry of the pattern. What are the similarities and what are the differences between **T** and $\mathbf{T}^{-1}$. By $\mathbf{T}^{-2}$ we mean $(\mathbf{T}^{2})^{-1}$, and by $\mathbf{T}^{-3}$ we mean $(\mathbf{T}^{3})^{-1}$ etc..

In the frieze below, each L has been labelled with the name of the symmetry that takes the L labelled **I** to it.

…	L	L	L	L	L	L	L	L	…
	$\mathbf{T}^{-3}$	$\mathbf{T}^{-2}$	$\mathbf{T}^{-1}$	**I**	**T**	$\mathbf{T}^{2}$	$\mathbf{T}^{3}$	$\mathbf{T}^{4}$	

3 Friezes

In which we look at some different friezes, and see where friezes may arise.

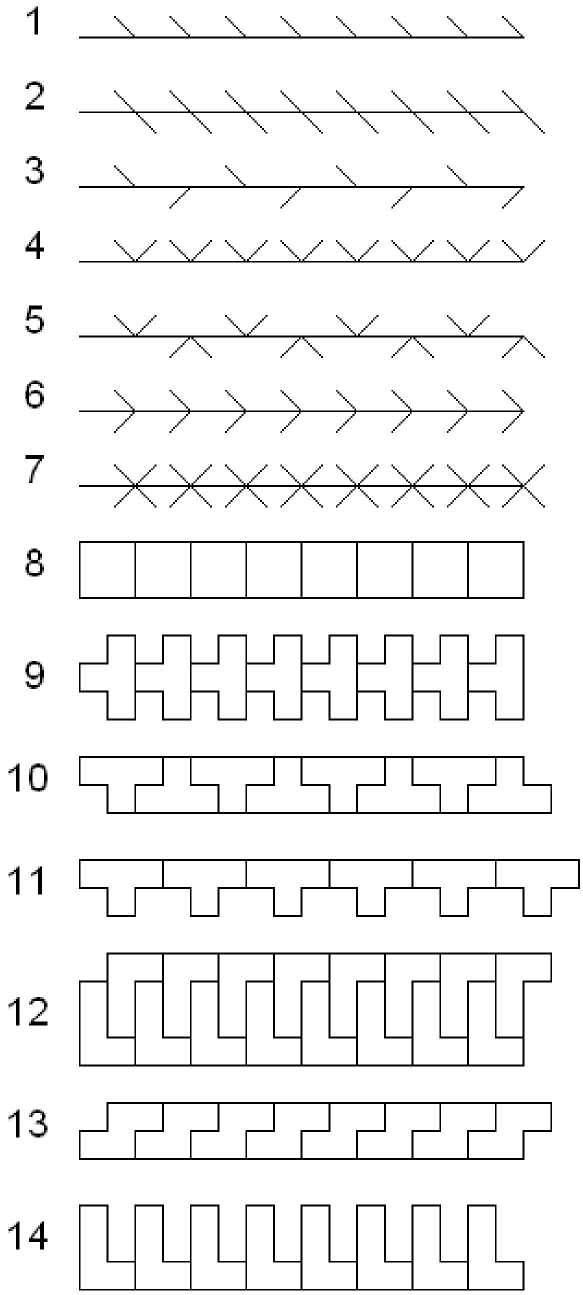

Figure 3.1

(a) Observing

- If the patterns labelled **1** to **14** in figure 3.1 were presumed to extend indefinitely to left and right, what kind of symmetry do all 14 share?
- Is this kind of symmetry to be found on
    - Ribbons and bangles, if you pretend that they can be straightened;
    - Some Greek vases, flattened;
    - Wall-paper borders;
    - Railings;
    - Sine wave;
    - Photographs in Chapter XI of *Symmetry* by István and Magdolna Hargittai;
    - 'Ironwork patterns', *Mathematics Teaching*, **151**, between 22 and 23, (1995)?
- More examples to be found in 'The seven frieze types' Heather McLeay and Jan Abas, *Mathematics in School*, **20**, 3, 24-28 (1991)
- If you look at pattern **14** closely, can you find a pattern in the set **1** to **7** with the same symmetries? Try the same exercise with pattern **13**, then pattern **12**, etc. down to **8**.

(b) Making

Make some more strip patterns.

- Footprints in the sand.
- If you can run a wheel along a straight bit of Spirograph you can make a cycloid.
- Make strip patterns on geoboard or spotty paper [Squared paper is very good for making frieze patterns because you are not only given the tramlines to work between, but also the spacing to get a regular pattern. If each person in a class makes a few friezes for homework, you can get a fine collection to sort.] Try matching your own friezes with patterns **1 – 14**, opposite, by their symmetries.
- Make strip patterns using one capital letter, reflected or not, from L, V, N, D, X . Use symmetries to compare the strips you make, with patterns **1 – 7**. Make strip patterns using two of the letters {p, q, b, d} at a time**.** Again, use symmetry to compare the strips you make, with patterns **1 – 7**.
- Make patterns with LOGO
  Try a procedure such as
  REPEAT 10 [FD :A RT 90 FD :B RT 90 FD :C RT 90 FD :D RT 90]
  Choose numbers for :A, :B, :C and :D.
  In this procedure, try putting :A = :B and :C = :D.

Also see *Micromath* 11.1 and 11.3.

- Make a stencil, or template, from a rectangular block or card. Sliding, turning and flipping the stencil will let you make copies. The sequence of moves must be copied faithfully. Advice on ways of doing this is given in ‘Algorithms for the seven distinct frieze types’, Jan Abas and Heather McLeay, *Mathematics in School*, **20**, 2, 2-6 (1991).

(c) Sharpening

The patterns we have been looking at are called **frieze patterns**. What distinguishes them from other patterns is that

(i) they have translation symmetry;

(ii) all the translations of a frieze pattern are parallel;

(iii) the pattern has a shortest translation ($\neq \mathbf{I}$).

The symmetry group of a frieze pattern is called a **frieze group**.

A single infinite straight line is not a frieze pattern because it does not satisfy (iii). But if a single straight line is added to an existing frieze pattern, you may still have a frieze pattern. All the translations in the symmetry group of a frieze pattern are multiples of the least translation. So if **T** is the shortest translation, all other translations are either $\mathbf{T}^n$ or $\mathbf{T}^{-n}$ for some positive integer n. ($\mathbf{T}^{-1}$ is just as short as **T**.) A proof follows, which you can skip for now, if you want.

If **S** were a translation through a distance s in the symmetry group of a frieze pattern, with shortest translation **T** through a distance t, and $\mathbf{S} \neq \mathbf{T}^n$ for any integer n, then s would not be equal to nt for any n, and s would lie between two consecutive multiples of t, say, $nt < s < (n + 1)t$. But then $0 < s - nt < t$, and it would follow that $\mathbf{ST}^{-n}$ was a translation in the symmetry group of the frieze (by closure and inverses in *Centre* **10**) through a smaller distance than t, and that is impossible.

4 Friezes with vertical reflections

In which we look again at the friezes of **Section 1** and see how more reflections in parallel axes combine.

(a) Observing

Either make a tracing of frieze **v11** or make a photocopy on a transparency.

Then use your tracing to test all the ways in which you can make the tracing match the original. First try *not* turning the tracing over, and later try *after* you have turned the tracing over. Describe the ways you can get the tracing to match the original.

(b) Making

- Draw a frieze on squared paper as follows. Choose an unsymmetrical motif, such as a LOGO flag (FD 100 REPEAT 3 [FD 30 RT 120] BK 100). Copy your motif across the paper by repeating the shortest translation to give a (**t**-style) pattern like those in Section **2**. Now select one axis perpendicular to the direction of the translations and reflect the whole of the pattern that you have drawn in it. Does the resulting pattern have reflections about many axes?
- Is the shortest translation of **v11** equal to the product of two reflection symmetries of **v11**?
- If 1, 2, 3 and 4 are parallel lines, and the reflections with these lines as axes are **1**, **2**, **3**, and **4**, respectively, how can you choose the lines 1, 2, 3 and 4 so that the translation **12** = the translation **34**?

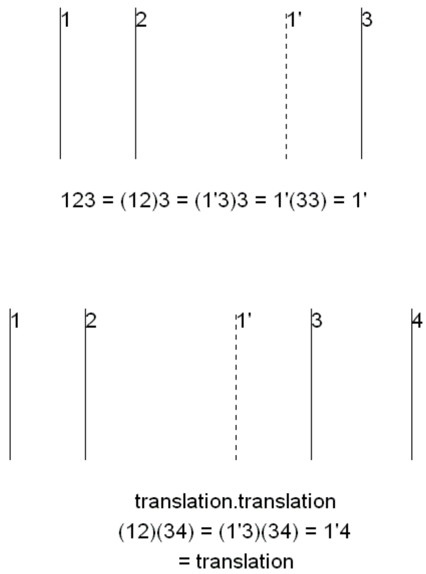

Figure 4.1

(c) Sharpening

Investigate the product of the three reflections **1**, **2** and **3**, where 1, 2 and 3 are parallel lines, by choosing a parallel line 1' such that **12** = **1'3**. See Figure 4.1.

Then **123 =** (**12**)**3** = (**1'3**)**3** = **1'.** This is another version of the *Theorem of the three reflections* (*Centre* **12**, where we had the three axes passing through one point).

If the translations and reflections are all symmetries of the frieze **v11**, show that whichever particular reflections are chosen,

reflection . reflection . reflection = reflection
= translation . reflection
= reflection . translation

Suppose **T** and **S** are two parallel translations and lines 1, 2, 3 and 4 are lines perpendicular to the direction of the translations such that **T** = **12** and **S** = **34.** Suggest how to find a line 1' such that **12** = **1'3**, and deduce that **TS** = **1234** = **1'334** = **1'4**. This shows that when two parallel translations are combined, the result is another parallel translation. See Figure 4.1.

So a bird's eye view of the symmetry group of the frieze **v11** is like this.

	translation	reflection
translation	translation	reflection
reflection	reflection	translation

5 Friezes with half-turns

In which we look at friezes with half-turns and translations, and see how these symmetries combine.

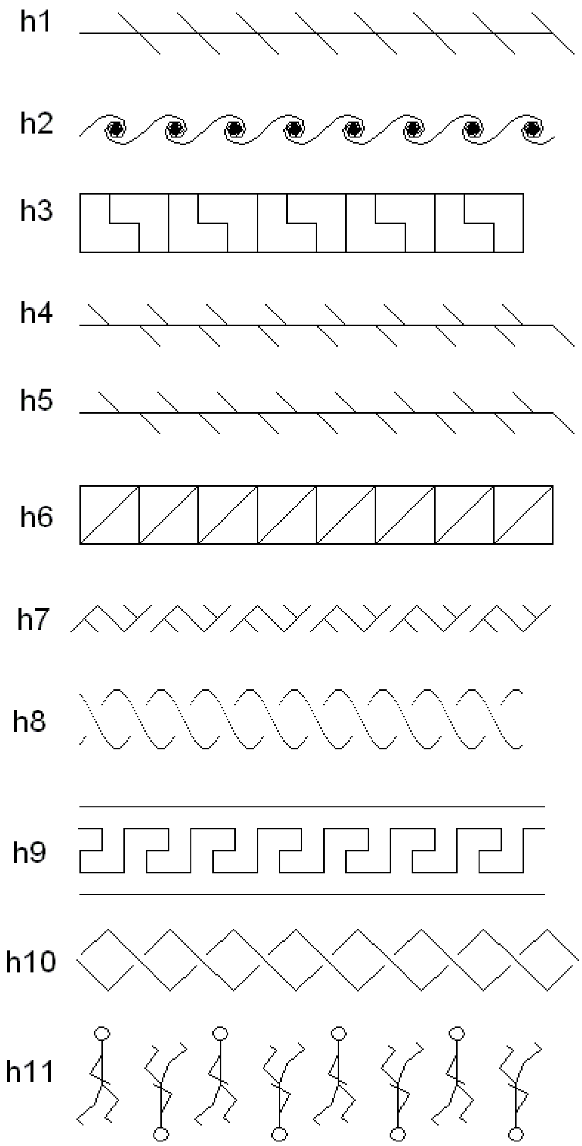

Figure 5.1

(a) Observing

Make a tracing of one or more of the friezes **h1** - **h11** in figure 5.1. Supposing each pattern continues indefinitely to left and right, find what kind of movements of the tracing paper allow the original and the copy to match. Try turning the tracing paper over to see if you can get a match that way. (The letter **h**, used in naming these patterns is the first letter of the term ‘half-turn’.) Figure 5.2 shows half-turn centres of **h3**.

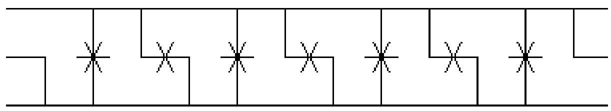

Figure 5.2

Look at the graph of $y = \tan x$ on a graphics calculator.

(b) Making

- Draw a frieze on squared paper as follows. Choose an unsymmetrical motif, such as a LOGO flag. Copy your motif across the paper by repeating the shortest translation to give a (**t**-style) pattern like those in Section **2**. Now select one point and draw the image of the whole of the pattern by a half-turn about that point. Does the resulting pattern have many centres of half-turn symmetry?
- Use squared paper to explore the combination of two half-turns with different centres. Choose two centres, D and E, on the same grid line. Then choose a triangle of grid points $A_1B_1C_1$. Find the image of $A_1B_1C_1$ under the half-turn with centre D and label its vertices $A_2B_2C_2$, with A_2 the image of A_1, etc.. Then find the image of $A_2B_2C_2$ under the half-turn with centre E and label its vertices $A_3B_3C_3$, with A_3 the image of A_2, etc.. Now look at the direction and length of A_1A_3, B_1B_3 and C_1C_3. What does the combination of two half turns with different centres appear to make? See Figure 5.3.
- Explore the combination of two half-turns with different centres with dynamic geometry software. Start with a triangle (or any polygon) and two half-turns, **H** and **K**. Show the image of the triangle under **H**, **HK**, **HKH**, **HKHK**,… and under **K**, **KH**, **KHK**, **KHKH**, …
- Try investigating, using squared paper in a similar way, the combination of a half-turn and a translation, in either order.

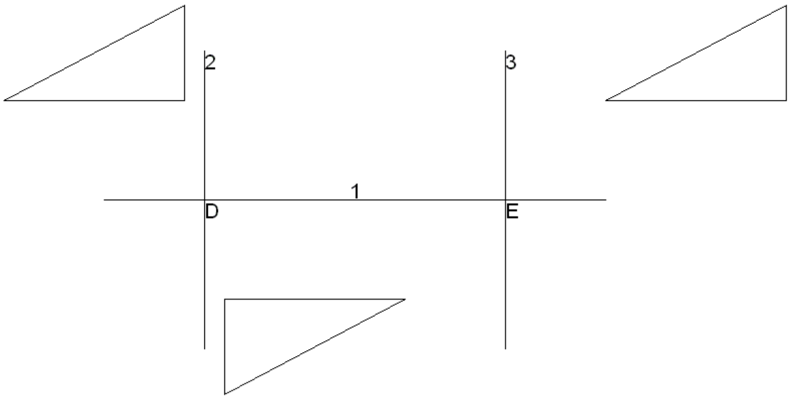

Figure 5.3

(c) Sharpening

Look back to *Centre* **8** where we found how to make a rotation by combining two reflections. Under what circumstances can the combination of two reflections with intersecting axes make a 180° rotation? Do you get the same rotation if you take these two reflections in the opposite order? If the half-turn is given, is there a choice of which two reflections combine to give that half-turn?

Take two half-turns $\mathbf{H_D}$ and $\mathbf{H_E}$ with centres *D* and *E* respectively. Let 1 be the line *DE*, let 2 be the line through *D* perpendicular to 1, and let 3 be the line through *E* perpendicular to 1. Letting **1, 2** and **3** denote the reflections with axes 1, 2 and 3 respectively, say why $\mathbf{H_D} = \mathbf{12}$ and $\mathbf{H_E} = \mathbf{13}$ (see *Centre* **8**). Does $\mathbf{H_D} = \mathbf{12} = \mathbf{21}$? Does $\mathbf{H_E} = \mathbf{13} = \mathbf{31}$? The combination $\mathbf{H_D H_E} = (\mathbf{12})(\mathbf{13}) = (\mathbf{21})(\mathbf{13}) = \mathbf{2}(\mathbf{11})\mathbf{3} = \mathbf{23}$ is a combination of two reflections in parallel lines, which we have already found (in **1(c)**) is a translation through twice the distance between them. See Figure 5.3.

<u>Combining two half-turns:</u> When two half-turns are combined, the result is a **translation** through twice the distance between the centres.

What happens if we combine a half-turn with a translation?

Let $\mathbf{H_D}$ be a half-turn with centre *D* and **T** a translation. Let 1 be the line through *D* in the direction of **T**, and let 2 be the line through *D* perpendicular to 1. Choose the line 3, so that $\mathbf{T} = \mathbf{23}$.

Then $\mathbf{H_D T} = (\mathbf{12})(\mathbf{23}) = \mathbf{1}(\mathbf{22})\mathbf{3} = \mathbf{13}$. Now 1 and 3 are perpendicular lines so **13** is a half-turn, by *Centre* **8**.

Put together a similar argument to show that if the translation had been taken first and the half-turn second, the combination would still have been a half-turn.

So a bird's eye view of the symmetry group of the frieze **h1** is like this.

	translation	half-turn
translation	translation	half-turn
half-turn	half-turn	Translation

In the figure for Section **1(c)**, in the investigation at the end of Section **2(b)**, and here in looking at the combination of two half turns, triangles have been drawn to expose the symmetries. You may imagine that this has just been done for the sake of illustration, but there is more to it than that. Suppose that **S** was some unknown symmetry that took *A* to *A'*, *B* to *B'* and *C* to *C'*, and that *ABC* were the vertices of a triangle (that is, not on one line). If there is some well-known symmetry (in these cases a translation **T**) which also takes *A* to *A'*, *B* to *B'* and *C* to *C'*, then $\mathbf{ST}^{-1}$ would fix *A*, *B* and *C*. From *Centre* **14(c)** [the theorem that any symmetry which fixes the three vertices of a triangle must fix every point of the plane], this would make $\mathbf{ST}^{-1}$ the identity. $\mathbf{ST}^{-1} = \mathbf{I}$ gives $\mathbf{S} = \mathbf{T}$. So a plane symmetry is completely determined by a triangle and its image.

If $\Delta ABC \equiv \Delta A'B'C'$, there is at most one symmetry which takes *A* to *A'*, *B* to *B'* and *C* to *C'*. Note. *A*, *B* and *C* must not lie on one line.

6 Friezes with a horizontal reflection

In which we look at friezes with a mirror line running right through the middle.

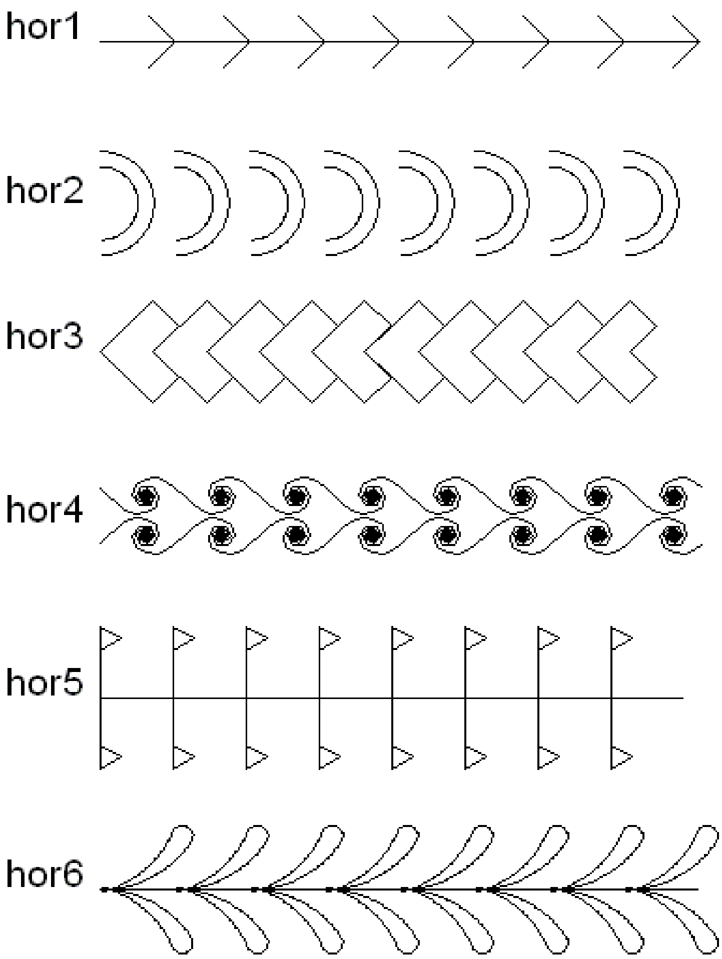

Figure 6.1

(a) Observing

- Do the friezes **hor1 – hor6** in Figure 6.1 have mirror symmetry? (The code **hor**, used in naming these patterns begins the word ‘horizontal’.)
- Make a tracing of one or more of the friezes **hor1 - hor6**. Supposing each pattern continues indefinitely to left and right, find what movements of the tracing paper allow the original and the copy to match. First try *not* turning the tracing over, and later try *after* you have turned the tracing over. Describe the ways you can get the tracing to match the original.
- Look for friezes like those of Figure 6.1 in Figure 3.1.

(b) Making

- On squared paper, design your own frieze with a single axis of reflection symmetry.
- If you place a hand mirror, with the mirror parallel to the translations of any one of the friezes **t1 – t10**, does the original frieze together with its mirror image still make a frieze pattern?
- If **T** is a translation parallel to the axis of a reflection **R**, use squared paper to explore the combinations **TR** and **RT**.

(c) Sharpening

Let **T** be a translation taking the point *D* to the point *E* in Figure 6.2. Let **R** be the reflection in the axis *DE*. Let the translation **T** = **VW**, where **V** and **W** are reflections with axes perpendicular to *DE*. Use *Centre* **8** (or Section **5(c)** above) to say why **VR** = **RV** and **WR** = **RW**. Use these equations to show that (**VW**)**R** = **R**(**VW**) and deduce that **TR** = **RT**.

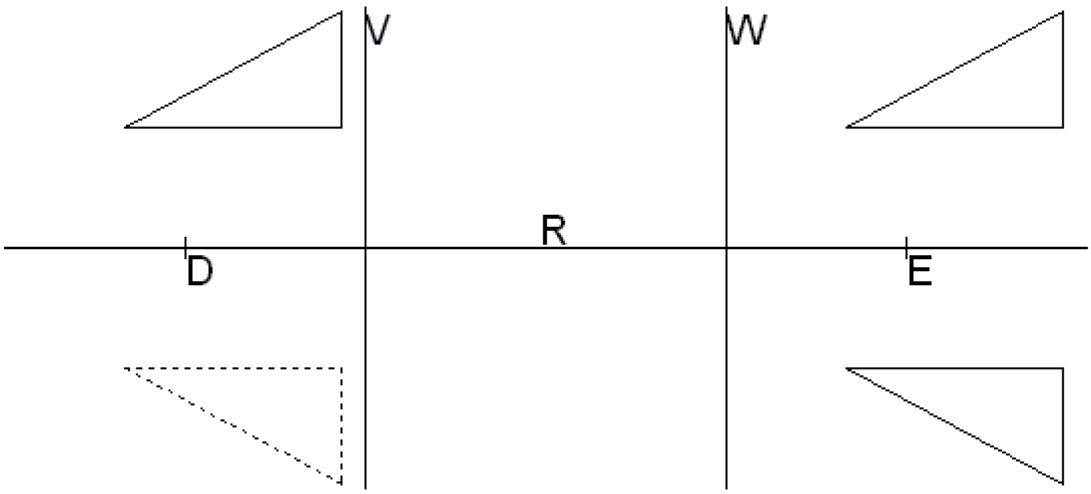

Figure 6.2

Check that the combination **TR** is not a translation, a reflection or a rotation.

Look at one of the friezes **hor1 – hor6.** Let **T** be a translation symmetry of this frieze and let **R** be its single reflection. Examine the effect of the combined symmetry **TR** on the frieze.

7 Glide-reflection

In which we find a kind of frieze that was hiding in **Section 6**, and name the fourth kind of symmetry.

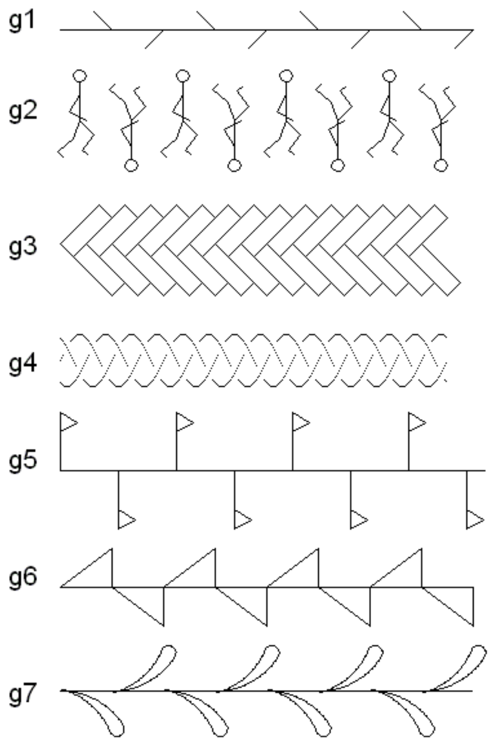

Figure 7.1

(a) Observing

- Look at the friezes **g1 – g7** and suppose that each pattern extends indefinitely to left and right. Use a mirror to check that none of them have reflection symmetry.
- Make a tracing of one of these friezes. Find what movements of the tracing paper allow the original and the copy to match. First try *not* turning the tracing over, and later try *after* you have turned the tracing over.
- We have previously found a matching *after* turning the tracing over, only when the frieze had reflection symmetry in Sections **4** and **6**. Double check that these friezes (**g1 – g7**) do *not* have reflection symmetry, as, if not, we have found a new kind of symmetry.
- A classic example of this kind of frieze is made by the footprints of someone walking on sand. The friezes **g6** and **g7** have been drawn with this example in

mind. Some artists design friezes of this kind with leaves on either side of a central stem. (The letter **g**, used in naming these patterns is the first letter of the word ‘glide’.)

- Look for friezes like those of Figure 7.1 in Figure 3.1.

(b) Making

Use squared paper to make a frieze, without mirror symmetry, but for which a flipped over tracing may be made to match the original. Can this matching be done in more than one way?

(c) Sharpening

Identify a shortest translation of a frieze for which you have made a tracing in **(a)** or **(b)**. How far must you slide the tracing along before a flip over gives an exact match? For this purpose, your slide should have been exactly half the length of the shortest translation. Combining a translation with a reflection with axis along the direction of the translation is a new kind of symmetry called a **glide-reflection**. The axis of the reflection (lying along the direction of the translation) is known as the *glide axis*. Can you think of a way of making a glide-reflection by combining three reflections? If so, you have another proof that a glide-reflection is a symmetry.

If **G** is a glide-reflection, what is $\mathbf{G}^2$? And what are $\mathbf{G}^3$ and $\mathbf{G}^{-1}$?

Of the four symmetries that we have named, a glide-reflection is usually the hardest to recognise. The difficulty is finding the glide axis. There is a rather counter-intuitive trick you can use to find a glide axis when you know there must be one and you cannot see what it is. Let *P* be any point of the plane (your choice), and let the glide-reflection **G** take *P* to *P’*. Where is the middle point of *PP’*? You can work out the answer using the frieze with glide-reflections that you have been working with. If you choose another point *Q* and the same glide-reflection **G** takes *Q* to *Q*’, the glide axis is the line joining the midpoints of *PP’* and *QQ’*.

A glide-reflection has a *direction*, parallel to the glide axis, and a *length*, the distance it moves a point on its glide axis, and is completely determined by the glide axis, the direction and the length.

8 Fixed lines

In which we look at the lines fixed by each kind of symmetry in order to decide what symmetries may be found in a frieze group.

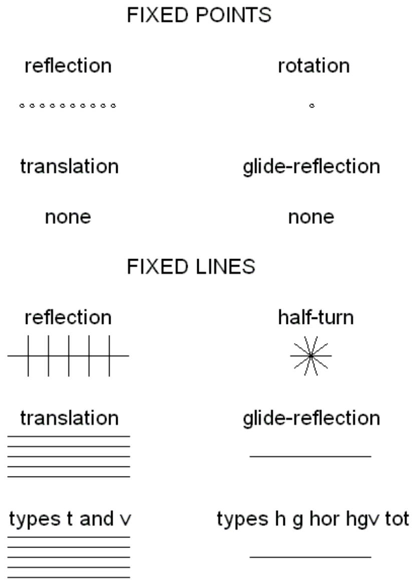

Figure 8.1

(a) **Observing**

Of the four kinds of symmetry that we know: reflection, rotation, translation and glide-reflection, which have one fixed point? Which have many fixed points and which have no fixed points?

- If, under a translation **T**, a point *A* is translated to *A'*, and *B* is any other point on the line *AA'*, must *B* be translated to a point on *AA*' by **T**? Although **T** does not fix any points on the line *AA'*, **T** fixes the line as a whole. Does every point of the plane lie on a fixed line of **T**? Are all the fixed lines of **T** parallel?
- If, under a reflection **R**, a point *A* is reflected to *A'* ≠ *A*, and *B* is any other point on the line *AA'*, where is *B* reflected to by **R**? Although **R** fixes just one point on the line *AA'*, **R** fixes the line as a whole. Does every point of the plane lie on a fixed line of **R**? Are all the fixed lines of **R**, other than its axis, parallel?
- Examine the possibility of fixed lines of a glide-reflection. Check that there is just one – the glide axis.
- Most rotations have no fixed lines. What is the exception and where are its fixed lines?

So far we have examined the fixed lines of individual symmetries. Now we examine whether all the symmetries in a group may fix the same line or lines. Look at each of the following symmetry groups and identify the lines fixed by every symmetry in the group.

- First look at the groups of symmetry of the friezes of type **t** in Section **2**.
- Next look at the groups of symmetry of the friezes of type **v** in Section **1**.
- Now look at the groups of symmetry of the friezes of type **h** in Section **5**.
- Look at the groups of symmetry of the friezes of type **hor** in Section **6**.
- Look at the groups of symmetry of the friezes of type **g** in Section **7**.

(b) Making

For those who know about coordinates.

Identify and name the following symmetries, assuming a is constant and $a \neq 0$.

If you find this exercise puzzling, just put $a = 1$.

$(x, y) \to (x, -y)$;

$(x, y) \to (-x, -y)$;

$(x, y) \to (x + a, y)$;

$(x, y) \to (x + a, -y)$;

$(x, y) \to (-x + 2a, y)$;

$(x, y) \to (-x + 2a, -y)$.

Identify fixed points or fixed lines for each of these symmetries.

(c) Sharpening

Exercises using *Centre* **10**, **Closure**.

(i) If a symmetry group contains at least one translation, why must it contain many translations with the same fixed lines?

(ii) If a symmetry group contains at least one glide-reflection, why must it contain many translations and many glide-reflections with the same glide axis?

(iii) If a symmetry group contains at least one translation and one half-turn, why must it contain many half-turns?

(iv) If a symmetry group contains at least one translation and one reflection with axis perpendicular to the direction of the translation, why must it contain many reflections?

In **3(c)** we specified frieze patterns only using translations.

> Equivalently, we can specify **frieze patterns** as those patterns all of whose symmetries fix a common line and include a shortest translation.

Notice that the fixed line in this specification incorporates condition **3(c)** (ii). More abstractly, a **frieze group** is a symmetry group in which all the symmetries share a fixed line, there is no shared fixed point and no infinitesimal transformations.

Just as a matter of convention we will always take the fixed line of our friezes horizontal.

To prove that we have found *all* the symmetries with a fixed line we will need to use the results of Sections **13** and **14**.

9 Friezes with half-turn centres not on reflection axes

In which we see one way in which half-turns and reflections may coexist in a frieze group.

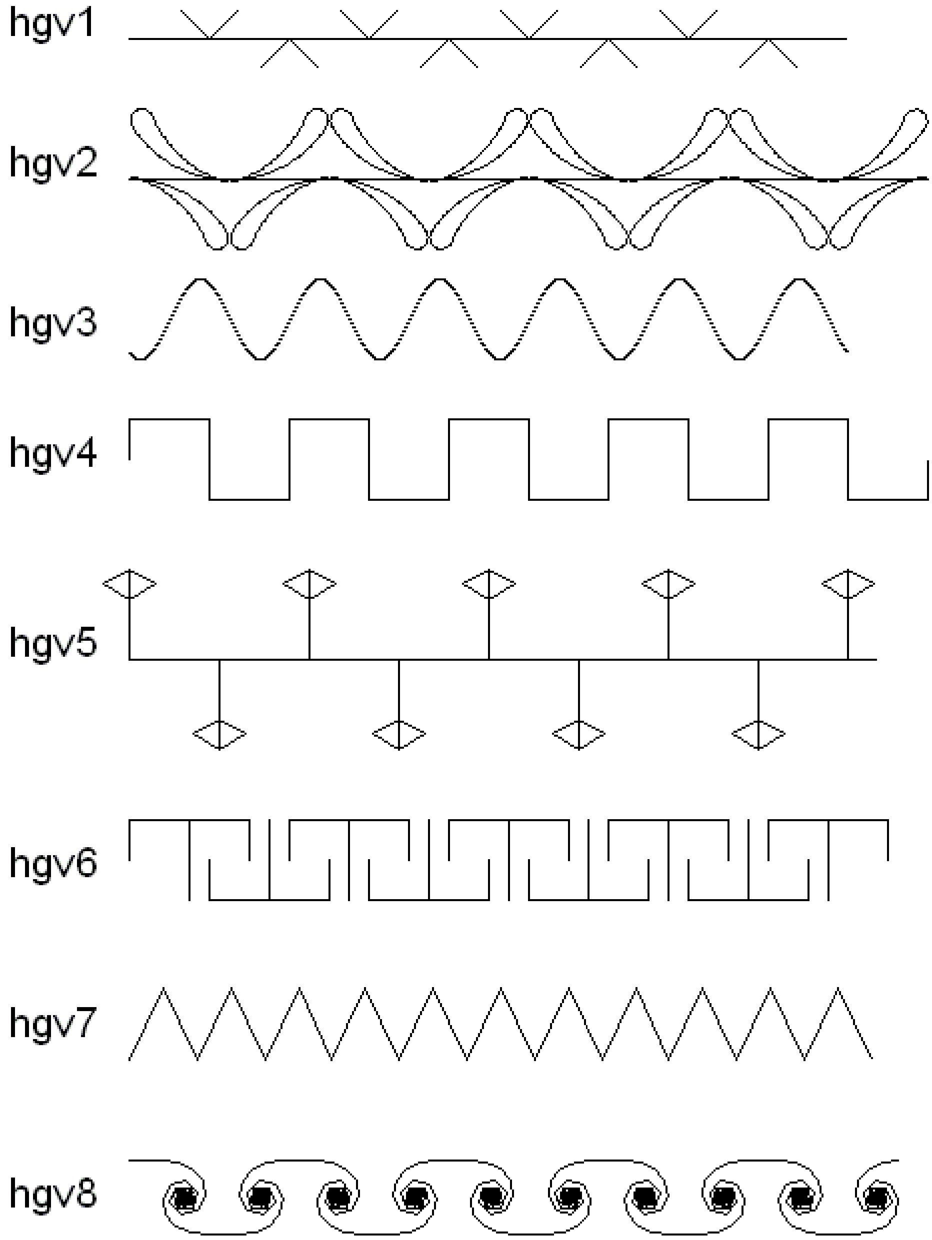

Figure 9.1

(a) Observing

Look at the friezes **hgv1 – hgv8** in Figure 9.1. Locate their centres of half turn symmetry and their axes of reflection symmetry.

What other symmetries do these friezes have?

Figure 9.2 shows the half-turn centres and mirror lines of **hgv4**.

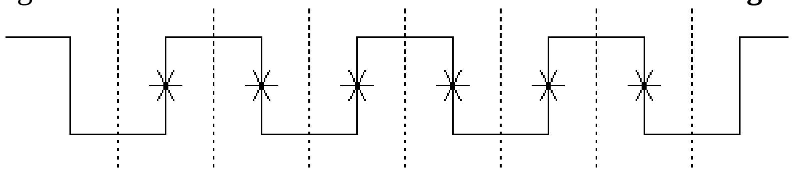

Figure 9.2

Look for friezes like those of Figure 9.1 in Figure 3.1.

(b) Making

- Use squared paper to explore the combination of a half-turn and a reflection with axis *not* through the centre of the half-turn, by choosing an arbitrary triangle of grid points ($A_1B_1C_1$), and looking at its image, first under the half turn (to get $A_2B_2C_2$) and then looking at the image of $A_2B_2C_2$ under the reflection (to get $A_3B_3C_3$). What symmetry carries A_1 to A_3, B_1 to B_3 and C_1 to C_3?

  If D is the centre of the half-turn and a is the axis of the reflection, how does the line through D perpendicular to a (which we will call ℓ) relate to this symmetry?

  If for A_1, B_1, C_1, A_3, B_3, and C_3 we denote the feet of their perpendiculars to ℓ by A_1', B_1', C_1', A_3', B_3', and C_3' respectively, is $A_1'A_3' = B_1'B_3' = C_1'C_3' = 2 \times$ the distance from D to a? See the first diagram in Figure 9.3.

  How would the result differ if the combination were in the opposite order, reflection first, half turn second?

- Explore the combination of a half-turn and a reflection with axis not through the centre of the half-turn with dynamic geometry software. Start with a triangle (or any polygon) a half-turn, **H**, and a reflection, **V**. Show the image of the triangle under **H**, **HV**, **HVH**, **HVHV**,… and under **V**, **VH**, **VHV**, **VHVH**, …

(c) Sharpening

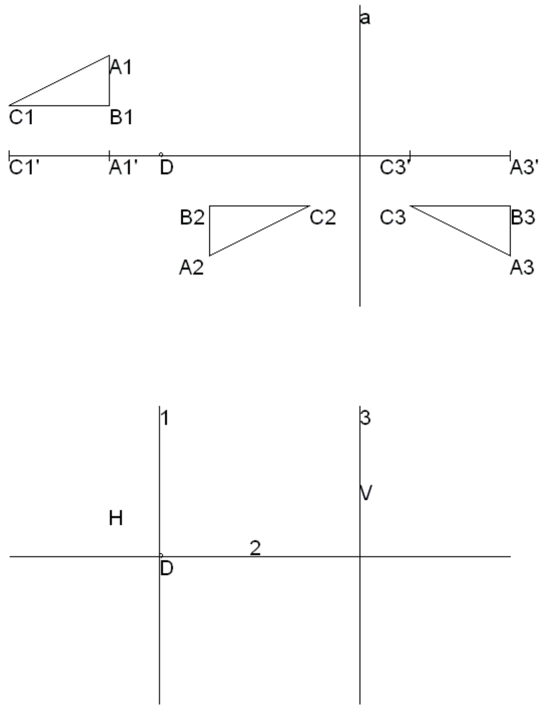

Figure 9.3

Let **H** be a half turn with centre *D* and **V** be a reflection with axis 3. *D* does not lie on 3.

Let 1 be the line through *D* parallel to 3, and 2 be the line through *D* perpendicular to 3. Let **1**, **2** and **3** denote the reflections with axes 1, 2 and 3 respectively. Then **H** = **12** = **21** (check with Section **5(c)**), and **3** = **V**.

HV = (**12**)**3** = **1**(**23**) = **1**(**32**) = (**13**)**2**. See the second diagram in Figure 9.3.

What is the combined symmetry **13** (see Section **1(c)**) and what is its direction?

What is the symmetry **2** and what is the direction of its axis?

So from Section **7(c)** what is the combination (**13**)**2**?

We have proved that **HV** = a glide reflection.

> The combination of a half-turn and a reflection with axis not through the centre of the half-turn is a glide-reflection.

What about the converse problem? If you have been given a glide-reflection **G**, can you find a half turn, **H**, and a reflection, **V**, such that **G** = **HV**? Clearly the centre of **H** would have to be on the glide axis of **G**, and the axis of **V** would have to be perpendicular to the glide axis. What more would be needed? The distance from the centre of **H** to the axis of **V** would have to be half the length of the glide and in the right direction. But this is enough, and so there are many pairs (**H**, **V**) such that **G** = **HV**. *Either*, any point on the glide axis may be chosen as the centre of **H**, *or*, any line perpendicular to the glide axis may be chosen as the axis of **V**.

This leads to the rather surprising result that if the symmetry group of a frieze pattern contains any two of

(i) a glide-reflection,

(ii) a half-turn, with centre not on the axis of (iii),

(iii) a reflection with axis perpendicular to the fixed line,

then that symmetry group contains all three.

We have already seen that (ii) and (iii) give (i) because **HV** = a glide-reflection.

(i) and (ii) give (iii) because if a half turn **H** and a glide reflection **G** are in the group, there is a **V** (which, to start with, we do not assume to be in the group) such that **G** = **HV**, then **HG** is in the group and so **H**(**HV**) = **V** is in the group.

Similarly (i) and (iii) give (ii) because if a reflection **V** and a glide reflection **G** are in the group, with the axis of **V** perpendicular to the glide axis, there is an **H** such that **G** = **HV**, then **GV** is in the group and so (**HV**)**V** = **H** is in the group.

Hence the naming of such frieze patterns as **hgv**!

10 Friezes with all possible symmetries

In which we see another way in which half-turns and reflections may coexist in a frieze group, with centres of half-turns *on* axes of reflections.

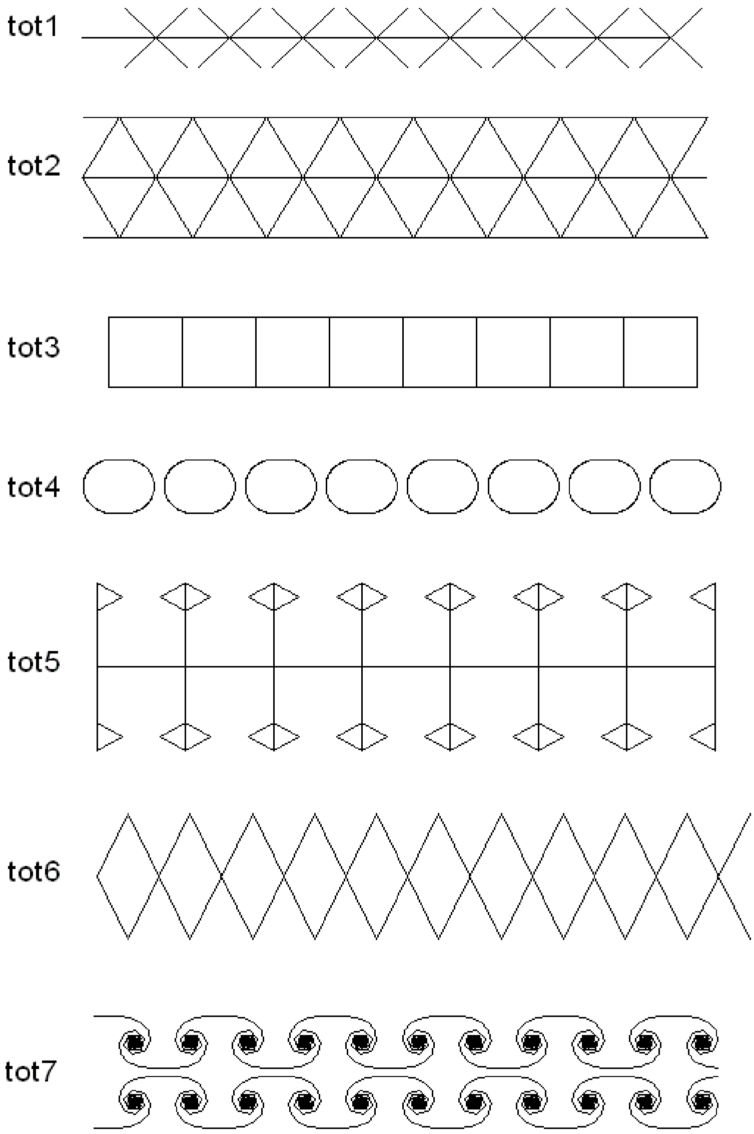

Figure 10.1

(a) Observing

Look at the friezes **tot1 – tot7** in Figure 10.1. Locate their centres of half turn symmetry and their axes of reflection symmetry.

What other symmetries do these friezes have? (The code **tot**, used in naming these patterns begins the word ‘total’.)

The half-turn centres and the axes of reflection of any **tot** frieze are illustrated in Figure 10.2.

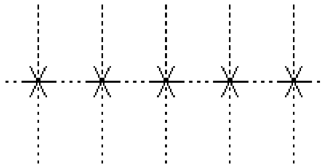

Figure 10.2

(b) Making

Draw a frieze on squared paper as follows. Choose an unsymmetrical motif, such as a LOGO flag, and choose a horizontal line which will be the fixed line of your frieze. Copy your motif to left and right across the paper along the fixed line by repeating the shortest translation to give a (**t**-style) pattern like those in Section **2**. Now reflect this pattern in the fixed line (to give a **hor**-style pattern). Finally select a point on the fixed line and add what is needed so that this point is a centre of half-turn symmetry of the pattern. (i) Does the resulting pattern have many centres of half-turn symmetry? (ii) Does the pattern have reflections in vertical axes. (iii) Does every vertical axis of a reflection pass through a half-turn centre?

(c) Sharpening

If a pattern has two perpendicular axes of reflection, why must it have a centre of half-turn symmetry at their point of intersection?

If a pattern has a reflection with axis *a* and a half turn centre *C* on *a*, why must it have a second reflection with axis through *C*?

11 Generators

In which we see how all the symmetries of a frieze group are combinations of one, two or three basic symmetries in the group.

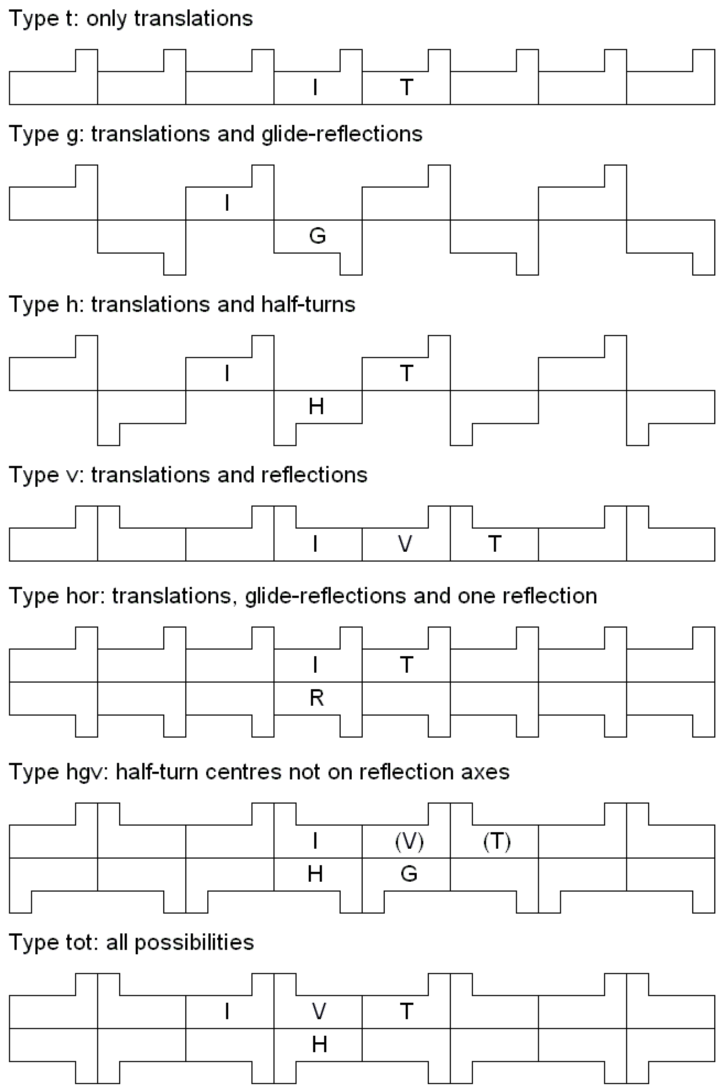

Figure 11.1

The diagrams in Figure 11.1 show frieze patterns constructed using L-shaped blocks. Some **blocks** have been labelled with symmetries of the pattern: **I** for the identity, **T** for a shortest translation, **V** for a reflection in a vertical axis, **R** for a reflection in a horizontal axis, **G** for a glide-reflection and **H** for a half-turn.

The task is to label all the blocks. For each pattern, one of the blocks has been labelled I. This is an “origin”. To find the label for another block, one finds the symmetry that takes the I block to the other block. If the symmetry **S** takes the I block to the other block, then that other block is labelled S.

- For the type **t** pattern the unlabelled blocks each need a label of the kind $\mathbf{T}^2$, $\mathbf{T}^3$, $\mathbf{T}^{-1}$, etc.. Because every block in the type **t** pattern is labelled with a power of **T**, the translation **T** is said to **generate** the symmetry group of a type **t** pattern which is then denoted by <**T**>.

- Likewise for the type **g** pattern, the unlabelled blocks each need a label of the kind $\mathbf{G}^2$, $\mathbf{G}^3$, $\mathbf{G}^{-1}$, etc.. Because every block in the type **g** pattern is labelled with a power of **G**, the glide-reflection **G** is said to **generate** the symmetry group of a type **g** pattern which is then denoted by <**G**>.
- For the type **h** pattern, label those blocks which you can, with powers of **T**. Mark the centre of the half-turn **H**. Then see where the T-labelled blocks go to under the half-turn **H**. Because every block in the type **h** pattern is labelled with a power of **T**, or with a combination of **H** with a power of **T** the two symmetries **T** and **H** are said to **generate** the symmetry group of a type **h** pattern, which is then denoted by <**T**, **H** >.
- For the type **v** pattern, label those blocks which you can with powers of **T**. Mark the axis of the reflection **V**. Then see where the T-labelled blocks go to under the reflection **V**. Because every block in the type **v** pattern is labelled with a power of **T**, or with a combination of **V** with a power of **T** the two symmetries **T** and **V** are said to **generate** the symmetry group of a type **v** pattern, which is then denoted by <**T**, **V** >.
- For the type **hor** pattern, label those blocks which you can, with powers of **T**. Mark the axis of the reflection **R**. Then see where all the T-labelled blocks go to under the reflection **R**. Because every block in the type **hor** pattern is labelled with a power of **T** or with a combination of **R** with a power of **T**, the two symmetries **T** and **R** are said to **generate** the symmetry group of a type **hor** pattern, which is then denoted by <**T**, **R**>.
- For the type **hgv** pattern, label those blocks which you can with powers of **G**. Mark the centre of the half-turn **H**. Complete the labelling by seeing where all the G-labelled blocks go to under the half-turn **H**. Because every block in the type **hgv** pattern is labelled with a power of **G**, or with a combination of **H** with a power of **G**, the two symmetries **G** and **H** are said to **generate** the symmetry group of a type **hgv** pattern, which is then denoted by <**G**, **H**>.
- For the type **tot** pattern, label those blocks which you can with powers of **T**. Mark the centre of the half-turn **H** and the axis of the reflection **V**. Complete the labelling of the upper row of blocks by seeing where all the T-labelled blocks go to under the reflection **V**. Label the lower row of blocks by seeing where the blocks in the upper row go to under the half-turn **H**. Because every block in the type **tot** pattern is labelled with some combination of the symmetries **H**, **V** and powers of **T**, the three symmetries **T**, **H** and **V** are said to **generate** the symmetry group of a type **tot** pattern, which is then denoted by <**T**, **H**, **V**>.

At the end of **5(c)**, we found that there was at most one symmetry matching the corresponding vertices of two congruent triangles. The same holds for L-shaped blocks, and so if there are two possible labels for one block, this establishes equal symmetries.

(i) For the type **g** pattern, what is the shortest translation?

(ii) For the type **h** pattern, locate the centres of the half-turns **HT**, $\mathbf{HT}^2$, **TH**.

(iii) For the type **h** pattern check that $\mathbf{HT} = \mathbf{T}^{-1}\mathbf{H}$.

(iv) For the type **v** pattern locate the axes of the reflections **VT**, $\mathbf{VT}^{-1}$, $\mathbf{VT}^{-2}$.

(v) For the type **v** pattern check that $\mathbf{VT} = \mathbf{T}^{-1}\mathbf{V}$.

(vi) For the type **hor** pattern check that **TR** = **RT**. What kind of symmetry is **RT**?

(vii) For the type **hgv** pattern check that $(\mathbf{T}) = \mathbf{G}^2$ and that $(\mathbf{V}) = \mathbf{HG}$. Locate the centres of the half-turns $\mathbf{G}^2\mathbf{H}$ and $\mathbf{G}^{-2}\mathbf{H}$ and the axes of the reflections **GH**, $\mathbf{G}^3\mathbf{H}$ and $\mathbf{G}^{-1}\mathbf{H}$. Check that $\mathbf{HG} = \mathbf{G}^{-1}\mathbf{H}$. What kind of symmetry is **HV**? What kind of symmetry is **VH**?

(viii) For the type **tot** pattern check that **HV** = **VH** and that (**HV**)**T** = **T**(**HV**). What kind of symmetry is **HV**?

12 The family of friezes

In which we identify the relationships between the different frieze groups.

The symmetries in frieze groups that we have found so far are

1) translations parallel to the fixed line,

2) reflections with axes perpendicular to the fixed line,

3) half-turns,

4) a reflection with axis on the fixed line, and

5) glide-reflections.

In the left-hand column below is a list of the different kinds of symmetries which we have noticed occurring in frieze patterns.

Use the columns of this array to indicate which of the symmetries are to be found in particular types of friezes.

FRIEZE TYPE	**t**	**g**	**h**	**v**	**hgv**	**hor**	**tot**
Translations							
Glide-reflections							
Half-turns							
Reflections with axis perpendicular to translations							
Reflection with axis parallel to translations							

Once I spent a morning in an Indian bazaar going round the bangle stalls. I looked at several thousand manufactured bangles. The only types of frieze which I found were **h**, **hgv** and **tot**. Have you any idea why that might have been?

Remembering that a frieze pattern has a fixed line and a shortest translation ($\neq$ **I**), the symmetry group of a frieze pattern contains, at least, a group of translations. The other possibilities come by adding one new symmetry at a time and seeing what group you get.

The seven different symmetry groups for friezes are shown below. An arrow points from a subgroup to a larger group which contains all the symmetries of the subgroup.

Translation frieze **t**
...L L L L L L L L L L L...
translations $\{\mathbf{T}^n: n \in \mathbf{Z}\}$

↙ ↓ ↘

half turn frieze
...Z Z Z Z Z Z Z Z Z Z...
translations $\{\mathbf{T}^n: n \in \mathbf{Z}\}$

half-turns
$\{\mathbf{T}^n\mathbf{H}: n \in \mathbf{Z}\}$, $\mathbf{HT} = \mathbf{T}^{-1}\mathbf{H}$

vertical mirror frieze
...V V V V V V V V V V...
translations $\{\mathbf{T}^n: n \in \mathbf{Z}\}$

vertical reflections
$\{\mathbf{T}^n\mathbf{V}: n \in \mathbf{Z}\}$, $\mathbf{VT} = \mathbf{T}^{-1}\mathbf{V}$

glide frieze
...LΓLΓLΓLΓLΓLΓ...
translations $\{\mathbf{T}^n: n \in \mathbf{Z}\}$
$= \{\mathbf{G}^{2n}: n \in \mathbf{Z}\}$
glide-reflections
$\{\mathbf{T}^n\mathbf{G}: n \in \mathbf{Z}\}$
$= \{\mathbf{G}^{2n+1}: n \in \mathbf{Z}\}$

↘ ↙ ↙ ↓

hgv frieze
...↑ ↓ ↑ ↓ ↑ ↓ ↑ ↓ ↑ ↓...
translations $\{\mathbf{T}^n: n \in \mathbf{Z}\}=\{\mathbf{G}^{2n}: n \in \mathbf{Z}\}$
glide-reflections $\{\mathbf{G}^{2n+1}: n \in \mathbf{Z}\}$
half-turns $\{\mathbf{G}^{2n}\mathbf{H}: n \in \mathbf{Z}\}$, $\mathbf{HG} = \mathbf{G}^{-1}\mathbf{H} = \mathbf{V}$
vertical reflections $\{\mathbf{G}^{2n+1}\mathbf{H}: n \in \mathbf{Z}\}=\{\mathbf{T}^n\mathbf{V}: n \in \mathbf{Z}\}$

horizontal mirror frieze
...D D D D D D D D D D...
translations $\{\mathbf{T}^n: n \in \mathbf{Z}\}$
one reflection $\{\mathbf{R}\}$
glide-reflections
$\{\mathbf{T}^n\mathbf{R}: n \in \mathbf{Z} \setminus \{0\}\}$
$\mathbf{TR} = \mathbf{RT}$

↘ ↙

total frieze
...X X X X X X X X X X...
translations $\{\mathbf{T}^n: n \in \mathbf{Z}\}$
half-turns $\{\mathbf{T}^n\mathbf{H}: n \in \mathbf{Z}\}$, $\mathbf{HT} = \mathbf{T}^{-1}\mathbf{H}$
vertical reflections $\{\mathbf{T}^n\mathbf{V}: n \in \mathbf{Z}\}$, $\mathbf{VT} = \mathbf{T}^{-1}\mathbf{V}$
horizontal reflection and glide-reflections$\{\mathbf{T}^n\mathbf{HV}: n \in \mathbf{Z}\}$, $\mathbf{HV} = \mathbf{VH} = \mathbf{R}$.

The names used here for these frieze groups are not standard. A class may want to invent its own names for the seven groups on the basis of what they felt was distinctive and representative about each of them.

13 One, two, three reflections

The analysis of frieze groups that we have given is based on the assumption that the only symmetries with a fixed line are either translations, half-turns, reflections or glide-reflections, and that is in turn based on the assumption that every symmetry is either a reflection, a rotation, a translation or a glide-reflection. In this section and the final one we will prove this.

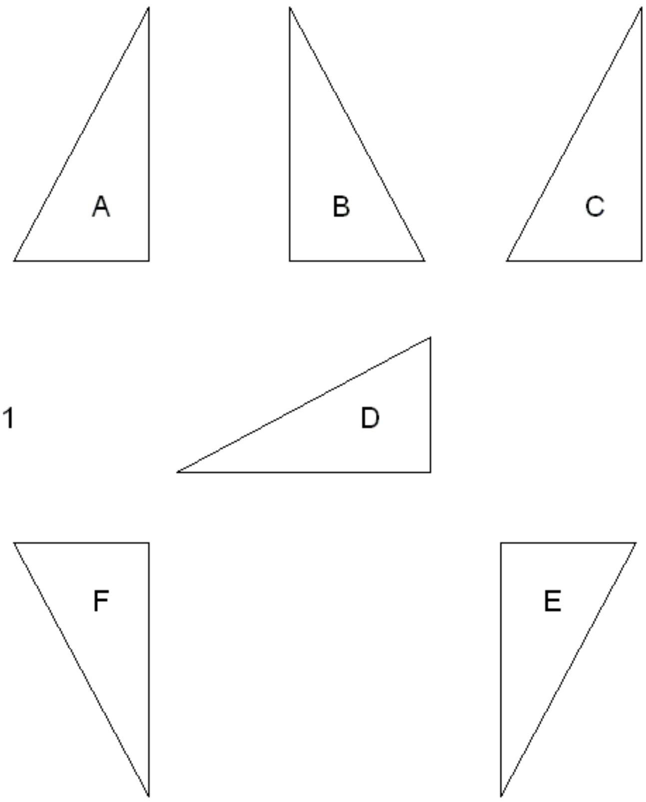

Figure 13.1

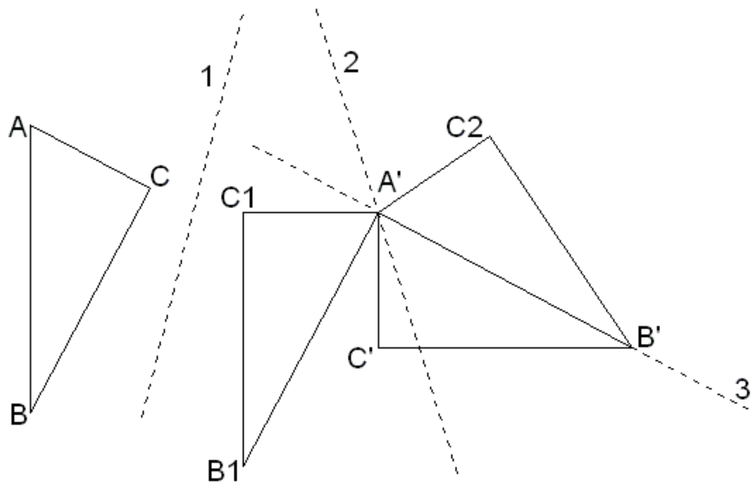

Figure 13.2

The key result we need for this was obtained in Section **5(c),** namely that there is at most one symmetry carrying the vertices of a triangle to the corresponding vertices of a congruent triangle. We also used this result in Section **11** to get equalities of symmetries in (iii), (v), (vi), (vii) and (viii).

(a) **Observing**

Look at the triangles in Figure 13.1. They are all congruent. There is a sequence of reflections which takes A to B to C to D to E to F to A. What symmetry takes triangle A to C? What symmetry takes triangle C to E? (Harder) What symmetry takes triangle B to D? The answers to these questions have to be symmetries which can be made by combining two reflections. If you have difficulty in recognising, say, the rotation taking *B* to *D*, remember that the centre of the rotation is where the two axes of reflection intersect.

Now see if you can recognise a symmetry taking B to E, and one taking C to F. Find the glide axis and the length of the glide. It is harder to recognise the glide-reflection taking A to D. In such a case use the method of finding the glide axis described in Section **7(c)**. For any two congruent triangles in the plane do you think you can find either a reflection, a rotation, a translation or a glide-reflection which carries one to the other? In **(b)** we will start to find out how to do this.

(b) Making

We are given two congruent triangles *ABC* and *A'B'C'*, and will construct a symmetry that carries the first to the second. Together with the result in Section **5(c)** this will establish a unique symmetry carrying *A* to *A'*, *B* to *B'* and *C* to *C'*. See figure 13.2.

<u>Step 1</u> : to be ignored if $A = A'$.

Find a line (1, the axis of reflection **1**) such that the reflection in that line as axis matches *A* with *A'*. If the image of *ABC* under this reflection is $A_1B_1C_1$, draw a figure showing the three triangles *ABC*, *A'B'C'* and $A_1B_1C_1$. Are you sure that $A_1 = A'$?

<u>Step 2</u> : prerequisite $A_1 = A'$, to be ignored if $B_1 = B'$.

Find a line (2, the axis of the reflection **2**) such that the reflection in that line as axis matches B_1 with *B'*. Why must this line pass through *A'*? If the image of $A_1B_1C_1$ under this reflection is $A_2B_2C_2$ draw a figure to illustrate $A_1B_1C_1$ and $A_2B_2C_2$. Check that $A_2 = A_1 = A'$. Are you sure that $B_2 = B'$?

<u>Step 3</u> : prerequisites $A_2 = A'$ and $B_2 = B'$, to be ignored if $C_2 = C'$.

Find a line (3, the axis of the reflection **3**) such that the reflection in that line as axis matches C_2 with *C'*. In fact the line 3 has to be *A'B'*. Check that the image of $A_2B_2C_2$ under this reflection is *A'B'C'*.

When all three steps are needed to match *ABC* to *A'B'C'*, the combination **123** gives a symmetry that carries the first triangle to the second.

(c) Sharpening

From **(b)**, if *ABC* and *A'B'C'* are congruent triangles then either a reflection or a combination of 2 or 3 reflections matches $A \to A'$, $B \to B'$ and $C \to C'$.

Now every symmetry matches a triangle to a congruent triangle. But from Section **5(c)**, there is at most one symmetry matching corresponding vertices of two congruent triangles. So every symmetry is either a reflection or may be expressed as a combination of 2 or 3 reflections.

Combinations of two reflections have been shown to give a rotation when the axes intersect, in *Centre* **8,** and to give a translation when the axes are parallel, in Section **1(c)**.

From *Centre* **7**, a reflection interchanges clock-wise and anti-clockwise. What is the effect of a combination of two reflections on clockwise and anti-clockwise?

What is the effect of a combination of three reflections on clockwise and anti-clockwise?

What is the effect of a combination of four reflections on clockwise and anti-clockwise? If a combination of four reflections has to equal a combination of one, two or three, which must it be?

When you are sure how to reduce a combination of four reflections, you may like to think how you might reduce a combination of five, six or more reflections.

The question which is still outstanding is how to understand a combination of three reflections. This is what **Section 14** is about.

14 Three reflections

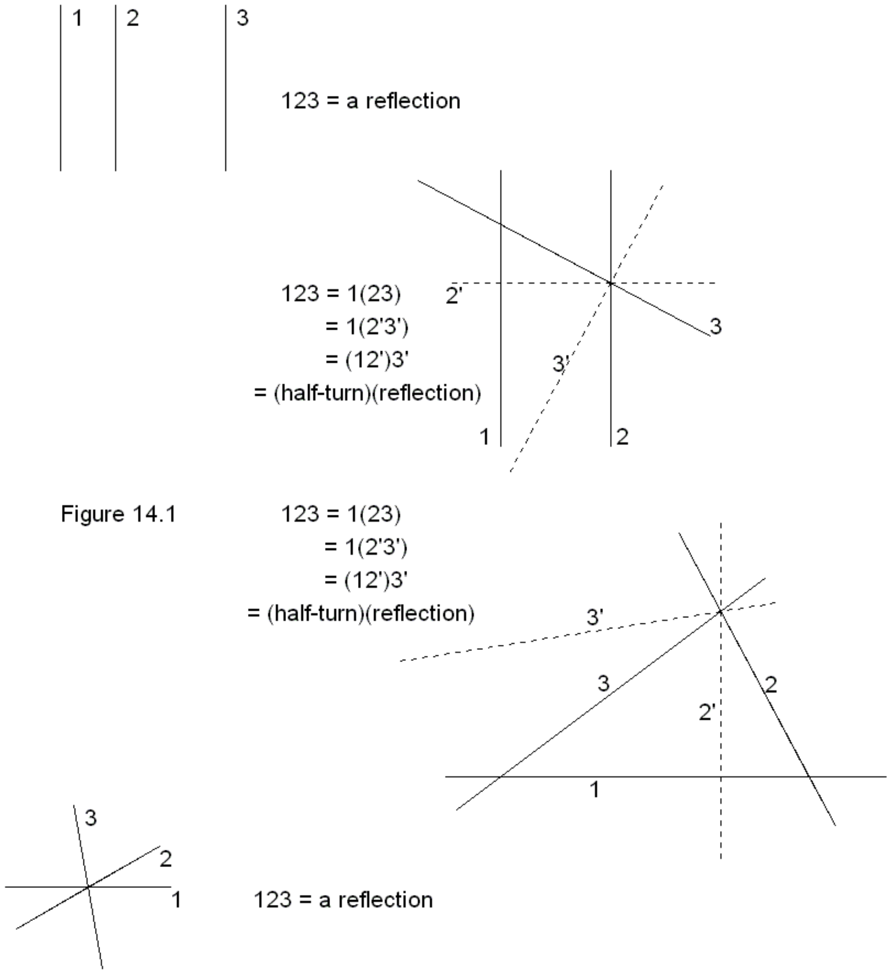

Figure 14.1

Let three reflections **1**, **2** and **3** have axes 1, 2 and 3.

- If 1, 2 and 3 are parallel, then **123** = a reflection by Section **4(c)**. See Figure 14.1.
- If 1 is parallel to 2, and 3 intersects them both, choose 2’ through the point of intersection of 2 and 3, perpendicular to 1, and then choose 3’ so that **23** = **2’3’**, which may be done by choosing 3’, again through the intersection of 2 and 3, in such a way that the angle from 2 to 3 is equal to the angle from 2’ to 3’, using *Centre* **8** with the method used in *Centre* **12(c)**. Then **123** = **12’3’** = (**12’**)**3’.** Since **12’** is a half-turn, **12’3’** is a glide-reflection by Section **9(c)**. The same argument holds if 1 is parallel to 3, and 2 intersects them both.
- If 1, 2 and 3 form the sides of a triangle, choose 2’ through the point of intersection of 2 and 3, perpendicular to 1, and then choose 3’ so that **23** = **2’3’**, which may be done using *Centre* **8** with the method used in *Centre* **12(c)**. Then **123** = **12’3’** = (**12’**)**3’.** Since **12’** is a half-turn, **12’3’** is a glide-reflection by Section **9(c)**.

- If 1, 2 and 3 pass though a common point, then **123** = a reflection, by *Centre* **12(c)**.

Exercise. If the lines 2 and 3 are parallel and the line 1 intersects them both, show that the combination **123** gives a glide-reflection.

Thus a combination of three reflections is either a single reflection or a glide-reflection.

This completes the proof that every symmetry of the plane is either a reflection, a rotation, a translation or a glide-reflection. So our analysis of the possible fixed lines of a symmetry given in Section **8** was complete, and we have determined all possible symmetry groups of a frieze pattern.

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Greek Vases, Dyfri Williams, British Museum Publications, 1985, ISBN 0-7141-2030-8 (for friezes)

The New Book of Chinese Lattice Designs, Daniel Sheets Dye, Dover Publications, 1981, ISBN 0-486-24128-9 (Mostly D_n, some frieze, some tessellations)

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Studies in Design, Christopher Dresser, Studio Editions, 1988, ISBN 1-85170-174-5

The decorative art of Arabia, Prisse d'Avennes, Studio Editions, 1989, ISBN 1-85170-189-3 (originally 1873) (all kinds of patterns)

The Grammar of Chinese Ornament, Owen Jones, Studio Editions, 1987, ISBN 1-85170-237-7 (originally 1867) (mostly central and frieze)

(b) Software

For CABRI see http://www.chartwellyorke.com

GEOMETER'S SKETCHPAD
http://www.keypress.com/sketchpad/java_gsp/ind>ex.html

MSWLogo (distinct from MSW LOGO, mind the gap) can be downloaded freely from http://www.softronix.com

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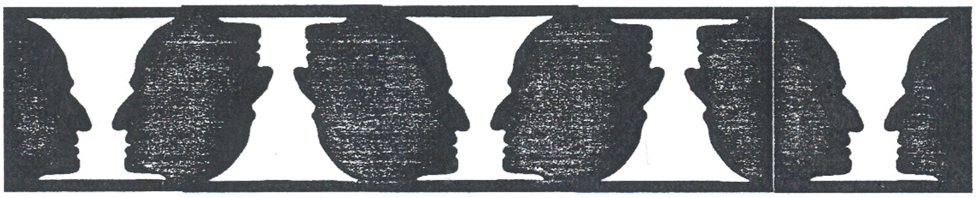

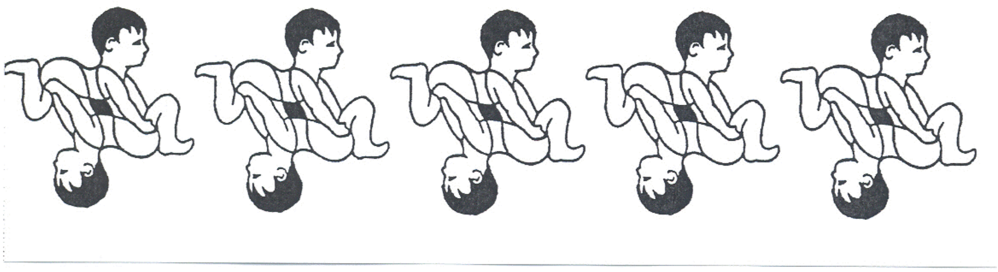

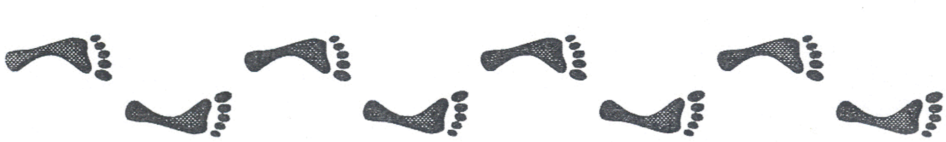

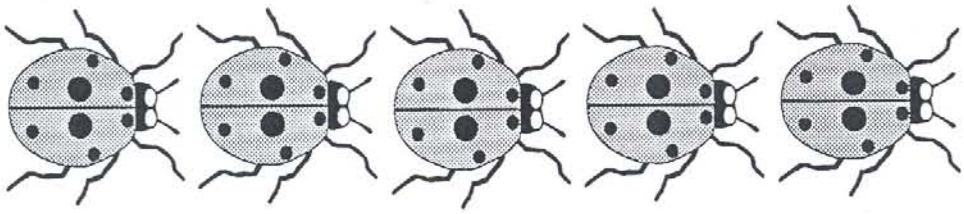

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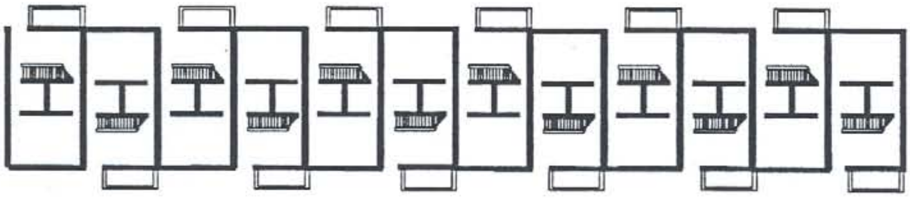

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