alpha

COLLAPSE

Günter Rote

Freie Universität Berlin, Institut für Informatik

joint work with

Uri Zwick

Tel Aviv University, Department of Computer Science

COLLAPSE

Günter Rote
Freie Universität Berlin, Institut für Informatik

joint work with

Uri Zwick
Tel Aviv University, Department of Computer Science

COLLAPSE

A Computational Geometer Tries to Learn Basic Mechanics

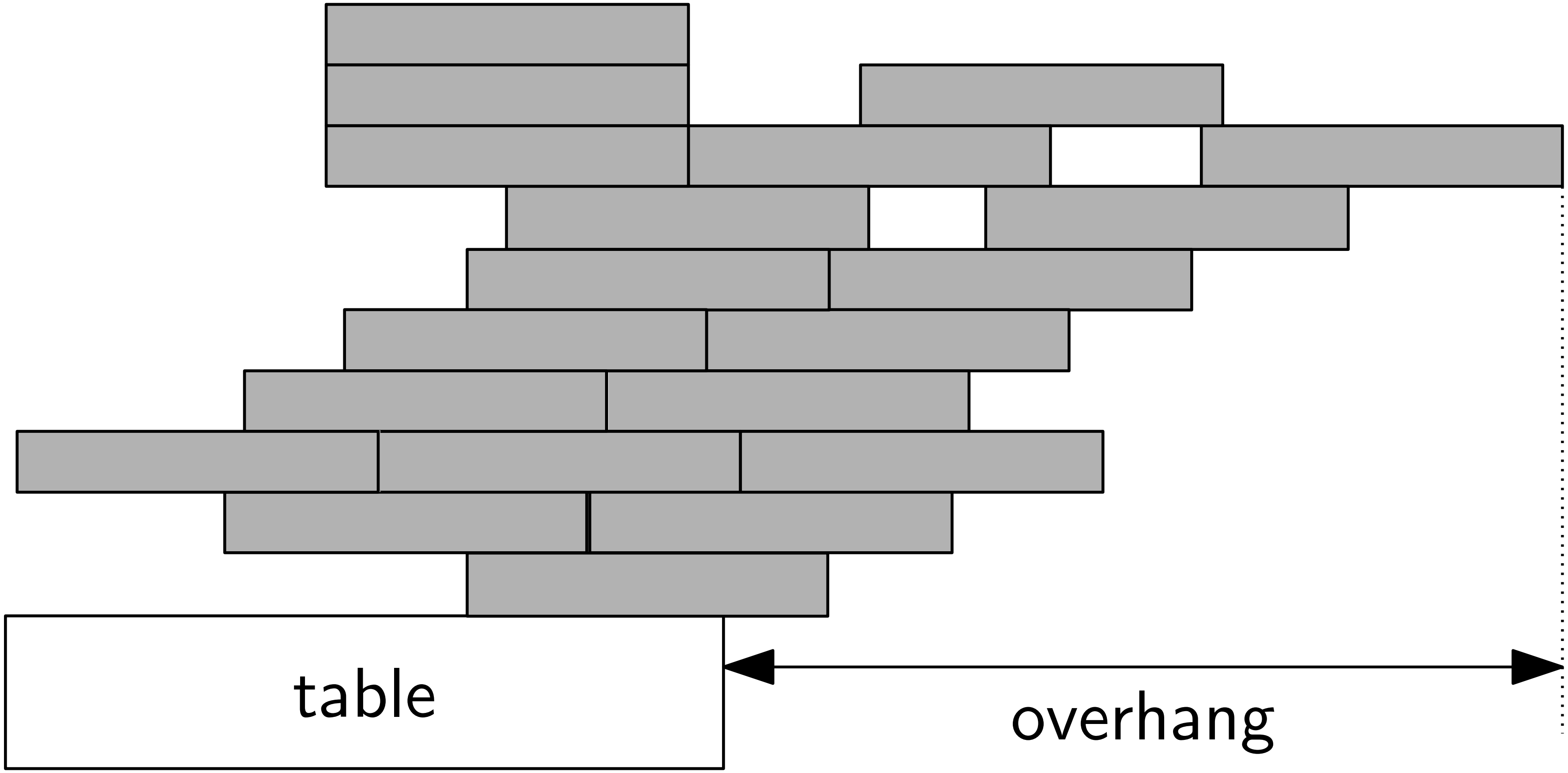
Günter Rote
Freie Universität Berlin, Institut für Informatik

joint work with

Uri Zwick
Tel Aviv University, Department of Computer Science

1. Introduction: OVERHANG

Freie Universität Berlin

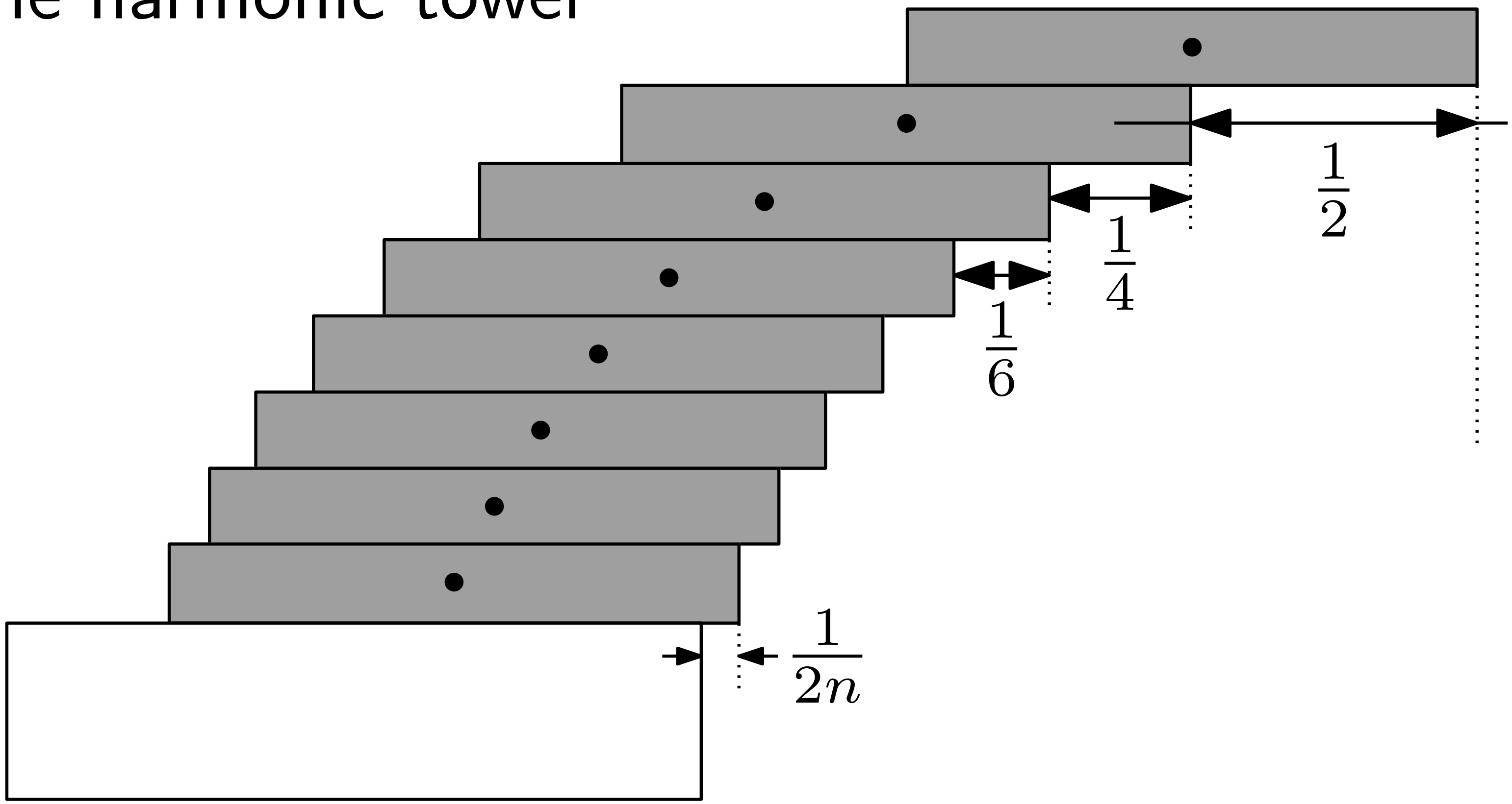


What is the largest overhang that can be achieved with n identical bricks?

The "standard" solution

Freie Universität Berlin

the harmonic tower



$$\text{overhang} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n} = \frac{1}{2} \cdot H_n = \Theta(\log n)$$

Maximum Overhang

Freie Universität Berlin

The maximum overhang achievable with n bricks is

$$\Theta(\sqrt[3]{n}).$$

Lower bound: Mike Paterson and Uri Zwick 2006
to appear in *American Mathemathical Monthly*

Upper bound:
Paterson, Y. Peres, M. Thorup, P. Winkler and Zwick 2007

Is it stable?

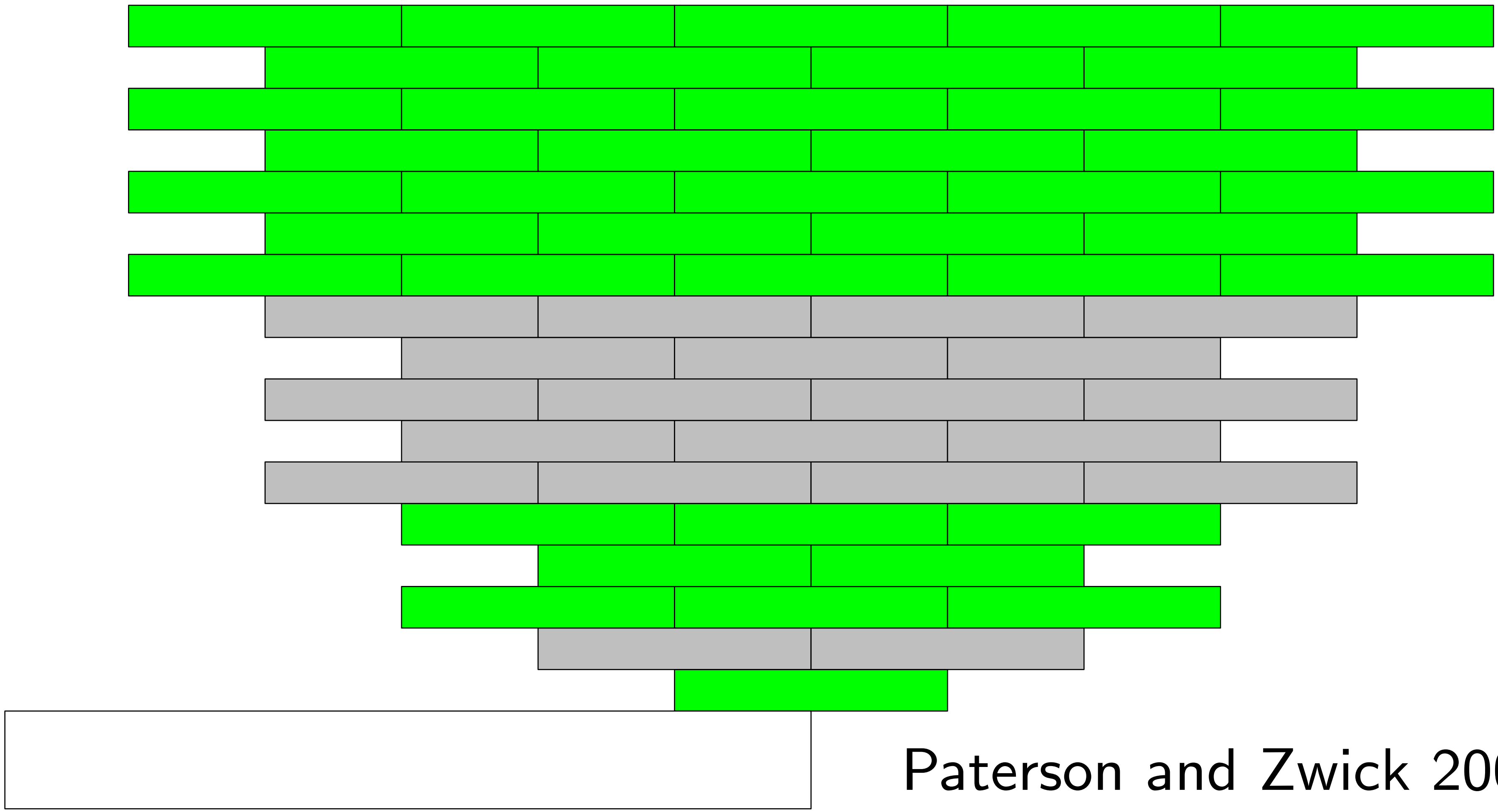
[photograph by Uri Zwick]

Is it stable?

[photograph by Uri Zwick]

The 5-stack

Freie Universität Berlin



Paterson and Zwick 2006

$$n \approx 1^2 + 2^2 + 3^2 + \cdots + k^2 = \Theta(k^3)$$

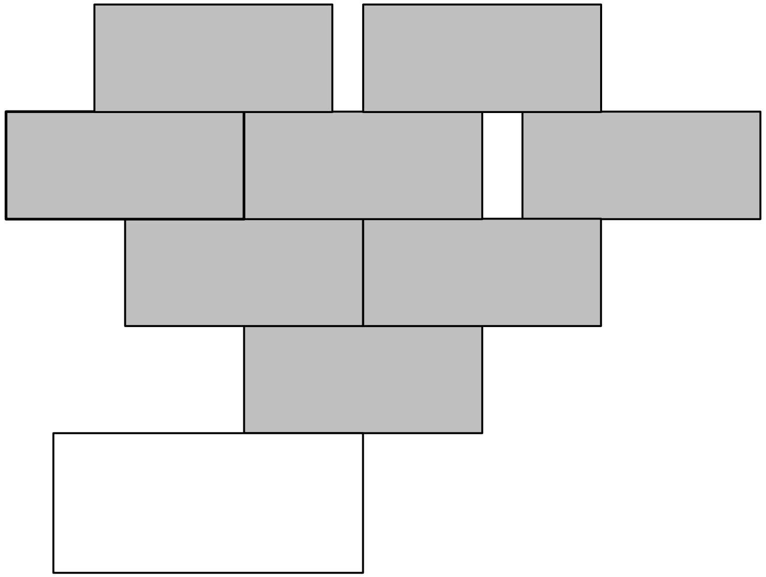
Outline

Freie Universität Berlin

1. Introduction: Overhang
2. Stability and instability
3. Global motion
4. Collisions
5. Implementation issues
6. Friction

2. Stability and instability

Freie Universität Berlin



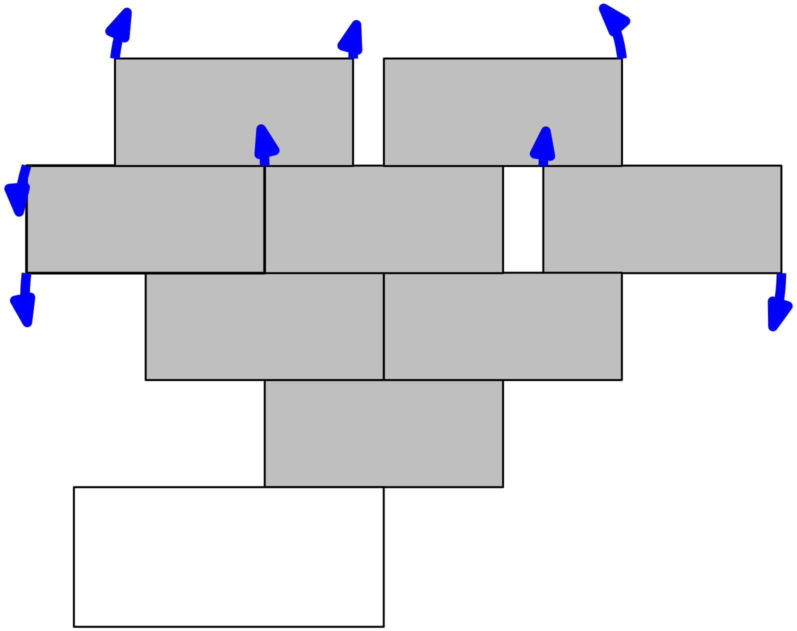
- Is it stable?

Compute a system of forces in equilibrium

$\longrightarrow$ a system of linear inequalities (linear optimization)

[KNOWN]

2. Stability and instability



- Is it stable?

Compute a system of forces in equilibrium

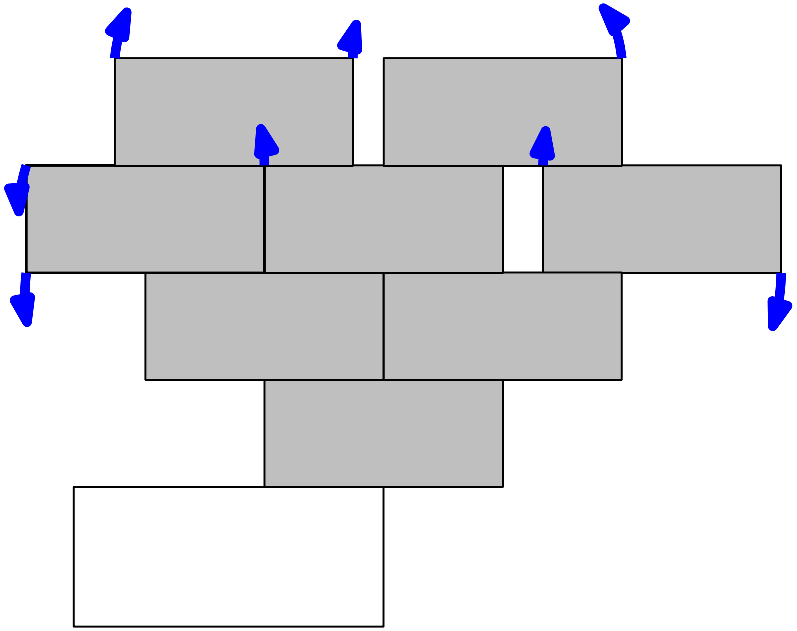
$\longrightarrow$ a system of linear inequalities (linear optimization)

[KNOWN]

- If it is unstable, how will it fall?

2. Stability and instability

Freie Universität Berlin



- Is it stable?

Compute a system of forces in equilibrium

→ a system of linear inequalities (linear optimization)

[KNOWN]

- If it is unstable, how will it fall?

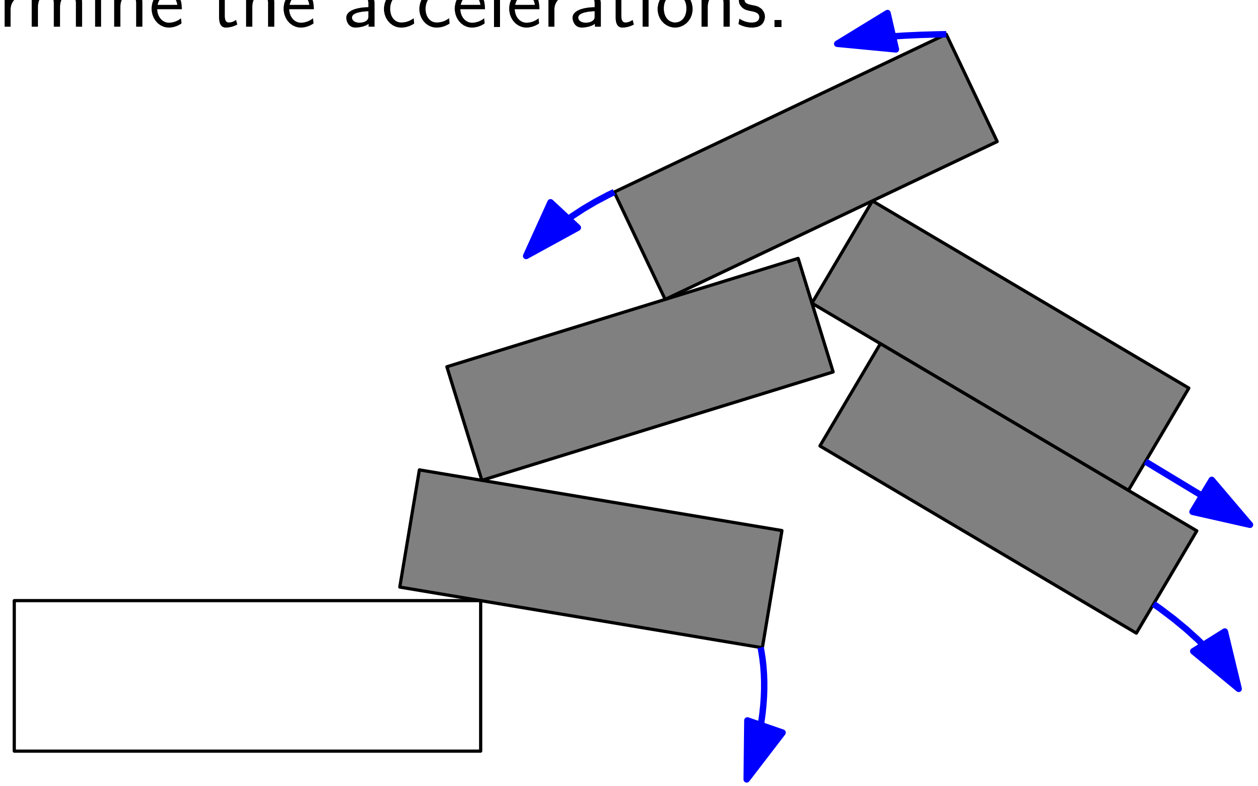
→ a convex quadratic optimization problem (quadratic objective function, linear constraints)

[THIS TALK]

Instability

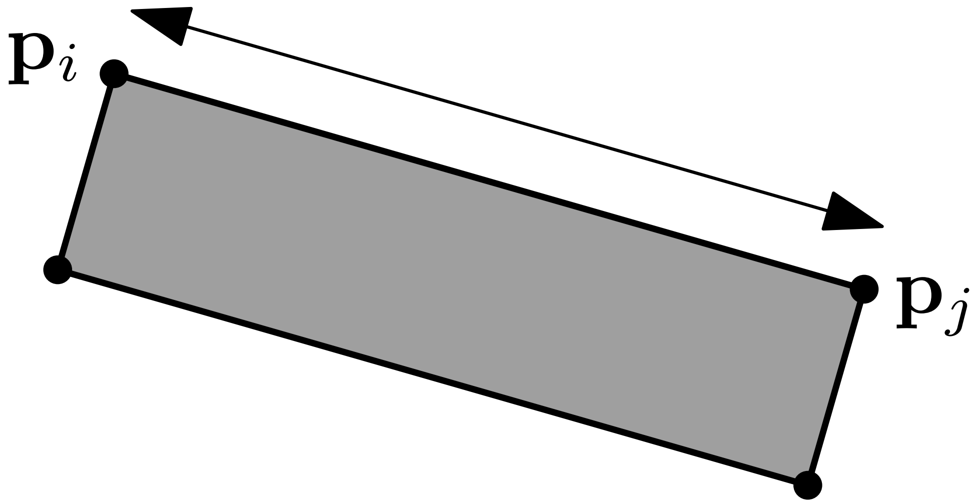
Freie Universität Berlin

More generally:
Given all locations and velocities at a given instant, determine the accelerations.



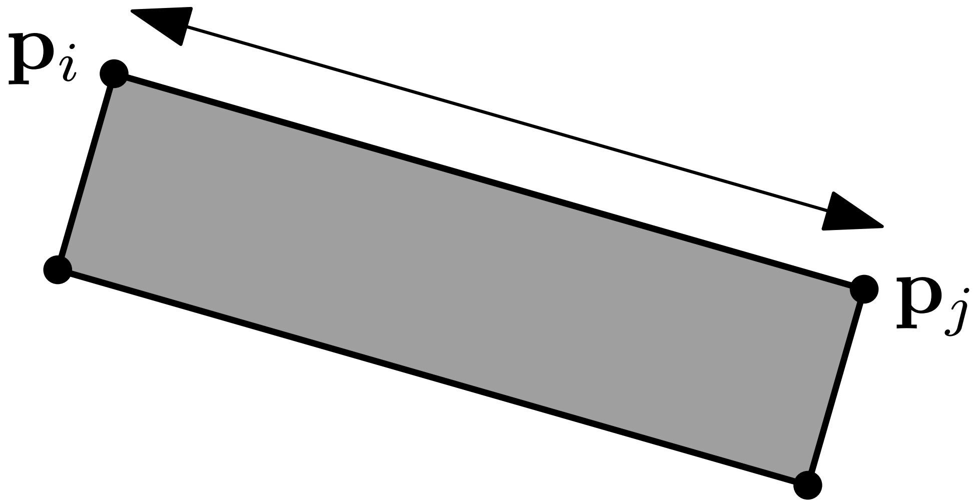
Assumption: NO FRICTION

Conditions I: Lengths

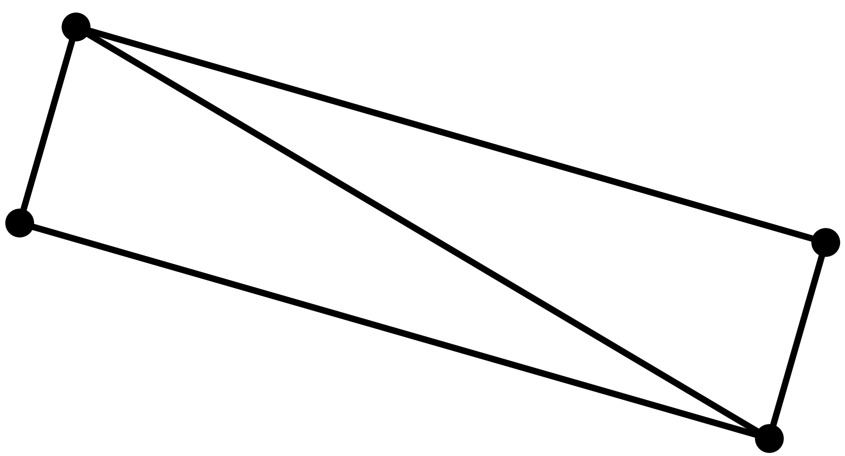


The length $\|\mathbf{p}_i - \mathbf{p}_j\|$ of the bar (i, j) remains constant.

Conditions I: Lengths



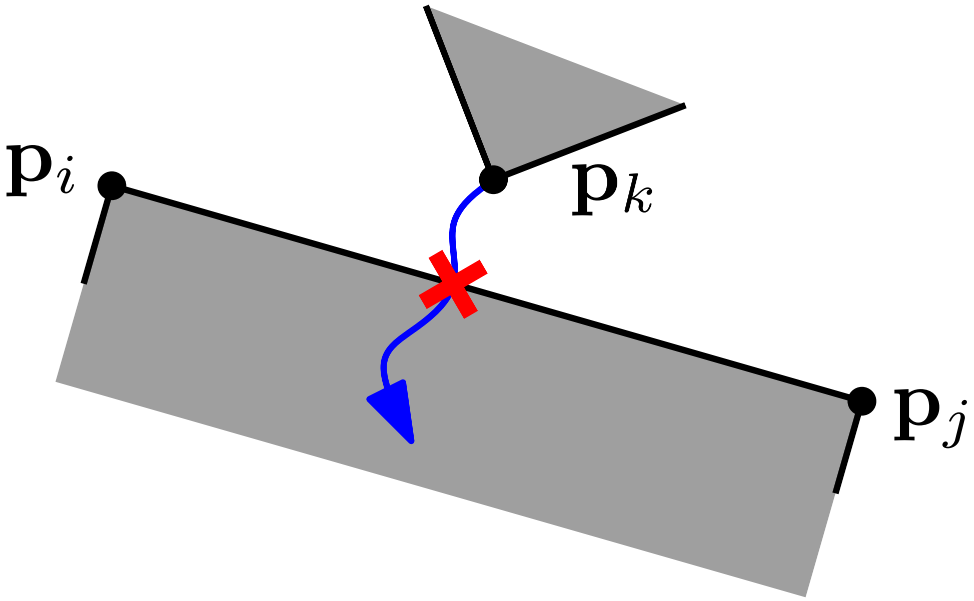
The length $\|\mathbf{p}_i - \mathbf{p}_j\|$ of the bar (i, j) remains constant.



A rigid block can be simulated by a framework of fixed-length bars.

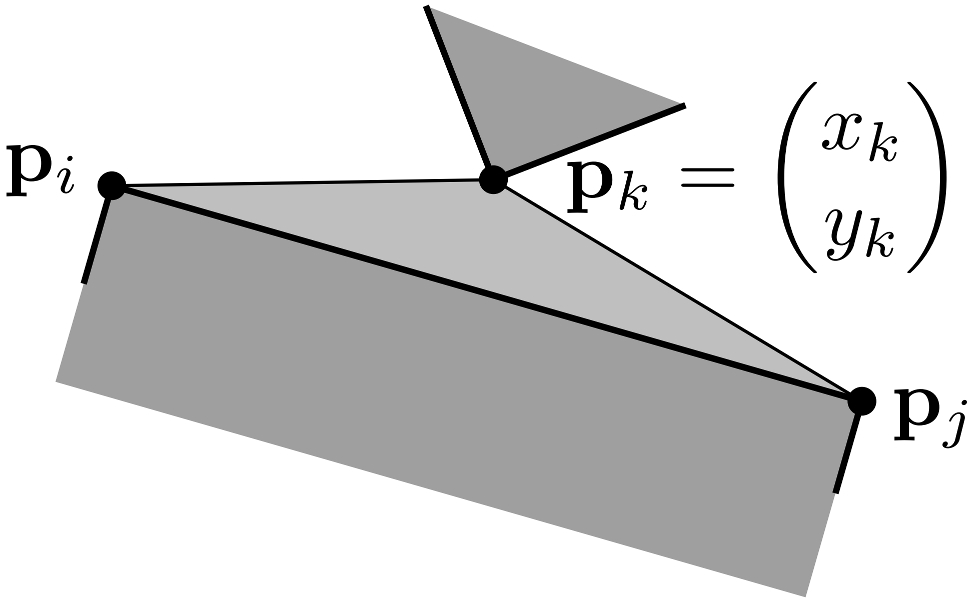
Conditions II: Sidedness

Collisions must be prevented.



Conditions II: Sidedness

Collisions must be prevented.

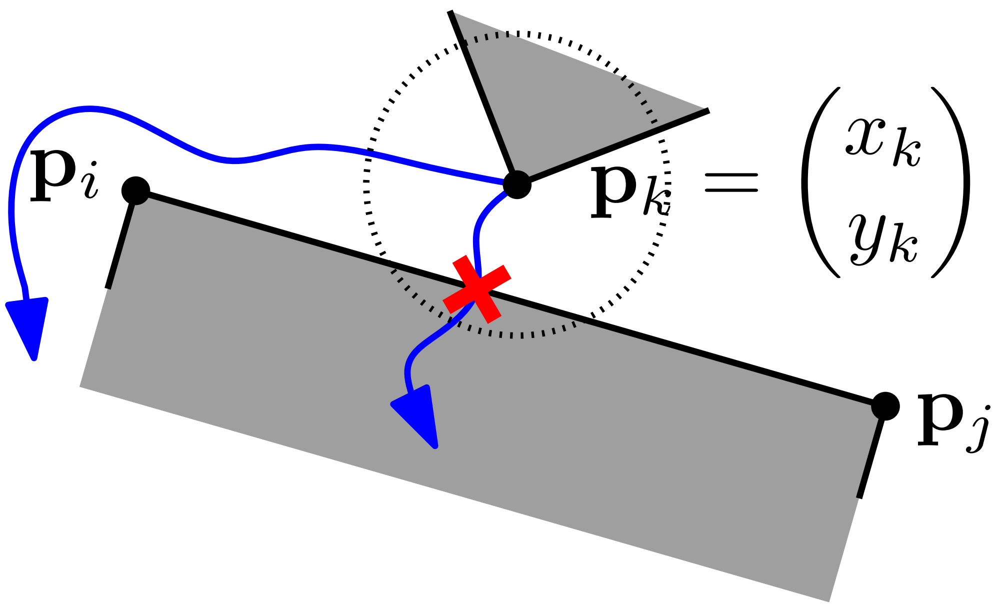


Area $A(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k) \geq 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{vmatrix} \geq 0$$

Conditions II: Sidedness

Collisions must be prevented.



Area $A(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k) \geq 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{vmatrix} \geq 0$$

This condition holds only *locally*, in the vicinity of the current configuration.

Equilibrium of forces (static)

Freie Universität Berlin

At every mass point $\mathbf{p}_i$ with mass m_i,

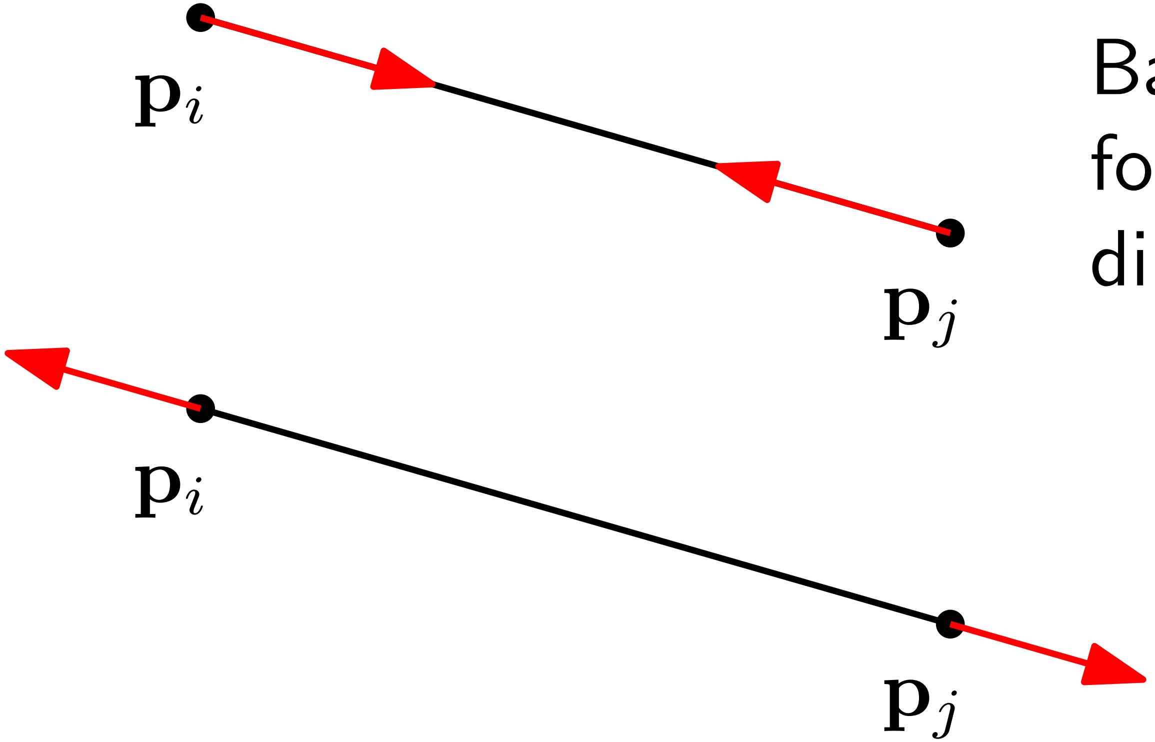
$$\underbrace{\sum \text{internal forces}}_{\substack{\text{forces}\\ \text{transmitted}\\ \text{through the}\\ \text{mechanism}}} + \underbrace{\sum \text{external forces}}_{\substack{m_i \cdot \mathbf{g}\\ \text{(gravity)}}} = \mathbf{0}$$

unknown *given*

Forces I: Lengths

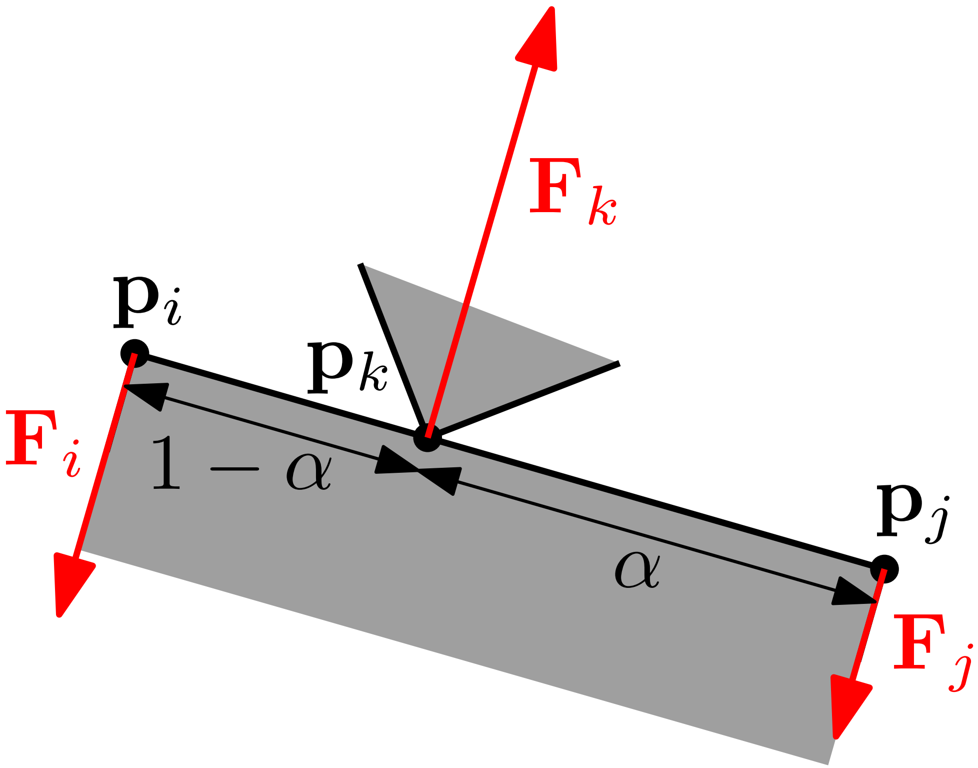
Freie Universität Berlin

The length $\|\mathbf{p}_i - \mathbf{p}_j\|$ remains constant.



Bars may transmit forces parallel to the bar directions.

Forces II: Sidedness



Collisions must be prevented.

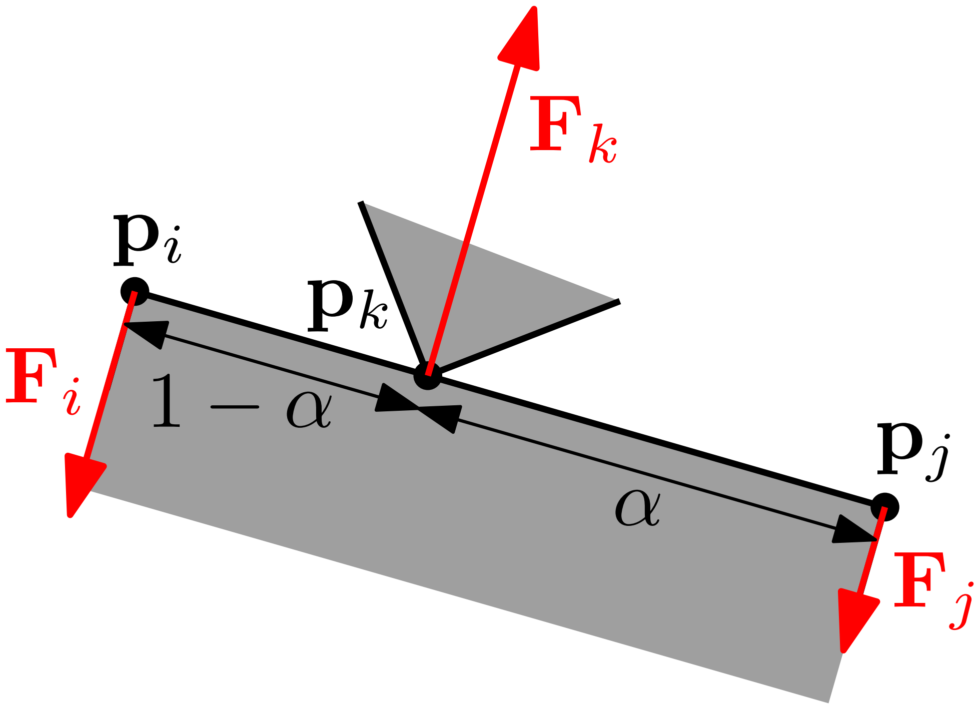
When $\mathbf{p}_k$ touches the bar $\mathbf{p}_i\mathbf{p}_j$, forces perpendicular to the bar direction may be transmitted, but *in one direction* only.

$\rightarrow$ linear inequalities

The force at $\mathbf{p}_k = \alpha\mathbf{p}_i + (1-\alpha)\mathbf{p}_j$ is distributed proportionally to $\mathbf{p}_i$ and $\mathbf{p}_j$:

$$\mathbf{F}_i = \alpha\mathbf{F}_k \text{ and } \mathbf{F}_j = (1-\alpha)\mathbf{F}_k$$

Forces II: Sidedness



Collisions must be prevented.

When $\mathbf{p}_k$ touches the bar $\mathbf{p}_i\mathbf{p}_j$, forces perpendicular to the bar direction may be transmitted, but *in one direction* only.

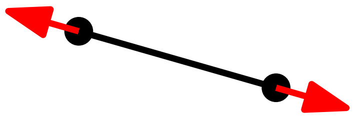
$\longrightarrow$ linear inequalities

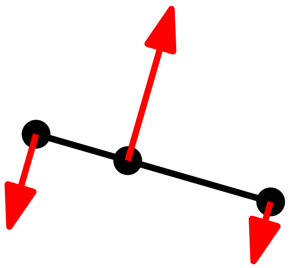
The force at $\mathbf{p}_k = \alpha\mathbf{p}_i + (1-\alpha)\mathbf{p}_j$ is distributed proportionally to $\mathbf{p}_i$ and $\mathbf{p}_j$:

$$\mathbf{F}_i = \alpha\mathbf{F}_k \text{ and } \mathbf{F}_j = (1-\alpha)\mathbf{F}_k$$

Internal forces

internal forces =

$\sum$ linear combinations of 

$+\sum$ nonnegative linear combinations of 

Testing stability

Freie Universität Berlin

Static equilibrium of forces:

At every mass point $\mathbf{p}_i$ with mass m_i,

$$\underbrace{\sum \text{internal forces}}_{\textit{unknown}} + \underbrace{\sum \text{external forces}}_{m_i \cdot \mathbf{g}} = \mathbf{0}$$

unknown *given*

Find a system of internal forces (subject to linear equations and inequalities) that balance the given external forces.

$\rightarrow$ linear programming

Equilibrium of forces (dynamic)

Newton's Laws: Forces must be in equilibrium

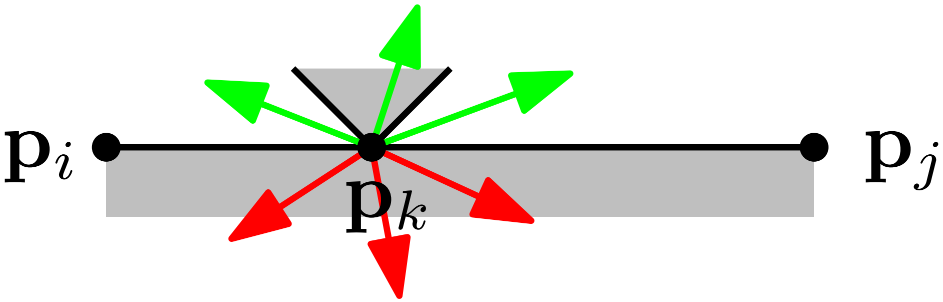
At every point $\mathbf{p}_i$ with mass m_i and acceleration $\mathbf{a}_i$,

$$\underbrace{\sum \text{internal forces}}_{} + \underbrace{\text{external force}}_{\substack{m_i \cdot \mathbf{g} \\ \text{(gravity)}}} + \underbrace{\text{inertial force}}_{-m_i \cdot \mathbf{a}_i} = \mathbf{0}$$

unknown *given* *unknown*

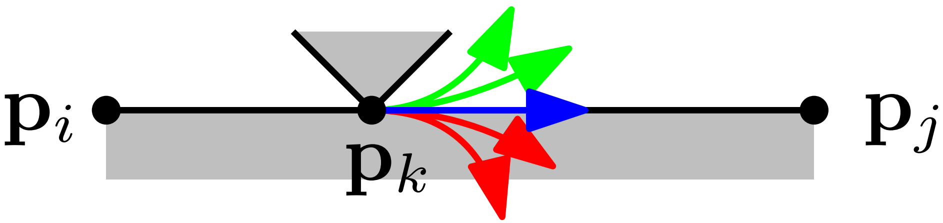
Determining the constraints (1)

Freie Universität Berlin



Sidedness constraint:
$A(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k) \geq 0$

If $A(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k) = 0$, then $\dfrac{\mathrm{d}}{\mathrm{d}t} A(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k) \geq 0$



If $A(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k) = 0$ and $\dfrac{\mathrm{d}}{\mathrm{d}t} A(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k) = 0$, then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} A(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k) \geq 0$$

Determining the constraints (2)

Length constraint:
$\|\mathbf{p}_i - \mathbf{p}_j\| = \text{const}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{p}_i - \mathbf{p}_j\| = 0 \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\|\mathbf{p}_i - \mathbf{p}_j\| = 0$$

The motion

location	$\mathbf{p}_i = \mathbf{p}_i(t)$, time parameter $t \ge 0$
velocity	$\mathbf{v}_i = \dot{\mathbf{p}}_i$
acceleration	$\mathbf{a}_i = \ddot{\mathbf{p}}_i$

Sidedness constraints:

(1) $$\frac{\mathrm{d}^2}{\mathrm{d}t^2} A(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k) \ge 0, \text{ for certain triples } (i, j, k).$$

$\longrightarrow$ linear inequalities in $\mathbf{a}$, for given $\mathbf{p}$ and $\mathbf{v}$.

Length constraints:

(2) $$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \|\mathbf{p}_i - \mathbf{p}_j\|^2 = 0, \text{ for all bars } (i, j).$$

$\longrightarrow$ linear equations in $\mathbf{a}$, for given $\mathbf{p}$ and $\mathbf{v}$.

Equilibrium of forces (continued)

Freie Universität Berlin

At every point $\mathbf{p}_i$ with mass m_i and acceleration $\mathbf{a}_i$,

$$\sum \text{internal forces} + \underbrace{\text{external force}}_{\substack{m_i \cdot \mathbf{g} \\ \text{(gravity)}}} + \underbrace{\text{inertial force}}_{-m_i \cdot \mathbf{a}_i} = \mathbf{0}$$

$$\underbrace{\sum \text{internal forces}}_{\text{unknown}} = m_i \cdot (\mathbf{a}_i - \mathbf{g})$$

unknown, subject to linear equations and inequalities (1) and (2)

Theorem

Let $\mathbf{p}_i$ and $\mathbf{v}_i$ be given for all points $i = 1, \ldots, n$.
The solution $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ of

$$\sum \text{internal forces} = m_i \cdot (\mathbf{a}_i - \mathbf{g}), \qquad \text{for all } i$$

subject to constraints (1) and (2), is given by the quadratic optimization problem

$$(*) \qquad \text{minimize} \sum_{i=1}^{n} m_i \cdot \|\mathbf{a}_i - \mathbf{g}\|^2$$

subject to (1) and (2).

Theorem

Let $\mathbf{p}_i$ and $\mathbf{v}_i$ be given for all points $i = 1, \ldots, n$. The solution $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ of

$$\sum \text{internal forces} = m_i \cdot (\mathbf{a}_i - \mathbf{g}), \qquad \text{for all } i$$

subject to constraints (1) and (2), is given by the quadratic optimization problem

$$(*) \qquad \text{minimize } \sum_{i=1}^{n} m_i \cdot \|\mathbf{a}_i - \mathbf{g}\|^2$$

assume $m_i \equiv 1$

subject to (1) and (2).

Intuition

$$\text{minimize} \sum_{i=1}^{n} \|\mathbf{a}_i - \mathbf{g}\|^2$$

Acceleration $\mathbf{a}_i$ tries to follow the gravity force $\mathbf{g}$ as closely as possible.

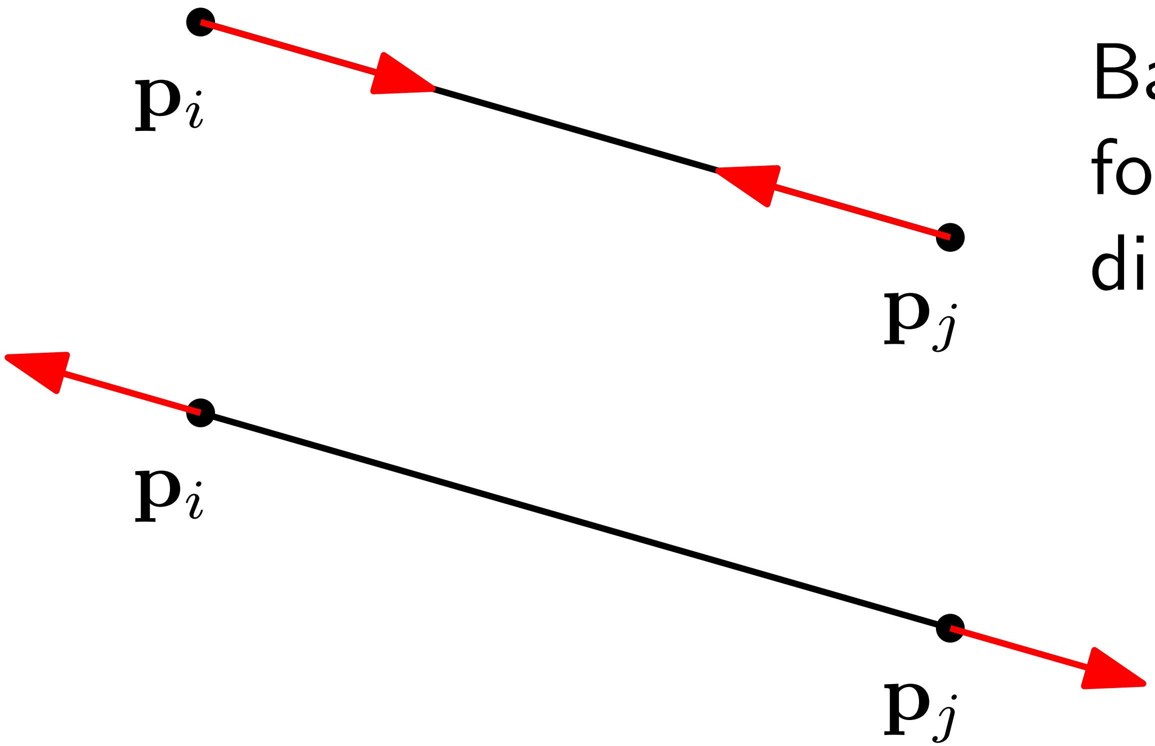
Any discrepancy $\mathbf{a}_i - \mathbf{g}$ must be *resolved* by internal forces:

$$\mathbf{a}_i - \mathbf{g} = \sum \text{internal forces at point } i$$

The internal forces are just the *dual variables* (Lagrange multipliers) corresponding to the constraints (1) and (2).

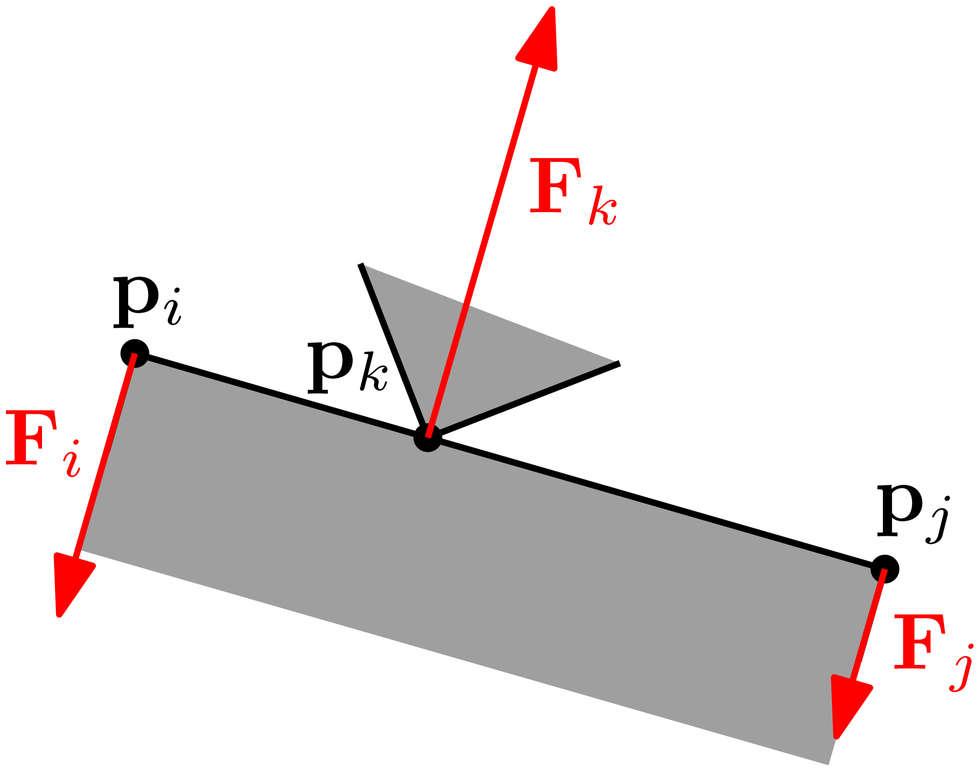
Forces as dual variables: Lengths

The length $\|\mathbf{p}_i - \mathbf{p}_j\|$ remains constant.



Bars may transmit forces parallel to the bar directions.

Forces as dual variables: Sidedness



Collisions must be prevented.

When $\mathbf{p}_k$ touches the bar $\mathbf{p}_i\mathbf{p}_j$, forces perpendicular to the bar direction may be transmitted, but *in one direction* only.

$\rightarrow$ linear inequalities

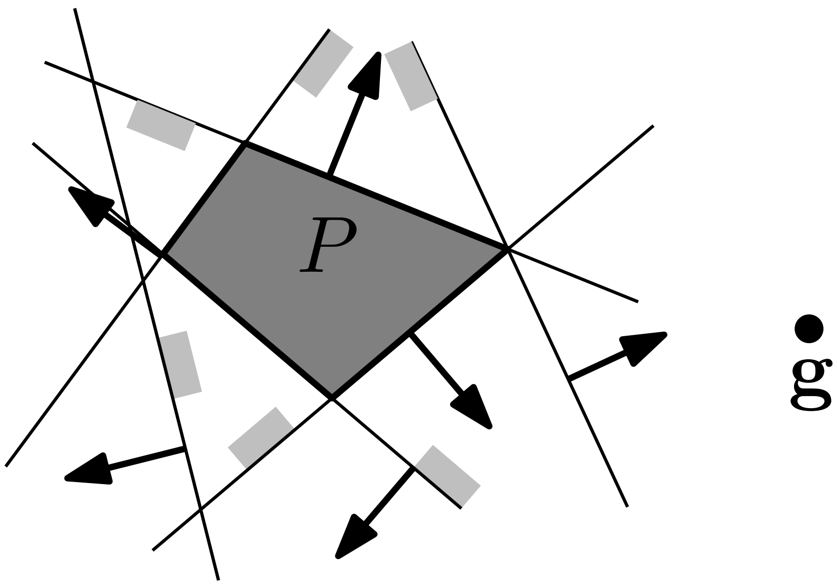
The force at $\mathbf{p}_k = \alpha\mathbf{p}_i + (1-\alpha)\mathbf{p}_j$ is distributed proportionally to $\mathbf{p}_i$ and $\mathbf{p}_j$:

$$\mathbf{F}_i = \alpha\mathbf{F}_k \text{ and } \mathbf{F}_j = (1-\alpha)\mathbf{F}_k$$

Geometric intuition (polytopes)

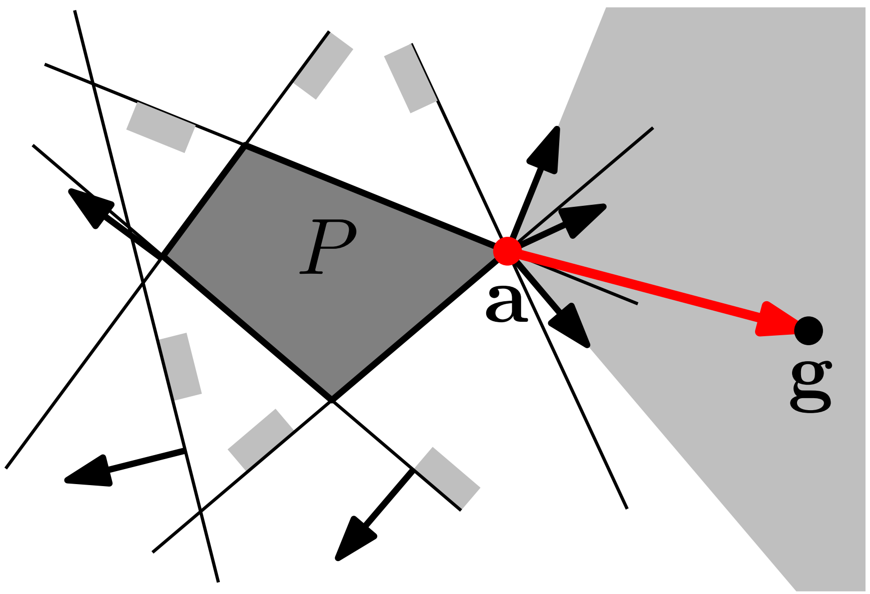
Freie Universität Berlin

For a polytope $P \subseteq \mathbb{R}^{2n}$ given by linear inequalities and a *target point* $\mathbf{g}$. Find a point $\mathbf{a} \in P$ such that



Geometric intuition (polytopes)

For a polytope $P \subseteq \mathbb{R}^{2n}$ given by linear inequalities and a *target point* $\mathbf{g}$. Find a point $\mathbf{a} \in P$ such that



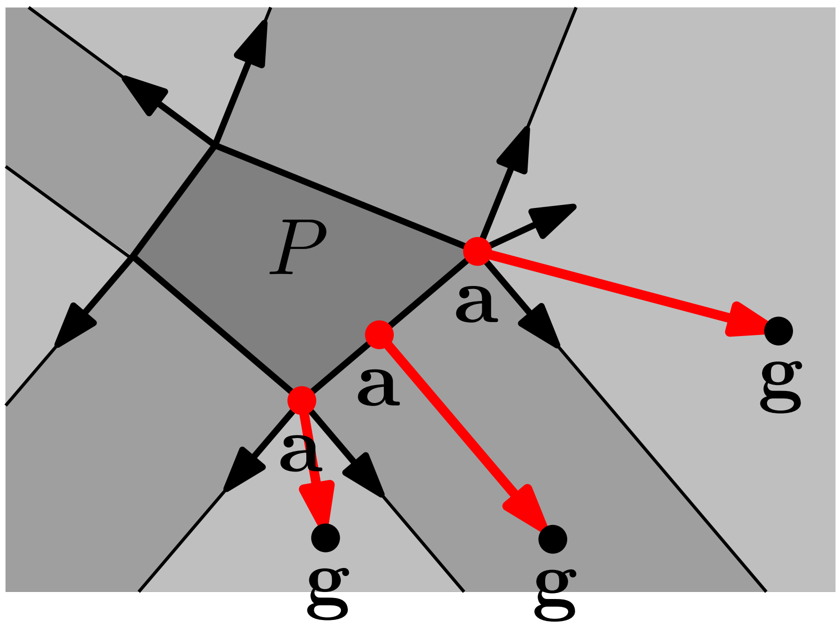
the difference vector $\mathbf{g} - \mathbf{a}$ lies in the cone spanned by the normal vectors of the tight inequalities at $\mathbf{a}$:

$$\mathbf{g} - \mathbf{a} = \sum \text{internal forces}$$

Geometric intuition (polytopes)

Freie Universität Berlin

For a polytope $P \subseteq \mathbb{R}^{2n}$ given by linear inequalities and a *target point* $\mathbf{g}$. Find a point $\mathbf{a} \in P$ such that



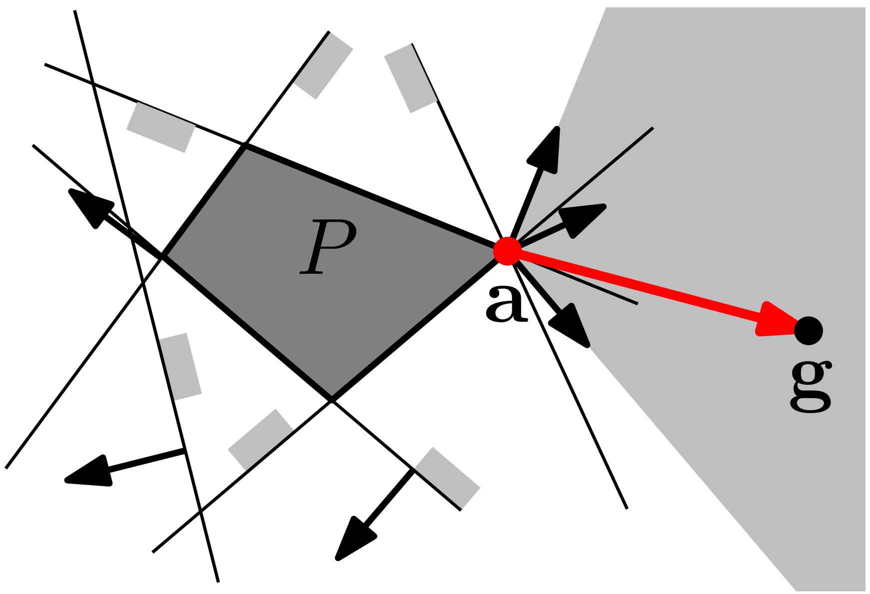
the difference vector $\mathbf{g} - \mathbf{a}$ lies in the cone spanned by the normal vectors of the tight inequalities at $\mathbf{a}$:

$$\mathbf{g} - \mathbf{a} = \sum \text{internal forces}$$

The point $\mathbf{a} \in P$ is uniquely determined by $\mathbf{g}$, since the *normal fan* of the polytope P partitions space.
$\mathbf{a}$ is the point in P closest to $\mathbf{g}$.

Geometric intuition (polytopes)

For a polytope $P \subseteq \mathbb{R}^{2n}$ given by linear inequalities and a *target point* $\mathbf{g}$. Find a point $\mathbf{a} \in P$ such that



the difference vector $\mathbf{g} - \mathbf{a}$ lies in the cone spanned by the normal vectors of the tight inequalities at $\mathbf{a}$:

$$\mathbf{g} - \mathbf{a} = \sum \text{internal forces}$$

The *internal forces* (the linear combination of normal vectors that produce $\mathbf{g} - \mathbf{a}$) are in general *not unique*.

The Principle of Least Constraint

Freie Universität Berlin

Über ein neues allgemeines Grundgesetz der Mechanik.

(Vom Herrn Hofrath und Prof. Dr. *Gaufs* zu Göttingen.)

(On a new general principle of mechanics.)

J. reine angew. Math. **4** (1829), 232–235

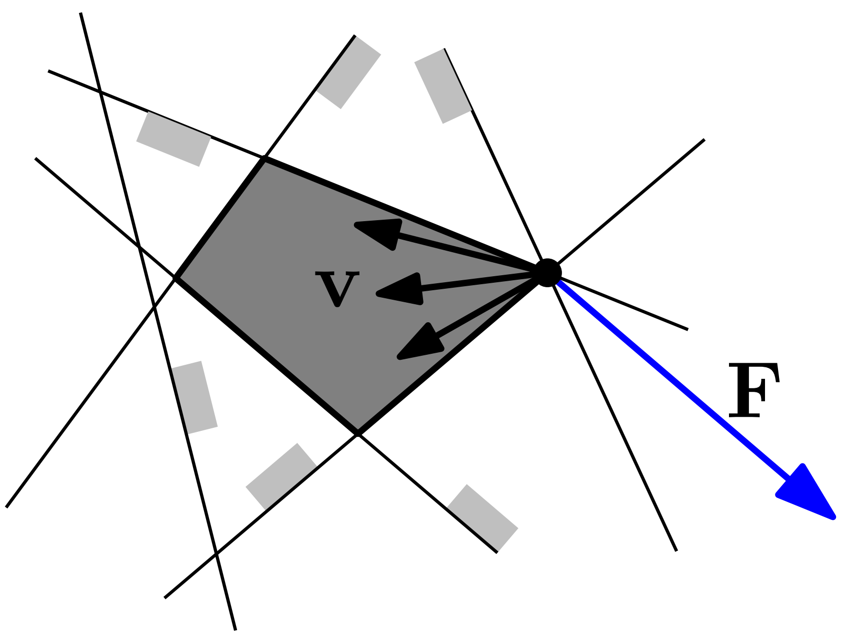
Prinzip des *kleinsten Zwanges* (principle of least restraint)

derivation from d'Alembert's *method of virtual velocities* (virtual displacements)

Virtual Velocities (Displacements)

Every *feasible direction* $\mathbf{v}$ (virtual velocity, virtual displacement) must make a negative inner product with the external and inertial forces $\mathbf{F}$:

$$\langle \mathbf{v}, \mathbf{F} \rangle = \sum_i \langle \mathbf{v}_i, \mathbf{F}_i \rangle \leq 0$$



D'Alembert's Principle

The *internal forces* are eliminated.

The Principle of Least Constraint

Freie Universität Berlin

Es ist sehr merkwürdig, dafs die freien Bewegungen, wenn sie mit nothwendigen Bedingungen nicht bestehen können, von der Natur gerade auf dieselbe Art modificirt werden, wie der rechnende Mathematiker, nach der Methode der kleinsten Quadrate, Erfahrungen ausgleicht, die sich auf unter einander durch nothwendige Abhängigkeit verknüpfte Gröfsen beziehen. Diese Analogie liefse sich noch weiter verfolgen, was jedoch gegenwärtig nicht zu meiner Absicht gehört.

It is very remarkable that the free motions, if they are not consistent with constraints, are modified by nature in just the same way, as the calculating mathematician, according to the *method of least squares*, adjusts measurements of quantities that are related to each other by a dependency.
This analogy could be pursued further, which is, however, currently not my intention.

C. F. Gauß, Über ein neues allgemeines Grundgesetz der Mechanik, 1829

Historical Notes

A. Prékopa: On the development of optimization theory
Amer. Math. Monthly **87** (1980), 527–542

Method of *Lagrange multipliers* for finding extrema of functions subject to equality constraints

(Lagrange, *Méchanique Analytique*, 1788)

Purpose:
Find the stable equilibrium state of a mechanical system.

Historical Notes

A. Prékopa: On the development of optimization theory
Amer. Math. Monthly **87** (1980), 527–542

Method of *Lagrange multipliers* for finding extrema of functions subject to equality constraints

(Lagrange, *Méchanique Analytique*, 1788)

Purpose:
Find the stable equilibrium state of a mechanical system.

Extension to mechanical systems with *inequalities*:

J. Fourier (1798): formulated *Fourier's Principle*
A.-A. Cournot (1827)
M. V. Ostrogradsky (1834)
Gy. Farkas (1894, 1898): *linear* inequalities

Historical Notes

A. Prékopa: On the development of optimization theory
Amer. Math. Monthly **87** (1980), 527–542

Method of *Lagrange multipliers* for finding extrema of functions subject to equality constraints

(Lagrange, *Méchanique Analytique*, 1788)

Nonlinear optimization (*nonlinear* inequalities):

W. Karush (1939, Master's thesis, Univ. Chicago)
F. John (1948)
H. W. Kuhn, A. W. Tucker (1951)

Applications of Gauß' Principle

many papers on *rigid-body simulations*:

Proc. IEEE Conf. Robotics and Automation (ICRA)
Proc. ACM SIGGRAPH

3. Global motion

Freie Universität Berlin

$$\mathbf{p}(t) = (\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t)),$$
$$\mathbf{v}(t) = (\mathbf{v}_1(t), \ldots, \mathbf{v}_n(t)) = \dot{\mathbf{p}}(t),$$
$$\mathbf{a}(t) = (\mathbf{a}_1(t), \ldots, \mathbf{a}_n(t)) = \ddot{\mathbf{p}}(t)$$

$$\mathbf{a}(t) = f(\mathbf{p}(t), \mathbf{v}(t))$$

$$\ddot{\mathbf{p}}(t) = f(\mathbf{p}(t), \dot{\mathbf{p}}(t))$$

$f(\mathbf{p}, \mathbf{v})$ is given as the solution of the quadratic optimization problem.

$\longrightarrow$ second-order system of differential equations

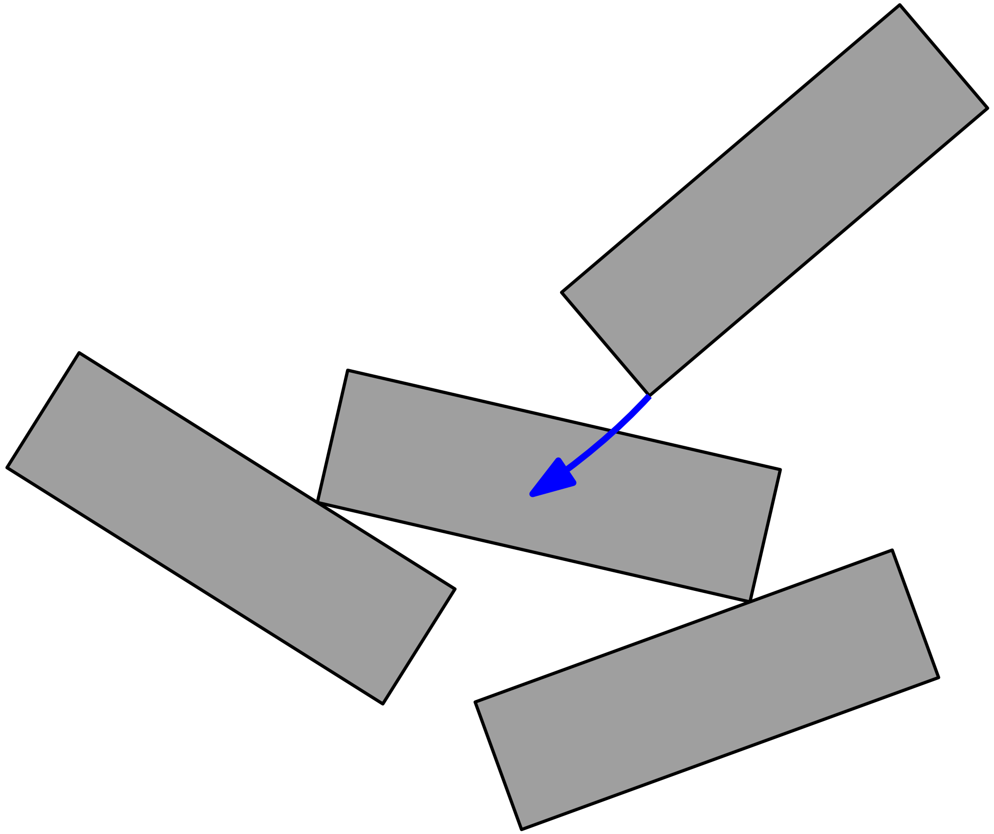
Implementation

Rough prototype implementation in a student project, using Euler integration

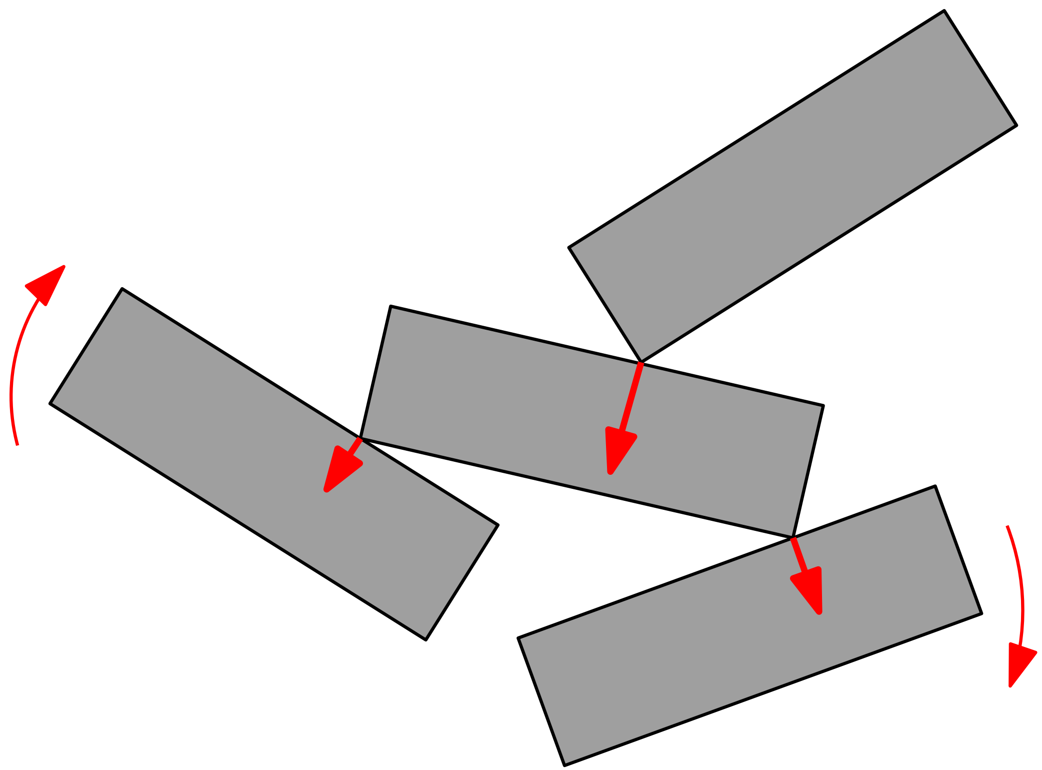
Simplex-type quadratic optimization algorithm implemented in CGAL (Computational geometry algorithms library).
CGAL focuses on exact algorithms, problems with few variables (no refined linear algebra procedures).
$\longrightarrow$ very slow

Desired:
- visualization of forces

4. Collision



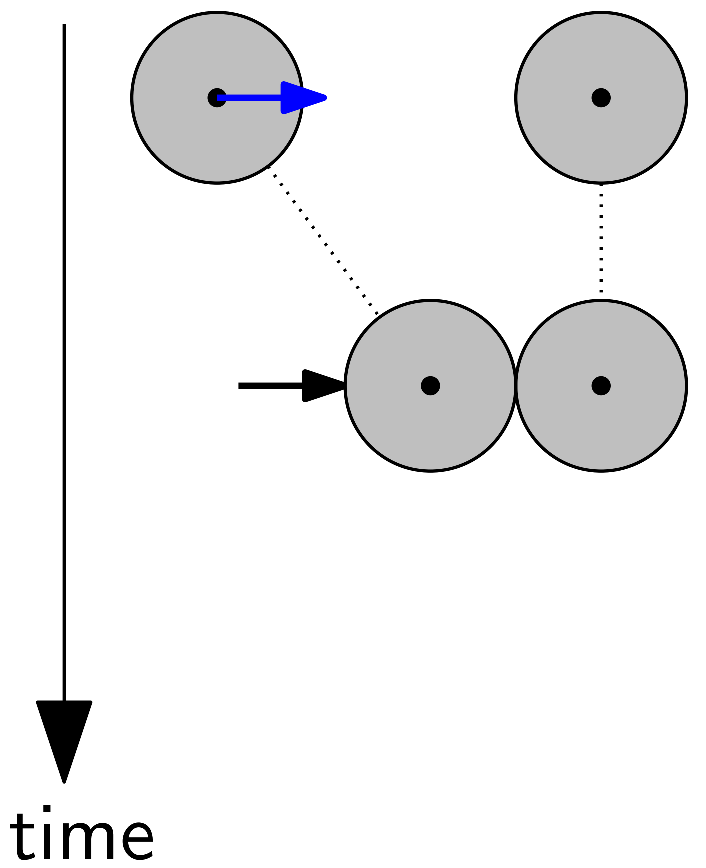
4. Collision



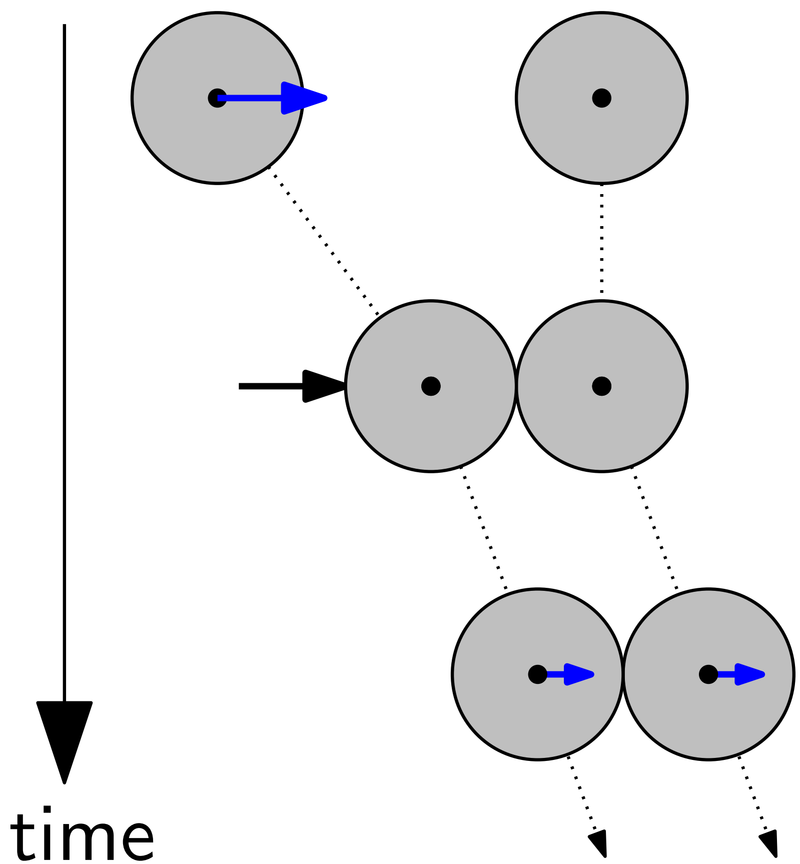
Velocities $\mathbf{v}_i$ change discontinuously.

"the ill-posed nature of simultaneous multiple collisions"
A. Chatterjee, A. Ruina, *J. Appl. Mech.* (1998)

Elastic and inelastic collision

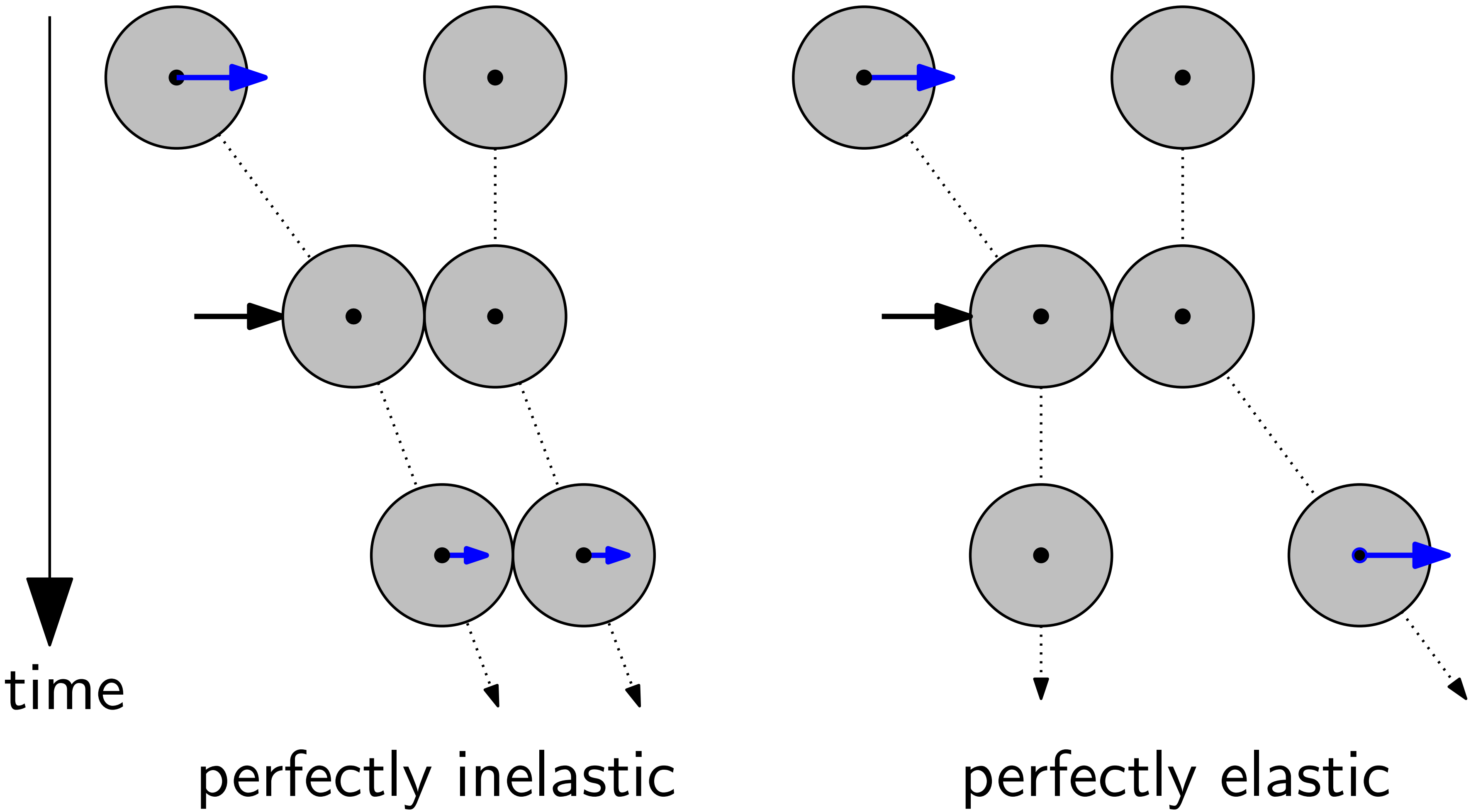


Elastic and inelastic collision



perfectly inelastic

Elastic and inelastic collision



Total momentum is always preserved.

Perfectly inelastic collision

$$\text{minimize} \sum_{i=1}^{n} m_i \cdot \|\mathbf{v}_i^{\text{new}} - \mathbf{v}_i^{\text{old}}\|^2$$

subject to length and sidedness constraints on $\mathbf{v}_i^{\text{new}}$.

Perfectly inelastic collision

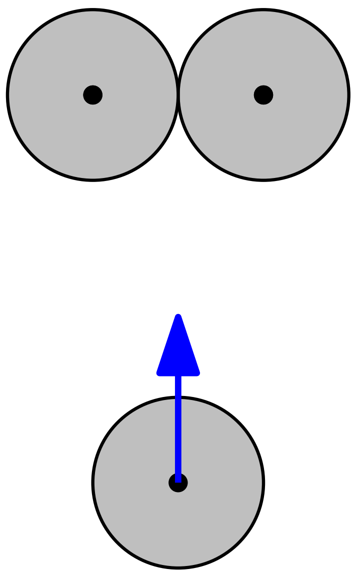
$$\text{minimize } \sum_{i=1}^{n} m_i \cdot \|\mathbf{v}_i^{\text{new}} - \mathbf{v}_i^{\text{old}}\|^2$$

subject to length and sidedness constraints on $\mathbf{v}_i^{\text{new}}$.

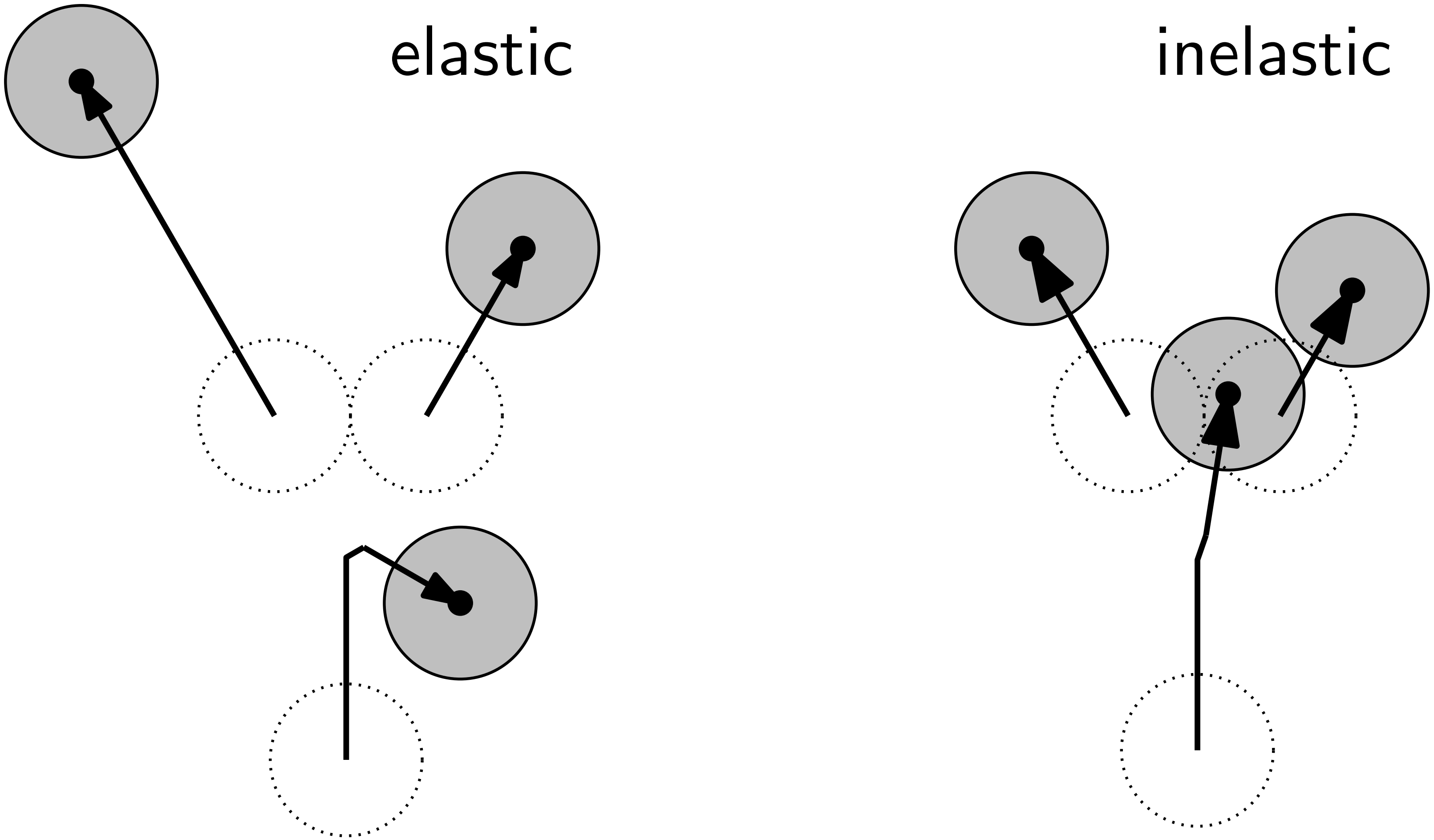
The case of two colliding disks is represented correctly.

"Proof" as above:
Exchange of momentum (impulse) can only happen if two bodies remain in contact afterwards (to first order).

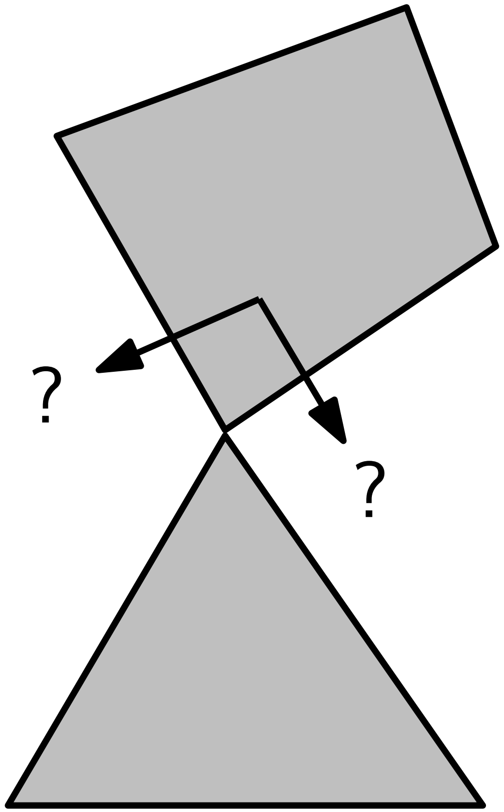
Nondeterministic Billiards



Nondeterministic Billiards



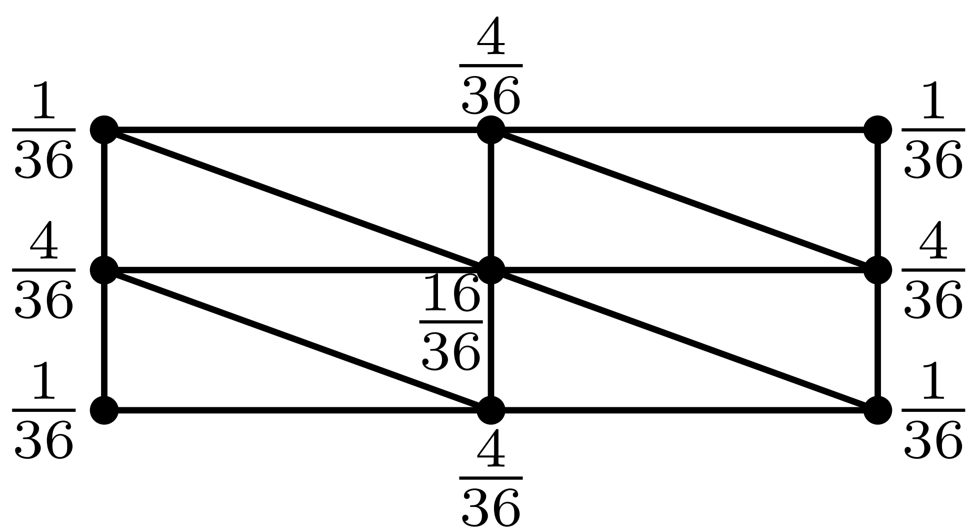
Nondeterministic Billiards



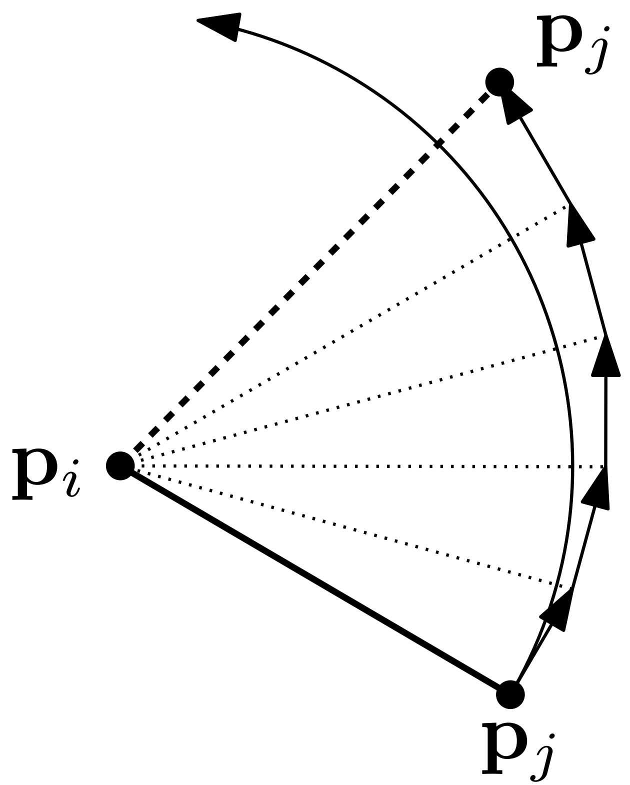
Homogeneous mass distributions

A homogeneous rectangle

can be replaced by a rigid framework of 9 discrete points:

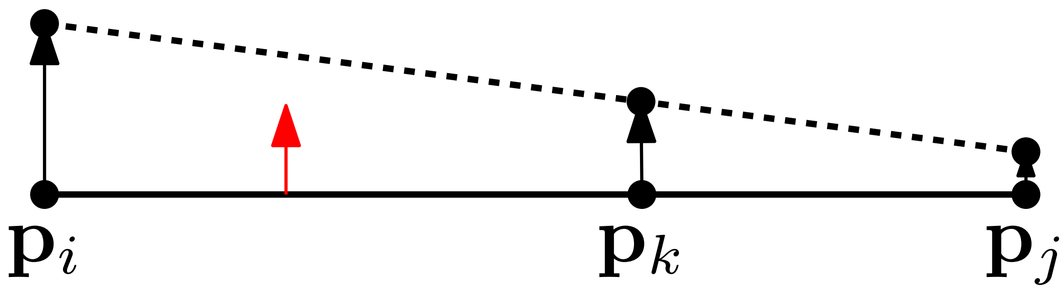


(A) Systematic growth of lengths (discretization error)

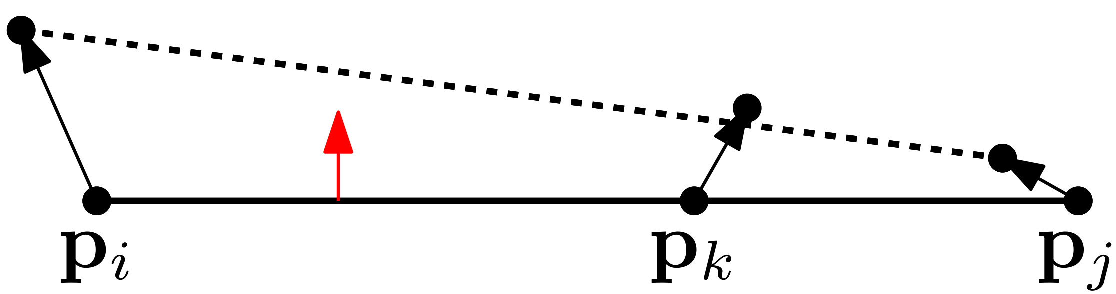


5. Implementation Problems

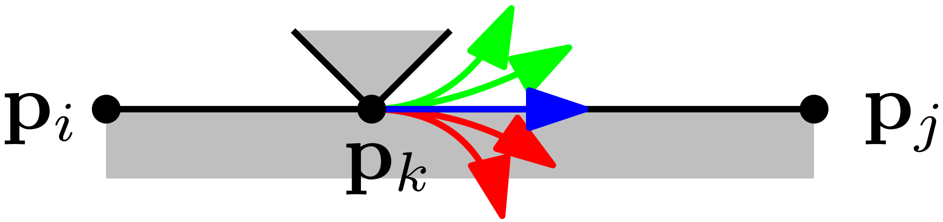
(B) Point may creep across an edge (discretization error)



(First-order) non-penetration condition looks only at the projection onto the normal vector.



(C) Non-penetration condition requires testing for zero.



If $A(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t} A(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k) = 0$, then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} A(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k) \geq 0$$

$\rightarrow$ linear inequalities in $\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k$, for given $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$ and $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$.

5. Implementation

Freie Universität Berlin

Eliminate velocities $\mathbf{v}_i$ and accelerations $\mathbf{a}_i$!

1. Linear extrapolation:

$$\bar{\mathbf{p}}_i := \mathbf{p}_i^{(k-1)} + [\mathbf{p}_i^{(k-1)} - \mathbf{p}_i^{(k-2)}]$$

2. Apply external forces: $\tilde{\mathbf{p}}_i := \bar{\mathbf{p}}_i + \mathbf{g} \cdot (\Delta t)^2$
3. Apply constraints:

$$(**) \qquad \text{minimize} \sum_{i=1}^{n} m_i \cdot \|\mathbf{p}_i - \tilde{\mathbf{p}}_i\|^2$$

subject to linearized constraints (1) and (2) on $(\mathbf{p}_i)$.

4. Use the solution $\mathbf{p}_i$ as $\mathbf{p}_i^{(k)}$. $\longrightarrow$ Gauß

5. Implementation

Eliminating velocities $\mathbf{v}_i$ and accelerations $\mathbf{a}_i$:

Objective function:

$$\mathbf{p}_i^{\text{new}} \approx \mathbf{p}_i^{\text{old}} + \mathbf{v}_i \cdot \Delta t + \mathbf{a}_i \cdot (\Delta t)^2$$

$$\tilde{\mathbf{p}}_i \approx \mathbf{p}_i^{\text{old}} + \mathbf{v}_i \cdot \Delta t + \mathbf{g} \cdot (\Delta t)^2$$

$\Longrightarrow$

$$\mathbf{p}_i^{\text{new}} - \tilde{\mathbf{p}}_i \approx (\mathbf{a}_i - \mathbf{g}) \cdot (\Delta t)^2$$

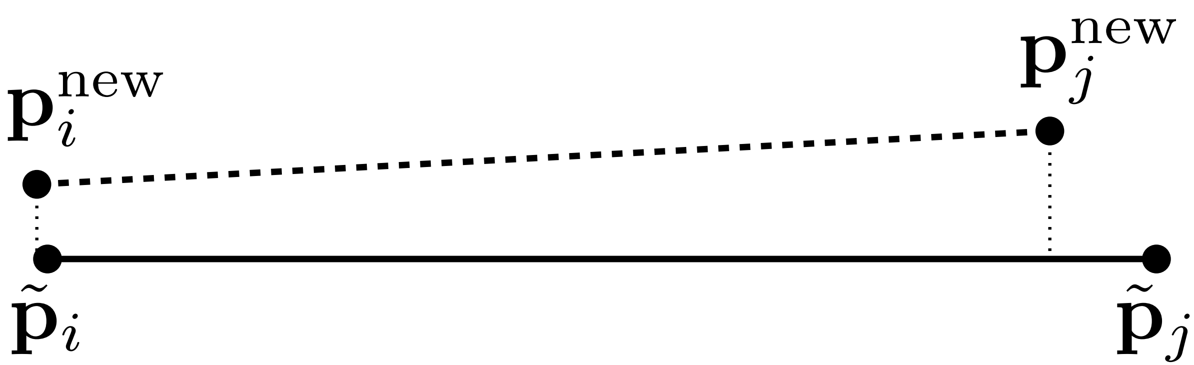
Therefore,

$$\text{minimize} \sum_i \|\mathbf{p}_i^{\text{new}} - \tilde{\mathbf{p}}_i\|^2 \;\equiv\; \text{minimize} \sum_i \|\mathbf{a}_i - \mathbf{g}\|^2$$

5. Implementation

Freie Universität Berlin

Linearized length constraint (1):



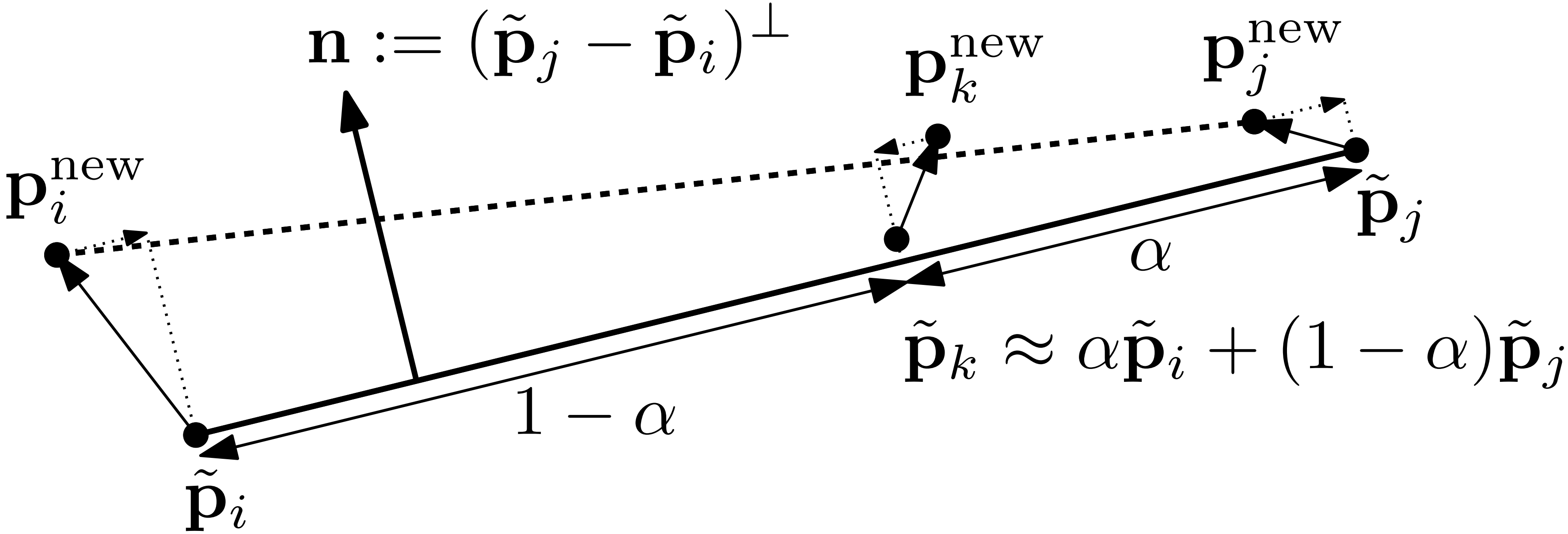
$$\langle \mathbf{p}_i^{\mathrm{new}} - \mathbf{p}_j^{\mathrm{new}}, \tilde{\mathbf{p}}_i - \tilde{\mathbf{p}}_j \rangle = \|\tilde{\mathbf{p}}_i - \tilde{\mathbf{p}}_j\| \cdot \ell_{ij}$$

ℓ_{ij} = the desired (correct) length of edge ij.

(No redundant bars!)

5. Implementation

Linearized non-penetration constraint (2):



$$\langle \mathbf{p}_k^{\mathrm{new}}, \mathbf{n}\rangle \geq \alpha\langle \mathbf{p}_i^{\mathrm{new}}, \mathbf{n}\rangle + (1-\alpha)\langle \mathbf{p}_j^{\mathrm{new}}, \mathbf{n}\rangle$$

5. Implementation

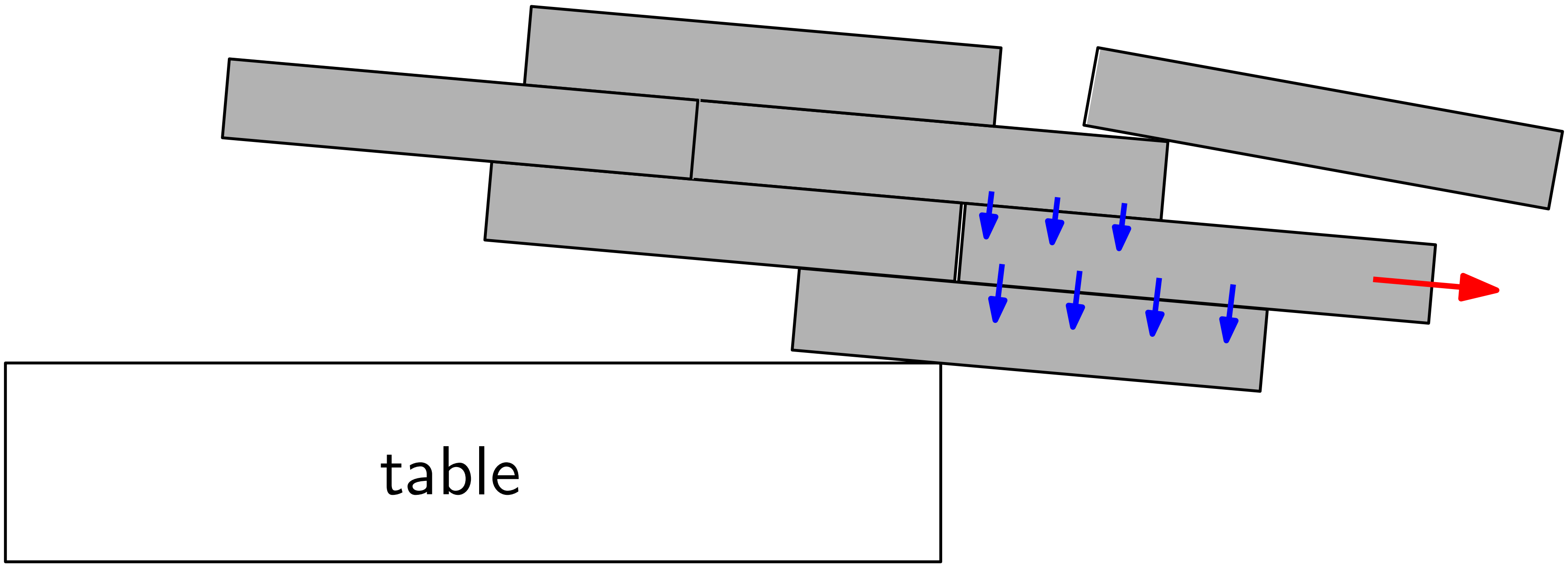
- Length and penetration is corrected at every step
- The same model solves (non-elastic) collisions
- No distinction between (slight) penetration and (sudden) collision

5. Implementation

- Length and penetration is corrected at every step
- The same model solves (non-elastic) collisions
- No distinction between (slight) penetration and (sudden) collision
- Quadratic optimization problems solved by CPLEX (commercial solver, uses interior-point method)
- Surrounding framework written in PYTHON
- Simulation written to a file;
  visualization by student project (JAVA)

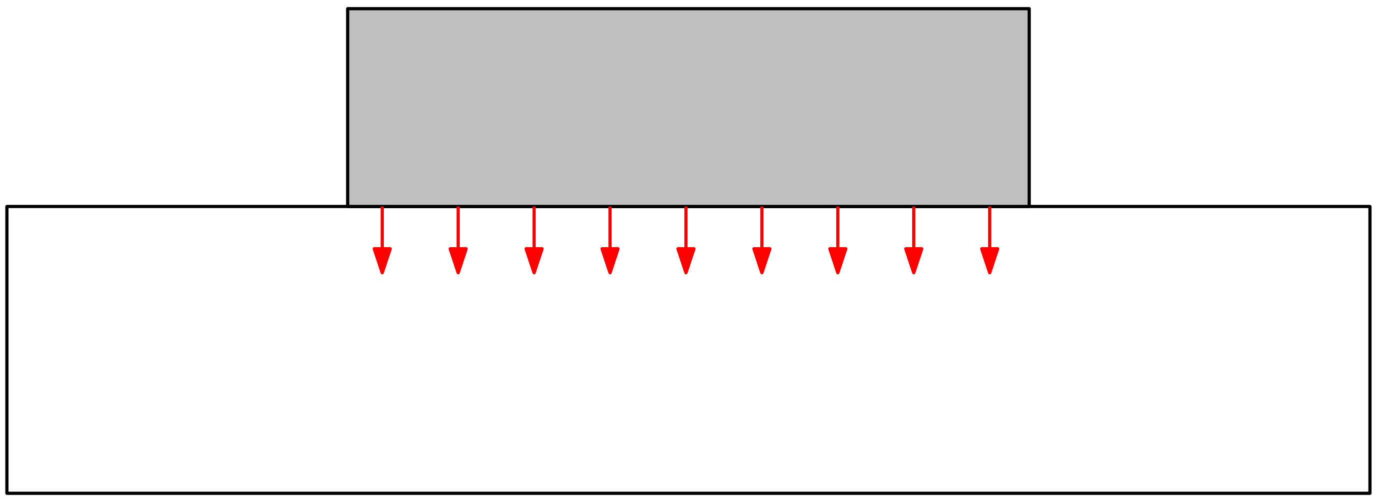
6. Friction

Friction is proportional to the force (pressure).

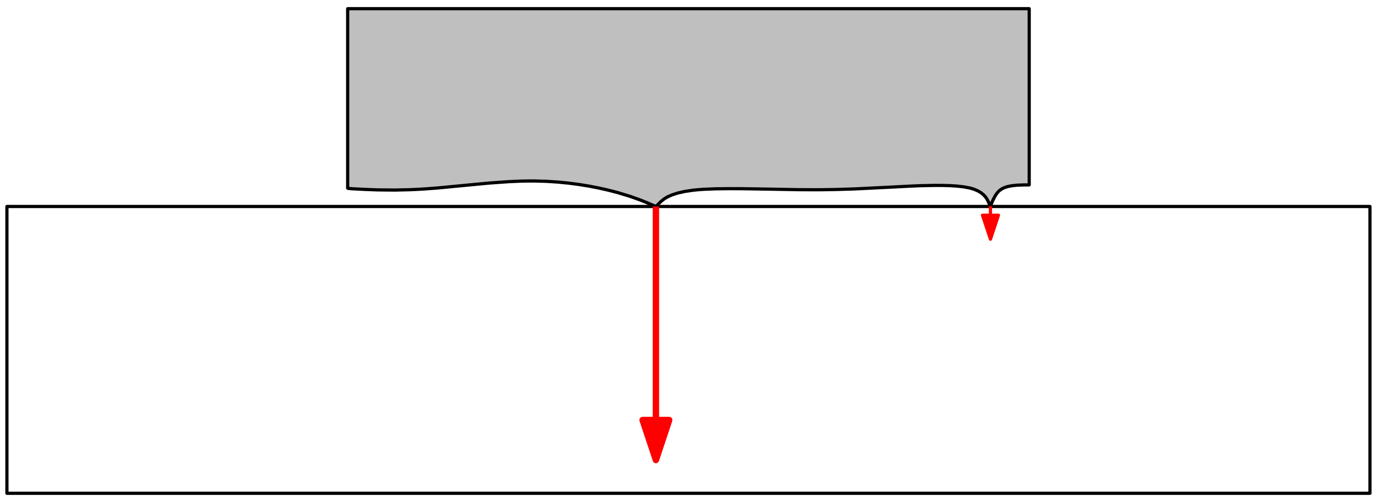


PROBLEM: Forces are not unique.

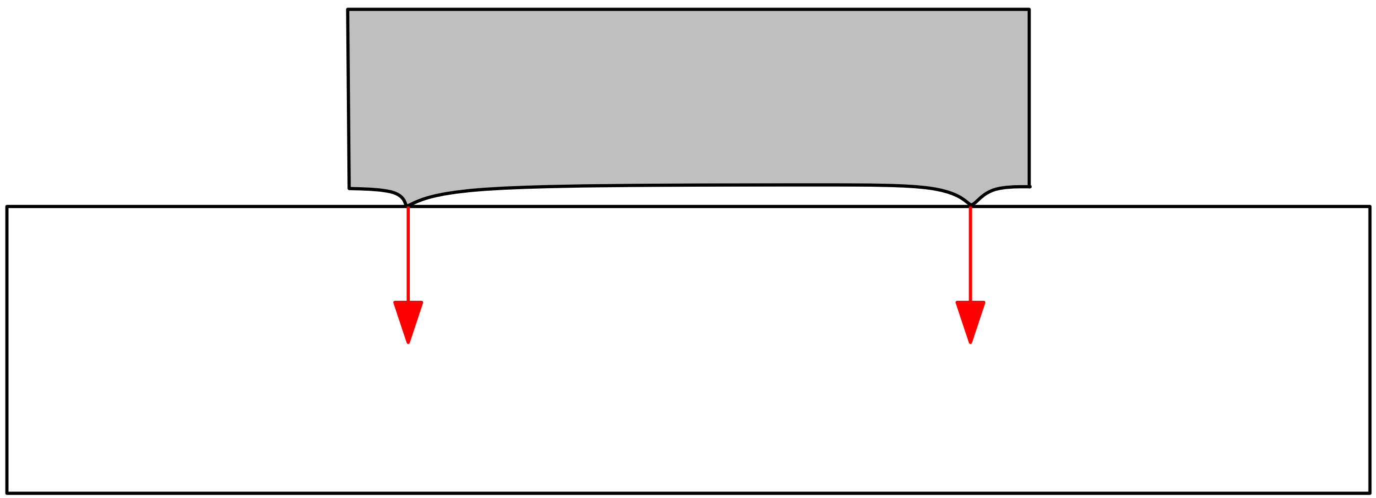
Forces are not unique



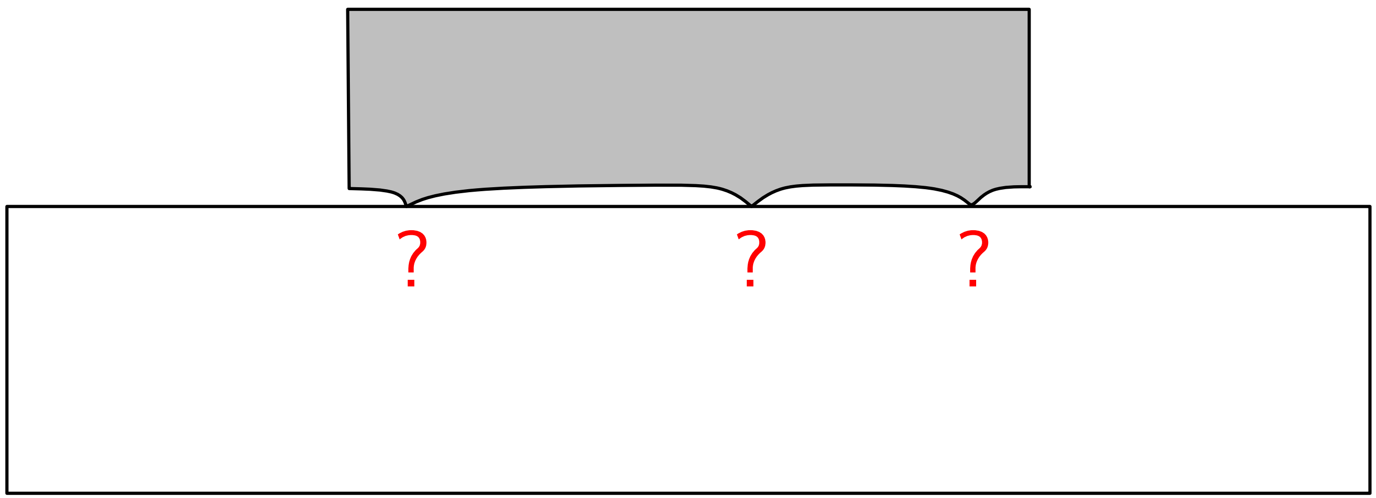
Forces are not unique



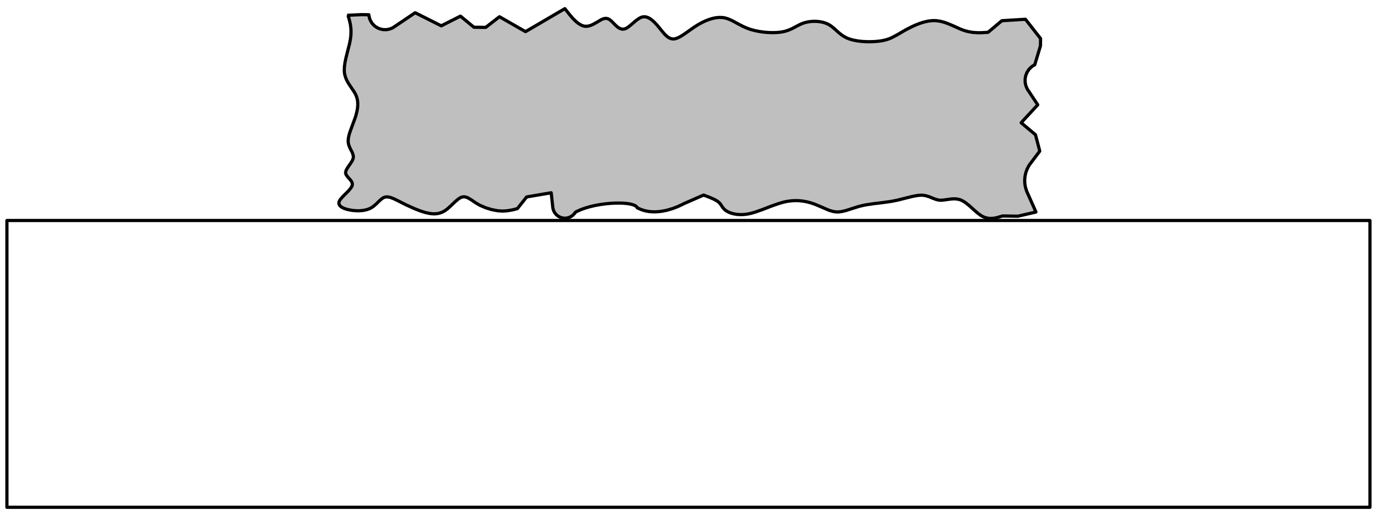
Forces are not unique



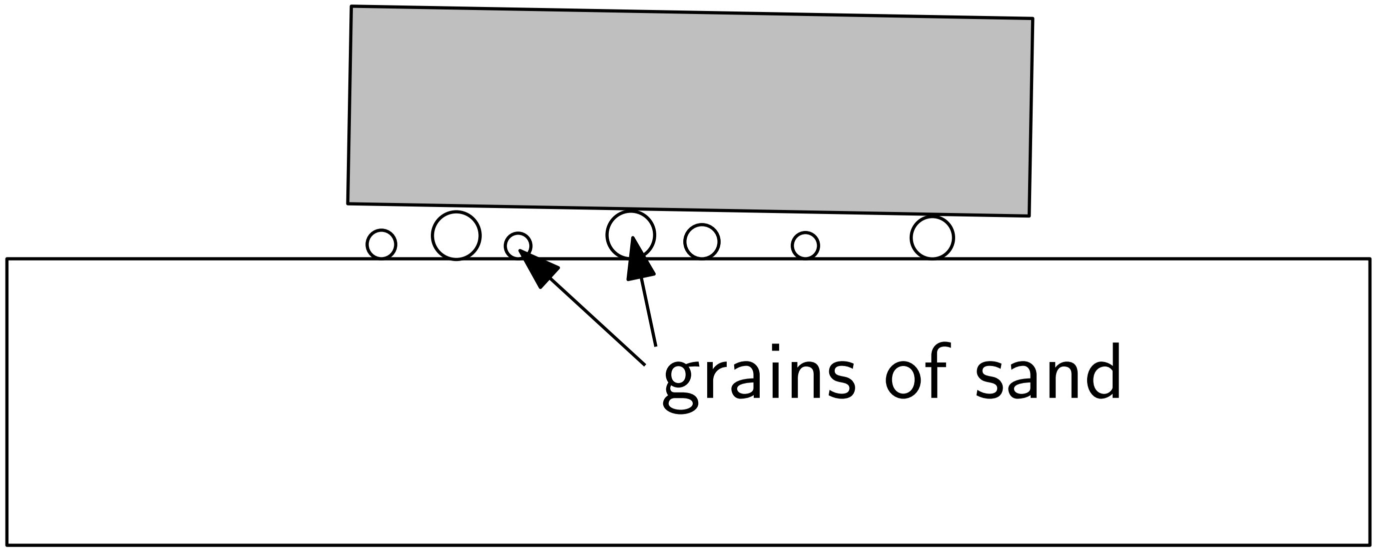
Forces are not unique



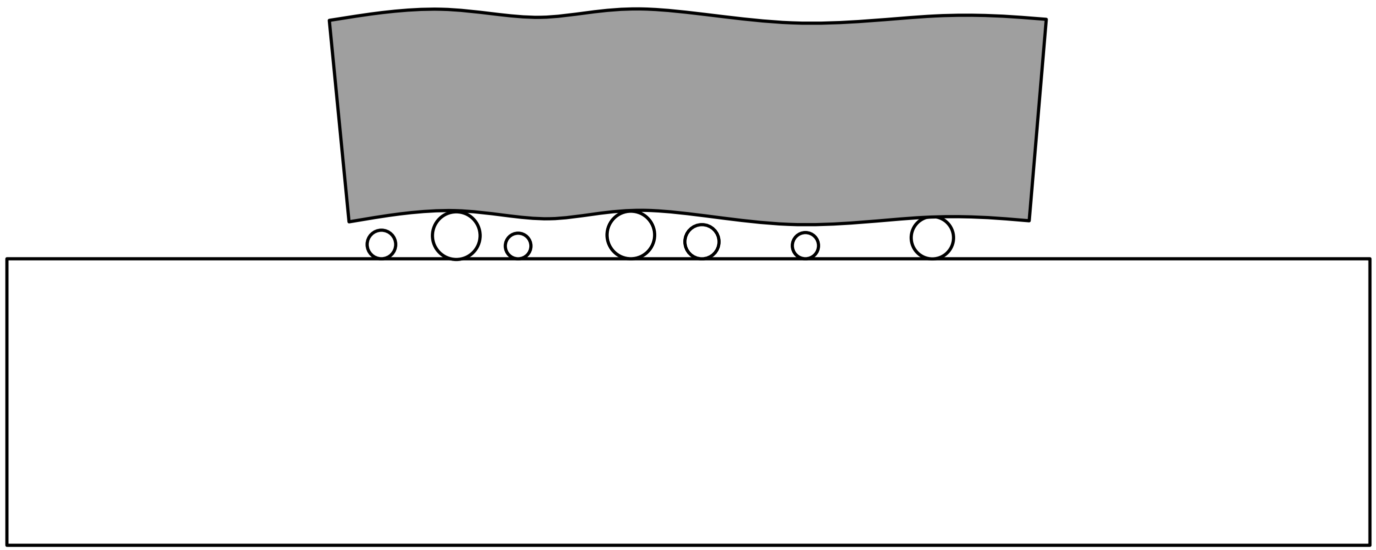
Forces are not unique



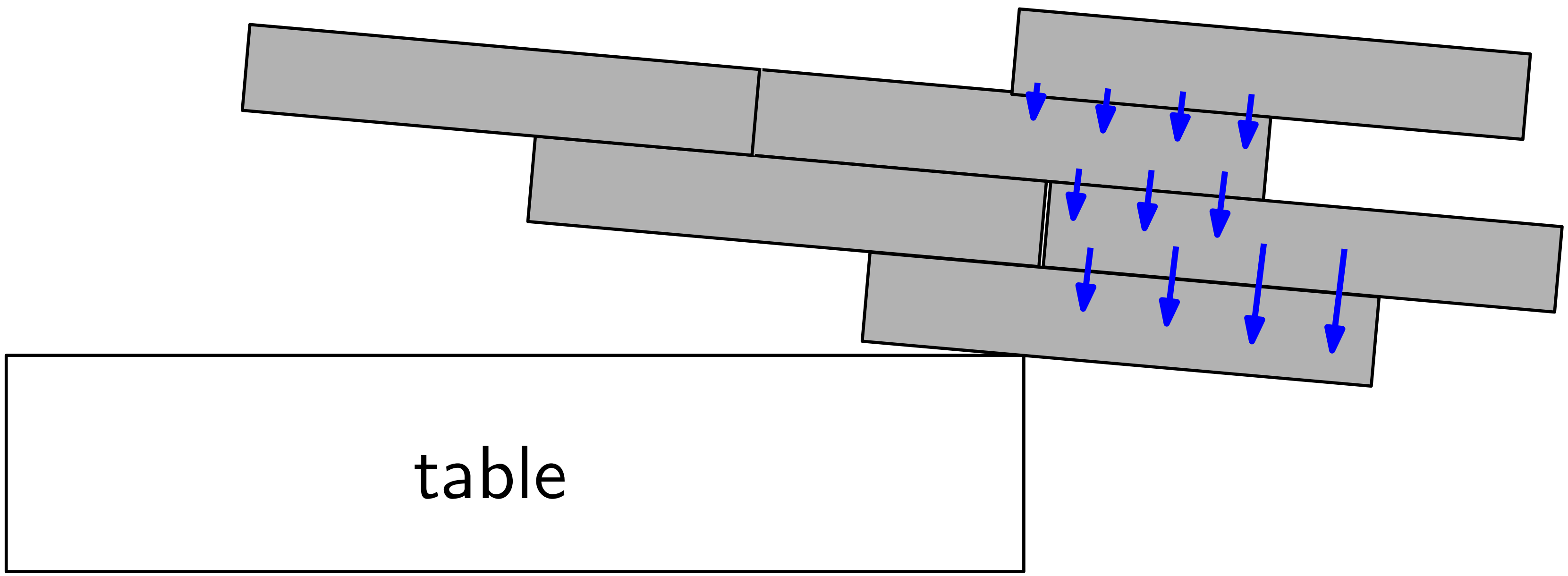
Forces are not unique



Forces are not unique



Unique forces ?



Calculating the distribution of forces requires looking into stiffness/elasticity, shock waves, etc.