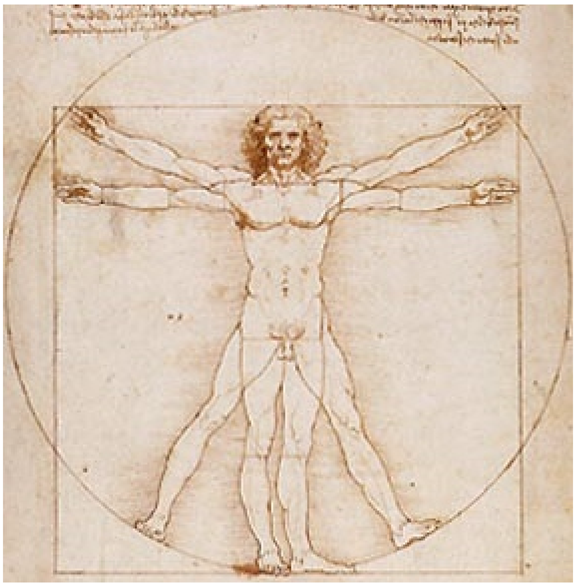
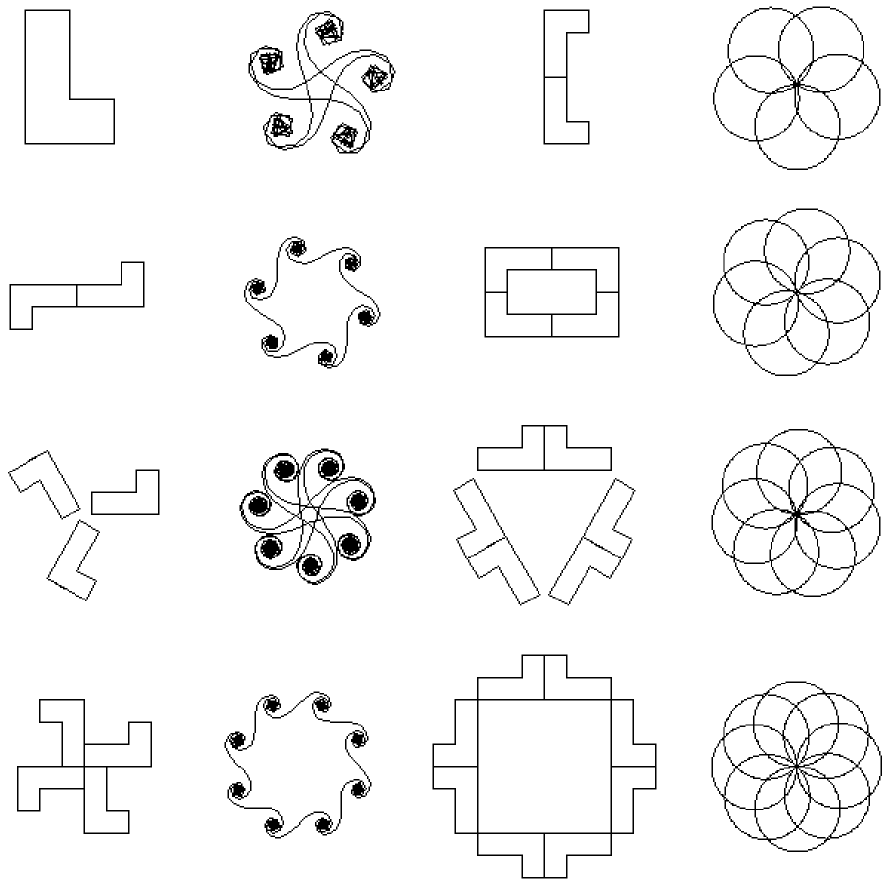
Sorting by Symmetry

Patterns with a Centre



Bob Burn

Looking at symmetrical patterns is intriguing and satisfying. Understanding the way symmetrical patterns are made is part of geometry. We will make a lot of patterns, each with a centre. Two different families will emerge. We will describe these families and show why, when looking at plane patterns with a centre, there are just these two families and no more.



All the symmetries of each of these patterns share a fixed point - a centre

Figure 1.0

Contents

Introduction

The ideas in this book have been developed in the form of a repeating cycle: ***observing, making, sharpening***. First you, the reader, look at some images and objects and notice similarities, next you make something that is similar or related, and then you reflect on what you have done and try to clarify the ideas. Then, when you start the cycle again, you may sometimes start by using some of the ideas that you have sharpened earlier. This repeated cycle gives you a sequence through which you learn by guided discovery, and develop the clarity needed to reason about symmetry.

The presentation presumes that you are familiar with length and angle, and with parallel and perpendicular lines. There are places where the word congruence is used, but if the word puts you off, you can probably skip those bits without much loss. If you have met symmetry before, well and good, but if you have not, what you find here will, I hope, make good sense. It would be useful to have a geoboard and access to LOGO (procedures are given in FMSLogo, which may be freely downloaded from the internet) and a dynamic geometry program. It is essential to have both squared and isometric paper, with either dots or lines, and also some kind of tracing paper - kitchen paper will do. In one place polar graph paper is needed. A minimal number of lines and circles on this graph paper will suffice.

The amount of 'sharpening' that you may be ready for will vary from person to person. Some of the most general arguments have been marked as such so that you can recognise them for what they are and grapple with them only when you feel ready.

We use the word *pattern* when we see some kind of copying or replication. We may see patterns when we look at brickwork, wall paper or floor tilings, or we may see pattern in an ornament. You can start gathering a scrap book of patterns right away.

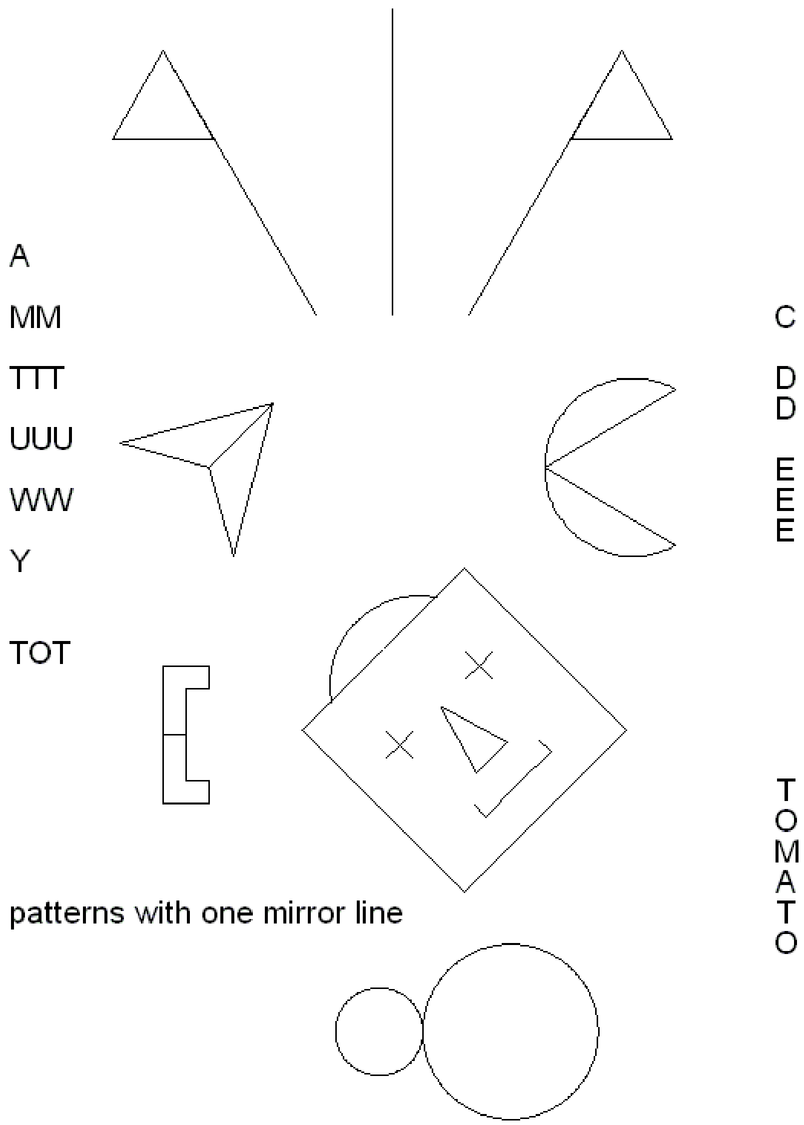

A
MM
TTT
UUU
WW
Y
TOT
C
D
D
E
E
E
TOMATO
patterns with one mirror line

Figure 1.1

1. <u>One reflection</u>, in which a mirror symmetry is linked to its axis, and is found to work all over the plane.

(a) Observing what you can match using one mirror

- Figure 1.1
- Orchids
- Leaves
- Butterfly wings
- Animal bodies (insects, fish, lizards, mammals)
- ↑, →, >, ▯, ∪, ⊃, ♣, ♥, ♠
- Kite
- Parabola
- Volkswagen, Rover, Mazda, Honda, Toyota, Daewoo, Citroën logos.

(b) Making

Here you are invited to *construct* figures which exhibit symmetry like those that you have been observing.

- Ink devils: take a plain sheet of paper and fold it in half; unfold it and then spread wet paint or ink (several colours if you like) on the paper to one side of the fold. Fold the paper again, and press. Then unfold and look at the result.
- Fold paper and use scissors to cut across the fold, continue to cut, wiggling your scissors a bit and then complete the cut by coming out across the fold again. Look at the piece you have cut out *and* the hole.
- Use an elastic band on a geoboard to make shapes with symmetry. Or else try to draw such shapes on dotty paper.
- Draw a hexagon which has just one mirror line.
- Construct a reflection image with a dynamic geometry program.
- Make patterns with LOGO. A flag

  ```
  FD 100 REPEAT 3 [RT 120 FD 30] BK 100
  ```

  has mirror image

  ```
  FD 100 REPEAT 3 [LT 120 FD 30] BK 100
  ```

A **mirror line** is the place where you put a mirror to make what you are looking at appear the same, when in fact, half of it is hidden by the mirror. Mark the mirror lines that you have found. When you copy by paper-folding, the fold in the paper is the mirror line.

(c) Sharpening

Now we are going to think about what you have seen and done, and use another method to look at reflections.
The method is that of pricking through a piece of folded paper. Unlike what you can see with a mirror, where drawings on half the sheet of paper can be seen directly and the other half appears as an image in the mirror, when paper is folded and pricked, both the original and its mirror image are available to look at, and there is no way of knowing which is which.

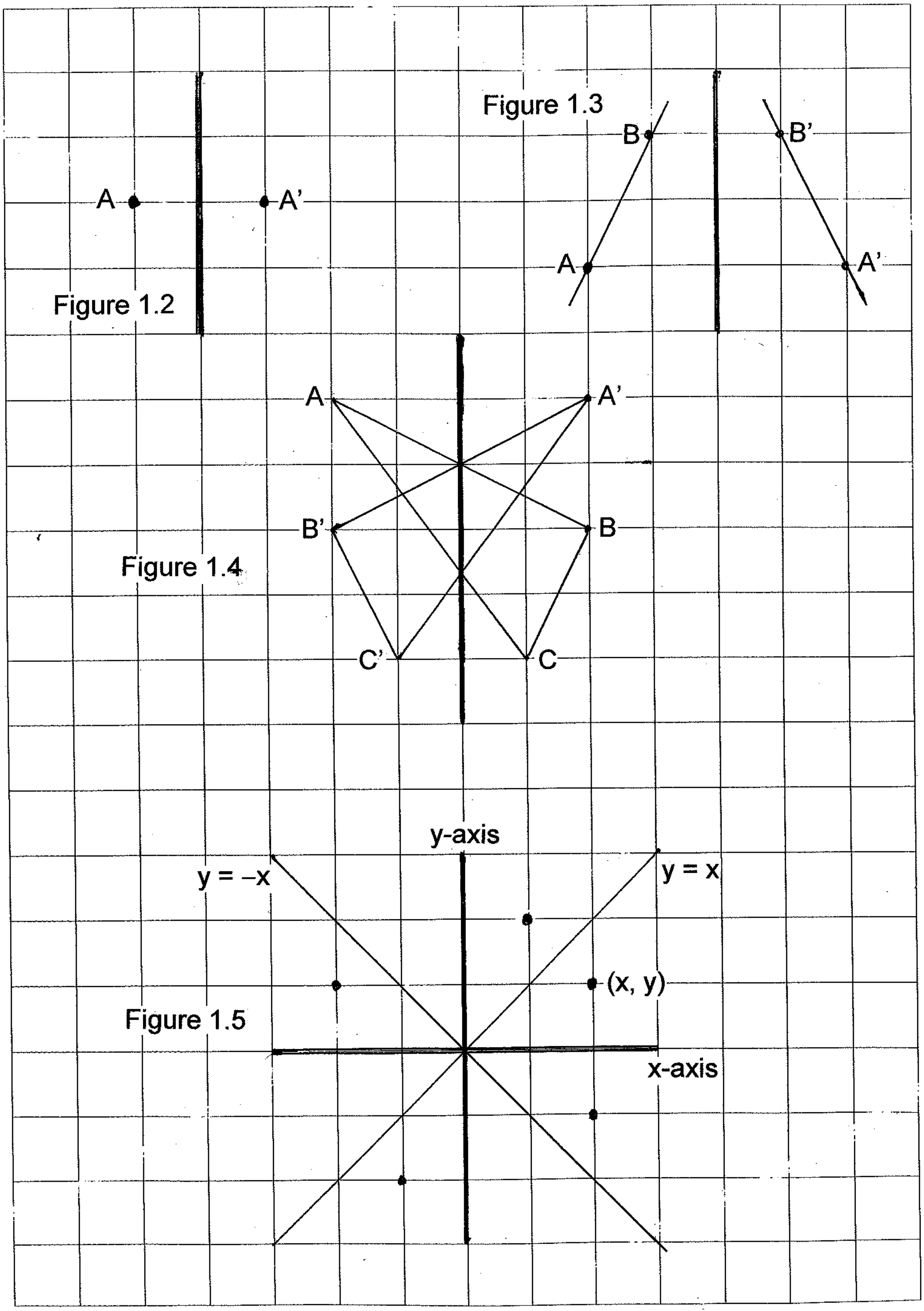


Figure 1.2

Figure 1.3

Figure 1.4

Figure 1.5

1. Fold a piece of unlined paper and press to make the fold precise.
Then take the point of a compass and prick through the folded paper.
Unfold the paper and mark the two pricked places A and A'. (Figure 1.2)
Use a ruler to join AA'. How does AA' relate to the fold line? (Some people call the fold line the *mediator* of AA'.) Notice how the folding matches certain lengths, and so guarantees their (unfolded) equality, and how the folding matches certain right-angles, and so guarantees *their* (unfolded) equality. Have you now got reasons for claiming that the fold line is the perpendicular bisector of AA'?

2. How does folding paper correspond to reflection in a mirror? Does the fold line correspond to the mirror line? The mirror line is called the **axis** of the reflection.

3. Now take a fresh piece of plain paper, fold it and press to make the fold precise. Prick through the paper at *two* points. You obviously get something rather special if the two pricks are along a line either parallel or perpendicular to the fold line and these two special cases should be investigated *after* looking at the general case. Unfold and label the two points from one prick A and A' and the two points from the other prick B and B'. What can you say about the lengths AB and $A'B'$? (Figure 1.3) Use a ruler to join AB. AB may cross the fold line or it may not. If it doesn't, extend the line AB until it does, then do the same with $A'B'$. Where do AB and $A'B'$ meet? How do the lines AB and $A'B'$ relate to the fold line? If AB meets the fold line at M, what is matched with the points A, B and M when the paper is folded?

4. Now take another piece of fresh plain paper, fold it and make the fold precise. Then prick through in three places to make a scalene triangle. (Figure 1.4)
Unfold the paper, mark three pricked points A, B, C, and the corresponding points A', B', C'. What can you say about these two triangles? Use what you found with the two pricks exercise. When you are convinced that they are congruent, look at how a line through A turns from the direction AB to AC. Clockwise or anti-clockwise? Then look at how a line through A' turns from the direction $A'B'$ to $A'C'$. Clockwise or anti-clockwise? Is one direction the opposite of the other?

5. If (x, y) is a point on a pattern with the x-axis as a reflection axis (or mirror line), what other point must be on the pattern? (Figure 1.5)
If (x, y) is a point on a pattern with the y-axis as a reflection axis, what other point must be on the pattern?
If (x, y) is a point on a pattern with $y = x$ as a reflection axis, what other point must be on the pattern?
If (x, y) is a point on a pattern with $y = -x$ as a reflection axis, what other point must be on the pattern?

patterns with two mirror lines

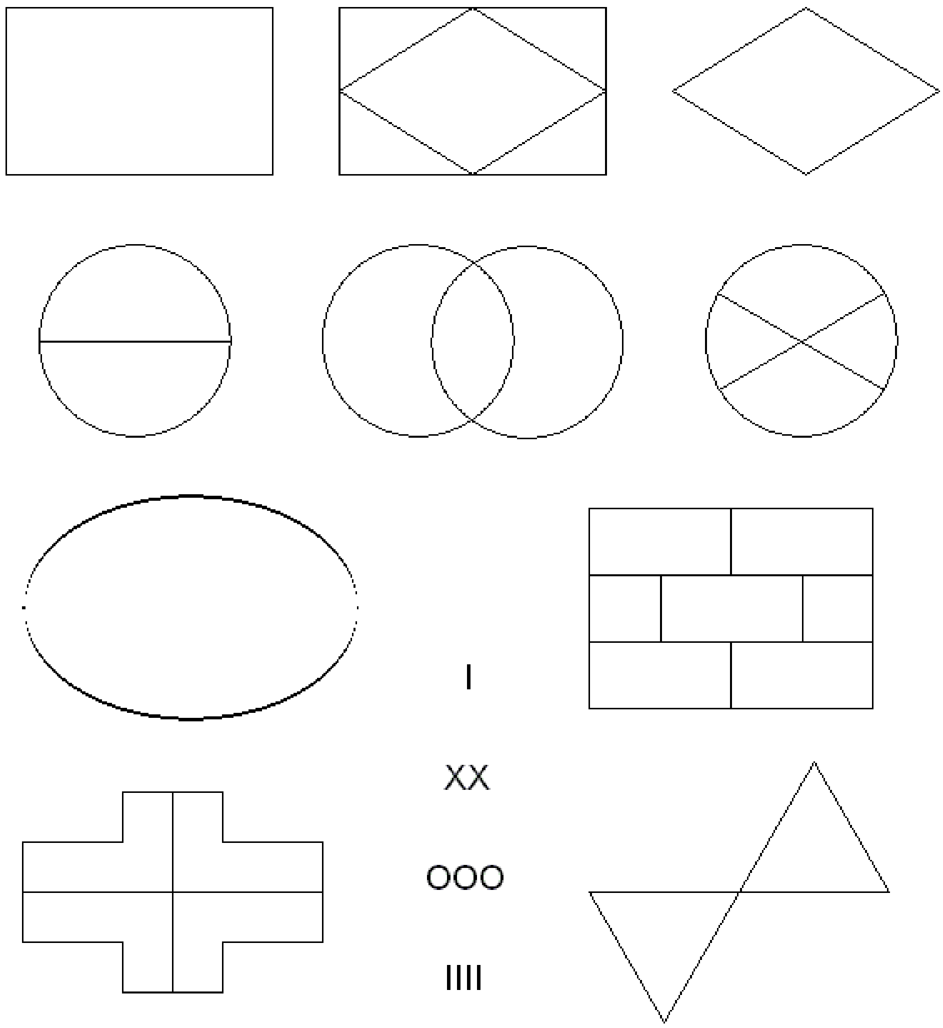


Figure 2

2. Two reflections (with intersecting axes), in which patterns with just two mirror lines are compared.

(a) Observing what you can match with a mirror

- Figure 2
- Hyperbola (two branches)
- −, =, ≡, [], {}, :, θ, ϕ
- BMW, Audi logos

Try to spot two mirror lines in each case.

(b) Making

- Make patterns with two reflections with rubber bands on a geoboard or with a pencil on dotty paper
- Fold paper and then fold again with the first fold lying on itself, then make a straight or wiggly cut with scissors across the double fold point. Look at both the hole and the cut-out.
- Draw a hexagon which has just two mirror lines
- Draw some patterns with LOGO, such as
  REPEAT 2 [FD :A RT 90 FD :B RT 90]; choose numbers for :A and for :B.

(c) Sharpening

1. With a fresh sheet of plain paper, do the double fold as in **2(b)Making,** second activity. Then, with a compass point prick right through the four overlying sheets. Unfold. What is the shape made by the four pricked points? What else can you call the two fold lines for this shape?

2. After making a wiggly cut, in the twice folded paper (as in **2(b)Making**, second activity), unfold the paper, and examine the cut-out and the hole. How many ways can you make the cut-out fit in the hole?
Try to describe the four ways in which you can replace the cut-out.
Two come directly from the two fold lines. How might you describe the other two ways?

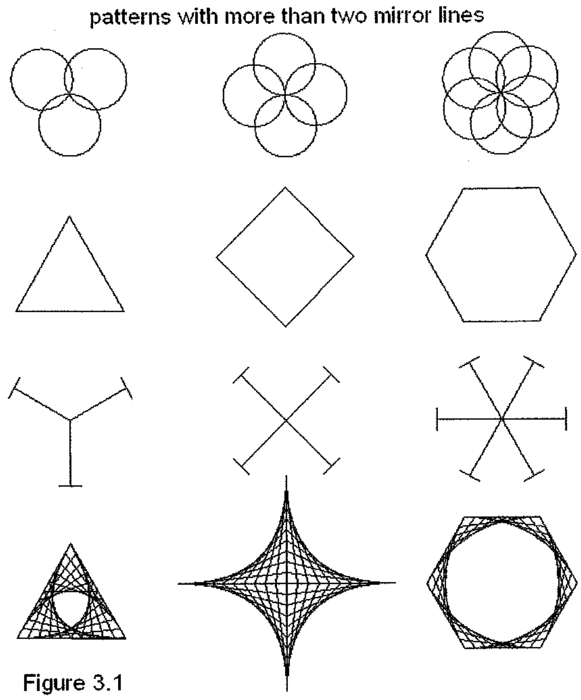
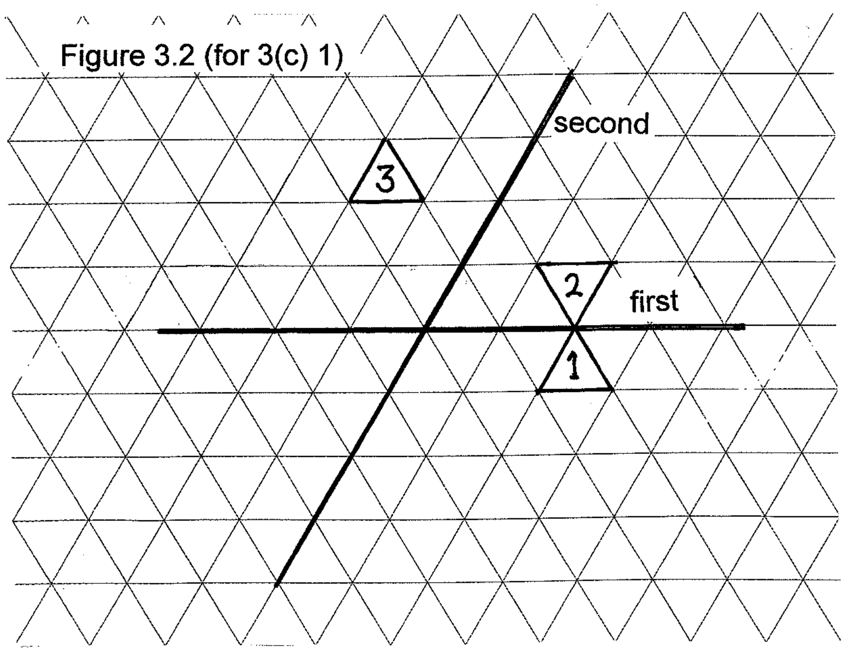


Figure 3.1



3. **<u>Two or more reflections (with axes through one point)</u>**, in which you get three (or more) mirror lines for the price of two.

(a) Observing - how many mirror lines can you find?

- Flower petals of monocotyledons (e.g. *iris*) and other flowers.
- Snowflakes
- Starfish
- Maltese Cross
- Figure 3.1
- Mitsubishi, Mercedes-Benz logos
- Some hubcaps
- The polar graph $r = 10 \sin 2\theta$ (if you have a program to sketch this)

(b) Making

- Use a compass to make equilateral triangles and regular hexagons. Keep the radius fixed. Draw one circle. Put the compass point on the circumference and draw another circle. Then put the compass point where the two circles intersect and draw another circle. That gives you the vertices of one equilateral triangle. Continue to make more. Sooner or later you get the vertices of a regular hexagon.
- Make patterns on squared, isometric or polar graph paper with at least two mirror lines.
- Draw a hexagon which has just three mirror lines
- Make patterns inside a Spirograph wheel.
- Sew patterns with coloured wool or thread on a cardboard base. To create curve stitching designs see *Curve Stitching* in **17.** Reading list.
- Make patterns with LOGO, especially regular polygons.
  REPEAT :N [FD 100 RT 360/:N] makes a regular polygon with *N* sides.
  REPEAT :N [FD 100 RT (360/:N)*:A] may make a regular or a star polygon depending on the value of :A. Choose numbers for :N and :A with :A < :N < 12.
- Explore the possibility of multiple folds of plain paper with the fold lines sharing a common point, cutting across near that point with scissors.
- Get two similar hand mirrors. Put their reflecting faces together and holding them in this position, join the two mirrors with sticky tape along one of the short edges. Then open the mirrors and rest them on some polar graph paper, so that you can check the angle between the mirrors. Use your homemade kaleidoscope to find what angle between the mirrors gives exactly six images to look at. What angle gives eight?
- More ideas in *Starting from Mirrors*, in **17.** Reading list.

(c) Sharpening

When a pattern has *two* mirror lines there are often some surprising consequences. The next activity is the paper and pencil counterpart of the kaleidoscope in **3(b)** just above.

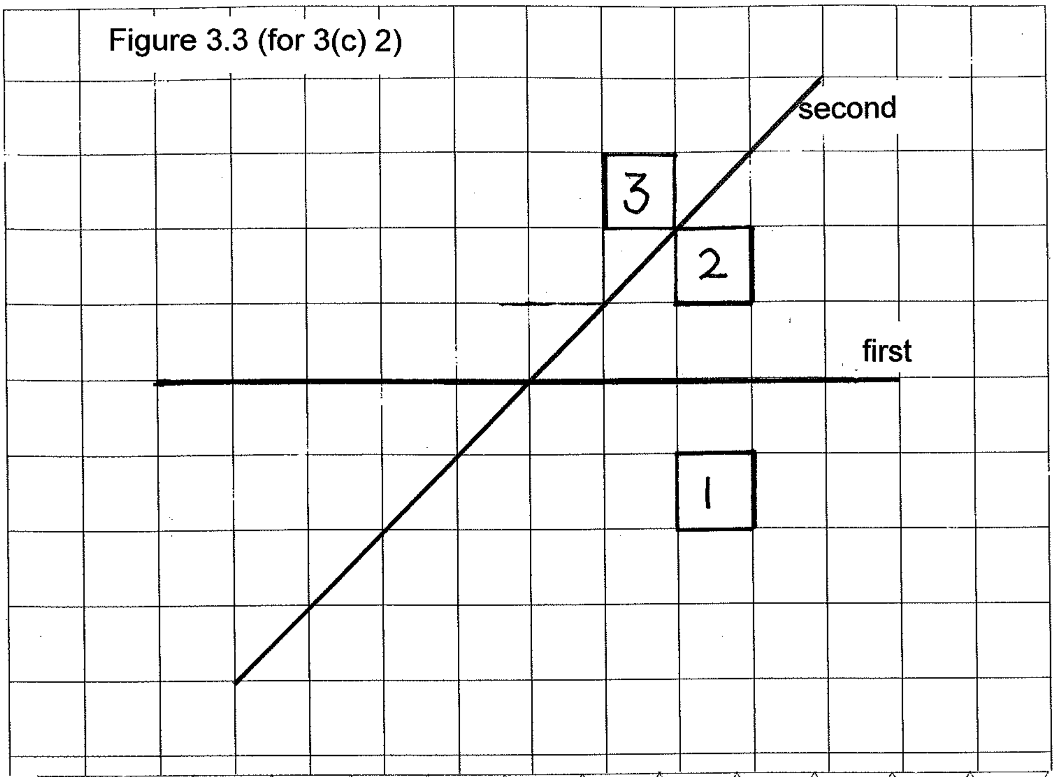


Figure 3.3 (for 3(c) 2)

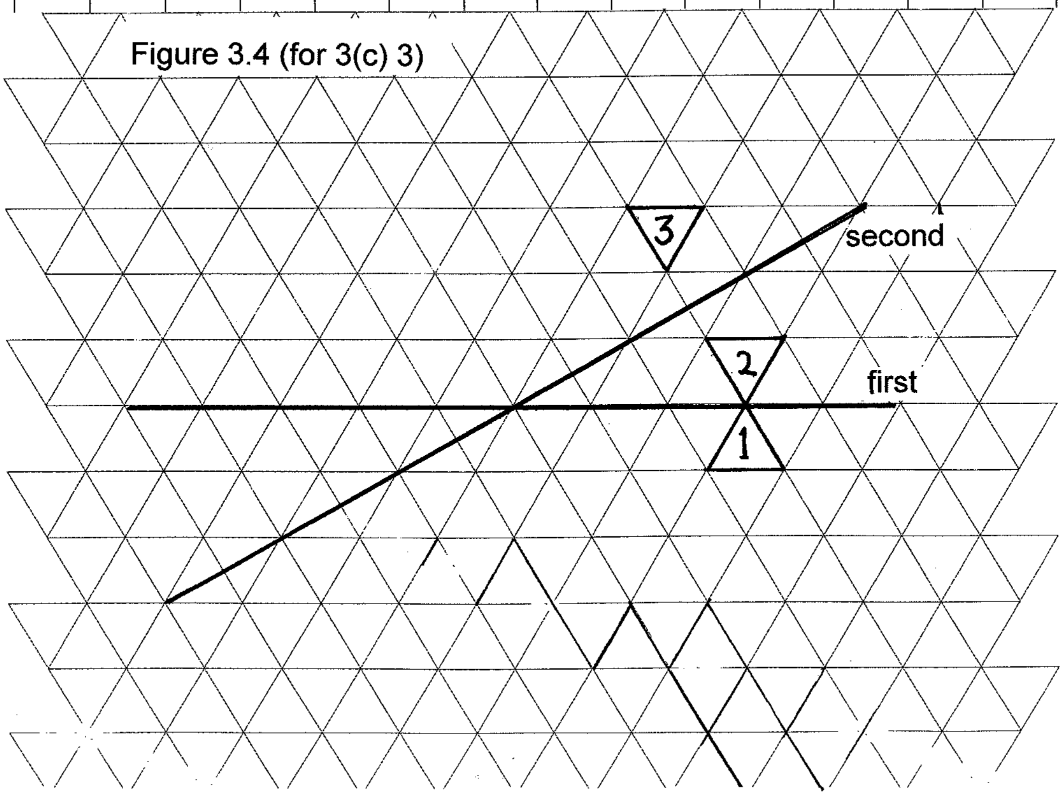


Figure 3.4 (for 3(c) 3)

1. (Figure 3.2) We will now look at what a pattern must be like if it has two reflections in axes at 60°. With some isometric paper boldly mark two lines at an angle of 60° (which will act as mirror lines, or axes of reflection). Label one of them **first** and the other of them **second**. Now choose one of the printed triangles on the paper, mark its outline clearly, and write a number **1** inside it. We are going to label some of the other printed triangles with the numbers **2**, **3**, **4**, **5** and **6**. Look for the reflection of triangle **1** in the **first** axis. If you are not sure whether you have found it you can check by folding along the **first** axis. Write a number **2** inside the reflection of **1** in the **first** axis.

Now find the reflection of **2** in the **second** axis. If you are not sure whether you have found it you can check by folding along the **second** axis. Write a number **3** inside the reflection of **2** in the **second** axis.

Now continue marking triangles, alternating between reflection in the **first** axis and reflection in the **second**. **1↔2**, **3↔4** and **5↔6** will be corresponding images under reflection in the **first** axis, and **2↔3**, **4↔5** will be corresponding images under reflection in the **second axis**.

(a) What do you notice about the relation between triangles **1** and **6**? A consequence of this relation is that if we went on repeating the process, six triangles are all we would get.

(b) Now stand back and look at all six triangles making a pattern together. The pattern must be symmetrical about the **first** and **second** axes because that is the way we made it. Is the set of triangles symmetrical about another axis? Although we started with just two mirror lines, you may have found more than you bargained for!

2. (Figure 3.3) A similar exploration can be made about what patterns have two reflections in axes at 45°. This time you need to use ordinary squared paper. Mark a **first** and **second** axis as before, but this time choose a printed *square* to label **1**. Then reflect **1** in the **first** axis to find **2**. Reflect **2** in the **second** axis to find **3**. Continue reflecting in **first** and **second** alternately, to make **1↔2**, **3↔4**, **5↔6**, **7↔8** images under reflection in the **first** axis and **2↔3**, **4↔5**, **6↔7** images under reflection in the **second** axis.

(a) What do you notice about the relation between squares **1** and **8**? A consequence of this relation is that if we went on repeating the process, eight squares are all we would get.

(b) Now stand back and look at all eight squares making a pattern together. The pattern must be symmetrical about the **first** and **second** axes because that is the way we made it. Is the set of squares symmetrical about another axis? Two more? Starting with reflection symmetry about two axes may create more symmetry automatically.

3. Optional supplementary problem. (Figure 3.4) Using isometric paper and two axes of reflections inclined at 30°, follow the structure of 1 and 2 above, to find what other symmetries are implied. If two printed equilateral triangles on isometric paper have a common side, then together they form a rhombus with angles of 60° and 120°. The long diagonal of the rhombus is at 30° to the sides. How many more mirror lines must there be if two mirror lines at an angle of 30° are given?

patterns with half turn symmetry and no mirror line
find the centre of each pattern

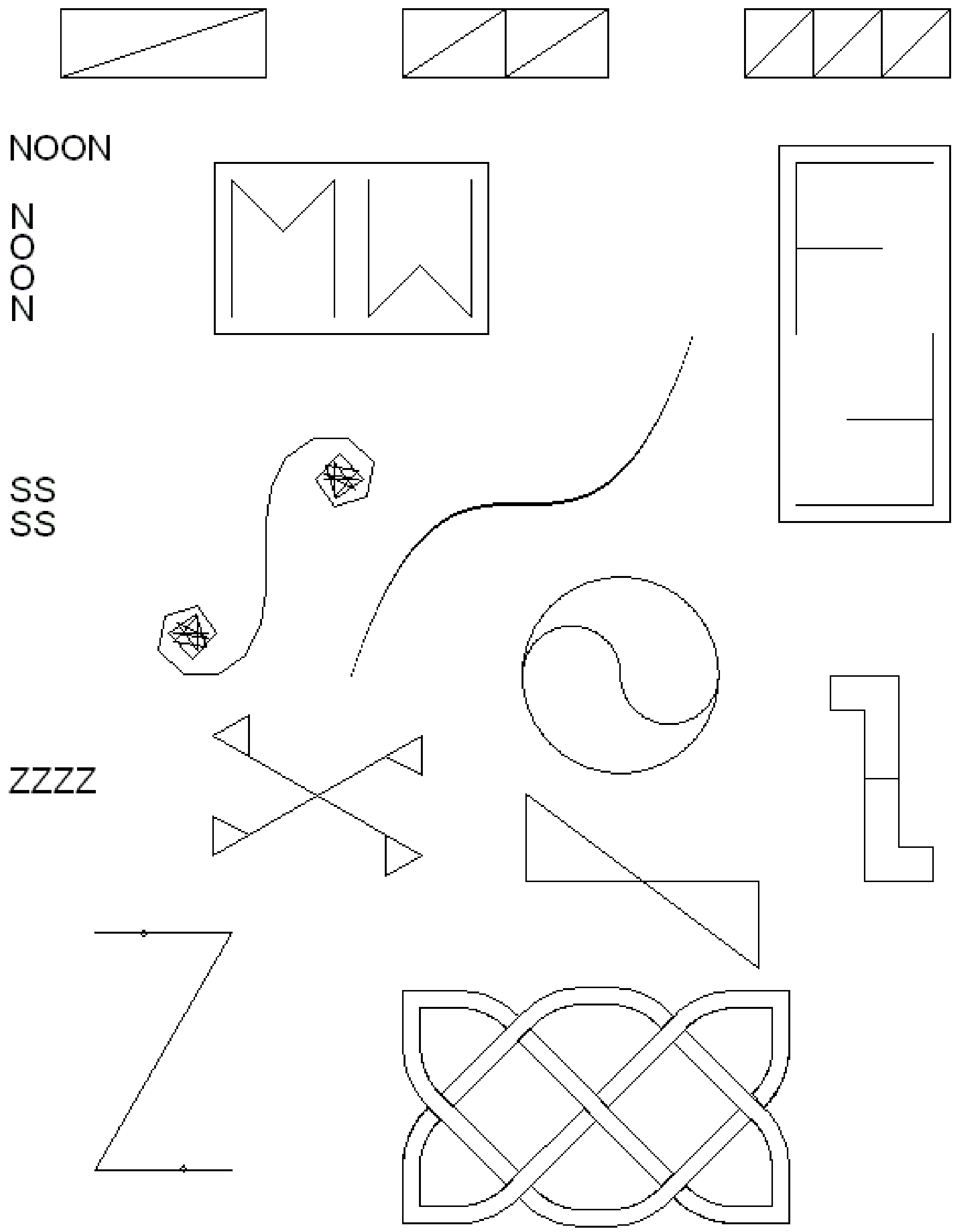


Figure 4

4. <u>**Half turn**</u>, in which we find symmetrical copying by turning.

(a) Observing

You can check a turning symmetry by using tracing paper or making a photocopy on a transparency. You need to pin the tracing to the original at the central point and then rotate the tracing.

- Figure 4
- National Rail sign.
- Parallelogram (but not a rhombus or rectangle if you want a bare half turn)
- Has been recycled sign
- Some Celtic knots
- %, $, ≠, #
- The graph of $y = x^3$
- Renault, Hyundai, Fiat, Suzuki, Seat logos

Try to find the half turn centre in each case and check that there are no mirror lines.

(b) Making

- Use a rubber band, stretched to look like a line segment that may have kinks in it, to divide a 9 pin or 16 pin geoboard into congruent halves. The rubber band should not look like a loop. The same exercise can be done with a pencil on square dotty paper.
- Cut out a parallelogram with scissors. In how many ways can it be put back in the hole it has left?
- Draw a hexagon that has half turn symmetry but no mirror lines.
- Use an unsymmetrical *motif* in LOGO, one in which the turtle starts and finishes in the same place and in the same direction (sometimes called 'state transparent') such as

```
TO FLAG
  FD 100
  REPEAT 3 [RT 120 FD 30]
  BK 100
  END
```

  Make two flags pointing in opposite directions, for example, drawn with a command such as REPEAT 2 [FLAG RT 180]

(c) Sharpening

1. If you have a large Z and a tracing of it, mark a point on the outer line of the Z and on the tracing of that point with a bold dot. Then do the half turn and where the dot on the tracing meets the original shape again, mark this point also with a dot. What is the mid-point of the two dots on the original? See Figure 4.

2. If a figure has half turn symmetry about the origin (0, 0), and (x, y) is one point of the figure, name another point of the figure.
3. Look for half turn symmetry in the shapes you looked at earlier with just two reflection axes, in Section **2**.

4. Do the points made by pricking the plain paper folded twice, as in **2(c)**, with the second fold taking the first fold onto itself, have half-turn symmetry?

Rotational symmetry

one-third turn

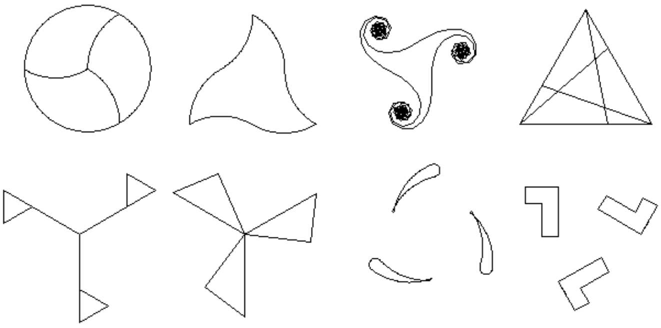


Figure 5

quarter turn

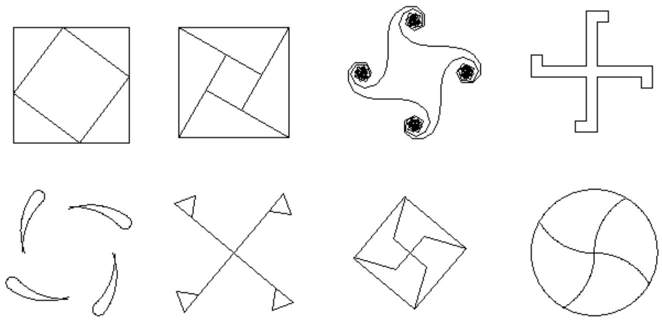
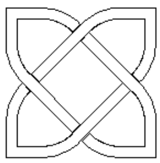


Figure 6



5. <u>One-third turn</u>, in which three copies are made by turning.

As with the half-turn, use tracing paper to check the existence of this kind of symmetry.

(a) Observing

- Figure 5
- Nat West sign
- Wool sign
- Should be recycled sign (three bent arrows)
- Isle of Man sign
- Some toy windmills

Try to find the centre of the one-third turn and check that there are no mirror lines in each case.

(b) Making

- With a compass at a fixed radius, draw one circle and then with the compass point on the circumference, join the centre of the first circle to the circumference. Do this three times. See the first pattern in Figure 5.
- Draw your own design which exhibits a one-third turn on isometric paper.
- Cut out the drawing you have just made on isometric paper. In how many ways may it be reinserted in the hole that it has left?
- With a state transparent motif in LOGO (for example, FLAG, as in Section **4 (b)**), try a procedure such as REPEAT 3 [FLAG RT 120]

(c) Sharpening

1. Which of the patterns with two or more reflections also have one-third turn symmetry?
2. If a pattern has symmetry through a one-third turn must it have symmetry through a two-thirds turn also?

6. <u>Quarter turn</u>, in which four copies are made by turning.

(a) Observing

- Figure 6, which includes the next two examples
- Square for Pythagoras
- A few Celtic knots

Find the quarter turn centre in each case and check that there are no mirror lines.

(b) Making

- On squared paper: one big square with four small squares outside the big square but with their sides on those of the big square near the four corners of the big square.
- Circle with four semicircles of half radius inside.
- Use LOGO to try REPEAT 4 [FD :A RT 90 FD :B RT 90 FD :C RT 90] using your own choice of the numbers :A, :B and :C.

(c) Sharpening

1. Which of the patterns with two or more reflections (Section **3**) also have quarter turn symmetry?

2. If a pattern has quarter turn symmetry must it have half turn symmetry and three-quarter turn symmetry also?

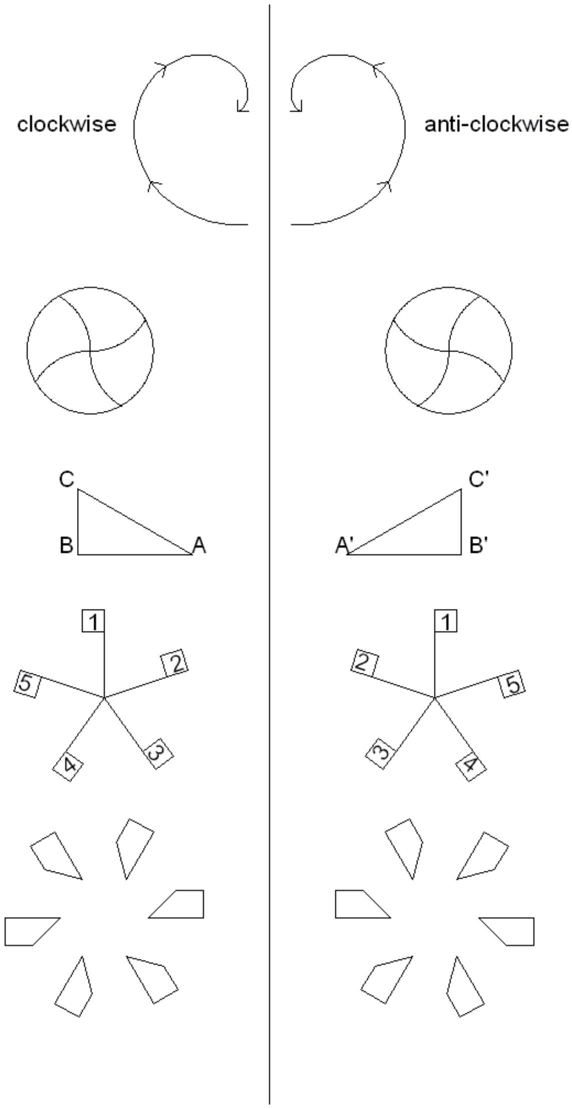


Figure 7

From this point on, we will be clarifying the words we have been using, until we are sure that we have found all the symmetrical kinds of pattern with a centre.

7. <u>Clockwise and anti-clockwise</u>, in which we see what reflections change.

When, after a paper fold, you used scissors to cut out a shape by cutting across the fold, you simulated a reflection by fitting the cut-out back in its hole with a flip-over. The cut-out was turned through 180° in three dimensions and reinserted back to front. This experience is handy for certain purposes, but misleading for others. It might make you think that a reflection was some form of rotation in disguise.

(a) Observing

Watch yourself writing in a large mirror. If you are right-handed, the person in the mirror is left-handed, and *vice-versa*.

Watch the second hand on a clock turning (clockwise) and then look at it in a mirror; the hand of the clock in the mirror turns the other way (anti-clockwise).

Look at the curl in Figure 7, and its mirror image.

Think of the circles with curvy radii (in Figure 7) as the view from above a chimney cap. Which way round will the cap move when the wind blows?

The labels by the vertices of the triangles (in Figure 7) are not part of the pattern; they are there to indicate which points are matched by the reflection $A \leftrightarrow A'$, $B \leftrightarrow B'$ and $C \leftrightarrow C'$. Now look at the circuits $A \rightarrow B \rightarrow C \rightarrow A$ and $A' \rightarrow B' \rightarrow C' \rightarrow A'$. Which way round do they go?

The numbers on the groups of five little flags (in Figure 7) are not part of the pattern, they are only there to mark which flag is matched with which under the reflection. Which way around do the circuits $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1$ go?

(b) Making

Cut out an irregular quadrilateral from cardboard. Use it as a template to make a pattern and its mirror image, as in the pattern with a 60° rotation at the bottom of Figure 7. Locate which of the quadrilaterals can be drawn from the template with one side up, and which can only be drawn with the other side up.

You will have found that under a reflection, when the template is turned upside down

clockwise → anti-clockwise, and
anti-clockwise → clockwise.

You can also do this exercise with a dynamic geometry program.

You will also have found that under a rotation in the plane

clockwise → clockwise
anti-clockwise → anti-clockwise.

(c) Sharpening

So there *is* a difference between a reflection and a rotation. What do you guess will happen to clockwise and anti-clockwise after *two* reflections?

Because the patterns we made with reflections generally have rotations as well, we will examine why that must be.

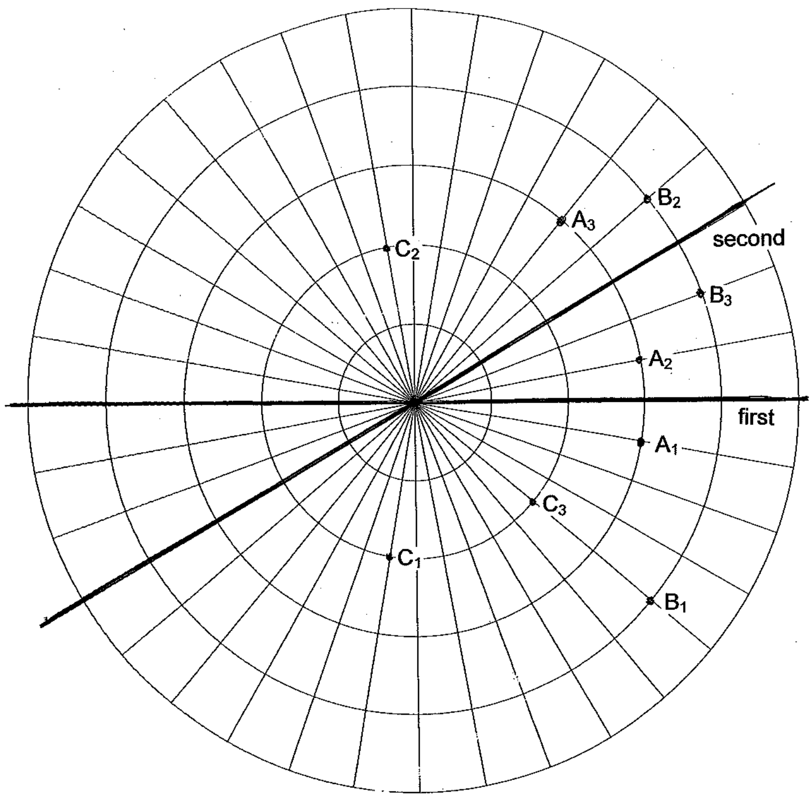
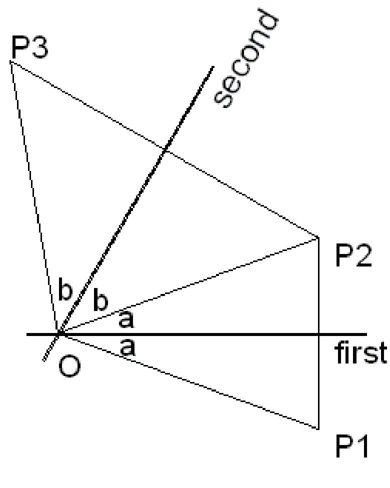

B₂
A₃
second
C₂
B₃
A₂
first
A₁
C₃
C₁
B₁

Figure 8.1


second
P3
P2
b
b
a
a
first
O
P1

8. <u>Combining two reflections</u>, in which a rotation is made from two reflections.

(a) Observing

Of the patterns with reflection symmetry that you have seen or made, which ones have a rotation (or turning) symmetry?

(b) Making

Take a piece of plain paper, fold it once and prick to give two points matching with a reflection: A_1 and A_2. Unfold the paper and label the two points on both sides of the paper. For each pricked point, the label must be the same on front and back. Then fold the paper again (from an unfolded state) in such a way that the second fold line clearly intersects the first fold line. Now prick through the point A_2 to give one new pricked point A_3. Unfold the paper again. You should see A_1 and A_2 matching across the first fold line and A_2 and A_3 matching across the second fold line. If you label the intersection of the two fold lines O, what can you say about the lengths OA_1, OA_2 and OA_3. How are these lines inclined to the two fold lines?
If we reflected in the first fold line *and then* reflected in the second fold line, what point is A_1 matched with? [$A_1 \to A_2 \to A_3$] How is the angle $\angle A_1OA_3$ related to the angle between the two fold lines?

Explore the combination of two reflections with a dynamic geometry program.

(c) Sharpening

Take a piece of polar graph paper. Mark two long lines through the centre boldly and clearly. Label one of them **first** and the other **second**. Now choose any grid point on the sheet, and mark it A_1. It is best to start with a grid point that is *not* on the lines labelled **first** or **second**, though such points should be considered later, when you have some confidence about what is going on. Mark the reflection of A_1 in **first** as A_2. Then mark the reflection of A_2 in **second** as A_3. Start again with a different grid point on the same polar graph paper and label it B_1. Mark the reflection of B_1 in **first** as B_2. Then mark the reflection of B_2 in **second** as B_3. Start again with a different grid point C_1 etc. See Figure 8.1.
Continue building up such triples of points until you think you can recognise the symmetry that carries P_1 (*via* P_2) to P_3.

<u>Combining two reflections</u>: When two reflections in intersecting axes are combined, the result is a rotation about the point of intersection of the axes, through twice the angle between the axes.

combining two reflections

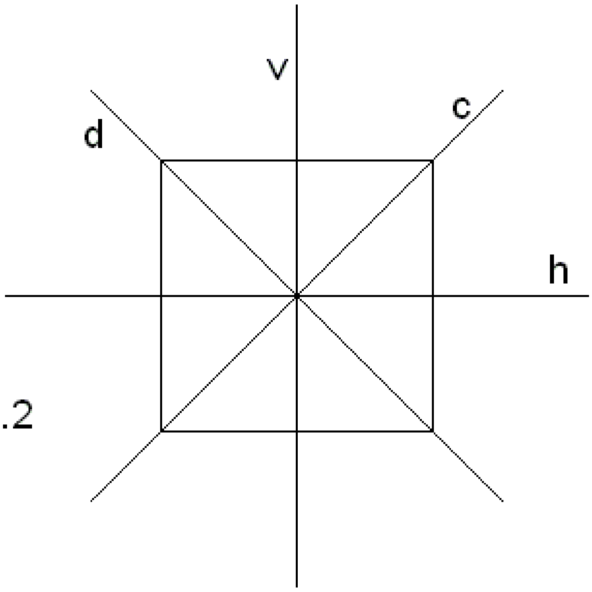


Figure 8.2

The letters *h*, *c*, *v*, *d* label axes of reflection of the square shown in Figure 8.2 (*h* for horizontal, *v* for vertical and *d* for diagonal).
In the table below are spaces to enter 0°, 90°, 180° or 270°, the anti-clockwise rotation symmetries of the square, obtained by combining two of the reflections. Use the combination result just found.

second reflection

first reflection		h	c	v	d
	h				
	c				
	v				
	d	90°	180°	270°	0°

It is interesting to note where the entries are symmetric in the table, giving **xy** = **yx**, so to speak. In all, eight entries appear symmetrically and eight do not.

The next step in our path to clarity, is to ask what is *the same* about rotations and reflections, that is, to give a meaning to the word 'symmetry'.

9. <u>What is a symmetry?</u>, in which we catch hold of what reflections and rotations share.

We saw in **1(c)** that a reflection matched a line segment with another line segment of the same length. This makes a reflection match a triangle with a congruent triangle, and match angles with equal angles.
In Section **8** we saw that a rotation is formed when two reflections are combined one after the other, and in fact any rotation may be decomposed into two reflections. So, like reflections, rotations match equal lengths, congruent triangles and equal angles. These properties, which reflections and rotations share, are taken as the defining properties of a **symmetry** (or isometry). Because the congruence of triangles and the equality of angles follow from the equality of lengths, we can define a symmetry as a matching (of the points of the plane) in which lengths are matched with equal lengths.

Definition of symmetry in the plane.
A symmetry is a matching of points of a figure or of the plane such that if A is matched with A' and B is matched with B', then

the length AB = the length $A'B'$.

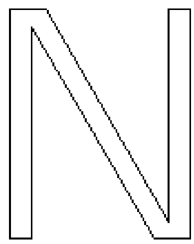
Reflections and rotations are examples of symmetries. There are more.

Because the word rotation also describes a movement, you might think that if the point A is matched with the point A' by a particular rotation, there would be a unique route along a circular arc under which A may be moved to A'. However the definition only specifies the start and finish points. Symmetry is something which you may recognise in a figure in which you only know the appearance before and after. It is as if you had turned your back while the process was happening. Possible intermediate positions are not part of the definition.

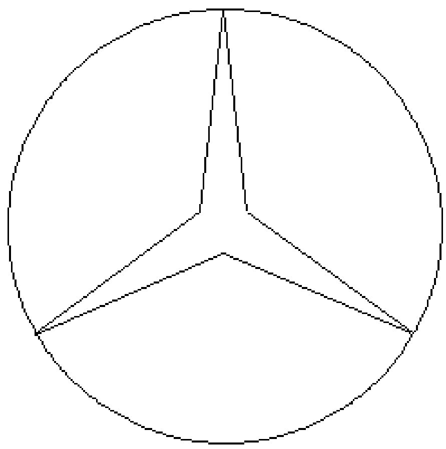
Now we are going to bring in a new word (the word *group*) to describe *all the symmetries* of whatever pattern we are looking at.

Identity

(i)



(ii)



(iii)

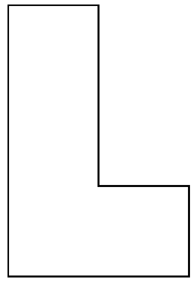


Figure 10.1

10. <u>Symmetry group</u>, in which we list all the symmetries of a pattern.

i. The identity
(a) Observing
Count the number of rotations and reflections you can see for
(i) a capital N, (see Figure 10.1)
(ii) the Mercedes-Benz logo, (see Figure 10.1)
(iii) F, G, L, P, Q, the Alpha-Romeo, Ford, Nissan, Peugeot and Vauxhall logos.

You may have found one for the N (i), five for the M-B logo (ii), and perhaps none for the capital letters and subsequent logos (iii); but there is a kind of non-event that counts as a symmetry once we have the definition of symmetry. Leaving everything where it is, leaves lengths unchanged. This symmetry is called **the identity** and may also be thought of as a rotation through 0° or 360°. If you add in this one, you should have found two symmetries for the N (i), six for the Mercedes-Benz logo (ii), and one each for the capital letters and subsequent logos (iii). What other capital letters only have the identity symmetry?

(b) Making

If you cut out a figure that you have drawn on a piece of paper, then the number of ways in which the figure may be replaced in the hole from which it has been cut, gives a method of counting its symmetries. Apply this idea to a square and make a list of its symmetries. You can simulate a reflection by turning the square over.

(c) Sharpening

> A complete list of all the symmetries which preserve the appearance of an object or pattern is called its **symmetry group**.

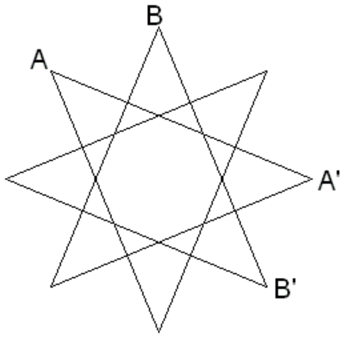
<u>The identity</u> (*a theorem*). Every symmetry group contains the identity. This is rather obvious, because if you leave something where it is, all distances on it are unchanged. The identity is usually called **I**.

ii Inverses
(a) Observing

Make a list of the symmetries of a square. Use the figure at the end of section **8** to help you.

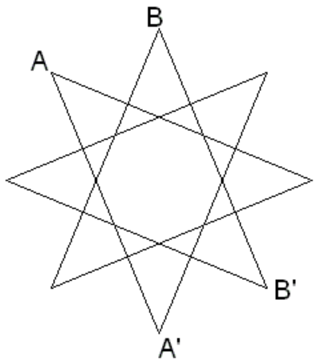
	anti-clockwise rotations				Reflections			
symmetry	0°	90°	180°	270°	**h**	**c**	**v**	**D**
inverse								

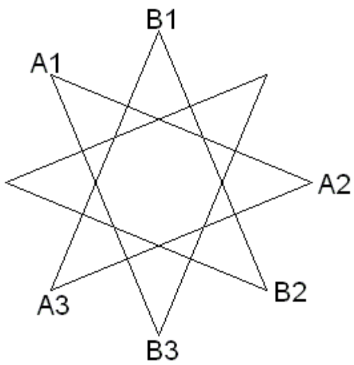
Below this list of eight symmetries, try to make another list in which each symmetry in the second list undoes its respective symmetry in the first list. If a symmetry in the first list matches *P* to *Q*, then the corresponding symmetry in the second list must match *Q* to *P*. [Beware! Some symmetries are undone by themselves!]



symmetry	inverse
A to A'	A' to A
B to B'	B' to B

symmetry	inverse
A to A'	A' to A
B to B'	B' to B





symmetry R	symmetry S
A1 to A2	A2 to A3
B1 to B2	B2 to B3

symmetry RS
A1 to A3
B1 to B3

Figure 10.2

Generally a rotation through $a°$ is undone by a rotation through $-a°$ or $(360 - a)°$ about the same centre, and a reflection is undone by itself.
The standard word for this is the matching of each symmetry with its **inverse**. The nice thing to notice is that all the inverses are in the original group. Actually this is guaranteed by the definition of symmetry group. [There is no **(b)** here.]

(c) Sharpening

Inverses (*a theorem*). If a symmetry **S** is in a symmetry group, then its inverse, denoted by $\mathbf{S}^{-1}$, is also in the group.
This is because if **S** (matching A with A' and matching B with B') is a symmetry, it matches lengths with equal lengths, $AB = A'B'$. But then $A'B' = AB$ and this makes $\mathbf{S}^{-1}$ a symmetry in the group. (Figure 10.2)

iii. Closure
(a) Observing
In Section **8** we put together two distinct reflections and found they made a rotation. This explained why symmetry groups with at least two reflections with intersecting axes always contained rotations.

Notice that $\mathbf{SS^{-1}} = \mathbf{I} = \mathbf{S^{-1}S}$. Also that when **R** is a reflection $\mathbf{RR} = \mathbf{I}$.

(b)Making

Practise this idea, by seeing how combining two reflection symmetries of a square gives one of its rotation symmetries, as in Section **8 (c)**. Try several pairs of reflection axes.

What do you get if you combine two rotations with the same centre through angles of $a°$ and $b°$ (both anti-clockwise)?

(c) Sharpening

If you do one thing that leaves an object looking the same and then another, overall, the object looks the same!

Closure (*a theorem*). If symmetries **R** and **S** are in a symmetry group, then the combined symmetry **RS** is also in the group. This is true even when **R** = **S**.
RR is usually written $\mathbf{R}^2$.

	R		**S**				**RS**	
A_1	→	A_2	→	A_3	gives	A_1	→	A_3
B_1	→	B_2	→	B_3	gives	B_1	→	B_3

Because **R** is a symmetry, $A_1B_1 = A_2B_2$. Because **S** is a symmetry, $A_2B_2 = A_3B_3$. Therefore $A_1B_1 = A_3B_3$ and so **RS** is a symmetry. (Figure 10.2)

Practice. If **R** and **S** are both reflection symmetries of a square with axes inclined at 45°, describe the difference between **RS** and **SR**.
Is it possible to have two different reflection symmetries **R** and **S** of a square such that **RS** = **SR**?

Our next step of clarification is to see what the symmetry groups which only contain rotations have to be like.

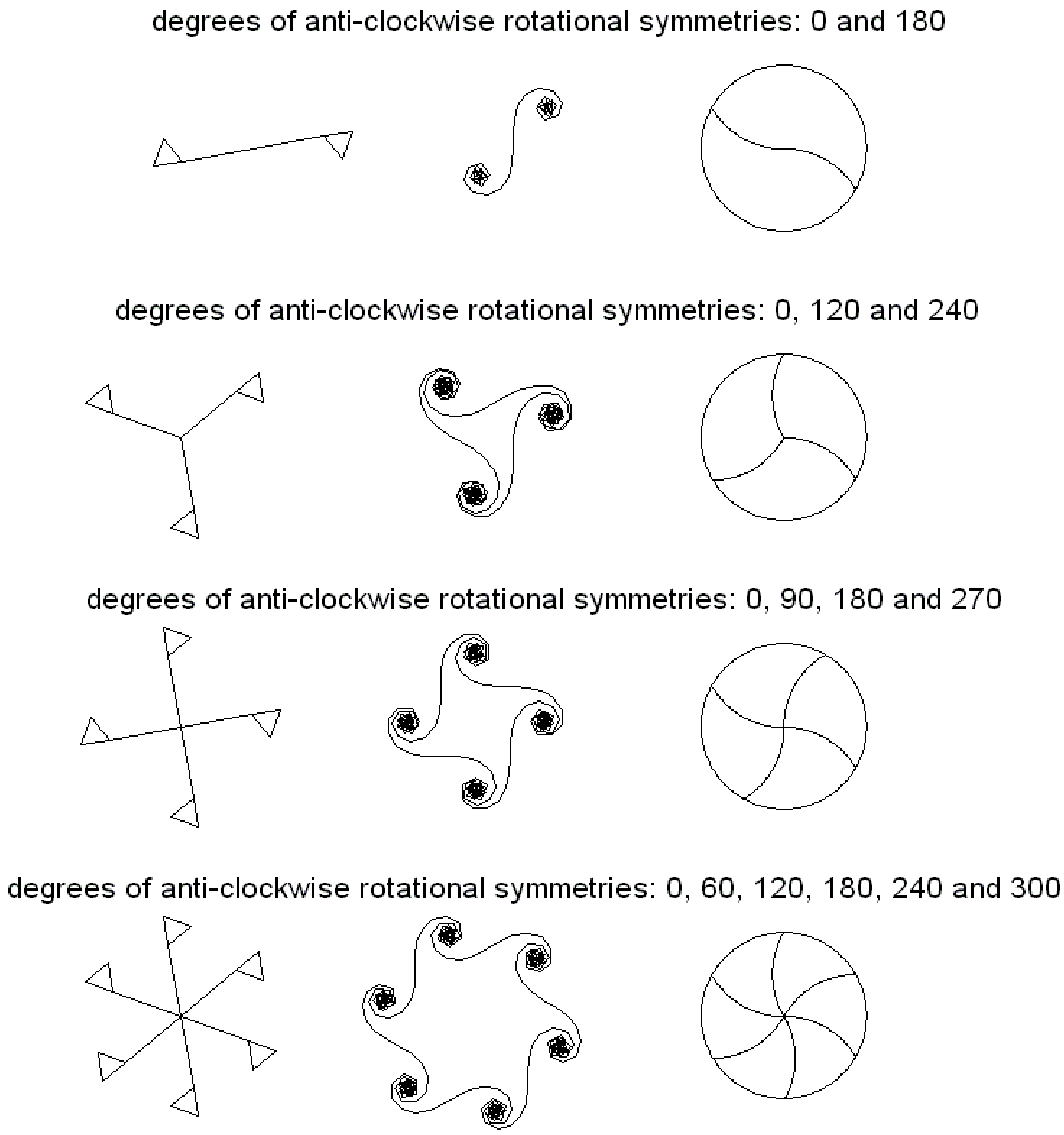

degrees of anti-clockwise rotational symmetries: 0 and 180
degrees of anti-clockwise rotational symmetries: 0, 120 and 240
degrees of anti-clockwise rotational symmetries: 0, 90, 180 and 270
degrees of anti-clockwise rotational symmetries: 0, 60, 120, 180, 240 and 300

Figure 11.1

11. Cyclic groups – groups just with rotations having the same centre, in which we pinpoint one of the families of patterns that we have come across.

(a) Observing

Look back at the patterns that only had rotation symmetries.
If they only had 2 rotation symmetries what were the angles of those rotations?
If they only had 3 rotation symmetries what were the angles of those rotations?
If they only had 4 rotation symmetries what were the angles of those rotations?
If they only had 6 rotation symmetries what were the angles of those rotations?
(Figure 11.1)

(b) Making

It is worth checking your findings against the three theorems for groups that we found in Section **10**.
(i) Does each of these groups contain the identity?
(ii) Is there an inverse in your list for each of the symmetries that you have found?
(iii) Checking closure is a bit more trouble, but you can do it by filling in these tables. You will have checked closure if you find that all the entries inside the table are in the list of rotations outside. (rotation a°).(rotation b°) = rotation $(a + b)$°.
The rotations are indicated just by their (anti-clockwise) angles. Remember, a rotational symmetry through 360° is indistinguishable from a rotational symmetry through 0°. So 180° + 180° = 360° = 0°, likewise, 180° + 270° = 360° + 90° = 90°.

C_2	0°	180°
0°		
180°	180°	0°

C_3	0°	120°	240°
0°			
120°			
240°	240°	0°	120°

C_4	0°	90°	180°	270°
0°				
90°				
180°				
270°	270°	0°	90°	180°

Decide how you can check for inverses by looking at the tables, after they have been filled in.

Once you have got the hang of it you can fill in the table for six rotational symmetries as well.

C_6	0°	60°	120°	180°	240°	300°
0°						
60°						
120°						
180°						
240°						
300°	300°	0°	60°	120°	180°	240°

(c) Sharpening

Suppose you had an object in the plane which had exactly five rotation symmetries (with the same centre) and no others. What would you expect the angles of those five rotations to be? See Figure 11.2.

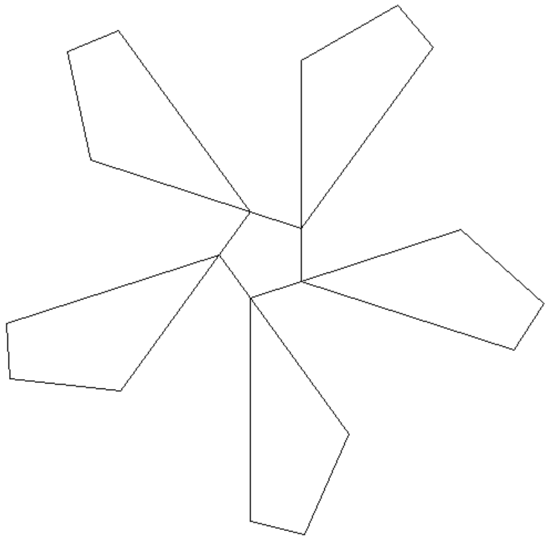


figure 11.2

Paragraphs (i), (ii) and (iii), below, show how to construct a proof that the five rotations have to be 0°, 72°, 144°, 216° and 288°. If you don't want to think about that now, then skip it. The three theorems of Section **10** have to be used: the **identity** is in every group; each symmetry in the group has an **inverse** in the group; and, combining two symmetries in the group always keeps you inside the group, so the group is **closed**.

If you would like to argue why the five angles you have guessed must be right, you could ask yourself these questions:

(i) Why must one of the angles be 0° (or 360°)? (Which theorem from **10**?)

(ii) If $a°$ is one of the angles, why must $2a°$, $3a°$, $4a°$, $5a°$ etc also be amongst the angles? (Which theorem from **10**?) Why must $-a°$, $-2a°$, $-3a°$, $-4a°$, $-5a°$ etc also be amongst the angles? (Which theorem from **10**?)

(iii) Now we suppose that $a°$ is the smallest positive angle. We would like to think that the five angles were $a°$, $2a°$, $3a°$, $4a°$, $5a° = 360°$ (which would tell us what $a°$ was, and we would then know everything). Just to be sure, let's pretend that a positive angle $b°$ was in the list and not equal to any of $a°$, $2a°$, $3a°$, $4a°$, $5a°$. Why can't b be less than a? If b lay between $2a$ and $3a$, would there have to be a rotation through an angle of $(b - 2a)°$ in the group? (Which theorems?) Would be $b - 2a$ be less than a? What is wrong with that? So b cannot fail to be a multiple of a.

Thus the angles $a°$, $2a°$, $3a°$, $4a°$, $5a°$ are the angles of rotation, and since one of them is 0° (or 360°), the five angles are 0°, 72°, 144°, 216° and 288°.

Could you run through a similar argument to show that if an object in the plane had exactly *six* rotation symmetries, they would have to be through angles which were all multiples of 60°?

Could the argument be applied to an object in the plane which had exactly n rotation symmetries? What would be the angles of the rotations then?

A finite symmetry group is called **cyclic** when all its symmetries are multiples of one member of the group. That one member is said to be the group's **generator**. If the word *cyclic* makes you think of a wheel, that is usually a helpful image.
Multiples of a **generator** give all the elements of a **cyclic** group.

The group consisting of {0°, 180°} is called C_2. Its generator is the 180° rotation. This is sometimes written <180°> = C_2. The group consisting of {0°, 120°, 240°} is called C_3. The group consisting of {0°, 90°, 180°, 270°} is called C_4. In general, the group consisting of exactly n rotations is called C_n, the cyclic group of order n.

The argument we used above, shows that the rotation with smallest angle is a generator of C_n. Thus <120°> = C_3; <90°> = C_4; <72°> = C_5; <60°> = C_6. In general, <360/n°> = C_n. But there are often other angles which also generate the group, in the sense that its multiples give all the rotations. Test out 240° in C_3, and 270° in C_4. More surprisingly test each of 144°, 216° and 288° in C_5.

<u>Excursion on generators</u> (This investigation is more about numbers than geometry.) Finding all the possible generators of C_8 and of C_{12} also forms an interesting investigation. [C_4 has two possible generators, C_5 has four, C_8 has four and C_{12} also has four. Make sure you know what these generators are before you launch into a general investigation. The LOGO procedure below may help you. Don't feel obliged to tackle the general investigation if it seems a bit of a mouthful.]
In C_n the rotation through $(360/n)°$ is always a generator, so all the rotations of the group are through angles of $((360/n)*a)°$, so these are the only angles to test if we are looking for other generators.
Here is a procedure in LOGO for exploring whether $((360/N)*A)°$ is also a generator of C_N.

```
TO TRY :N :A
ST
LT 90
REPEAT 360 [FD 2 RT 1]
RT 180*:A/:N
REPEAT :N [FD (720/PI)*SIN 180*:A/:N RT 360*:A/:N]
HT
END
```

For C_4, TRY 4 1 and TRY 4 3 illustrate the generators 90° and 270°, but TRY 4 2 shows that 180° is not a generator - it does not reach the four points around the circle. For a particular :N, you know that :A = 1 always gives a generator, so execute the procedure   TRY :N 1   in order to check the visual effect of choosing a generator.

When you have found the four generators of C_8 and the four generators of C_{12}, you may like to conjecture the values of a that make a rotation through $((360/n)*a)°$ a generator of C_n. The answer is that $((360/n)*a)°$ is a generator of C_n when and only when hcf $(n, a) = 1$. You may not be interested in a proof at this stage. You can safely skip the proof that follows if you wish.
The proof that a rotation through $((360/n)*a)°$ generates C_n when hcf $(n, a) = 1$ uses number theory. hcf $(n, a) = 1$ guarantees that there are integers x and y such that $xn + ya = 1$, and this in turn guarantees that when the rotation through $((360/n)*a)°$ has been repeated y times, $(360/n)*ay = (360/n)(1 - xn) = 360/n - 360x$, so that has

the same result as the rotation $(360/n)°$. Since this rotation generates the group, so must the rotation we started with.
When hcf $(n, a) = d > 1$, then the smallest angle which can be made by repeating rotations through $((360/n)*a)°$ is $((360/n)*d)°$ and that only generates the group $C_{n/d}$.

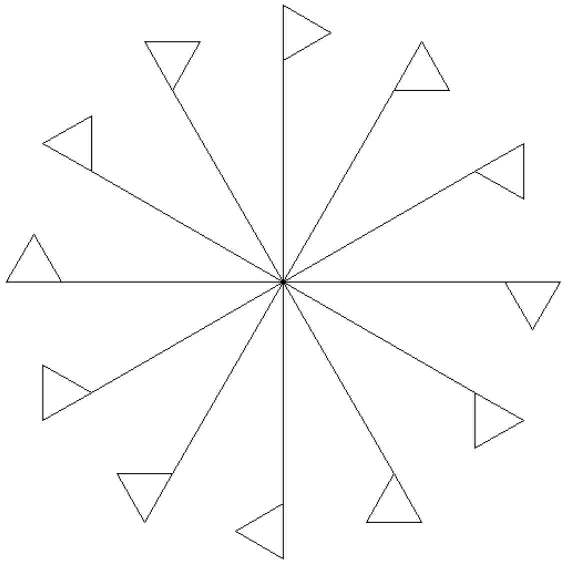


figure 11.3

Challenge Problem. (This is a number investigation.) If you look at Figure 11.3 with its C_{12} symmetry, it has the rotations of C_2, C_3, C_4 and C_6. Try turning this idea around. If a figure has the symmetries of C_2 and C_3, must it have all the symmetries of C_{12}? Try another pair. If a figure has C_3 and C_4 symmetry, must it have all the symmetries of C_{12}? Try other pairs: C_2 and C_4; C_3 and C_6; C_4 and C_6. The question about C_4 and C_6 is equivalent to asking whether a figure having a rotation symmetry of 90° and also 60°, must also have a rotation symmetry of 30°. The closure and inverse theorems in Section **10** about symmetry groups should be enough to make sure about that.

A figure with C_{18} symmetry has the rotations of C_2, C_3, C_6 and C_9. If a figure had the symmetries of C_2 and C_9 would that be enough to guarantee that it had all the symmetries of C_{18}? What about C_3 and C_6?

(A harder challenge) If a figure has both C_{10} and C_{18} symmetry, can you show that it has C_{90} symmetry? Convert the question into one about angles of rotation. This time you need two steps to get from the angles you have been given to the one you want. In all these questions you are trying to make a generator of the big group.

Here is a last question about cyclic groups. It is rather general, and the rest of the booklet does not build on it, so you can ignore it if you wish. The question is: "What is the full set of rotations of a figure which has both C_m and C_n symmetry?" The angles you can get from these two groups are all multiples of $360/m°$ combined with all multiples of $360/n°$, that is to say $x(360/m) + y(360/n)$ for all integers x and y. The number theory that you need is that the smallest value of $xa + yb$ is the hcf of a and b, and that lcm $(a, b) \times$ hcf $(a, b) = ab$.
In general, if a figure has both C_m and C_n symmetry, can you show that it has $C_{\text{lcm}(m, n)}$ symmetry?

Now you have a good vocabulary for groups of rotations, and we turn to look more carefully at reflections.

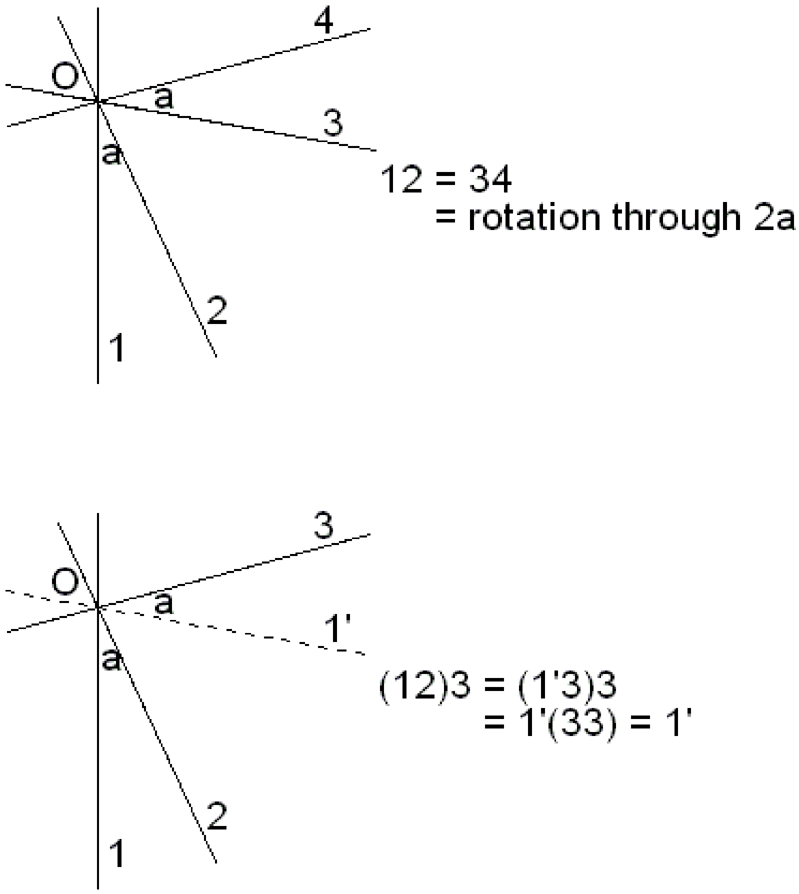


Figure 12

12. <u>Combining three reflections</u>, in which we go back to reflections and look more carefully at how they combine.

(a) Observing

Look back to the table you made at the end of Section **8.** Find the four entries equal to 90° in that table. List the pairs of reflections that gave this angle. What do you notice about them? Be careful about the order. We will use this list, to see how the rotation though 90° combines with a reflection.

(b) Making

hc = **cv** = **vd** = **dh**. Simplify **hcd**. Use **hc** = **vd** and **dd** = **I**.
Simplify **hvd**. Use **vd** = **hc** and **hh** = **I**.
If you wanted to combine a 90° rotation with **d**, what way of expressing the 90° rotation would be most useful?

(c) Sharpening Take four lines 1, 2, 3, and 4 all through a point *O*, such that the angle *a*° from 1 to 2 is equal to the angle *a*° from 3 to 4. (Figure 12) If we use boldface and

denote the reflection in 1 by **1**, etc., what is the combined result **12** (which means first reflect in the line 1, and then reflect in the line 2 and look at the overall result) and what is the combined result **34**?
You should have got a rotation through an angle $2a°$ about O in both cases.
So **12** = **34**.

Now take three lines 1, 2 and 3 all through a point O such that the angle from 1 to 2 is $a°$. Can you find a line 1′ through O such that the angle from 1′ to 3 is also equal to $a°$? If so, then, as before, **12** = **1′3**. So, **123** = (**12**)**3** = (**1′3**)**3** = **1′**(**33**) = **1′I** = **1′**.

This shows that a combination of three reflections, whose axes all pass through one point, is equal to a single reflection. This is sometimes called the theorem of the three reflections.

You can also explore the combination of three reflections in intersecting axes with a dynamic geometry program.

You can use the kind of argument we put together for the theorem of the three reflections to investigate the combination of a rotation and a reflection, when the centre of the rotation lies on the axis of the reflection? Don't forget you can decompose a rotation into two reflections. So a rotation followed by a reflection with axis through its centre = three reflections in axes through one point = one reflection. What answer might you have expected from thinking about clockwise and anti-clockwise?

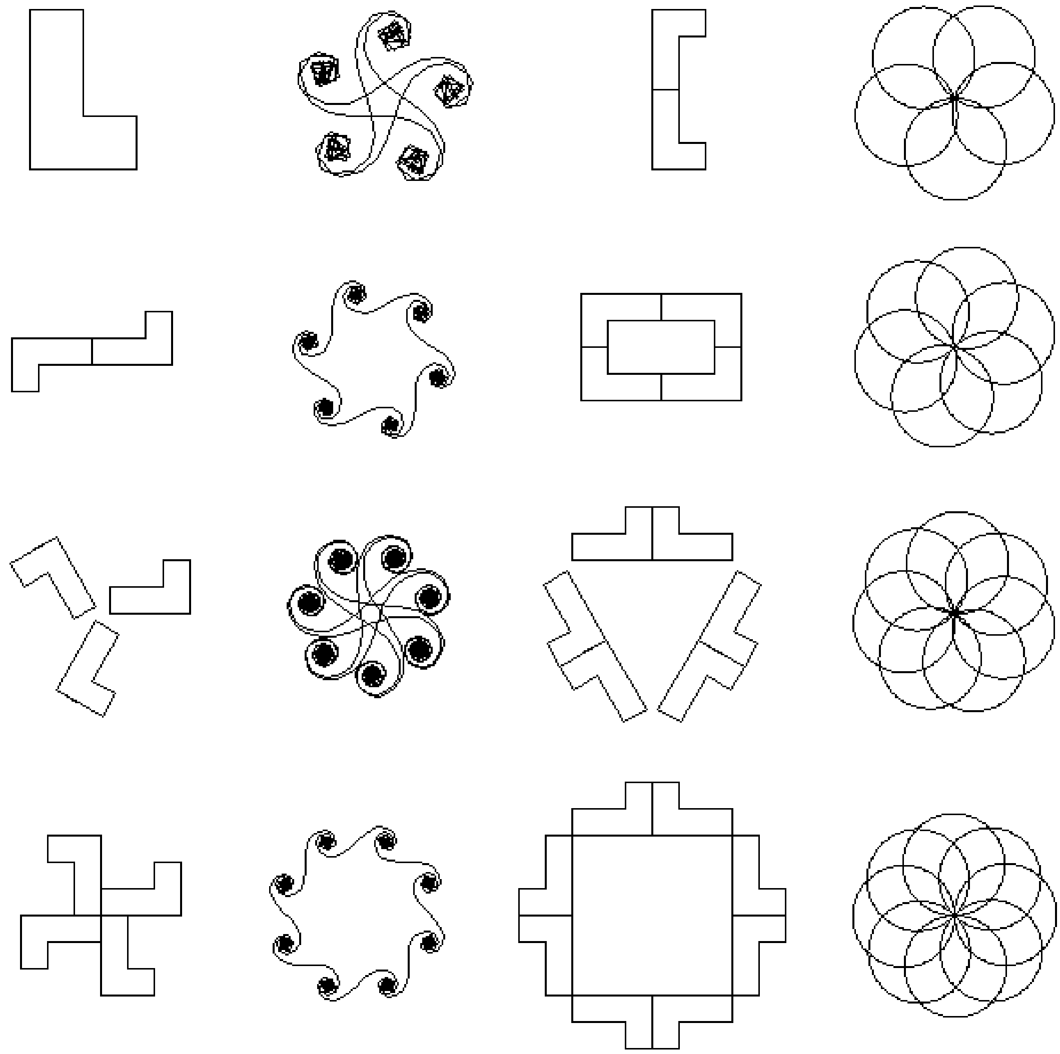
You can also use the theorem of the three reflections to investigate the combination of a reflection and a rotation (in the opposite order), when the axis of the reflection passes through the centre of the rotation? [Answer. *a reflection*.] What answer might you have expected from thinking about clockwise and anti-clockwise?

To summarise

rotation. reflection
= reflection. reflection. reflection
= reflection. rotation
= reflection.

So, from a bird's eye view, the table for combining the symmetries of a group of rotations and reflections must look like this.

	rotations	reflections
rotations	rotations	reflections
reflections	reflections	rotations



Count the number of rotations and reflections for each pattern

Figure 13.1

13. <u>Dihedral groups</u> - including a reflection, in which we identify the second family of patterns that we have come across.

The word "dihedral" is a Greek word meaning two-faced, and it is used here because turning something over, so that you can look at its back as well as its front, simulates a reflection.

(a) Observing

Look back at all the patterns you have, and count the number of rotations and the number of reflections in the symmetry groups of those patterns. (See Figure 13.1) Are all the possibilities you can track down recorded in this table?

		Number of reflections						
		0	1	2	3	4	5	6
Num-	1	poss	poss					
ber	2	poss		poss				
of	3	poss			poss			
rot-	4	poss				poss		
ations	5	poss					poss	
	6	poss						poss

(b) Making

Can you fill any of the gaps in the table here? Do you think it tells the whole story? That is, that a finite symmetry group with a centre, either consists entirely of rotations (and is then cyclic), or has the same number of rotations as reflection symmetries.

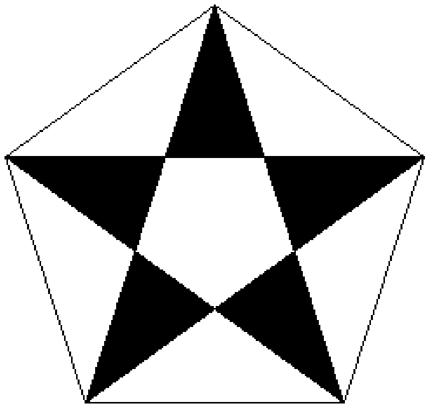
We are now going to build on Section **12**, to determine how many reflections there can be in a group with a certain number of rotations.

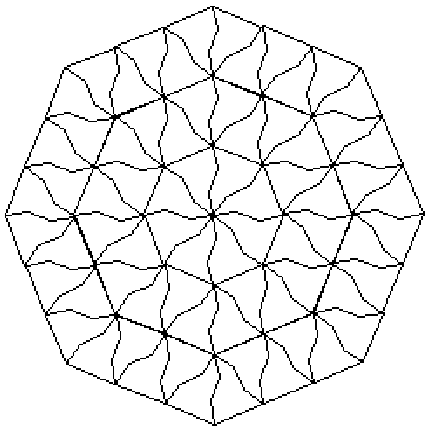
c) Sharpening - how many rotations and how many reflections?

Here is an argument which pins down the possible number of reflections in a plane symmetry group with exactly five rotations. While the argument itself is quite abstract, you may feel happier making sense of it if you are looking at a diagram of something which has a dihedral symmetry group containing just five rotations. A regular pentagon will do. See Figure 13.2.

Suppose that we have a dihedral group with exactly five different rotations called **A**, **B**, **C**, **D** and **E,** one of which actually has to be the identity, *and* at least one reflection **R**. From the third property of groups (closure in section **10**), **AR**, **BR**, **CR**, **DR** and **ER** are all in the group. Look at these five. From **12(c)** above, what kind of symmetries are they? [All reflections.]
Might two of them be equal? [If **AR** = **BR**, then **ARR** = **BRR**, so **A** = **B**, which is wrong**.**] Thus if there are any reflections in the group, there must be at least five. Might there be more than five reflections in the group?

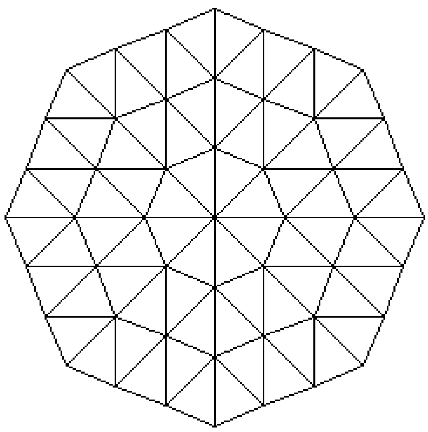
Figure 13.2





A cyclic pattern (here C8) may cover the whole plane

Figure 13.3



A dihedral pattern (here D8) may cover the whole plane

Let us suppose the group contains a reflection **Z**, which might be different from the five reflections **AR**, **BR**, **CR**, **DR** and **ER.** What could you say about the symmetry **ZR**? [It has to be in the group, and it has to be a rotation.] If **ZR** = **A, ZRR** = **AR,** so **Z** = **AR**, one of the reflections we already knew about. So now, we know that the group cannot have more than five reflections, so it has exactly five.

Could you run through a similar argument to show that if an object in the plane had exactly *six* rotation symmetries, and at least one reflection, then it would have exactly six reflections?

A **dihedral group** of symmetries in the plane is a finite group containing rotations and reflections. From the investigation we have just done, the number of rotations is the same as the number of reflections. The dihedral group with n rotations and n reflections is denoted by D_n.

The rotations in the **dihedral group** D_n match the rotations of the **cyclic group** C_n. If you match diagrams of patterns with symmetry groups D_1, D_2, D_3 etc. with patterns with symmetry groups C_1, C_2, C_3 etc. it should become obvious that each D group contains its corresponding C group. (Figure 13.3)

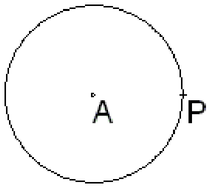
If you are sceptical and want to doubt whether the rotations in a dihedral group must match the rotations of a cyclic group, here are some questions to face.
(i) Must the identity be amongst these rotations?
(ii) Must the inverse of a rotation be a rotation with the same centre?
(iii) Must the combination of two rotations with the same centre be a rotation?
These were the three properties which we used to determine the structure of cyclic groups in Section **11**. So that structure holds for the rotations in a dihedral group.

Since the rotations of the dihedral group D_n form a cyclic group like C_n, the angles of the rotations in the dihedral group D_n are all multiples of $(360/n)°$.

If two of the axes of the reflections in D_n are inclined at $a°$, then their combination (which has to be in the group, by closure) is a rotation through $2a°$ from Section **8**. So $2a$ is a multiple of $360/n$, and therefore a is a multiple of $180/n$. If we use all the possible multiples of $180/n$, we get just n axes of reflection, and that is the right number for D_n.

Since, all along, we have been looking at patterns with a centre, the symmetries we have used have always had at least one fixed point. We need to check whether there might be any other symmetry, that we have not thought of so far, which might also have a fixed point.

Figure 14.1 (b)



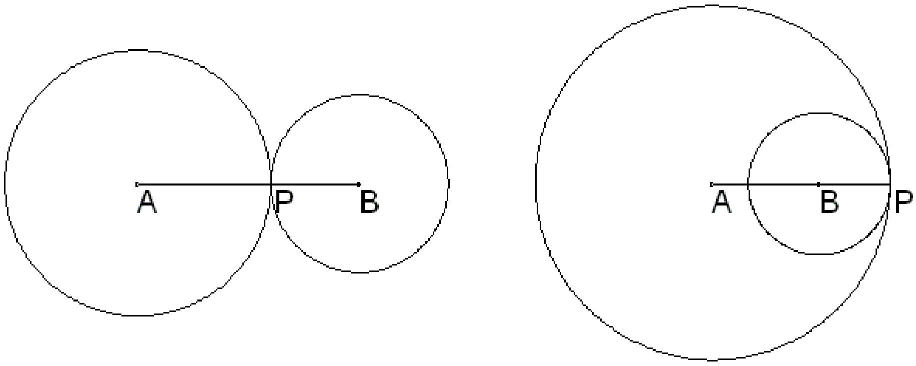
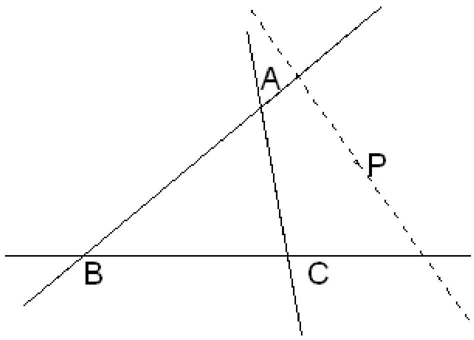


Figure 14.2 (c)



14. <u>Symmetries with a fixed point</u>, in which we see why a symmetry with a fixed point must be either a reflection, a rotation or the identity.

(a) Observing

What points are fixed by a reflection?
What points are fixed by a rotation?

(b) Making

If a symmetry **S** has a fixed point *A*, and the symmetry **S** takes the point $P \to P'$, what can you say about the lengths *AP* and *AP'*? If a circle is drawn with centre *A* and radius *AP*, what can you say about the images of points on the circle under the symmetry **S**? You may want to think through the answer by considering what happens to a circle when rotated about its centre, or reflected in a diameter. But that is not the point here. Go back to the definition of symmetry in Section **9** to answer this question, and try to pretend that you have never heard of rotations or reflections. (Figure 14.1)

(c) Sharpening

Our target is to show that a symmetry with a fixed point must be a reflection or a rotation. The first step is to pin down a symmetry with two fixed points.

If a symmetry **S** has two different fixed points *A* and *B*, and *P* is any third point on the line *AB*, what can you say about the circles centre *A*, radius *AP* and centre *B*, radius *BP*? At how many points do they meet? Only presume that the symmetry **S** satisfies the definition in Section **9**. Now think of the images of these two circles with fixed centres under **S**. If a point lies on both the circles must its image lie on both the images of the circles? So what is the image of *P* under **S**? (Figure 14.2)

This shows that if a symmetry fixes two points, it fixes all the points on the line joining them. [*A theorem*.] We use this repeatedly in the next paragraph.

Now suppose we have a symmetry **S** which fixes three points, *A*, *B*, *C*, at the vertices of a triangle (so that they are not just on one line). Use the theorem we have just proved to say why every point on the sides of the triangle *ABC* must be fixed by **S** (that is every point on *BC*, *CA*, and *AB*, extended indefinitely). Now choose a point, *P*, not on any side of the triangle. Draw a line through *P* which is not parallel to any of the sides of the triangle. Can you be sure that this line will meet the sides of the triangle in at least two different points? Thus two points on this line are fixed by **S**. So can you be sure that *P* must be fixed by **S**? And therefore **S** is the identity. [*Another theorem*: the identity is the only symmetry which fixes three points at the vertices of a triangle.] (Figure 14.2)

Suppose **S** is a symmetry which fixes the different points *A* and *B*. Let *P* be a point not on the line *AB*. (Figure 14.3) If **S** fixes *P*, then **S** is the identity. If **S** is not the identity, then every point off the line *AB* is moved by **S**. If **S** does not fix *P*, it must take *P* to a common point of the circles centre *A*, radius *AP*, and centre *B*, radius *BP*. These two

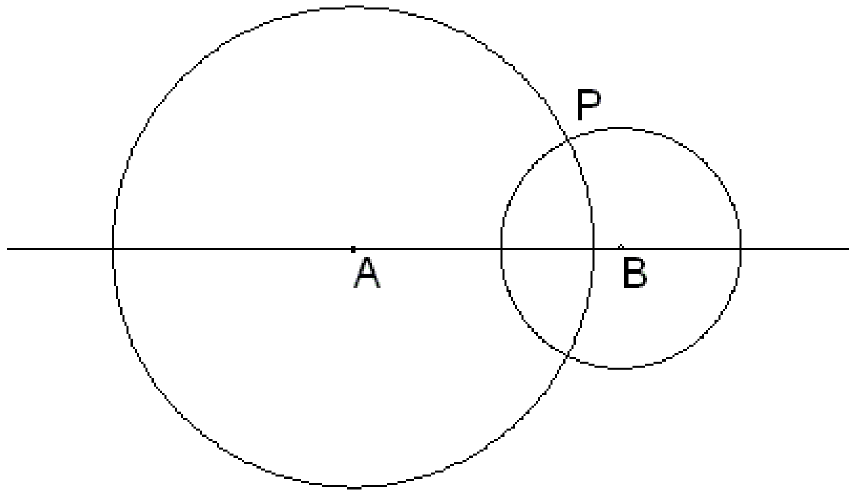


Figure 14.3

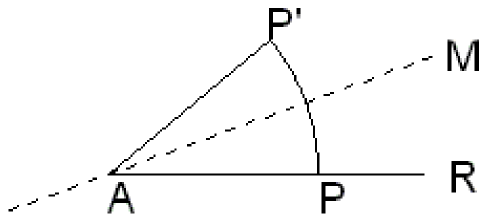


Figure 14.4

circles meet in exactly two points, one is *P*, the other is the reflection of *P* in the line *AB*.

A theorem. A symmetry fixing two different points *A* and *B* is either the identity, or the reflection with axis *AB*.

Here is an argument that a symmetry with only one fixed point must be a rotation. The argument is abstract because we are proving that it is not any weakness in our imaginations which has kept us working with reflections and rotations, and no other symmetries, so far.

Let **S** be the symmetry, A its unique fixed point, and P a point which is not fixed by **S**. Let **S** take the point $P \rightarrow P'$. Note that $AP = AP'$. Now draw a line through A and the mid-point of PP', the perpendicular bisector of PP' or bisector of $\angle PAP'$. Let **M** be the reflection with this line as axis (see Figure 14.4). Then **SM** fixes A (because A is fixed both by **S** and by **M**), and **SM** fixes P (because **S** takes P to P' and **M** takes it back again). So **SM** fixes the two different points A and P and therefore is either the identity or the reflection with axis AP, which we denote by **R**. **SM** = **I** or **R**, so **S** = **M** or **RM**, a combination of two reflections, whose axes intersect at A, which is a rotation. So a symmetry with only one fixed point is a rotation.

Thus a plane symmetry with one or more fixed points is either the identity, a reflection or a rotation. This means that a symmetry group with a centre can only have rotations and reflections in it, apart from the identity.

Every finite symmetry group with a centre is either cyclic (and consists only of rotations) or is dihedral (and consists of an equal number of rotations and reflections).

In fact, although a finite symmetry group has to have a centre, we are not in a position to prove that yet. The symmetry group of a circle (either a circumference or a disc) consists of rotations and reflections, but there is no smallest angle of rotation symmetry ($\neq 0°$) for the circle. That is why it does not count as a 'pattern' in the sense that we have been investigating. The symmetry group of a circle is continuous, rather than discrete. Perhaps more disconcerting is the group of rotations generated by a rotation through an angle $a°$ where no multiple of a is ever a multiple of 360, the number a being irrational (π or $\sqrt{2}$ or such like). The rotations in the group are through angles of $\pm na°$, for every integer n. You can illustrate this group with a LOGO procedure such as REPEAT :N [PU FD 100 PD FD 1 BK 1 PU BK 100 PD RT PI] which puts :N dots around the circumference of a circle, each one $\pi°$ further round than its predecessor. If :N < 115 the diagram is straightforward enough, but when :N > 1000 you will not be able to see a smallest angle of rotation. Even though the group is cyclic, it contains an infinity of rotations. Because this group has no smallest positive rotation it is said to be *not discrete*. The theorem we have stated above classifies discrete symmetry groups with a fixed point.

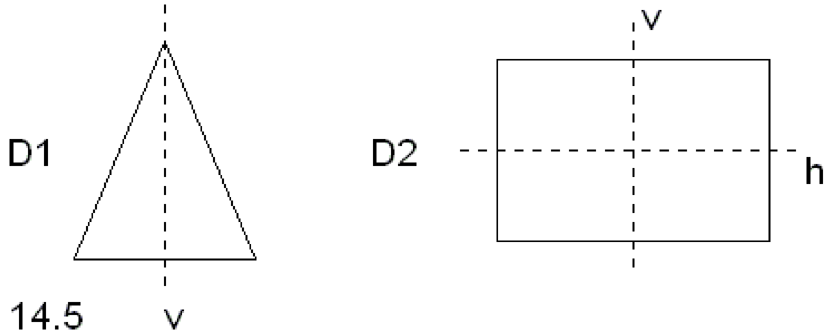


Figure 14.5

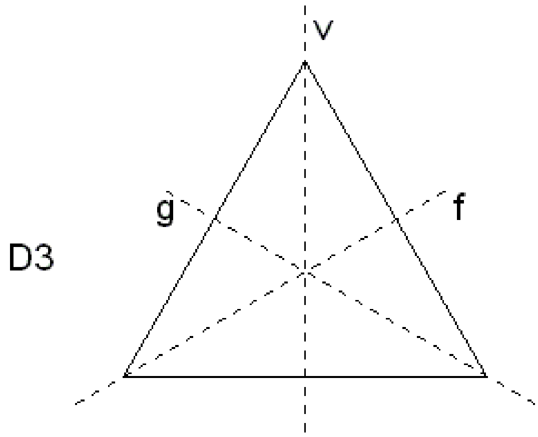


Figure 14.6

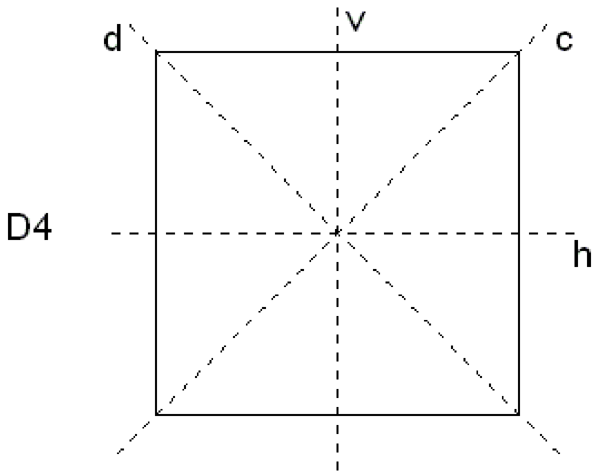


Figure 14.7

Complete the following tables for dihedral groups. Always take the rotations anti-clockwise. The first symmetry is in the left hand column. The second symmetry is in the top row. For each table, you worked out the top left quadrant in Section **11**. You also worked out how to fill the bottom right quadrant in Section **8**. Use your results in the bottom right quadrant to convert each of the remaining combinations into a convenient combination of three reflections, as in Section **12(b)**.

D_1	0°	**v**
0°		
v	**v**	0°

D_2	0°	180°	**v**	**h**
0°				
180°	180°	0°	**h**	**v**
v				
h				

See Figure 14.5 for the axes of the reflections **v** and **h**.

D_3	0°	120°	240°	**v**	**f**	**g**
0°						
120°						
240°	240°	0°	120°	**g**	**v**	**f**
v	**v**	**g**	**f**	0°	240°	120°
f						
g						

See Figure 14.6 for the axes of the reflections **v**, **f** and **g**.

D_4	0°	90°	180°	270°	**h**	**c**	**v**	**d**
0°								
90°								
180°								
270°	270°	0°	90°	180°	**c**	**v**	**d**	**h**
h	**h**	**c**	**v**	**d**	0°	90°	180°	270°
c								
v								
d								

See Figure 14.7 for the axes of the reflections **h**, **c**, **v** and **d**.

One nice property of a group table is that each row and each column contain all the symmetries of the group. This is because a row can contain no repetitions, since $\mathbf{xa} = \mathbf{xb} \Rightarrow \mathbf{x}^{-1}\mathbf{xa} = \mathbf{x}^{-1}\mathbf{xb} \Rightarrow \mathbf{a} = \mathbf{b}$. A column can contain no repetitions, since $\mathbf{ax} = \mathbf{bx} \Rightarrow \mathbf{axx}^{-1} = \mathbf{bxx}^{-1} \Rightarrow \mathbf{a} = \mathbf{b}$. The number of pigeons and the number of pigeon-holes is the same.

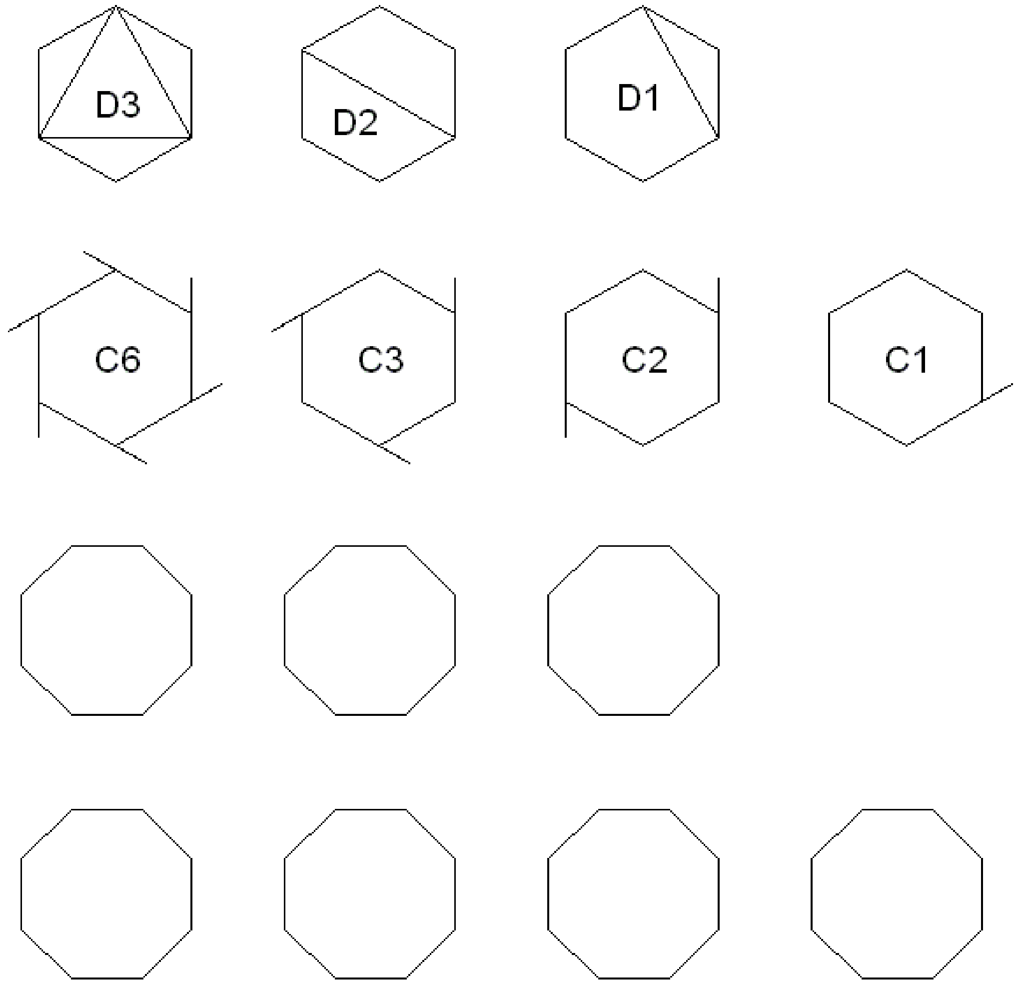


Figure 15.1

15. Subgroups, in which we look at patterns within patterns.

(a) Observing

In the tables you have made in Section **14** for D_1, D_2, D_3 and D_4, look for copies of the smaller groups inside the larger ones.

Inside D_2 you should find two copies of D_1 and a C_2.
Inside D_3 you should find three copies of D_1 and a C_3.
Inside D_4 you should find four copies of D_1 and a C_4. You should also find a copy of D_2, and, if you are very perceptive, a second copy of D_2, but with different reflections.

(b) Making

If you have a figure with a particular symmetry group and add some lines to it, but do not just copy the whole figure, the symmetry group that results will be a subgroup of the original group. Sometimes you can *increase* the number of symmetries by adding lines, but that possibility will be left for you to explore.

A regular hexagon has symmetry group D_6. By adding lines to the regular hexagon we can obtain figures with the subgroups D_3, D_2, D_1, C_6, C_3, C_2 and C_1. See Figure 15.1.

Regular octagons have also been drawn in Figure 15.1, which have the symmetry group D_8. Mark different octagons in such a way as to exhibit diagrams with the groups D_4, D_2, D_1, C_8, C_4, C_2 and C_1.

(c) Sharpening

When every symmetry of one group is a symmetry of another (usually bigger) group, the first group is called a **subgroup** of the second.

If you count the number of symmetries in a group and then count the number of symmetries in each of its subgroups, what do you notice? Make a conjecture about the number of symmetries in a group and the number in a subgroup.

16. Calculating with symmetries

Making

Here is a device to name and manipulate the symmetries in a dihedral group. It works for D_n, provided $n \geq 3$, but we will make it now, just for D_6.
Draw a regular hexagon. Join the centre of the hexagon to the six vertices. Then join the centre to the mid-points of the six sides. You now have twelve congruent triangles. They are going to be matched with the twelve symmetries of the group D_6.

(i) Label one of the twelve triangles I. It does not matter which.

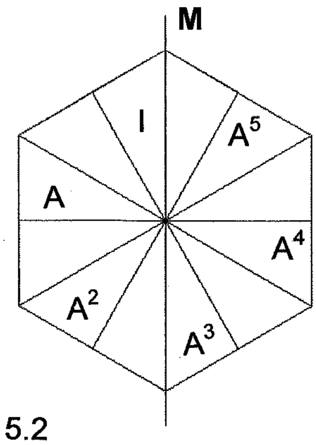


Figure 15.2

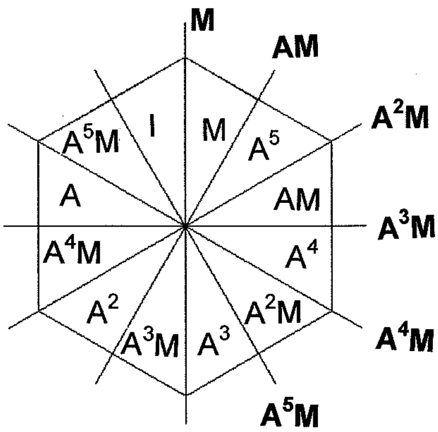


Figure 15.3

(ii) Name the rotations of D_6, **A** for 60° anti-clockwise, $\mathbf{A}^2$ for 120°, $\mathbf{A}^3$ for 180°, $\mathbf{A}^4$ for 240° and $\mathbf{A}^5$ for 300°. With this convention, can you see that $\mathbf{A}^6 = \mathbf{I}$? Now, label five more triangles, with the rotation that takes triangle I to that triangle. (Figure 15.2)

(iii) Choose one of the reflections of the hexagon and call it **M**. You should notice that the six unlabelled triangles are mirror images of the six labelled I, A, A^2, A^3, A^4, A^5. Now label the remaining six triangles M, AM, A^2M, A^3M, A^4M, A^5M, so that A^i and A^iM label triangles which are mirror images under **M**. (Figure 15.3)

Observing

Look carefully at your hexagon with twelve triangles labelled. The twelve triangles have been labelled with the twelve symmetries of the hexagon. Label the axis of **M** with **M**. Now look at the two triangles labelled I and AM. Is there a reflection symmetry of the hexagon which interchanges them? If so, mark its axis with the label, **AM**. Now look at the two triangles labelled I and A^2M. Is there a reflection symmetry of the hexagon which interchanges them? If so, mark its axis $\mathbf{A}^2\mathbf{M}$. Proceed similarly with the pairs of triangles I and A^3M, I and A^4M, I and A^5M. The labels on the twelve triangles are the twelve symmetries of the hexagon, labelling each triangle with the symmetry which carries the triangle labelled I to it.

Sharpening

The hexagon, with its twelve labelled triangles provides a tool for calculating combinations of symmetries of the hexagon. Some combinations do not need the tool.
(i) For example, to find $\mathbf{A}^2$ combined with $\mathbf{A}^3$ we just add the angles, as in a cyclic group, and get $\mathbf{A}^5$ (so $\mathbf{A}^2\mathbf{A}^3 = \mathbf{A}^5$). Less obviously, $\mathbf{A}^3$ combined with $\mathbf{A}^4$ gives **A** (since $\mathbf{A}^6 = \mathbf{I}$, $\mathbf{A}^3\mathbf{A}^4 = \mathbf{A}$).
(ii) The same idea works if you want to combine $\mathbf{A}^2$ with $\mathbf{A}^3\mathbf{M}$ and you get $\mathbf{A}^5\mathbf{M}$ (so $\mathbf{A}^2\mathbf{A}^3\mathbf{M} = \mathbf{A}^5\mathbf{M}$).
(iii) Which of the twelve symmetries is **MA**? If you can find where **MA** takes the triangle I to, you will know the answer. **M** takes I to M. Then **A** takes M to A^5M, so $\mathbf{MA} = \mathbf{A}^5\mathbf{M}$. Similarly work out $\mathbf{MA}^2$, $\mathbf{MA}^3$, $\mathbf{MA}^4$ and $\mathbf{MA}^5$. You should get $\mathbf{MA}^i = \mathbf{A}^{6-i}\mathbf{M}$, so $\mathbf{A}^i\mathbf{MA}^j = \mathbf{A}^{i+6-j}\mathbf{M}$ or $\mathbf{A}^{i-j}\mathbf{M}$.
(iv) The other tricky combination, for which you can use the hexagon, is for a combination of two reflections. Of course, if you know the angle between their axes, you can say that the combination is a rotation through twice that angle. But if you are trying to combine **M** with **AM**, the angle may not be obvious. You must first locate the axes of the two reflections, by finding the axis of symmetry for the triangles I and M, and the axis of symmetry for the triangles I and AM. Then you follow through what happens to the triangle I under the first and then the second symmetry. Under the reflection **M** the triangle I is taken to M and then under the reflection **AM**, M is taken to the triangle A^5 (so $\mathbf{M}.\mathbf{AM} = \mathbf{A}^5$). It is a good idea to practise this. Find $\mathbf{MA}^2\mathbf{M}$, $\mathbf{MA}^3\mathbf{M}$, $\mathbf{MA}^4\mathbf{M}$, and $\mathbf{MA}^5\mathbf{M}$. You should get
$\mathbf{MA}^i\mathbf{M} = \mathbf{A}^{6-i}$. So that in general one gets $\mathbf{A}^i\mathbf{MA}^j\mathbf{M} = \mathbf{A}^{i-j}$, which you could also have worked out from (iii).

The labelling of triangles that we have done on a hexagon to calculate with the symmetries of the dihedral group D_6 can be done with any regular n-gon to explore the symmetries of the dihedral group D_n provided $n \geq 3$.

17. Conclusion

When you are out and about and look at church windows, ornaments, hub caps, manhole-covers, badges, jewellery and such like, you should now be able to classify their symmetry groups as cyclic or dihedral.
The next step with symmetries is to look at friezes, that is, patterns that run along a line. You will find cyclic and dihedral patterns within friezes, on quilts and within wallpaper patterns. What you now know about cyclic and dihedral patterns will help you sort these other, potentially infinite, patterns.
If you go on to explore symmetry further, you will need to remember that the only symmetries with a fixed point (apart from the identity) are rotations and reflections, and that combining two reflections with axes inclined at an angle a° gives a rotation through $2a$°.
Keep looking.

18. Reading and sources list

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Software

MSWLogo (distinct from MSW LOGO, mind the gap) can be downloaded freely from http://www.softronix.com
The best known dynamic geometry programs are CABRI and GEOMETER'S SKETCHPAD. But now, GEOGEBRA is available freely from the web.
For CABRI see http://www-cabri.imag.fr/cabrijava/
For GEOMETER'S SKETCHPAD see http://www.dynamicgeometry.co.uk
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