Pseudotriangulations, Polytopes, and How to Expand Linkages

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[joint] work of/with Bob Connelly, Erik Demaine, Paco Santos, Ileana Streinu.

Unfolding of polygons

Theorem. *Every polygonal arc in the plane can be brought into straight position, without self-overlap.*

Every polygon in the plane can be unfolded into convex position.

Infinitesimal Motion

n vertices $p_1, \ldots, p_n$.

1. (global) *motion* $p_i = p_i(t)$, $t \geq 0$

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1. (global) *motion* $p_i = p_i(t)$, $t \geq 0$

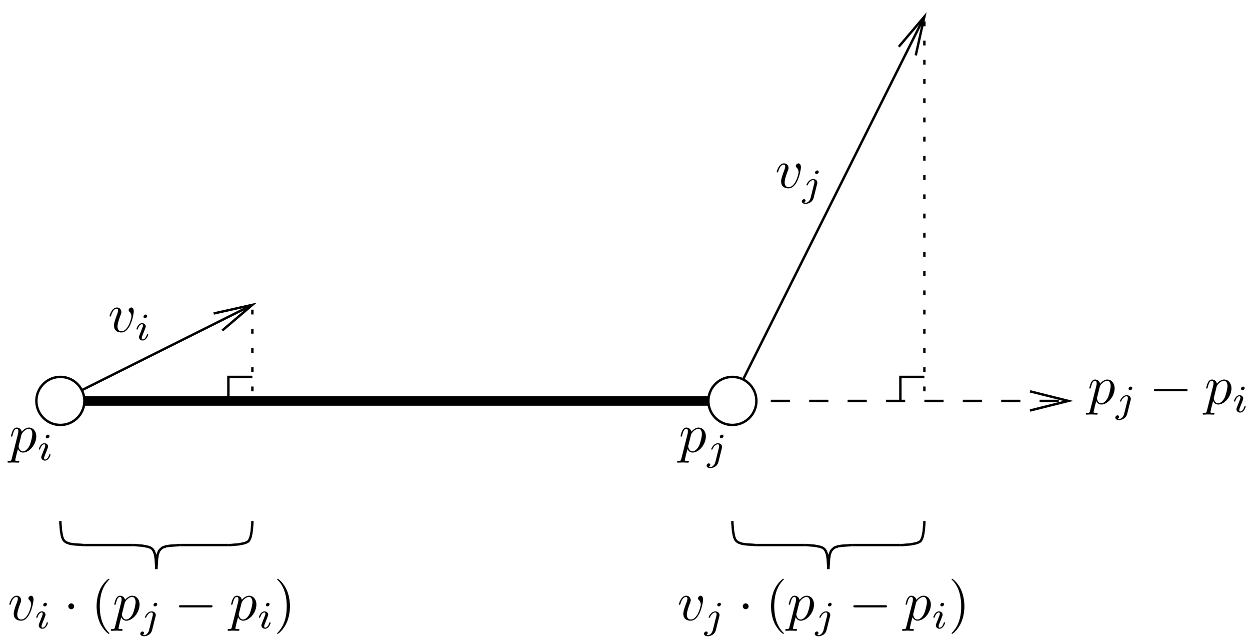
2. *infinitesimal motion* (local motion)

$$v_i = \frac{d}{dt} p_i(t) = \dot{p}_i(0)$$

   Velocity vectors $v_1, \ldots, v_n$.

Expansion

$$\tfrac{1}{2} \cdot \frac{d}{dt} |p_i(t) - p_j(t)|^2 = \langle v_i - v_j, p_i - p_j \rangle =: \exp_{ij}$$



expansion (or *strain*) $\exp_{ij}$ of the segment ij

The Rigidity Map

$$M\colon (v_1, \ldots, v_n) \mapsto (\exp_{ij})_{ij \in E}$$

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The rigidity matrix:

$$M = \underbrace{\begin{pmatrix} \text{the} \\ \text{rigidity} \\ \text{matrix} \end{pmatrix}}_{2|V|} \left.\vphantom{\begin{pmatrix} \text{the} \\ \text{rigidity} \\ \text{matrix} \end{pmatrix}}\right\} E$$

Expansive Motions

$\exp_{ij} = 0$ for all *bars* ij

(preservation of length)

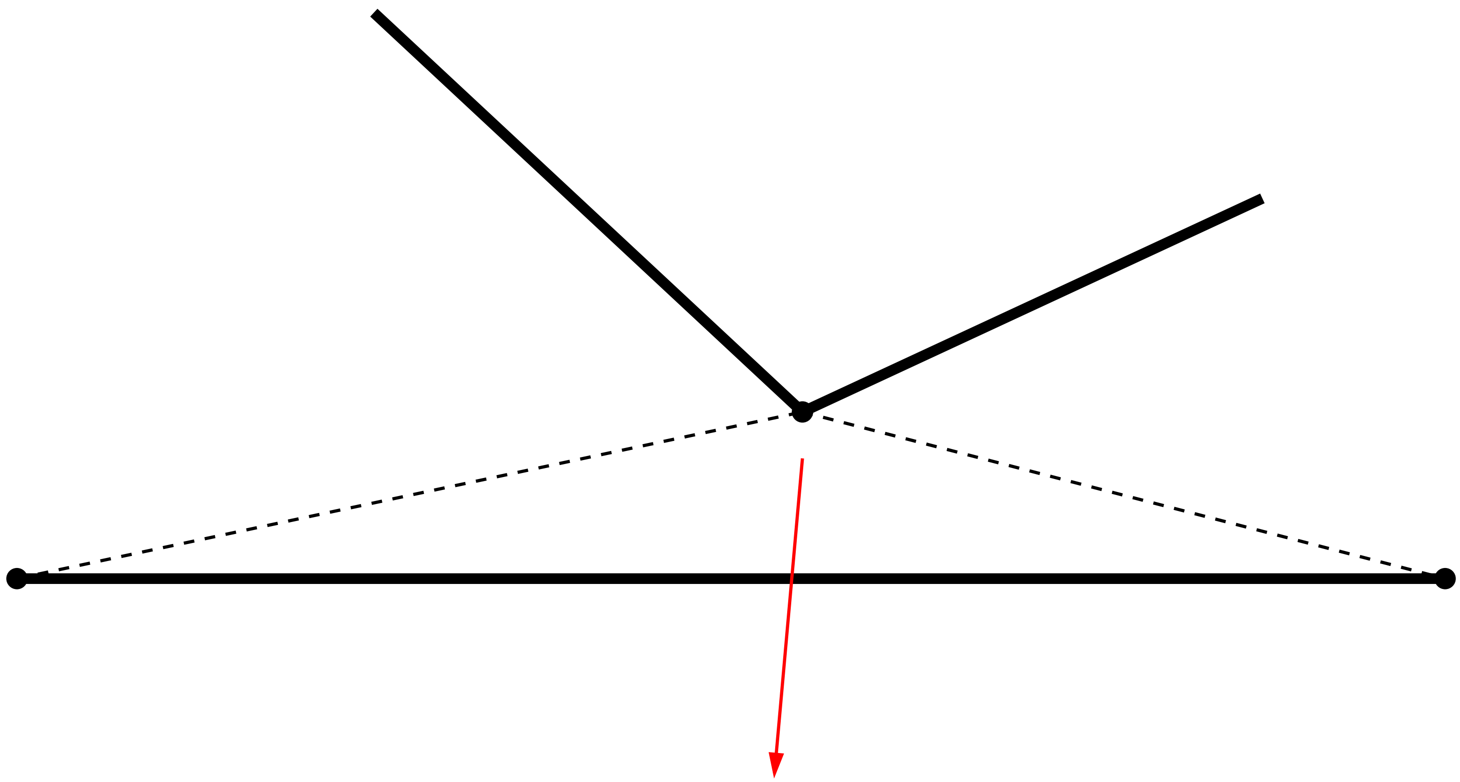
$\exp_{ij} \geq 0$ for all other pairs (*struts*) ij

(expansiveness)

$$[\quad \exp_{ij} > 0 \quad]$$

(strict expansiveness)

Expansive motions cannot overlap



Proof Outline

1. Prove that expansive motions *exist*.

2. Select an expansive motion and provide a global motion.

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Proof Outline

Existence of an expansive motion

$\Updownarrow$ (duality)

Self-stresses (rigidity)
Self-stresses on planar frameworks

$\Updownarrow$ (Maxwell-Cremona correspondence)

polyhedral terrains

[Connelly, Demaine, Rote 2000]

The Expansion Cone

The set of expansive motions forms a convex polyhedral cone $\bar{X}_0$ in $\mathbb{R}^{2n}$, defined by homogeneous linear equations and inequalities of the form

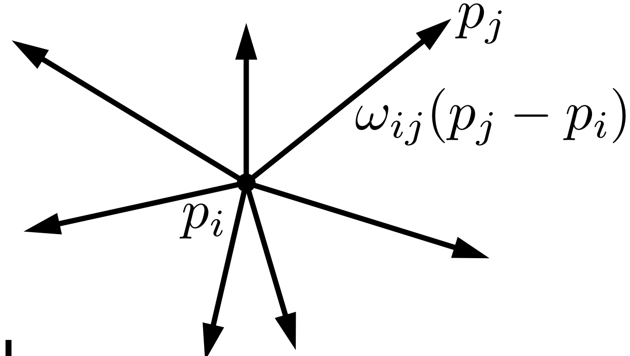
$$\langle v_i - v_j, p_i - p_j \rangle \left\{ \begin{array}{c} = \\ \geq \\ [>] \end{array} \right\} 0$$

Bars, Struts, Frameworks, Stresses

Assign a *stress* $\omega_{ij} = \omega_{ji} \in \mathbb{R}$ to each edge.

Equilibrium of forces in vertex i:

$$\sum_j \omega_{ij}(p_j - p_i) = 0$$



$\omega_{ij} \leq 0$ for struts: Struts can only push.
$\omega_{ij} \in \mathbb{R}$ for bars: Bars can push or pull.

Motions and Stresses

Linear Programming duality:

There is a strictly expansive motion if and only if there is no non-zero stress.

$$\langle v_i - v_j, p_i - p_j \rangle \begin{cases} = 0 \\ > 0 \end{cases}$$

$$\sum_j \omega_{ij}(p_j - p_i) = 0, \text{ for all } i$$

$$\omega_{ij} \in \mathbb{R}, \quad \text{for a bar } ij$$

$$\omega_{ij} \leq 0, \quad \text{for a strut } ij$$

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Linear Programming duality:

There is a strictly expansive motion if and only if there is no non-zero stress.

$$\langle v_i - v_j, p_i - p_j \rangle \begin{cases} = 0 \\ > 0 \end{cases}$$

$$\left[\; Mv \begin{cases} = 0 \\ > 0 \end{cases} \; \right]$$

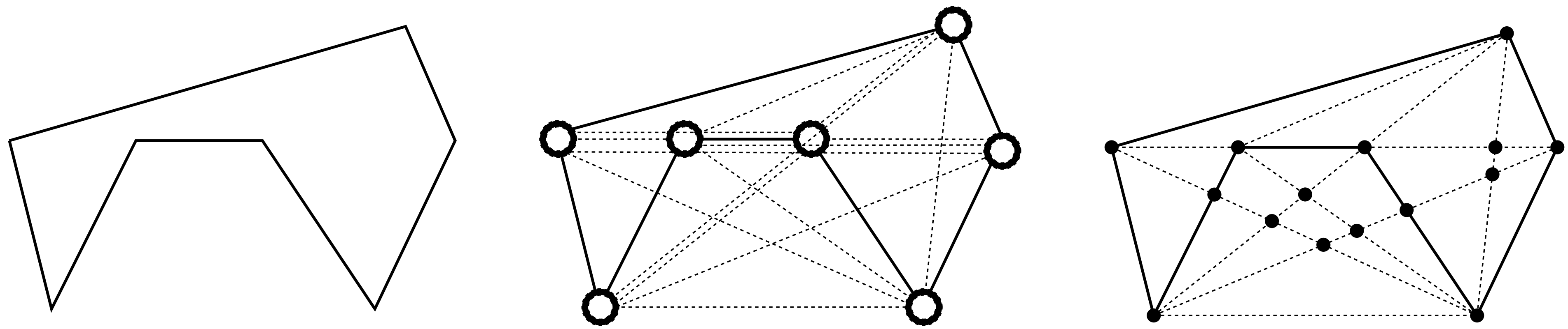
$$\sum_j \omega_{ij}(p_j - p_i) = 0, \text{ for all } i$$

$$[\; M^{\mathrm{T}}\omega = 0 \;]$$

$\omega_{ij} \in \mathbb{R}$, for a bar ij

$\omega_{ij} \le 0$, for a strut ij

Making the Framework Planar



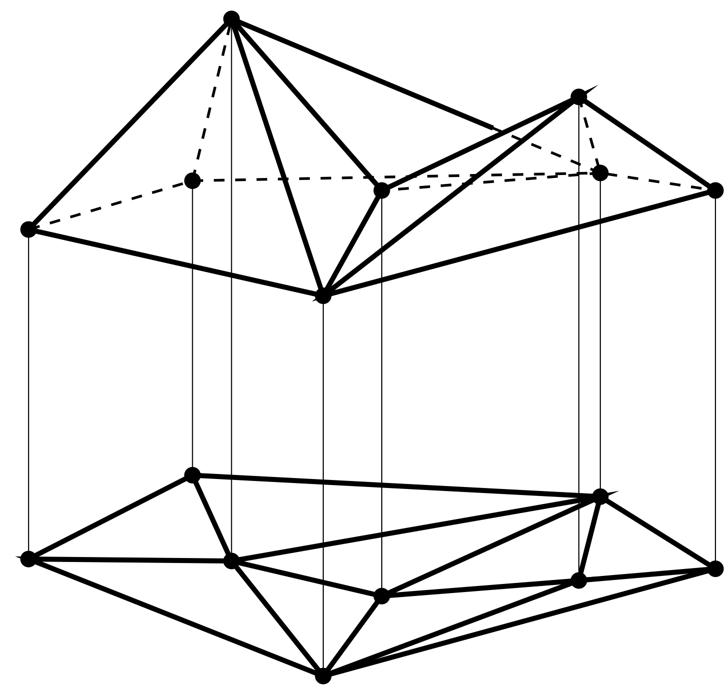
- subdivide edges at intersection points
- collapse multiple edges

The Maxwell-Cremona Correspondence [1850]

3-d lifting (polyhedral terrain)

$\Updownarrow$

self-stresses on a planar framework



The Maxwell-Cremona Correspondence [1850]

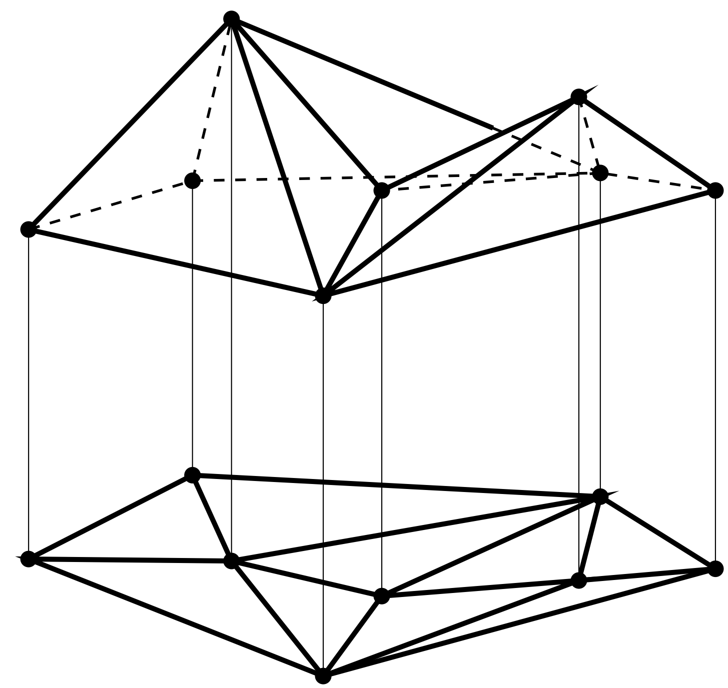
3-d lifting (polyhedral terrain)

$\Updownarrow$

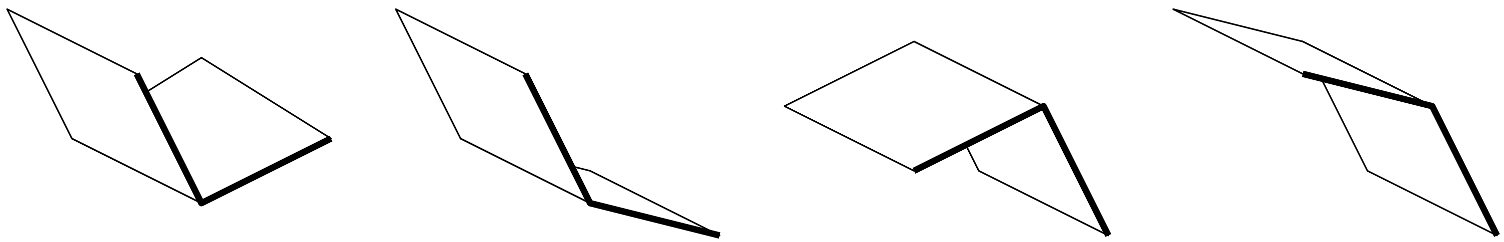
self-stresses on a planar framework

$\Updownarrow$

orthogonal dual

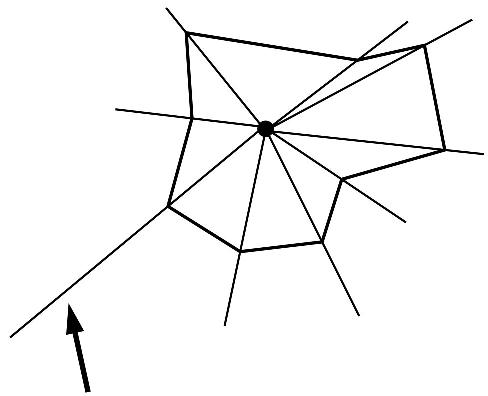


Valley and Mountain Folds



$\omega_{ij} > 0$	$\omega_{ij} < 0$
valley	mountain
bar or strut	bar

Look a the highest peak!

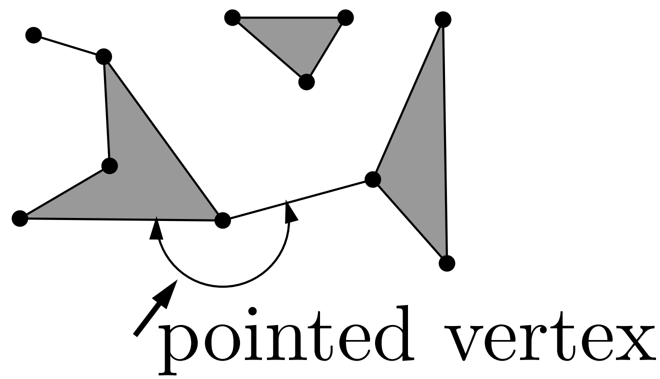


mountain → bar

Every polygon has > 3 convex vertices

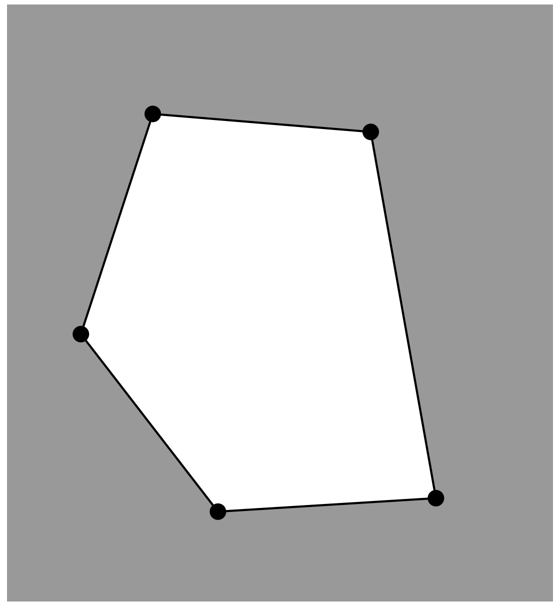
$\longrightarrow$ 3 valleys $\longrightarrow$ 3 bars.

The general case



There is at least one vertex with angle $> \pi$.

The only remaining possibility



a convex polygon

□

Constructing a Global Motion

[Connelly, Demaine, Rote 2000]

- Define a point $v := v(p)$ in the *interior* of the expansion cone, by a suitable non-linear convex objective function.
- $v(p)$ depends smoothly on p.
- Solve the differential equation $\dot{p} = v(p)$

Constructing a Global Motion

Alternative approach: Select an *extreme ray* of the expansion cone.

Streinu [2000]:
Extreme rays correspond to pseudotriangulations.

[show animation]

Part II: Pseudotriangulations

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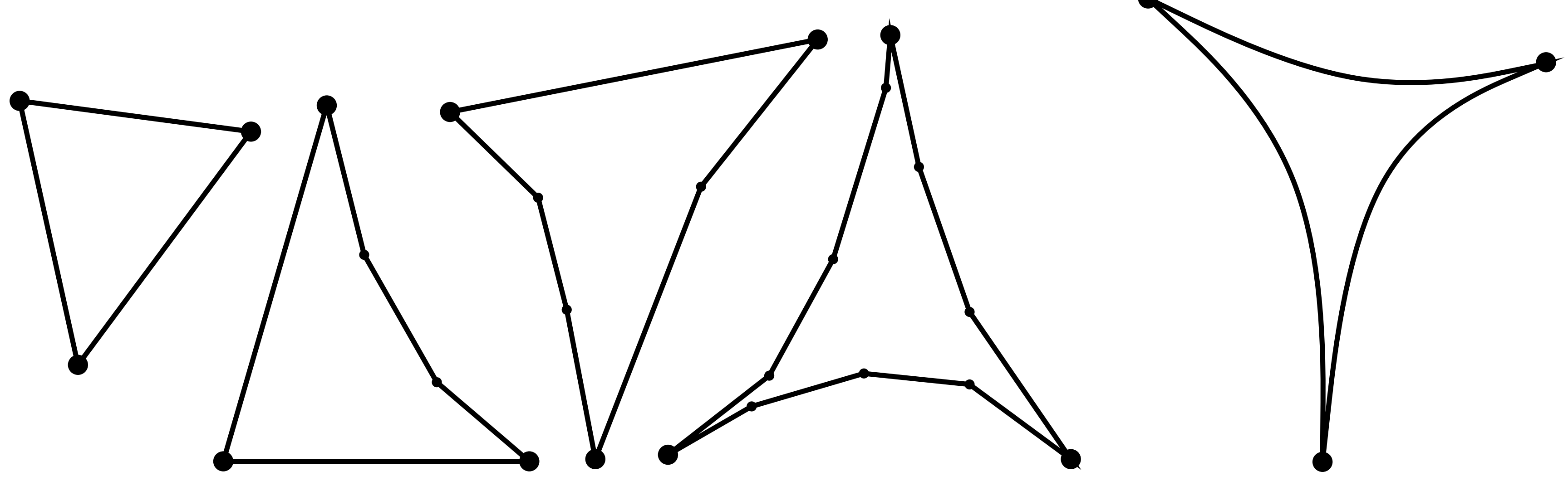
Pseudotriangulations!

Assumption: Points in general position.

Pseudotriangles

A pseudotriangulation has three convex *corners* and an arbitrary number of reflex vertices.



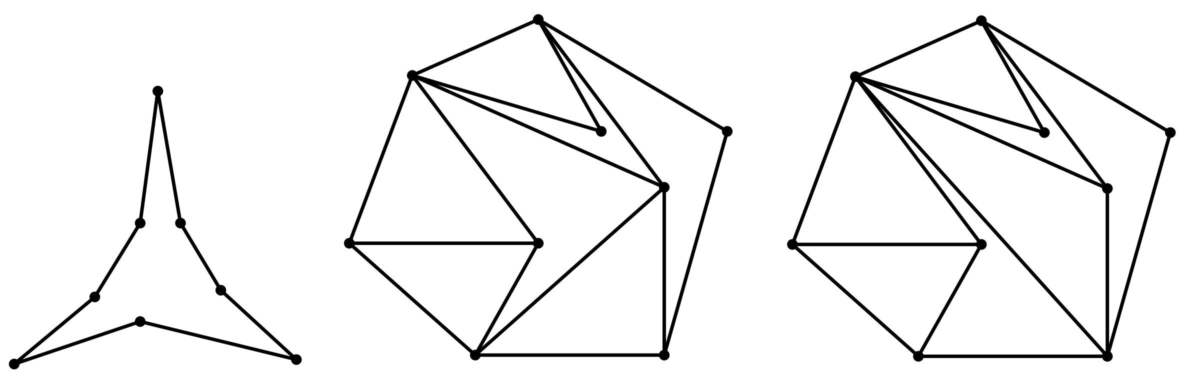
Pseudotriangulations/ Geodesic Triangulations

Other applications:

- data structures for ray shooting [Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, and Snoeyink 1994] and visibility [Pocchiola and Vegter 1996]
- kinetic collision detection [Agarwal, Basch, Erickson, Guibas, Hershberger, Zhang 1999–2001] [Kirkpatrick, Snoeyink, and Speckmann 2000] [Kirkpatrick & Speckmann 2002 this afternoon]
- art gallery problems [Pocchiola and Vegter 1996b], [Speckmann and Tóth 2001]

Minimum (or Pointed) Pseudotriangulations (PPT)

A *pointed* vertex is incident to an angle $> 180°$.
A *maximal* non-crossing and pointed set of edges decomposes the convex hull into $n-2$ pseudotriangles using $2n-3$ edges.



Characterization of Pointed Pseudotriangulations

An edge set with any two of the following properties:

- $2n - 3$ edges (or $n - 2$ faces)
- decomposition into pseudotriangles
- non-crossing, and every vertex is pointed.

[Streinu 2002]

Characterization of Trees

An edge set with any two of the following properties:

- $n - 1$ edges
- connected
- acyclic

Characterization of Pointed Pseudotriangulations

An edge set with any two of the following properties:

- $2n - 3$ edges (or $n - 2$ faces)
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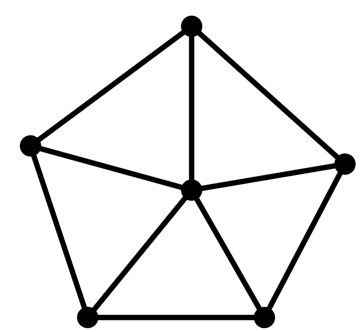
Characterization of Pointed Pseudotriangulations

An edge set with any two of the following properties:

- $2n - 3$ edges (or $n - 2$ faces)
- decomposition into pseudotriangles
- non-crossing, and every vertex is pointed.

Caveat: Removing edges from a triangulation does not necessarily lead to a pointed pseudotriangulation.



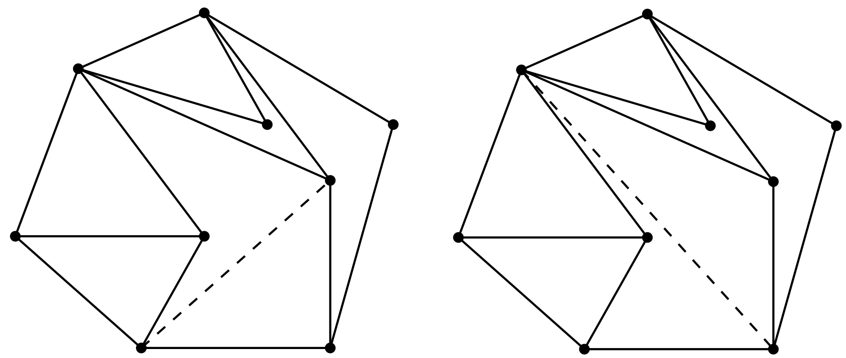
Rigidity Properties of Pseudotriangulations

- Pseudotriangulations are minimally rigid.
- a Henneberg-type construction
- Removing a hull edge gives an *expansive* mechanism with 1 degree of freedom.

[Streinu 2002]

Flipping of Edges

Any interior edge can be flipped against another edge. That edge is unique.



before after

Flipping of Edges

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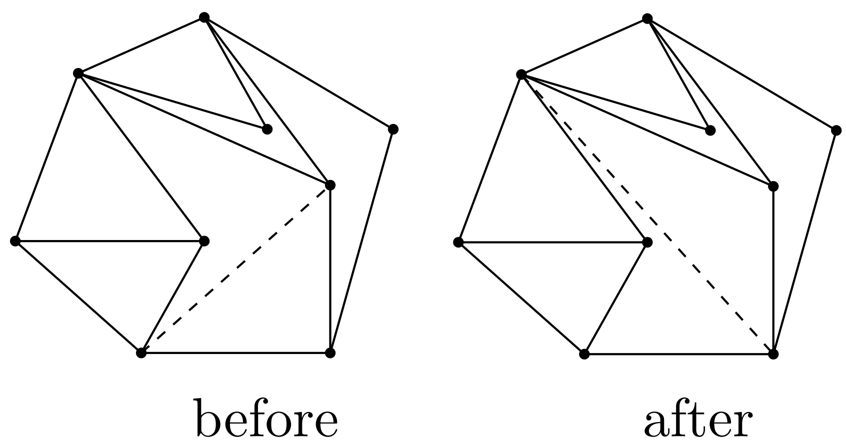


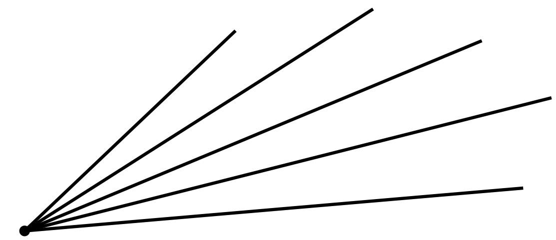
The flip graph is connected. Its diameter is $O(n^2)$.

[Brönnimann, Kettner, Pocchiola, Snoeyink 2001]

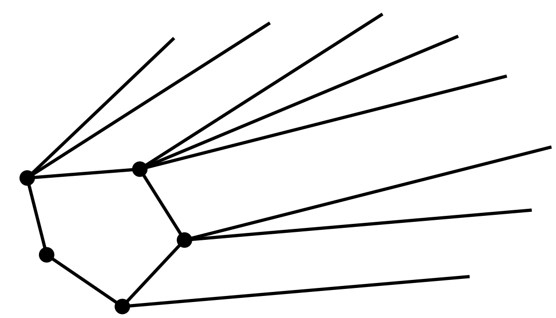
Part III: Cones and Polytopes

[Rote, Santos, Streinu 2002]

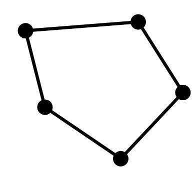
- The *expansion cone*
  $\bar{X}_0 = \{\, \exp_{ij} \geq 0 \,\}$



- The *perturbed expansion cone*
  $=$ the *PPT polyhedron*
  $\bar{X}_f = \{\, \exp_{ij} \geq f_{ij} \,\}$



- The *PPT polytope*
  $X_f = \{\, \exp_{ij} \geq f_{ij},$
  $\exp_{ij} = f_{ij}$ for ij on boundary $\}$



Pinning of Vertices

Trivial Motions: Motions of the point set as a whole (translations, rotations).

Pin a vertex and a direction. ("tie-down")

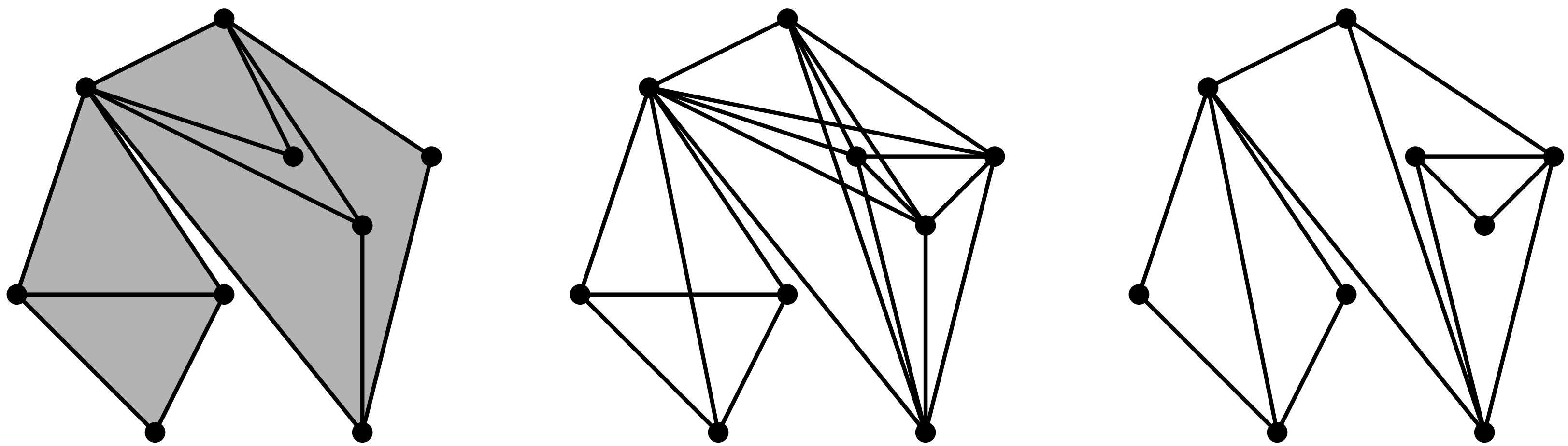
$$v_1 = 0$$

$$v_2 \parallel p_2 - p_1$$

This eliminates 3 degrees of freedom.

Extreme Rays of the Expansion Cone

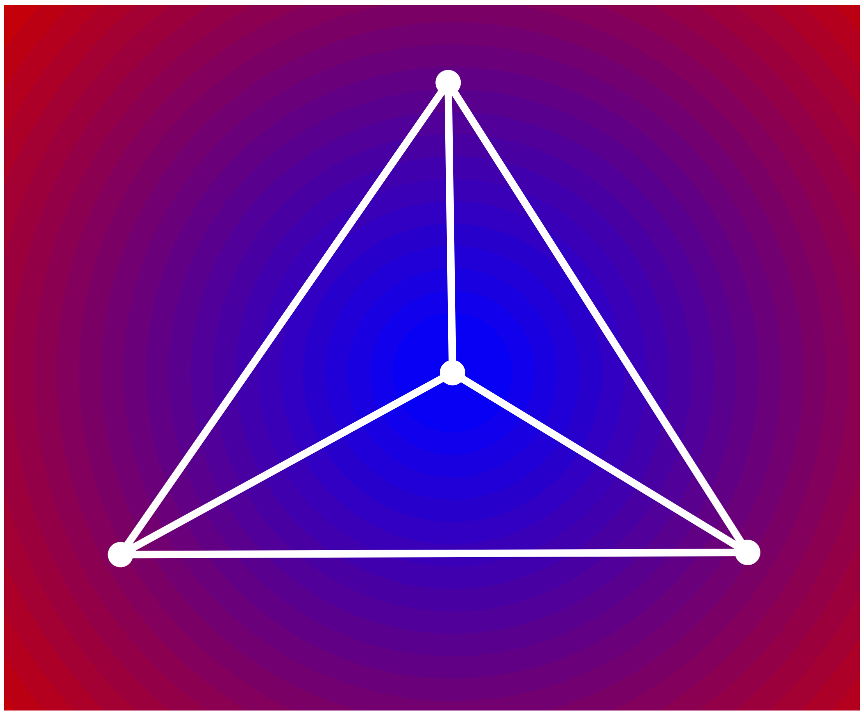
Pseudotriangulations with one convex hull edge removed yield expansive mechanisms. [Streinu 2000]
Rigid substructures can be identified.



A Polyhedron for Pseudotriangulations

Wanted:
A perturbation of the constraints "$\exp_{ij} \geq 0$" such that the vertices are in 1-1 correspondence with pseudotriangulations.

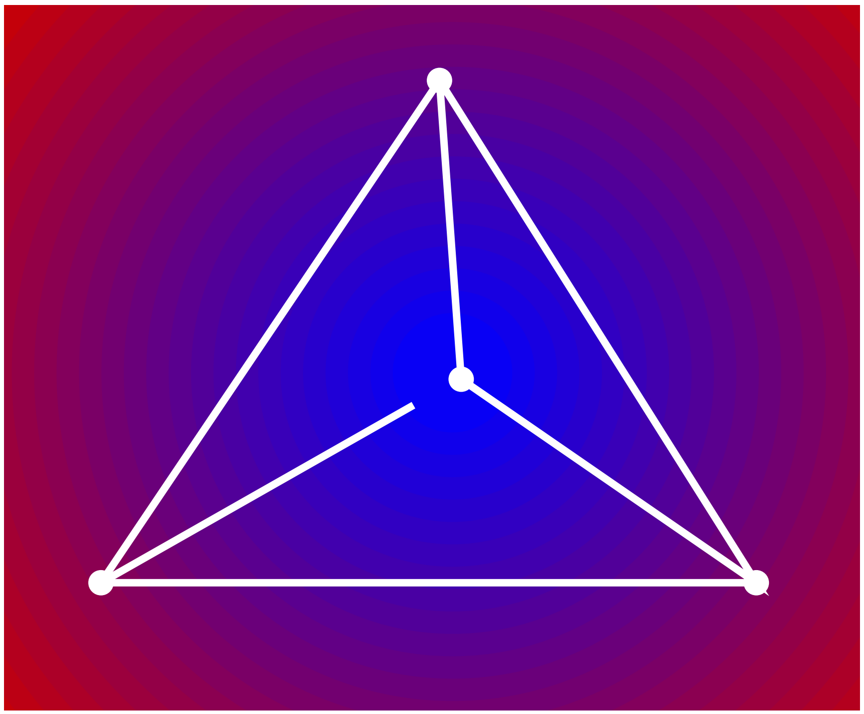
Heating up the Bars



$$\Delta T = |x|^2$$

Length increase $\geq \int_{x \in p_i p_j} |x|^2 \, ds$

Heating up the Bars



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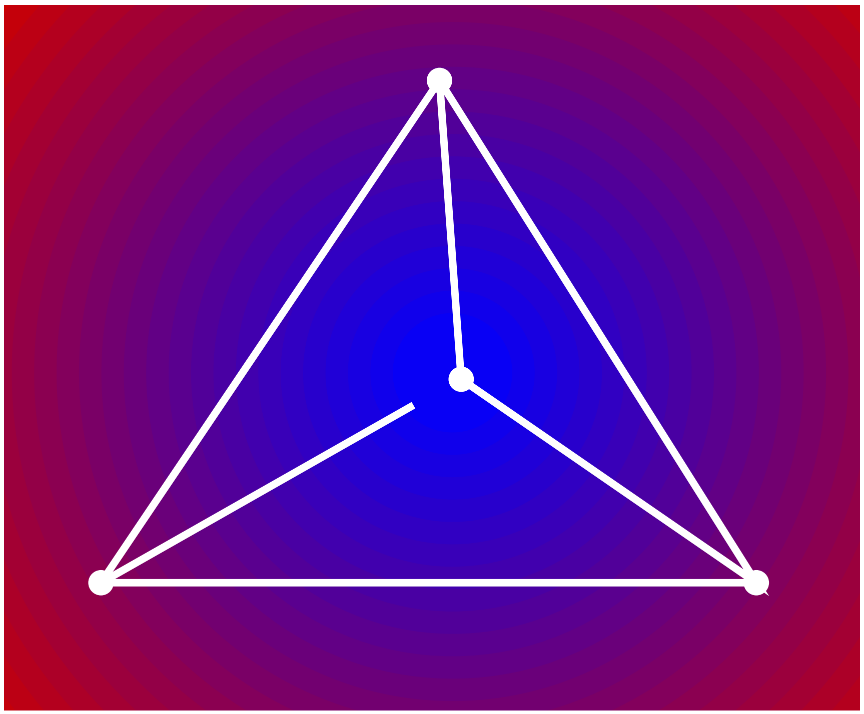
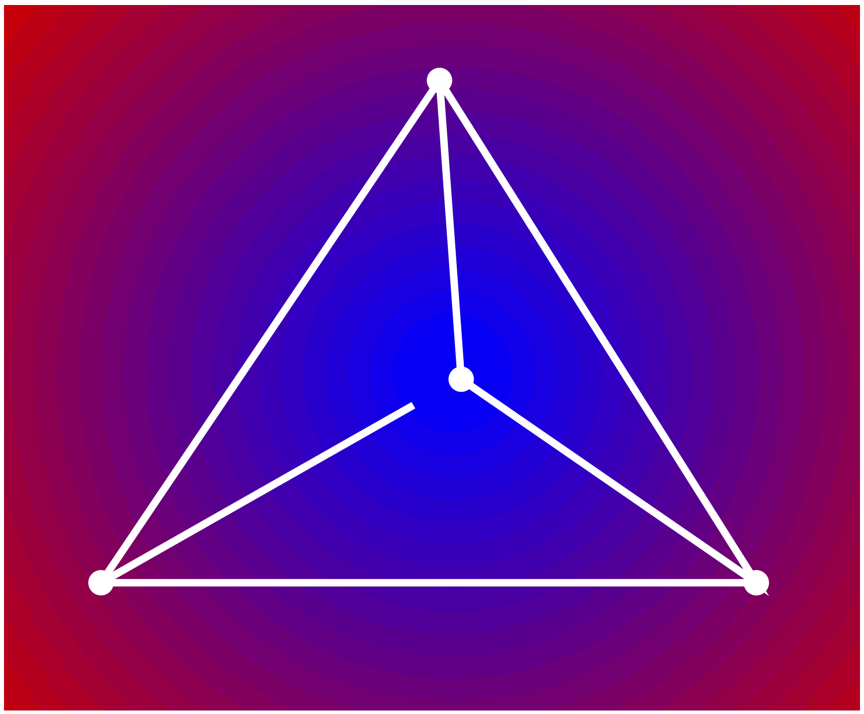


$$\Delta T = |x|^2$$

$$\text{Length increase } \geq \int_{x \in p_i p_j} |x|^2 \, ds$$

$$\exp_{ij} \geq |p_i - p_j| \cdot \int_{x \in p_i p_j} |x|^2 \, ds$$

Heating up the Bars



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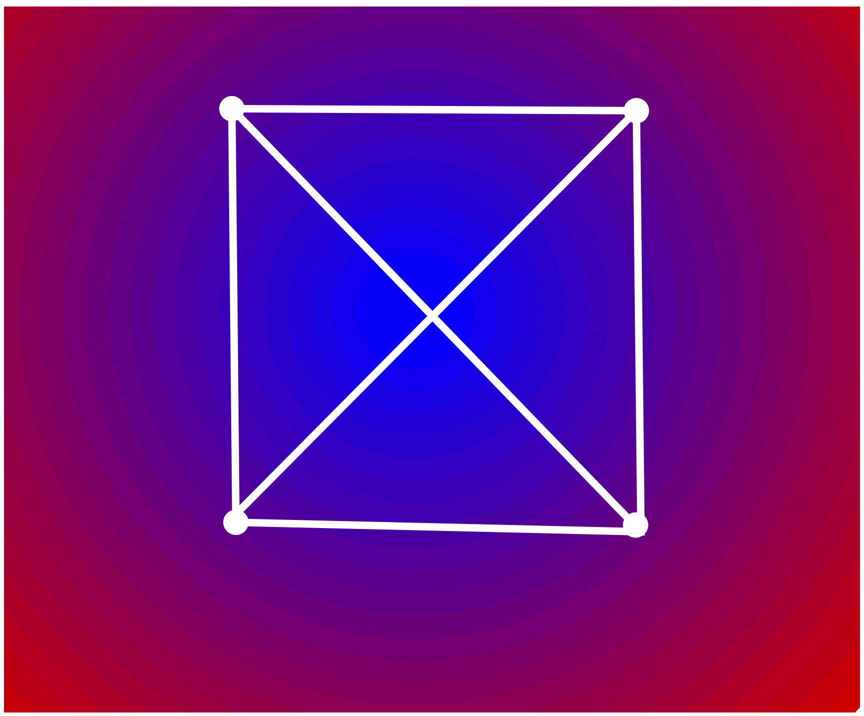
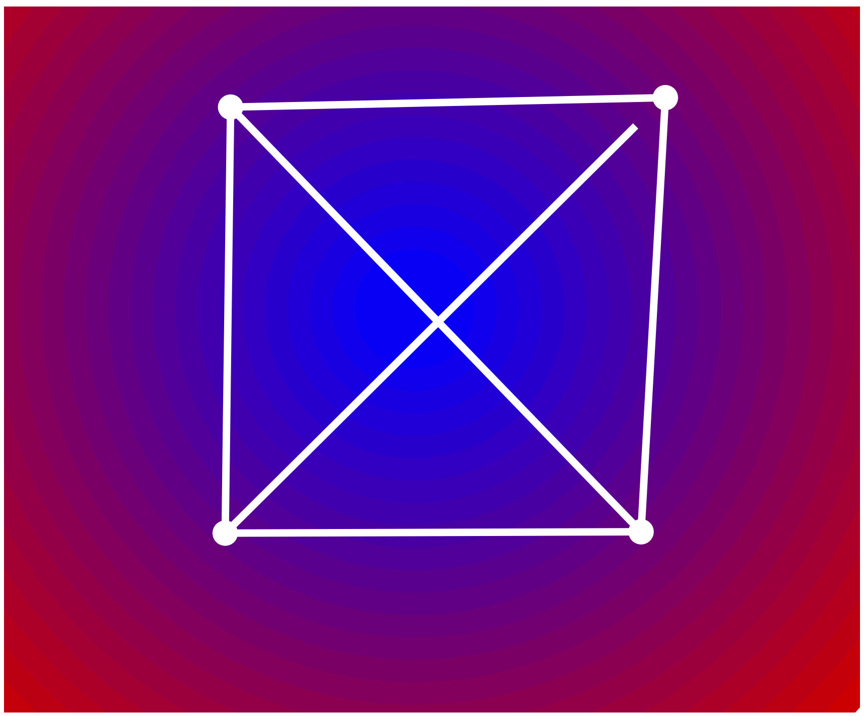
$$\exp_{ij} \geq |p_i - p_j| \cdot \int_{x \in p_i p_j} |x|^2 \, ds$$

$$\exp_{ij} \geq |p_i - p_j|^2 \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2) \cdot \frac{1}{3}$$

Heating up the Bars — Points in Convex Position



$\Rightarrow$



The Perturbed Expansion Cone = PPT Polyhedron

$$\bar{X}_f = \{\, (v_1, \ldots, v_n) \mid \exp_{ij} \geq f_{ij} \,\}$$

- $f_{ij} := |p_i - p_j|^2 \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2)$
- $f'_{ij} := [a, p_i, p_j] \cdot [b, p_i, p_j]$

  $[x, y, z]$ = signed area of the triangle xyz
  a, b: two arbitrary points.

Tight Edges

For $v = (v_1, \ldots, v_n) \in \bar{X}_f$,

$$E(v) := \{\, ij \mid \exp_{ij} = f_{ij} \,\}$$

is the *set of tight edges* at v.

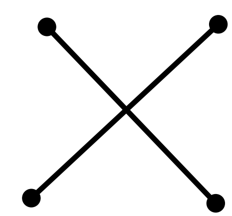
Maximal sets of tight edges $\equiv$ vertices of $\bar{X}_f$.

What are good values of f_{ij}?

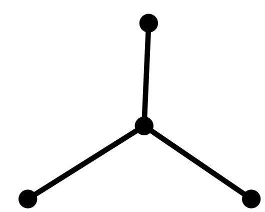
Which configurations of edges can occur in a set of tight edges?

We want:

- no crossing edges

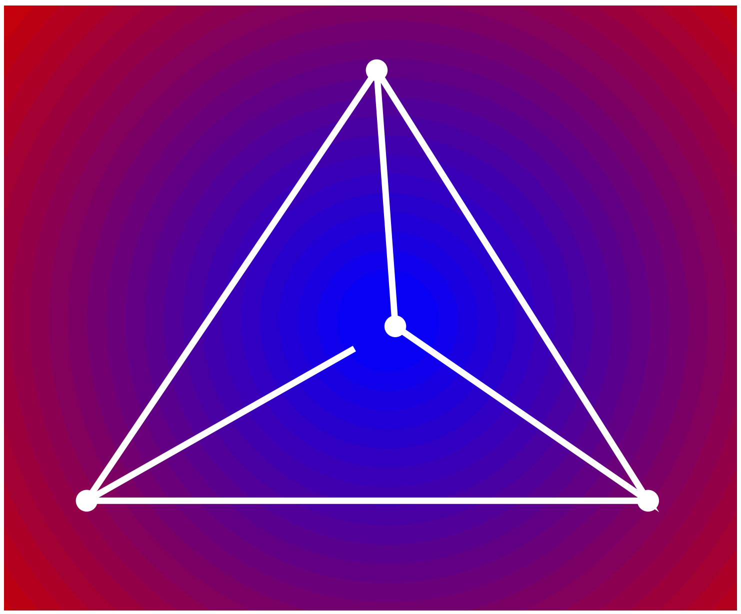


- no 3-star with all angles $\leq 180^\circ$



It is sufficient to look at 4-point subsets.

Good Values f_{ij} for 4 points

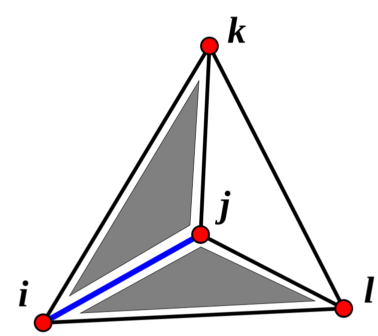


f_{ij} is given on six edges.
Any five values $\exp_{ij}$ determine the last one.
Check if the resulting value $\exp_{ij}$ of the last edge is feasible ($\exp_{ij} \geq f_{ij}$)
$\longrightarrow$ checking the sign of an expression.

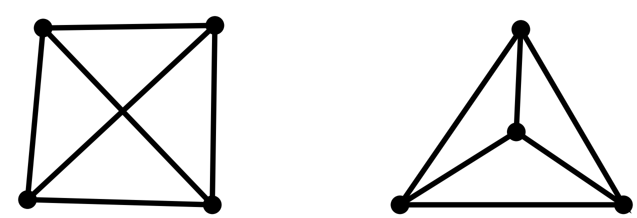
Good Values f_{ij} for 4 points

A 4-tuple p_1, p_2, p_3, p_4 has a unique self-stress (up to a scalar factor).

$$\omega_{ij} = \frac{1}{[p_i, p_j, p_k] \cdot [p_i, p_j, p_l]}, \text{ for all } 1 \le i < j \le 4$$



$\omega_{ij} > 0$ for boundary edges.
$\omega_{ij} < 0$ for interior edges.



Why the stress?

If the *equation*

$$\sum_{1 \le i < j \le 4} \omega_{ij} f_{ij} = 0$$

holds, then f_{ij} are the expansion values $\exp_{ij}$ of a motion (v_1, v_2, v_3, v_4).

Actually, "if and only if".

Why the stress?

If the *equation*

$$\sum_{1\le i<j\le 4} \omega_{ij} f_{ij} = 0$$

holds, then f_{ij} are the expansion values $\exp_{ij}$ of a motion (v_1, v_2, v_3, v_4).

Actually, "if and only if".

$[\ M^{\mathrm{T}}\omega = 0,\ f = \exp = Mv\ ]$

Good Perturbations

We need

$$\sum_{1 \leq i < j \leq 4} \omega_{ij} f_{ij} > 0$$

for all 4-tuples of points.

$\longrightarrow$ For every vertex v, $E(v)$ is non-crossing and pointed.

$\longrightarrow$ $\bar{X}_f$ is a simple polyhedron.

The PPT-polyhedron

Every vertex is incident to $2n-3$ edges.

Edge $\equiv$ removing a segment from $E(v)$.

Removing an interior segment leads to an adjacent pseudotriangulation (flip).

Removing a hull segment is an extreme ray. $\square$

Proof of

$$\omega_{12}f_{12}+\omega_{13}f_{13}+\omega_{14}f_{14}+\omega_{23}f_{23}+\omega_{24}f_{24}+\omega_{34}f_{34}>0$$

$$R(a,b) := \sum_{1\le i<j\le 4} \omega_{ij}\cdot [a,p_i,p_j][b,p_i,p_j]$$

$$R \equiv 1!$$

R is linear in a and linear in b. $R(p_i,p_j)=1$ is sufficient.
$R(p_1,p_2)$: all $f_{ij}=0$ except f_{34}

$$R(p_1,p_2)=\omega_{34}f_{34}=\frac{\det(p_1,p_3,p_4)\det(p_2,p_3,p_4)}{\det(p_3,p_4,p_1)\det(p_3,p_4,p_2)}=1. \qquad \square$$

The PPT polytope

Cut out all rays:

Change $\exp_{ij} > f_{ij}$ to $\exp_{ij} = f_{ij}$ for hull edges.

The PPT polytope

Cut out all rays:
Change $\exp_{ij} > f_{ij}$ to $\exp_{ij} = f_{ij}$ for hull edges.

The Expansion Cone $\bar{X}_0$:

collapse parallel rays into one ray. $\longrightarrow$ pseudotriangulations minus one hull edge. Rigid subcomponents are identified.

Expansive motions for a chain (or a polygon)

- Add edges to form a pseudotriangulation
- Remove a convex hull edge
- $\rightarrow$ expansive mechanism $\square$

Which f_{ij} to choose?

- $f_{ij} := |p_i - p_j|^2 \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2)$
- $f'_{ij} := [a, p_i, p_j] \cdot [b, p_i, p_j]$

Go to the space of the (exp_{ij}) variables instead of the (v_i) variables.

$$\mathrm{exp} = Mv$$

Characterization of the space $(\exp_{ij})_{i,j}$

A set of values $(\exp_{ij})_{1\le i<j\le n}$ forms the expansion values of a motion $(v_1, \ldots, v_n)$ if and only if the equation

$$\sum_{1\le i<j\le 4} \omega_{ij} \exp_{ij} = 0$$

holds for all 4-tuples.

SKIP

A canonical representation

$$\sum_{1 \le i < j \le 4} \omega_{ij} \, \mathrm{exp}_{ij} = 0, \text{ for all 4-tuples}$$

$$\mathrm{exp}_{ij} \ge f_{ij}, \text{ for all pairs } i, j$$

A canonical representation

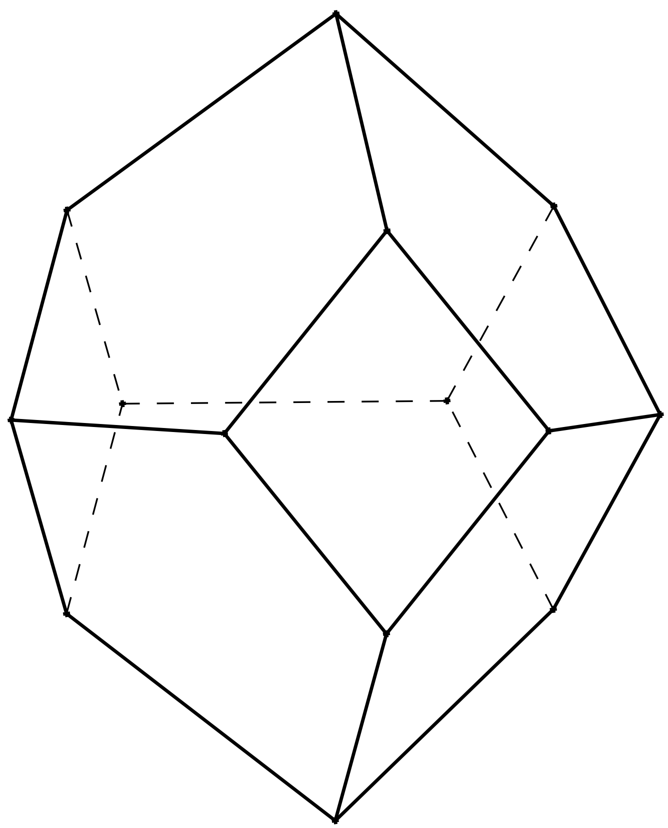
$$\sum_{1\le i<j\le 4} \omega_{ij} \exp_{ij} = 0, \text{ for all 4-tuples}$$
$$\exp_{ij} \ge f_{ij}, \text{ for all pairs } i,j$$

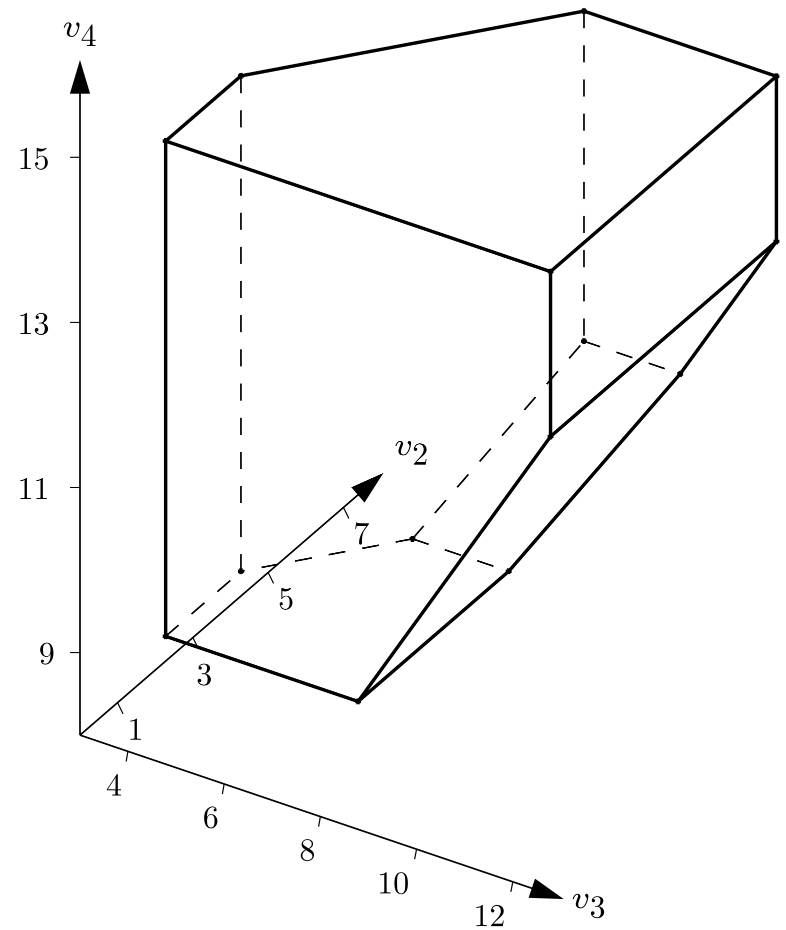
$$\sum_{1\le i<j\le 4} \omega_{ij} f_{ij} = 1, \text{ for all 4-tuples}$$

Substitute $d_{ij} := \exp_{ij} - f_{ij}$:

$$\textstyle\sum_{1\le i<j\le 4} d_{ij} \exp_{ij} = -1, \text{ for all 4-tuples} \qquad (1)$$
$$d_{ij} \ge 0, \text{ for all } i,j \qquad (2)$$

The Associahedron





Catalan Structures

- Triangulations of a convex polygon / edge flip
- Binary trees / rotation
- $(a * (b * (c * d))) * e$ / $((a * b) * (c * d)) * e$

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- Triangulations of a convex polygon / edge flip
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- non-crossing alternating trees
- . . . .

The Secondary Polytope

Triangulation $T \mapsto (x_1, \dots, x_n)$.

$$x_i := \text{total area of all triangles incident to } p_i$$

vertices $\equiv$ regular triangulations of $(p_1, \dots, p_n)$

$(p_1, \dots, p_n)$ in convex position:
pseudotriangulations $\equiv$ triangulations $\equiv$ regular triangulations.

$\longrightarrow$ two realizations of the associahedron.

These two associahedra are affinely equivalent.

Expansive Motions in One Dimension

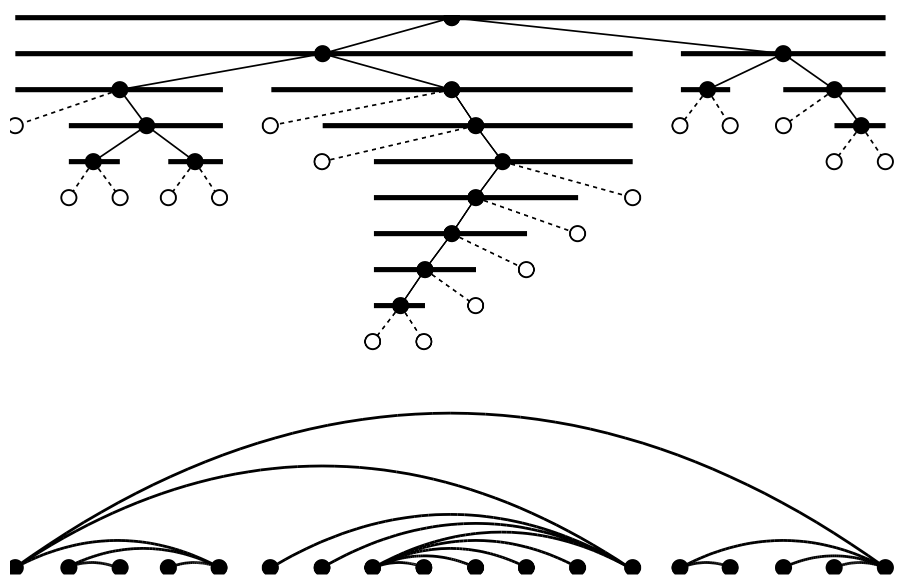
$$\{\, (v_i) \in \mathbb{R}^n \mid v_j - v_i \geq f_{ij} \text{ for } 1 \leq i < j \leq n \,\}$$

$$f_{il} + f_{jk} > f_{ik} + f_{jl}, \text{ for all } i < j < k < l.$$
$$f_{il} > f_{ik} + f_{kl}, \text{ for all } i < k < l.$$

For example, $f_{ij} := (i - j)^2$

related to the Monge Property.

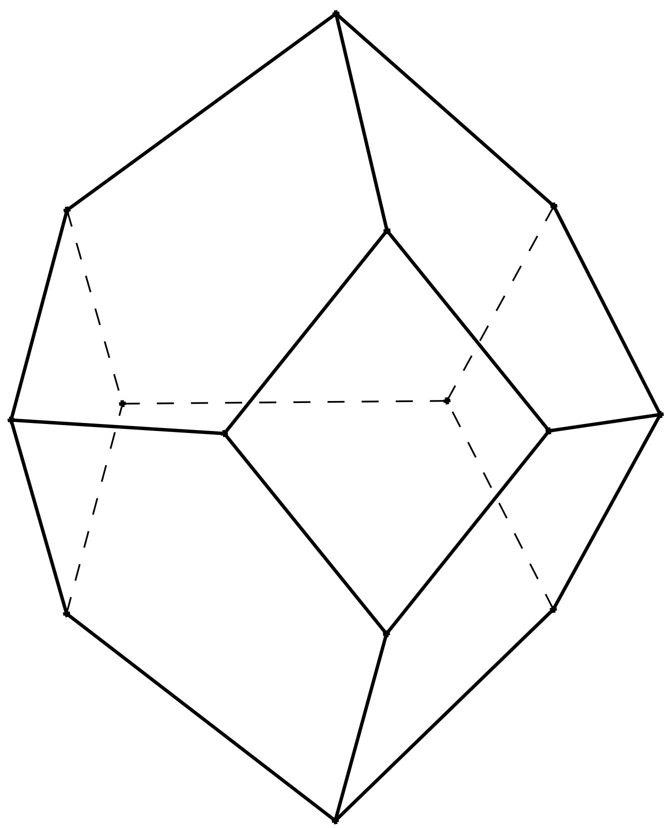
Non-crossing alternating trees

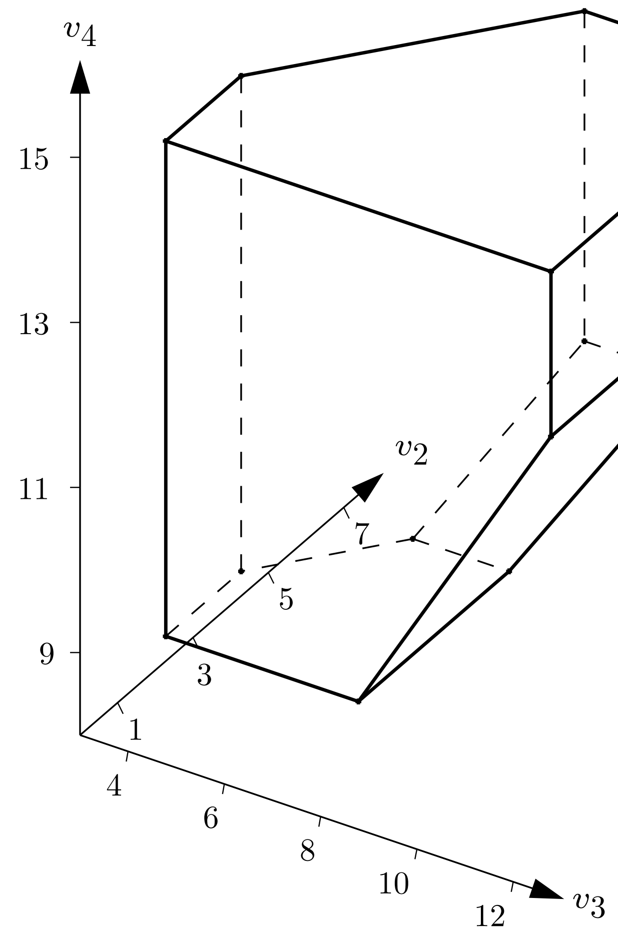


non-crossing: no two edges ik, jl with $i < j < k < l$.
alternating: no two edges ij, jk with $i < j < k$.

[Gelfand, Graev, and Postnikov 1997], in a dual setting.
[Postnikov 1997], [Zelevinsky ?]

The Associahedron





Open Questions

1. the meaning of $\sum \omega_{ij} f_{ij} = 1$
2. Is there essentially only one solution of $\sum \omega_{ij} f_{ij} > 0$?
3. canonical pseudotriangulations
4. pseudotriangulations in 3-space

The meaning of

$$\sum_{1\leq i<j\leq 4} \omega_{ij} f_{ij} = 1$$

"I believe there is some underlying homology in this situation. Given the fact that motions and stresses also fit into a setting of cohomology and homology as well, the authors might, at least, mention possible homology descriptions."

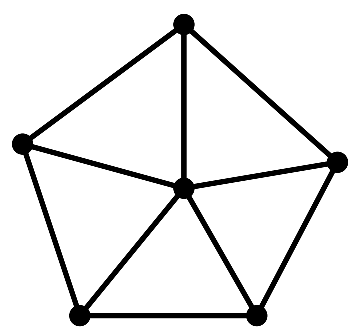
[a referee, about the definition of ω_{ij}]

The meaning of

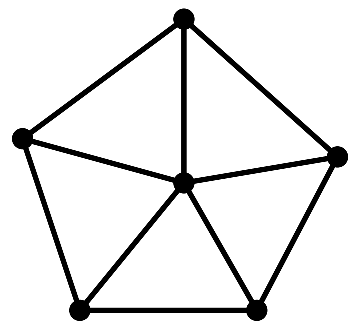
$$\sum_{1\leq i<j\leq 4} \omega_{ij} f_{ij} = 1$$

$$\omega_{ij} = \frac{1}{[p_i, p_j, p_k] \cdot [p_i, p_j, p_l]}$$

One can define a similar formula for ω for the k-wheel.



$\sum_{ij\in E}\omega_{ij}f_{ij}=1$ for the k-wheel



$$\omega_{i,i+1}=\frac{1}{[p_i,p_{i+1},p_0]\cdot[p_1,p_2,\ldots,p_k]}$$

$$\omega_{0i}=\frac{1}{[p_{i-1},p_i,p_0]\cdot[p_i,p_{i+1},p_0]}\cdot\frac{[p_{i-1},p_i,p_{i+1}]}{[p_1,p_2,\ldots,p_k]}$$

Open Questions

1. the meaning of $\sum \omega_{ij} f_{ij} = 1$
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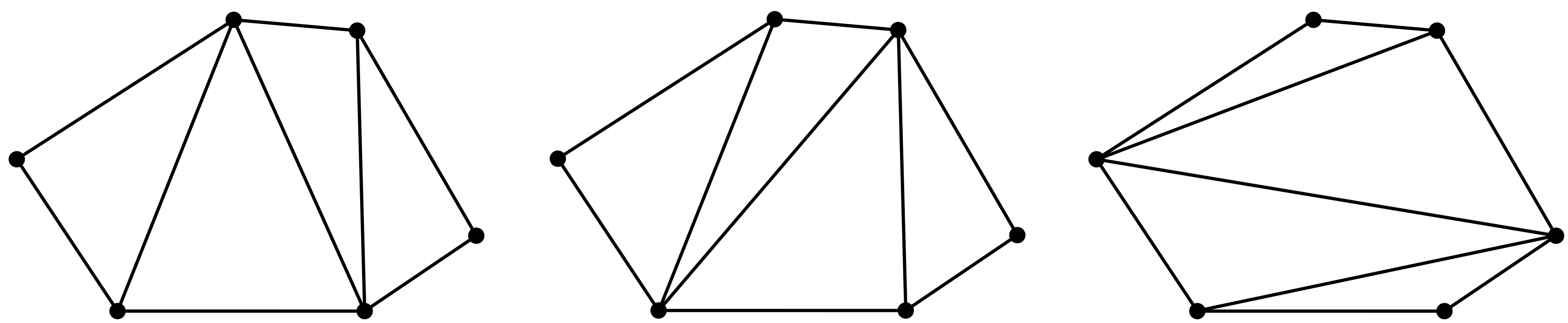
Canonical pseudotriangulations

Maximize/minimize $\sum_{i=1}^{n} c_i \cdot v_i$ over the PPT-polytope.

$c_i := p_i$:

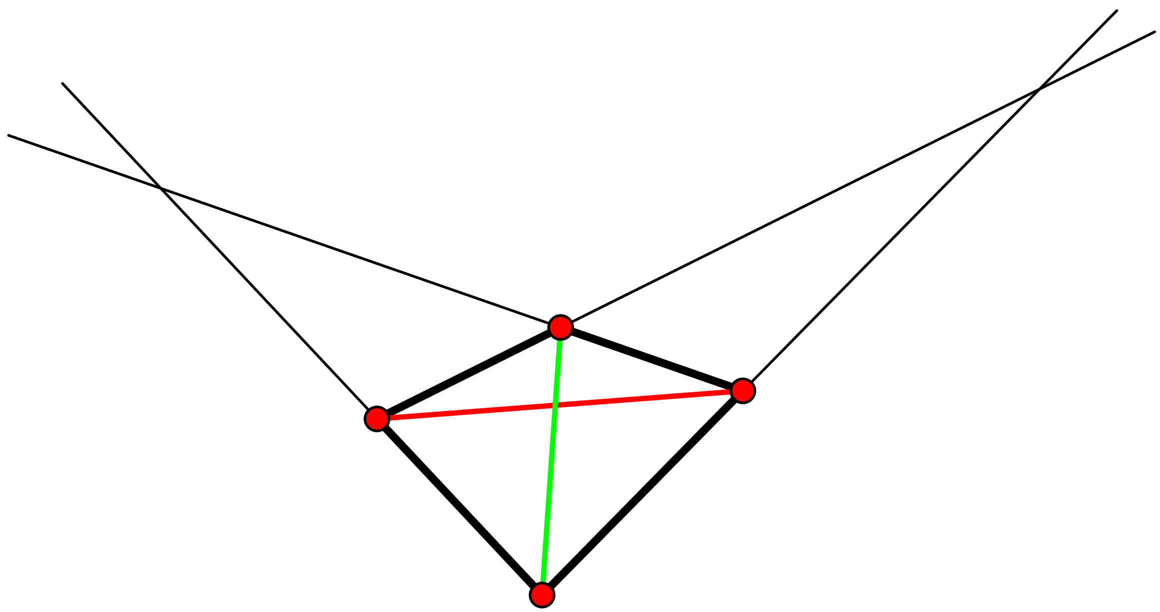


(a) (b) (c)

Delaunay triangulation

Max/Min $\sum p_i \cdot v_i$
(affine invariant)

Edge flipping criterion for canonical pseudotriangulations



Pseudotriangulations in 3-space?

Rigid graphs are not well-understood in 3-space.