# Polytopes and Plane Graphs with no Long Monotone Paths

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Freie Universität Berlin

joint work with

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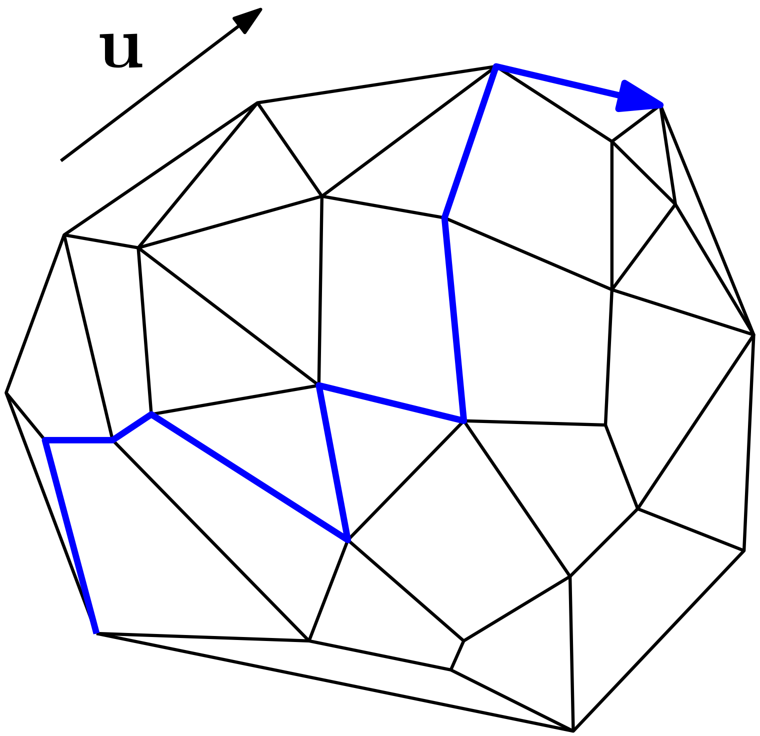
# Monotone Paths on Polytopes

**Conjecture:** Every 3D convex polytope with $n$ vertices has a monotone path of length $\Omega(\sqrt{n})$ in *some* direction.

[G. Rote, European Workshop on Computational Geometry, Dortmund March 2010]




(Motivation: Partial least-squares matching of point sets.)

$$\langle \mathbf{u}, p_1 \rangle < \langle \mathbf{u}, p_2 \rangle < \langle \mathbf{u}, p_3 \rangle < \cdots$$

THEOREM (2012-02-28). There is a family of triangulated polytopes with $n$ vertices, where the longest monotone path has length $O(\log n)$.

# Results on Polytopes

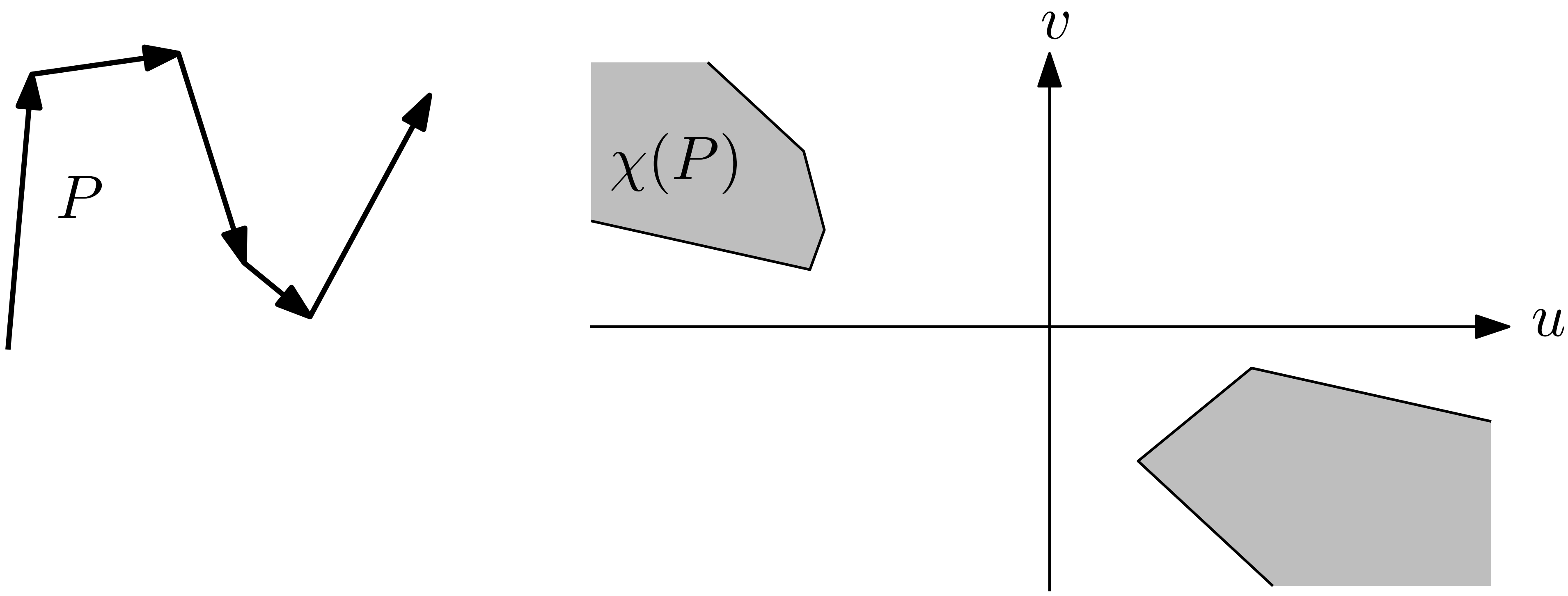
THEOREM (2012-02-28). There is a family of triangulated polytopes with $n$ vertices, where the longest monotone path has length $O(\log n)$. (L.B.: $\Omega(\log n/\log\log n)$)

THEOREM (2011). There is a family of triangulated polytopes with $n$ vertices and bounded degree $d$, where the longest monotone path has length $O(\log^2 n)$. (L.B.: $\Omega(\log n)$)

THEOREM (Chazelle, Edelsbrunner, Guibas 1989). Every polyhedral subdivision of the plane with $n$ vertices and degree $\le d$ contains a monotone path with $\ge \Omega(\log_d n + \log n/\log\log n)$ edges. This is tight.

# The characteristic region of a path

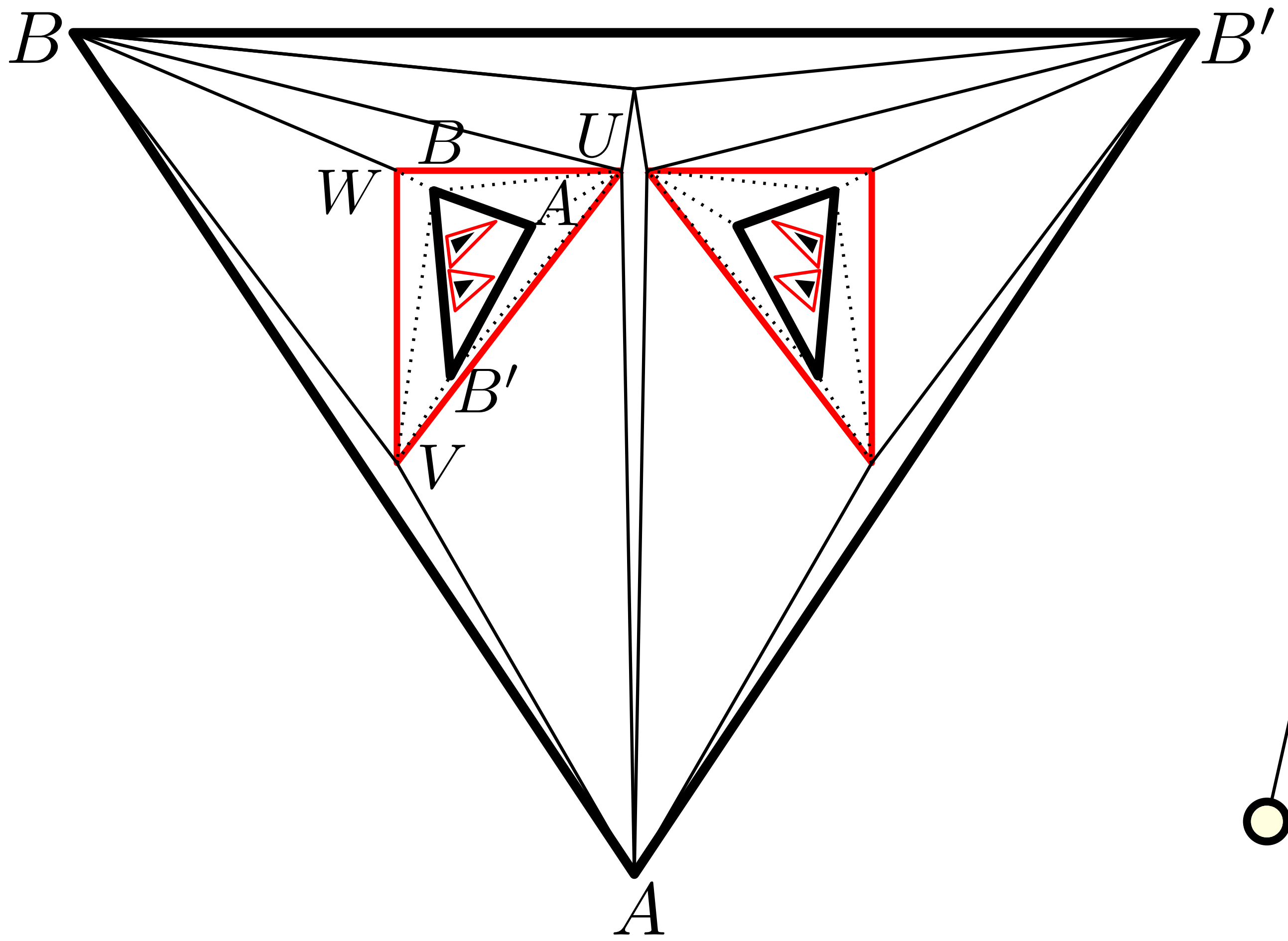


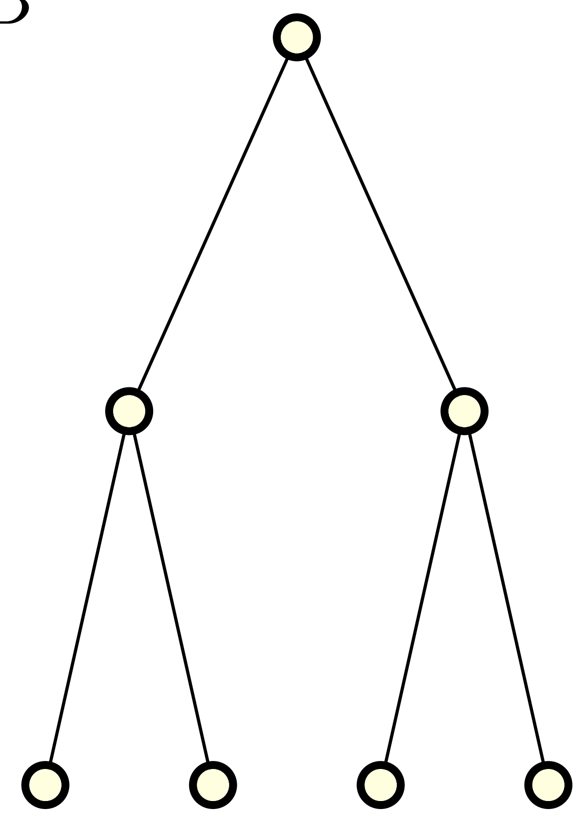



$\chi(P) =$
the set of directions $(u, v, 1)$ for which $P$ or its inverse is a monotone path.
$=$ two intersections of half-planes

# The $O(\log^2 n)$ construction

a hierarchical structure:



# The basic building block $\Delta$

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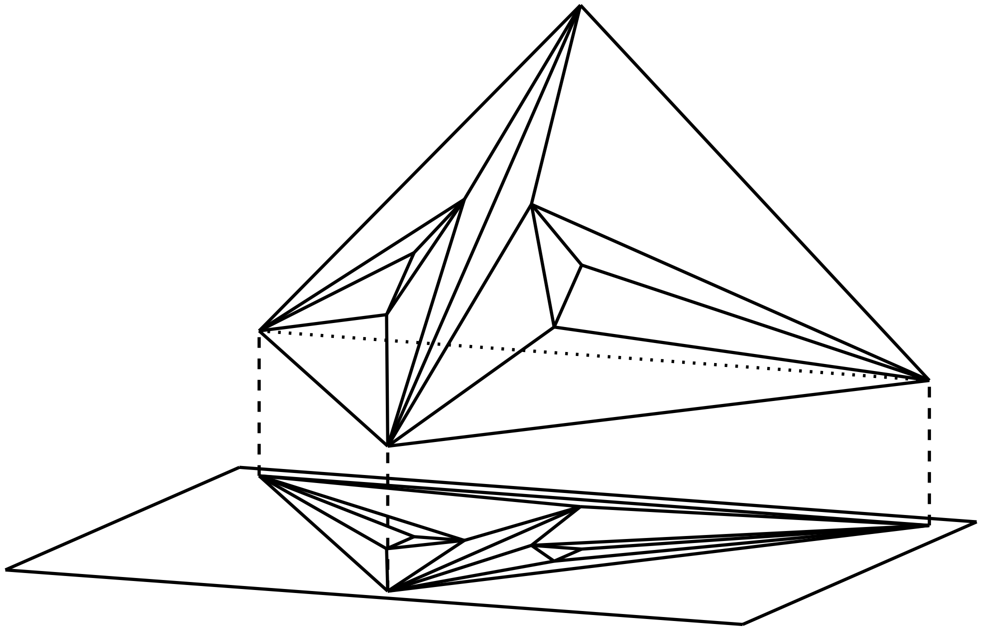

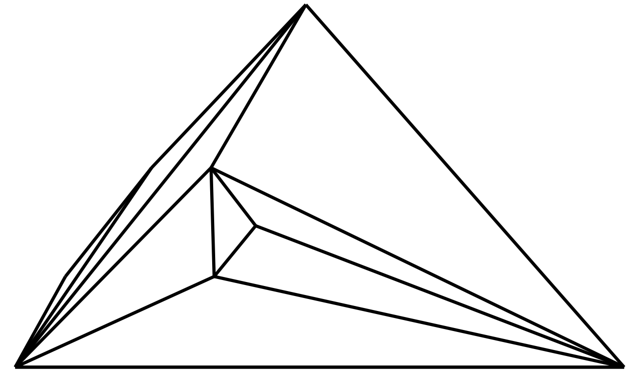

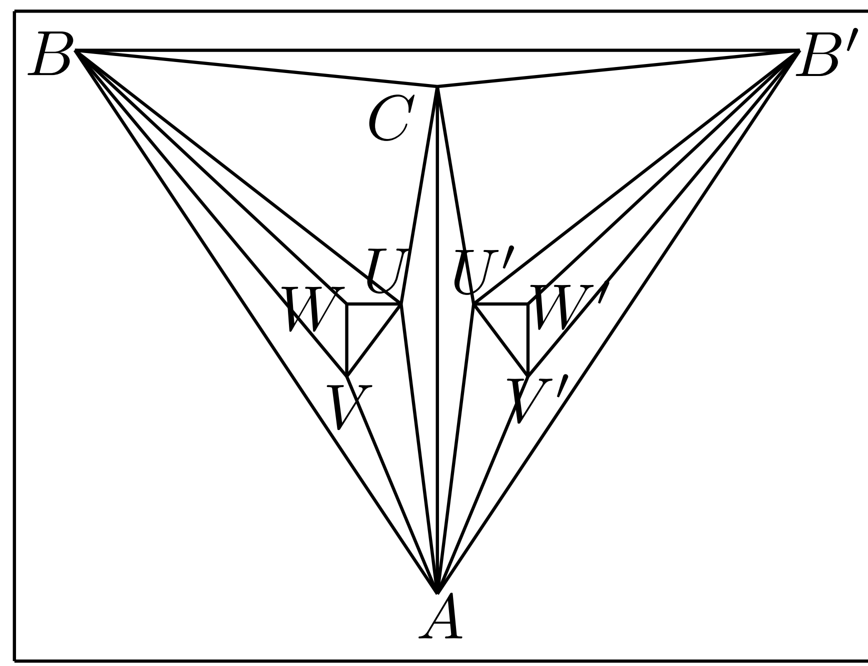


| point | $(x,\quad y,\quad z)$ |
|---|---|
| $A$ | $(0,\ 0,\ 0)$ |
| $B, B'$ | $(\mp 1,\ 1.5,\ 0)$ |
| $C$ | $(0,\ 1.4,\ 1)$ |
| $U, U'$ | $(\mp 0.1,\ 0.8,\ 0.55)$ |
| $V, V'$ | $(\mp 0.25,\ 0.6,\ 0.25)$ |
| $W, W'$ | $(\mp 0.25,\ 0.8,\ 0.39)$ |

# The characteristic region of $\Delta$

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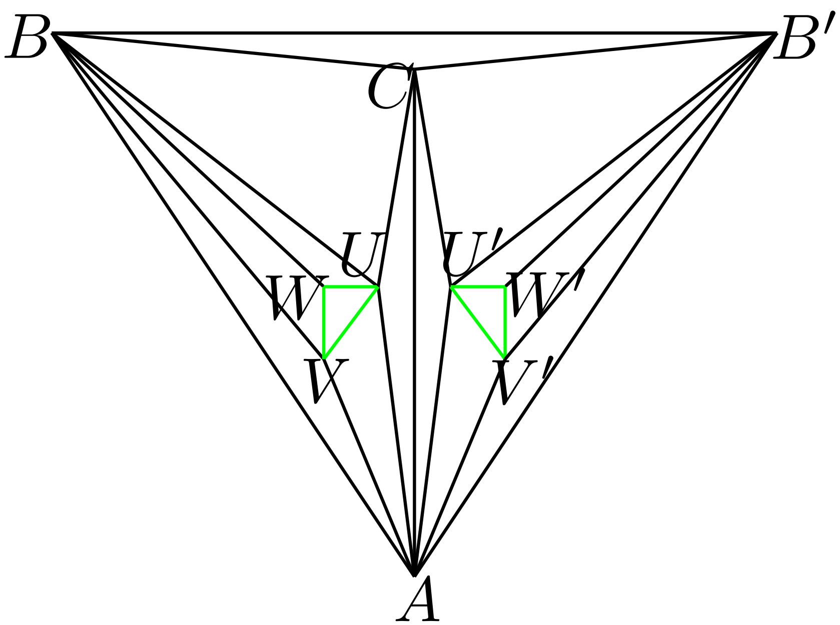
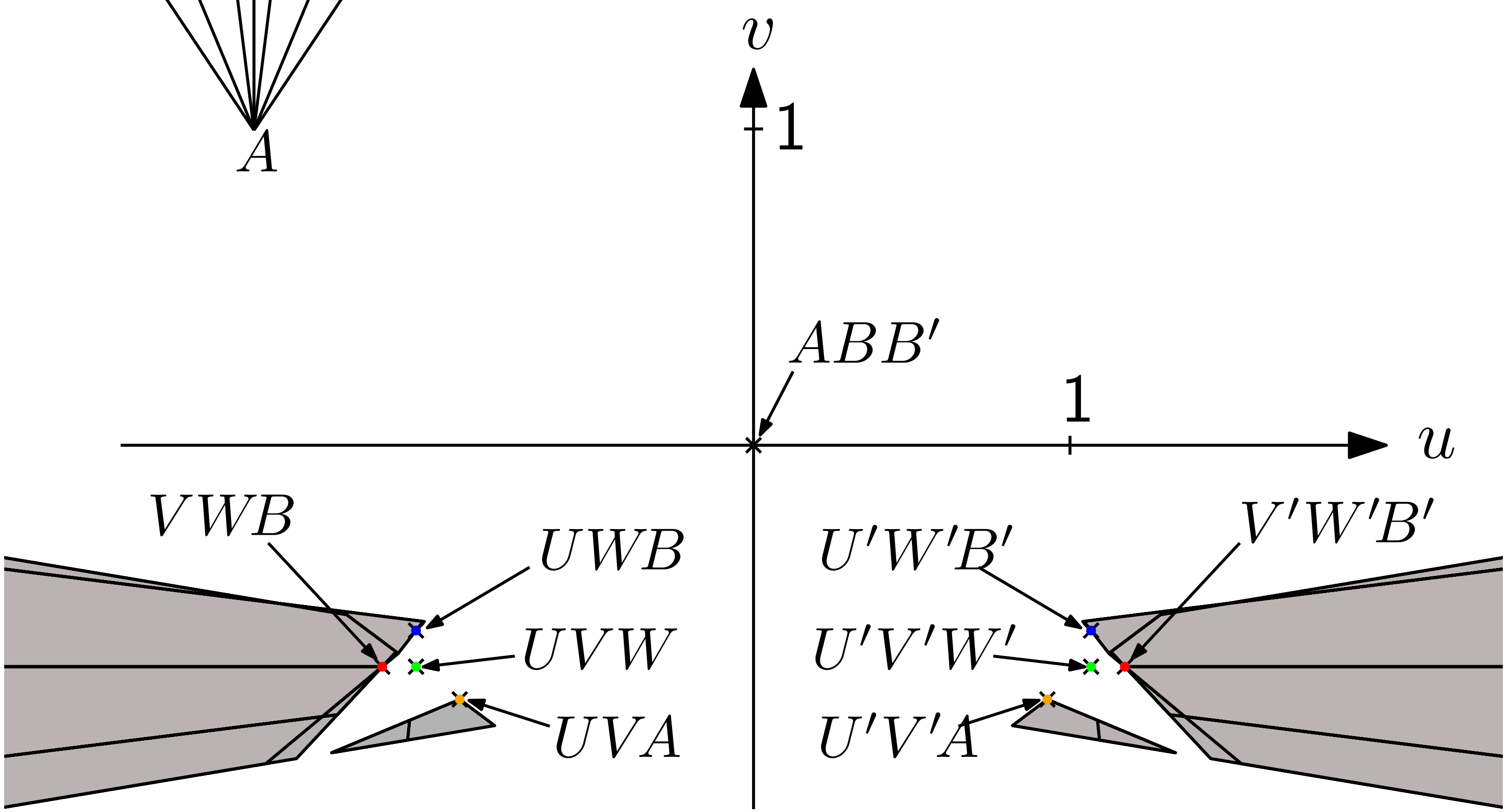
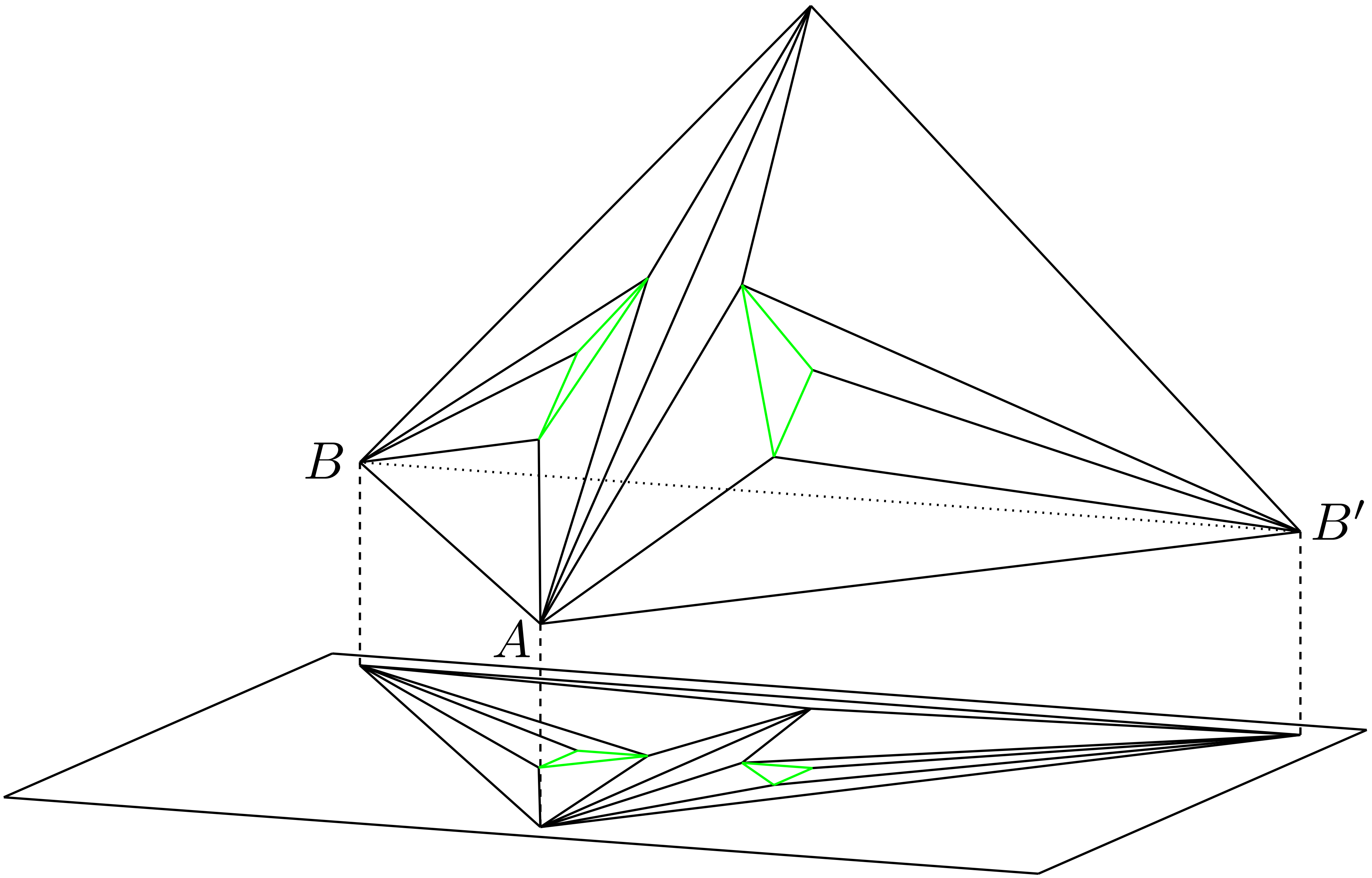



- start in $A$, $B$, or $B'$
- visit at least two vertices of $UVW$ and at least two vertices of $U'V'W'$ (in either order)
- end in $A$, $B$, or $B'$



# Placing the subcells



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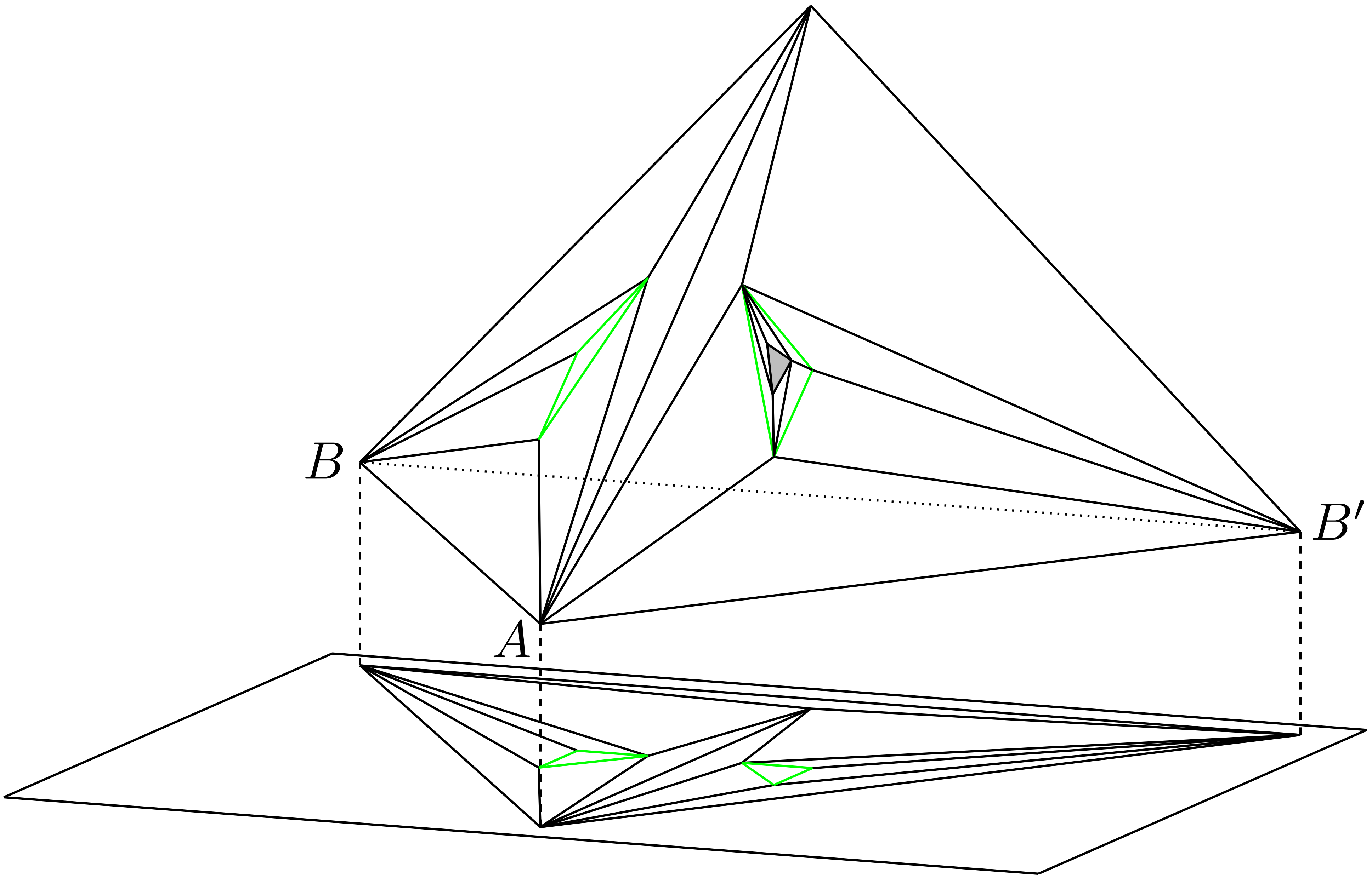
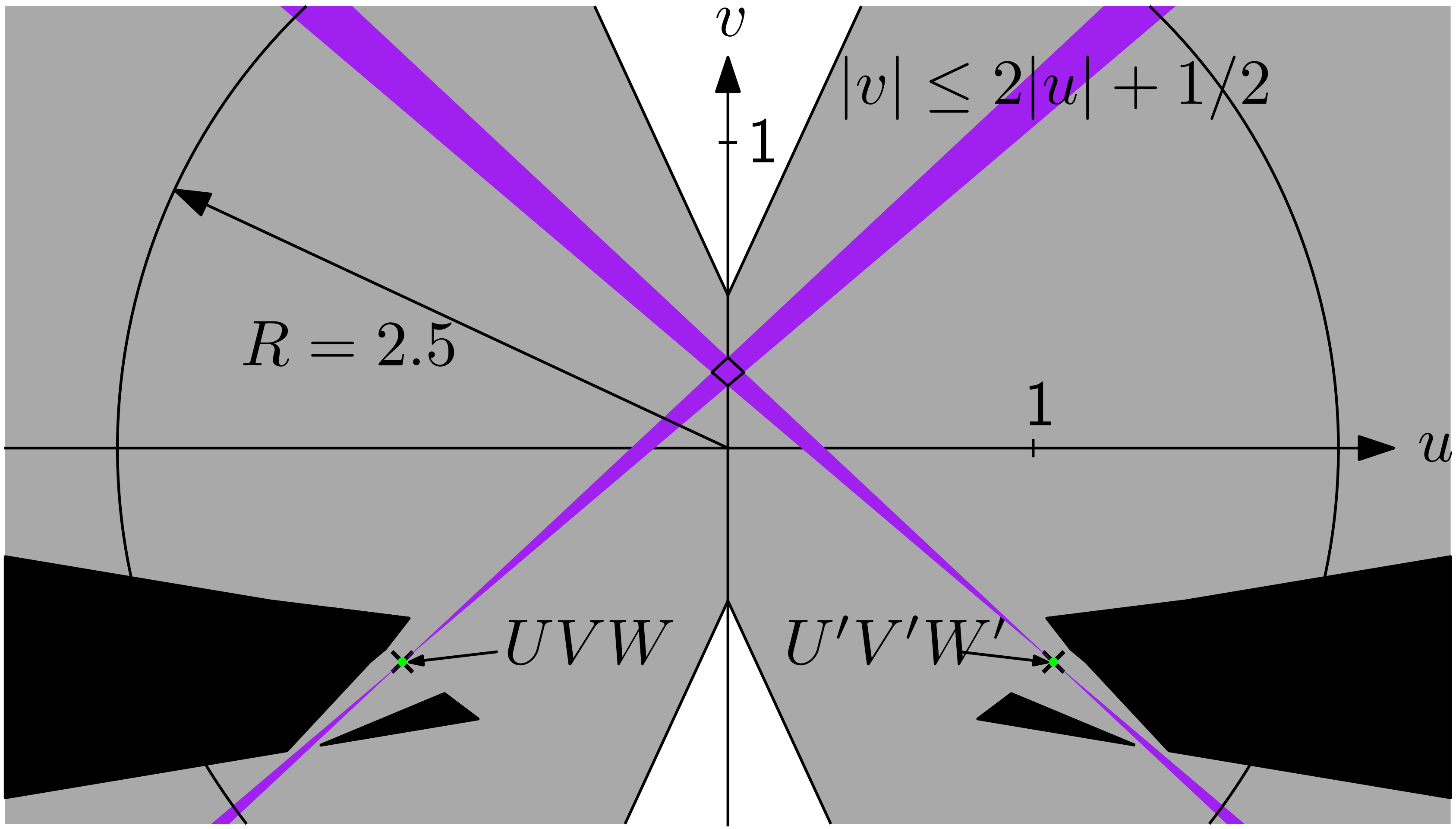


# Inductive construction

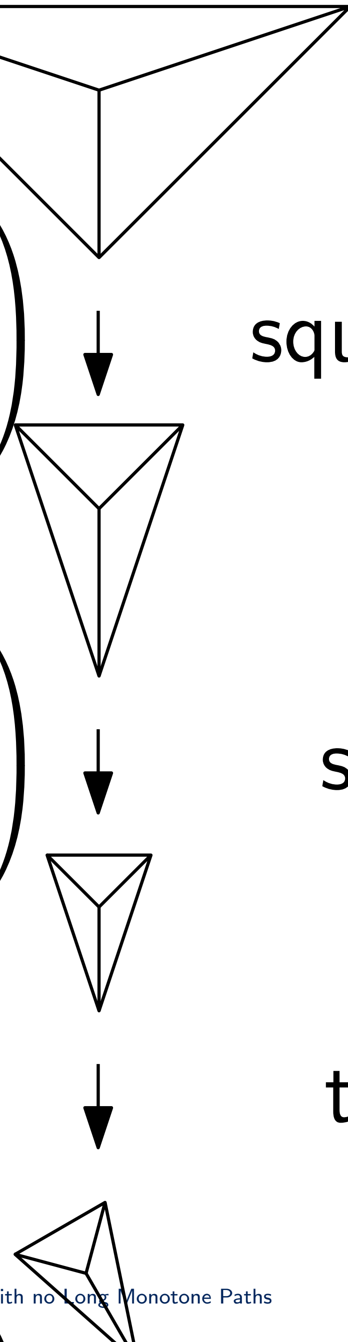
Characteristic regions: • lie in $|v| \le 2|u| + 1/2$

• have no triple intersections

• pairwise intersections lie within $\le R = 2.5$ of the origin



# Affine Transformations



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \varepsilon \cdot x \\ y \\ \varepsilon \cdot z \end{pmatrix}$$

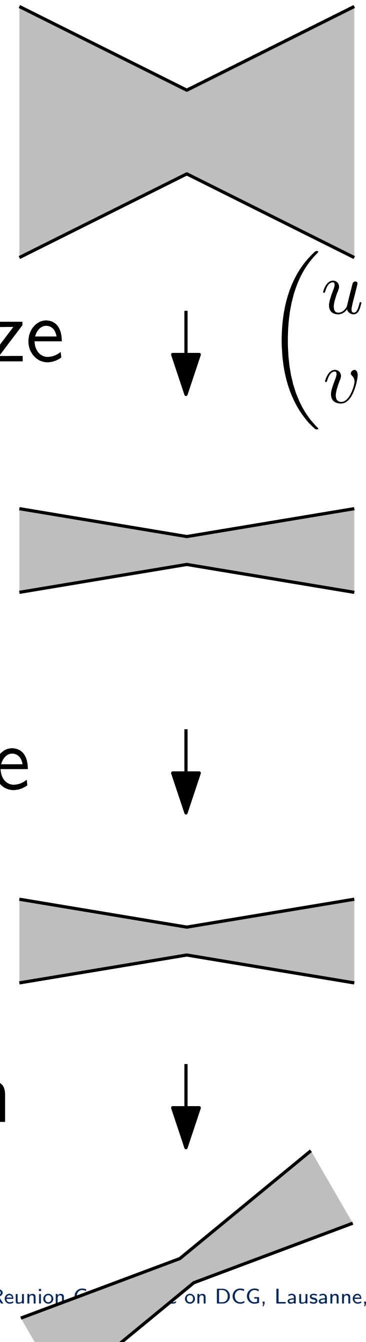
squeeze

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ \varepsilon \cdot v \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \varepsilon \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

scale

turn



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$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ \varepsilon \cdot z \end{pmatrix}$$

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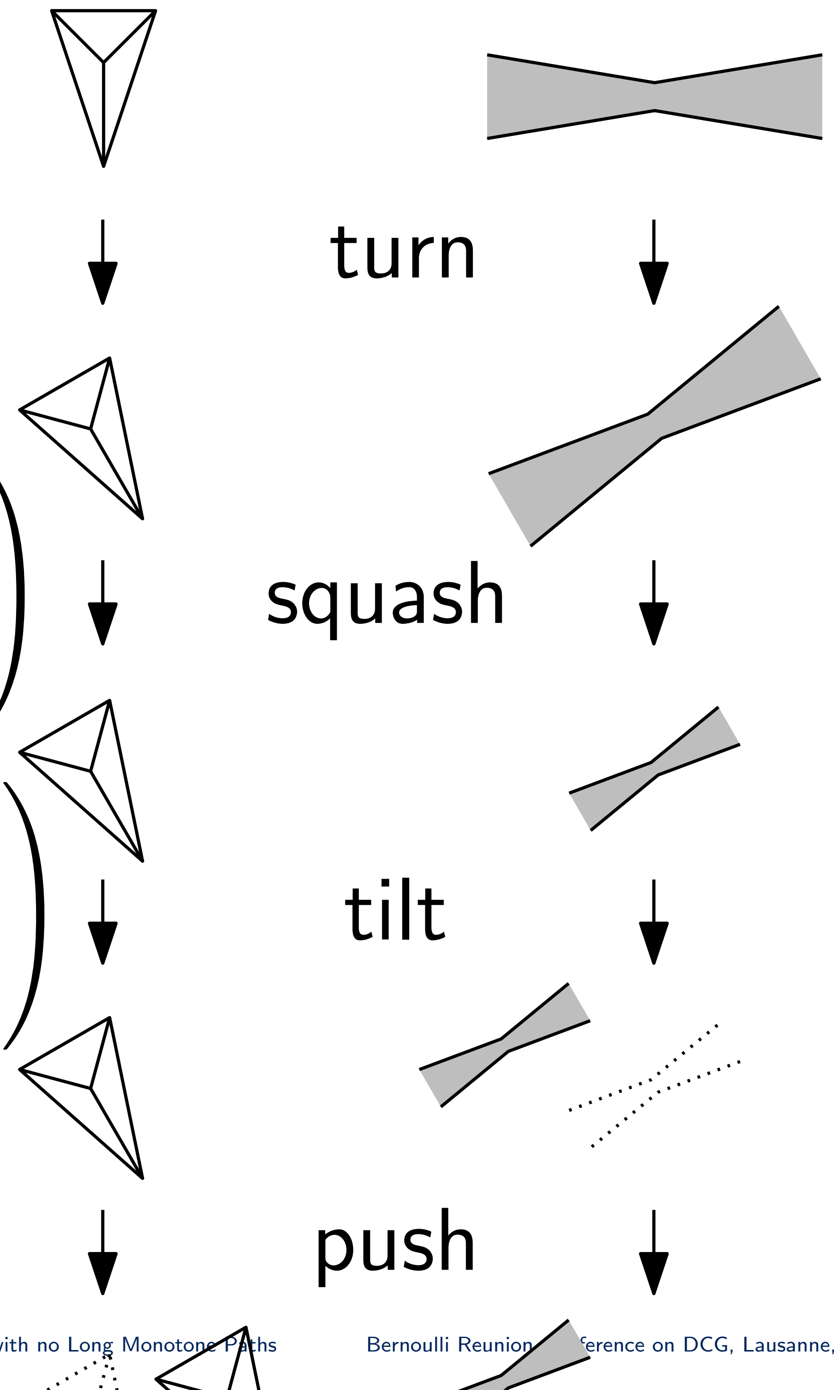

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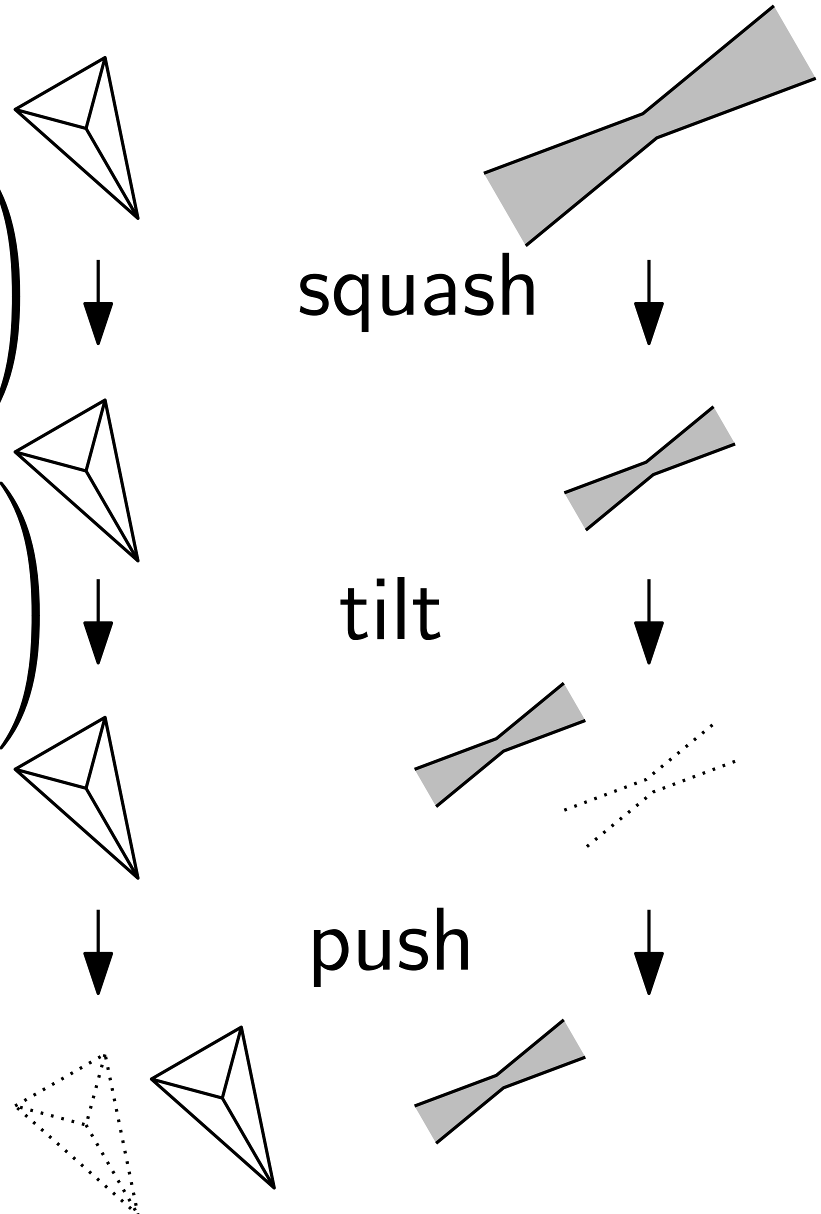

# The visited nodes

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A monotone path $P$ in direction $c$ can visit both children of a node $\Delta$ only if

- $c$ lies in $\chi(\Delta)$, or
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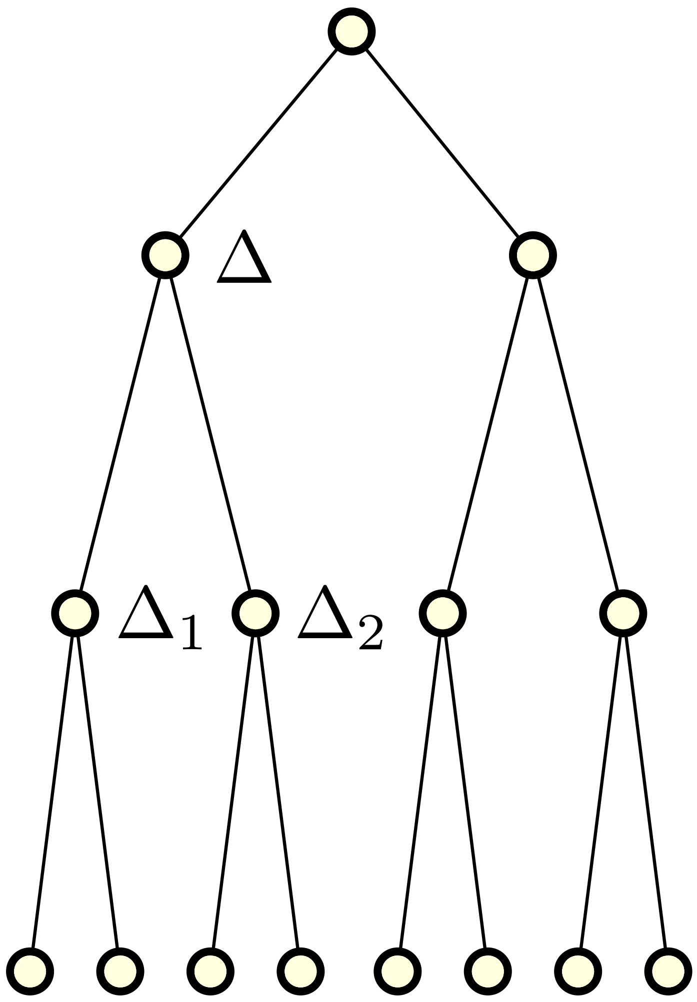


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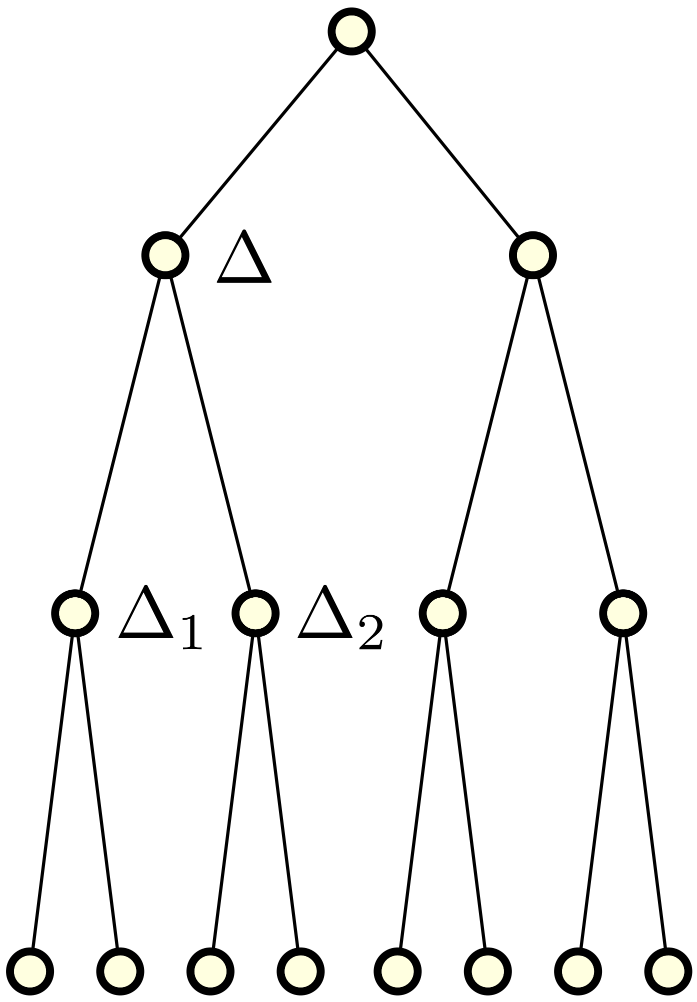

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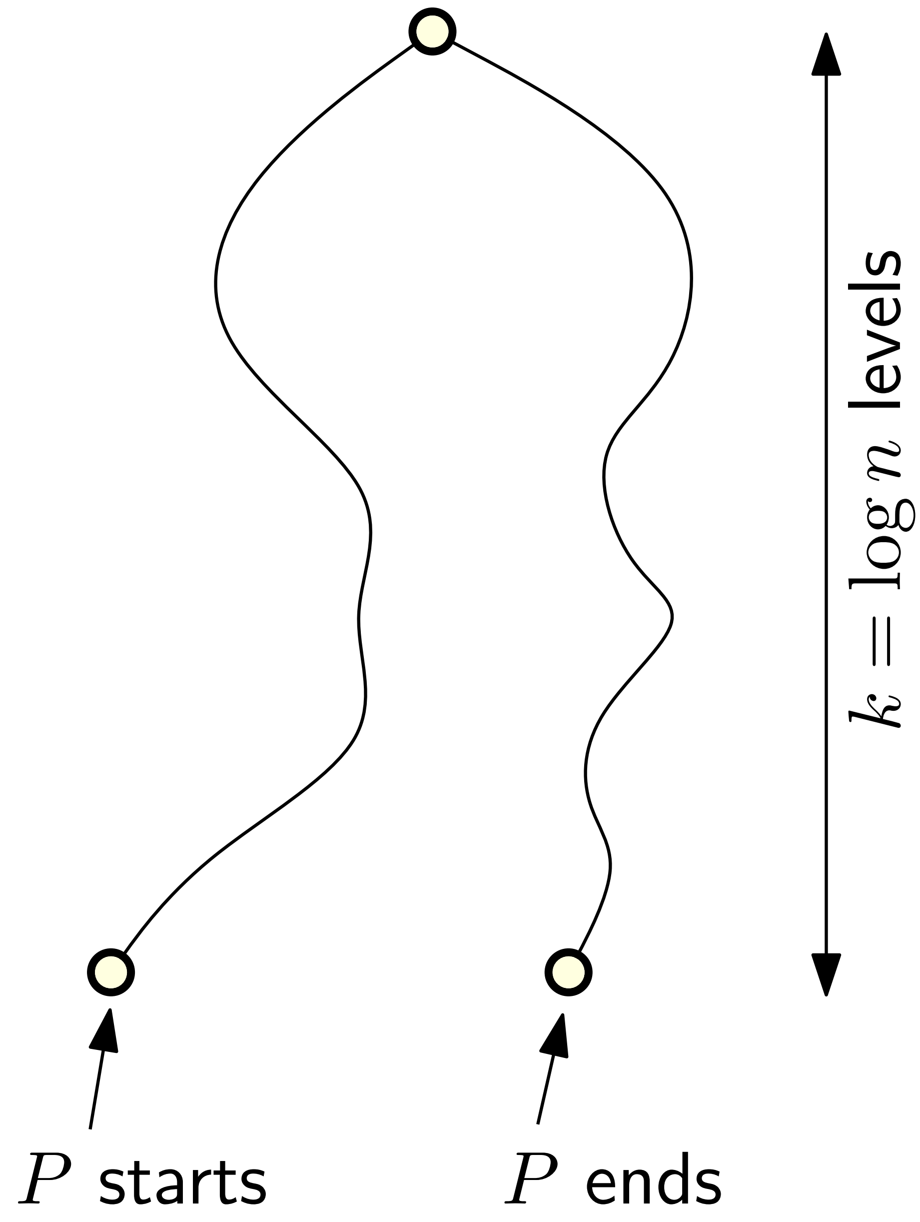

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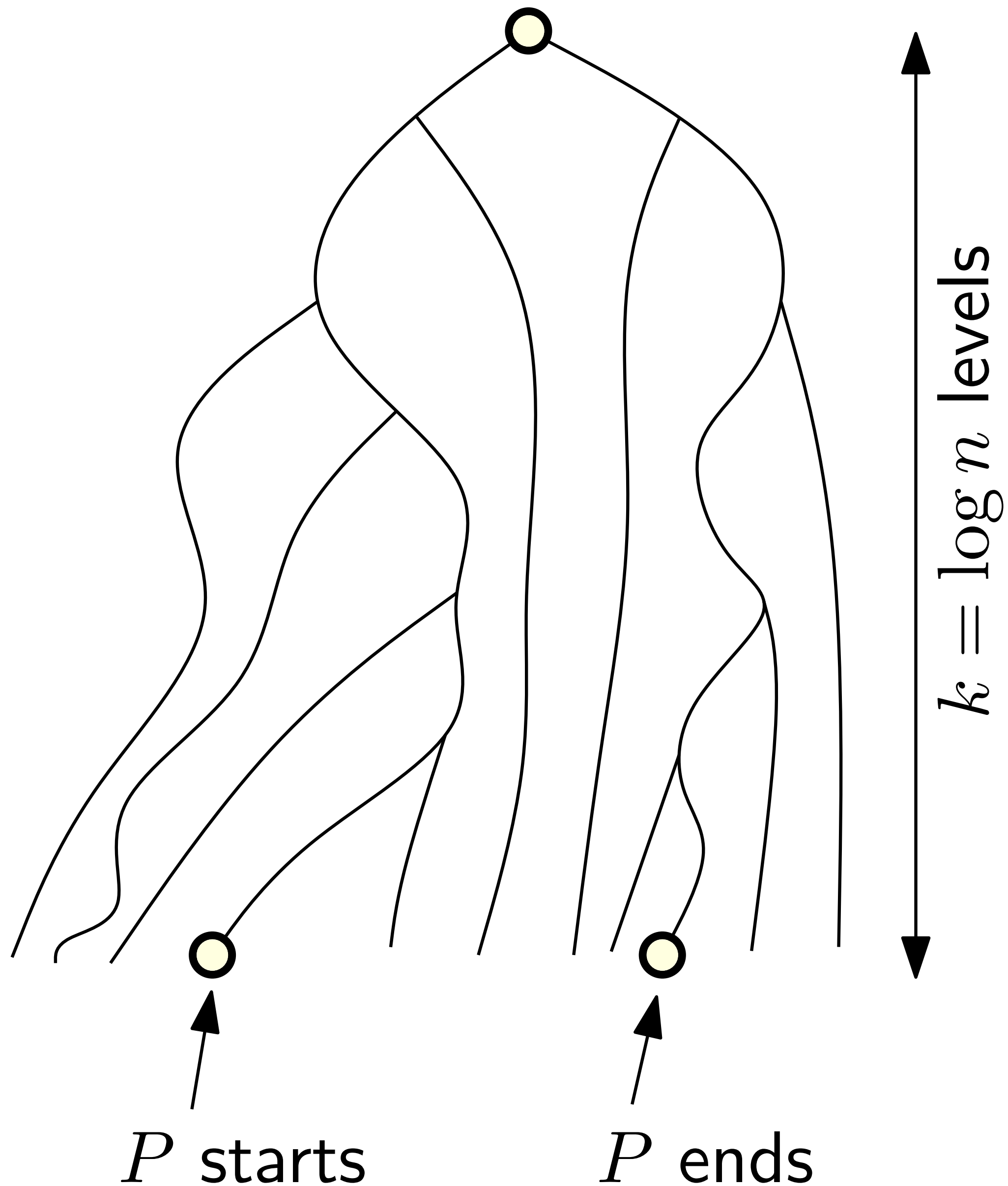

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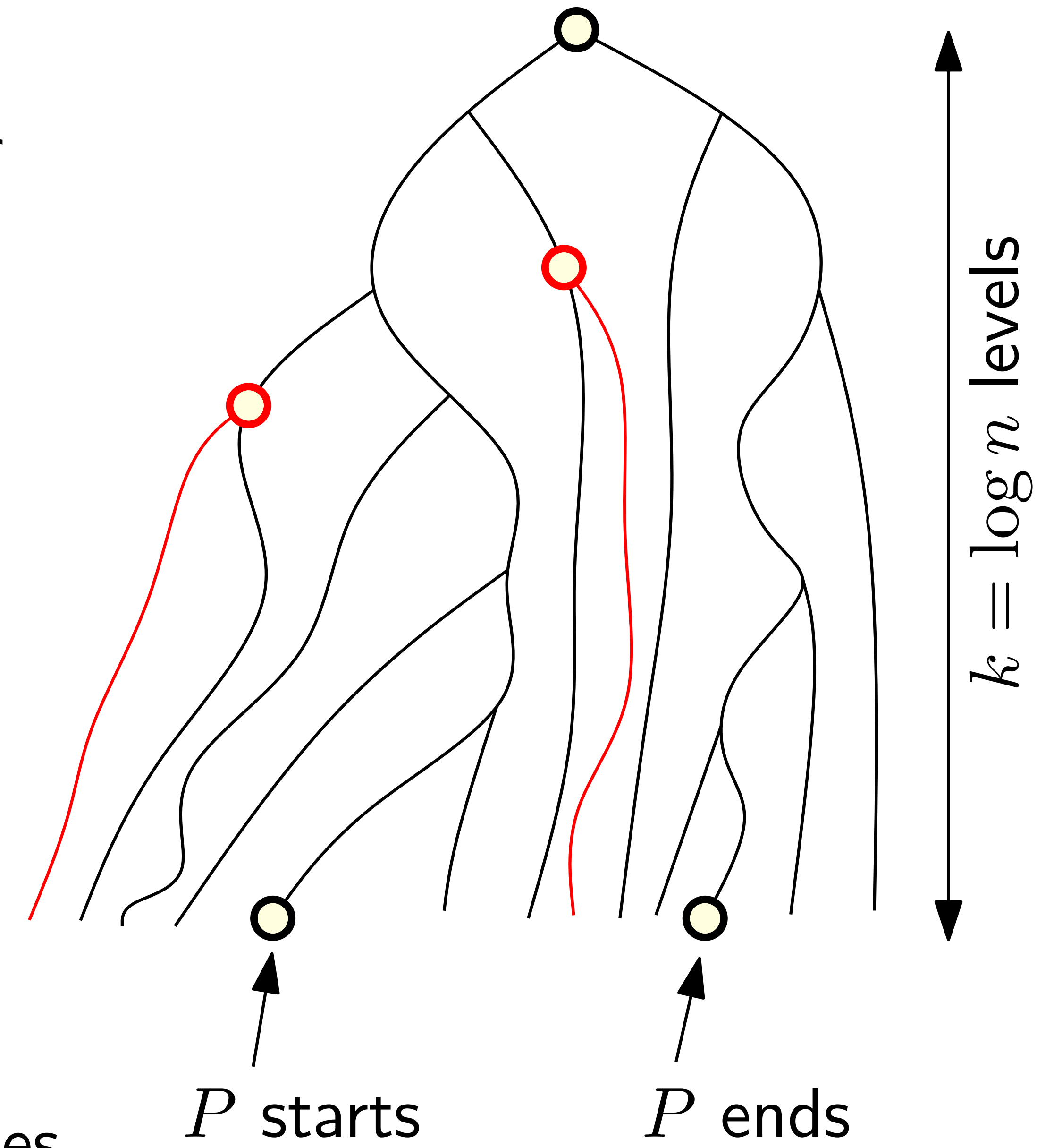
$2k$ paths of length $k$

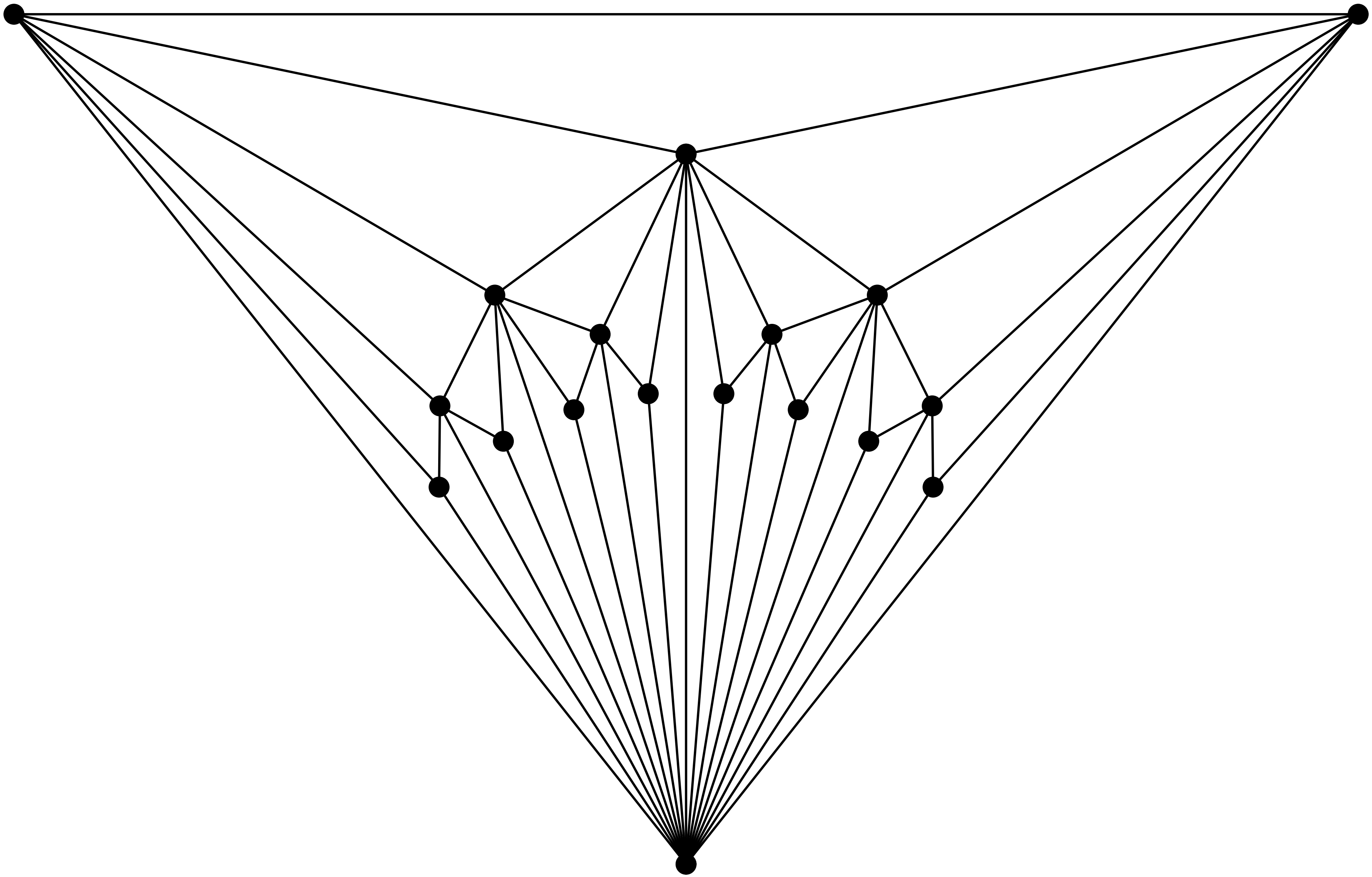
plus $2$ paths of length $k$

$\rightarrow O(k^2)$ nodes

$\rightarrow O(k^2) = O(\log^2 n)$ vertices



# The Construction for $O(\log n)$



# Results on Convex Planar Subdivisions



THEOREM. Let $v$ be a vertex in a convex subdivision of the plane with $n$ vertices and degree $\le d$. There is path starting in $v$ with $\ge \Omega(\log_d n)$ edges that is monotone in *some* direction. (This is best possible; Chazelle, Edelsbrunner, Guibas 1989.)

THEOREM. Let $G$ be a convex subdivision of the plane with $n$ vertices and $k$ unbounded faces. Then $G$ contains a path with $\ge \Omega(\log \frac{n}{k} / \log \log \frac{n}{k})$ edges that is monotone in *some* direction.
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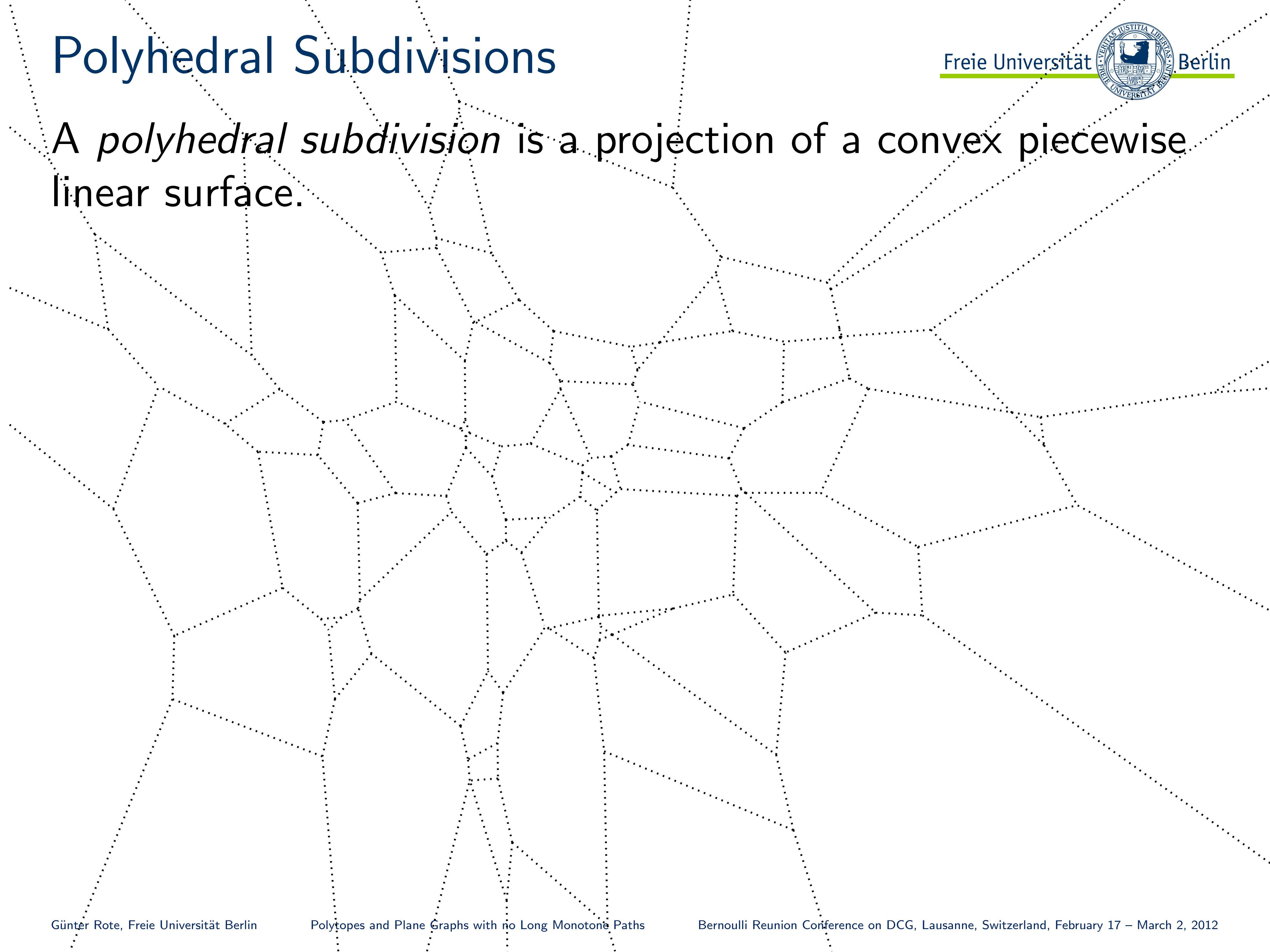
Every *polyhedral* subdivision of the plane with $n$ vertices and degree $\le d$ contains a monotone path with $\ge \Omega(\log_d n + \log n / \log \log n)$ edges. This is tight.

# Polyhedral Subdivisions



A *polyhedral subdivision* is a projection of a convex piecewise linear surface.



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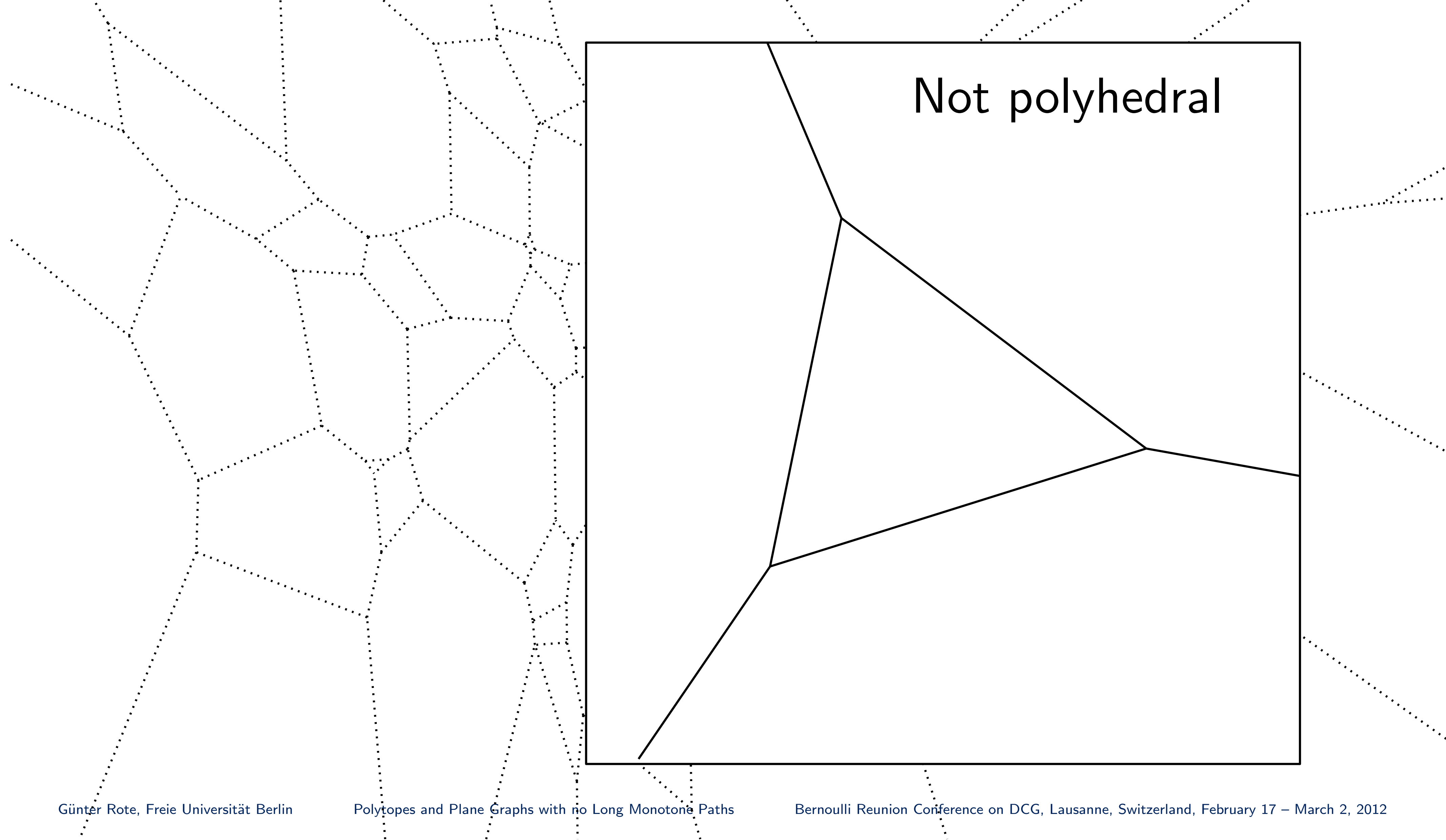

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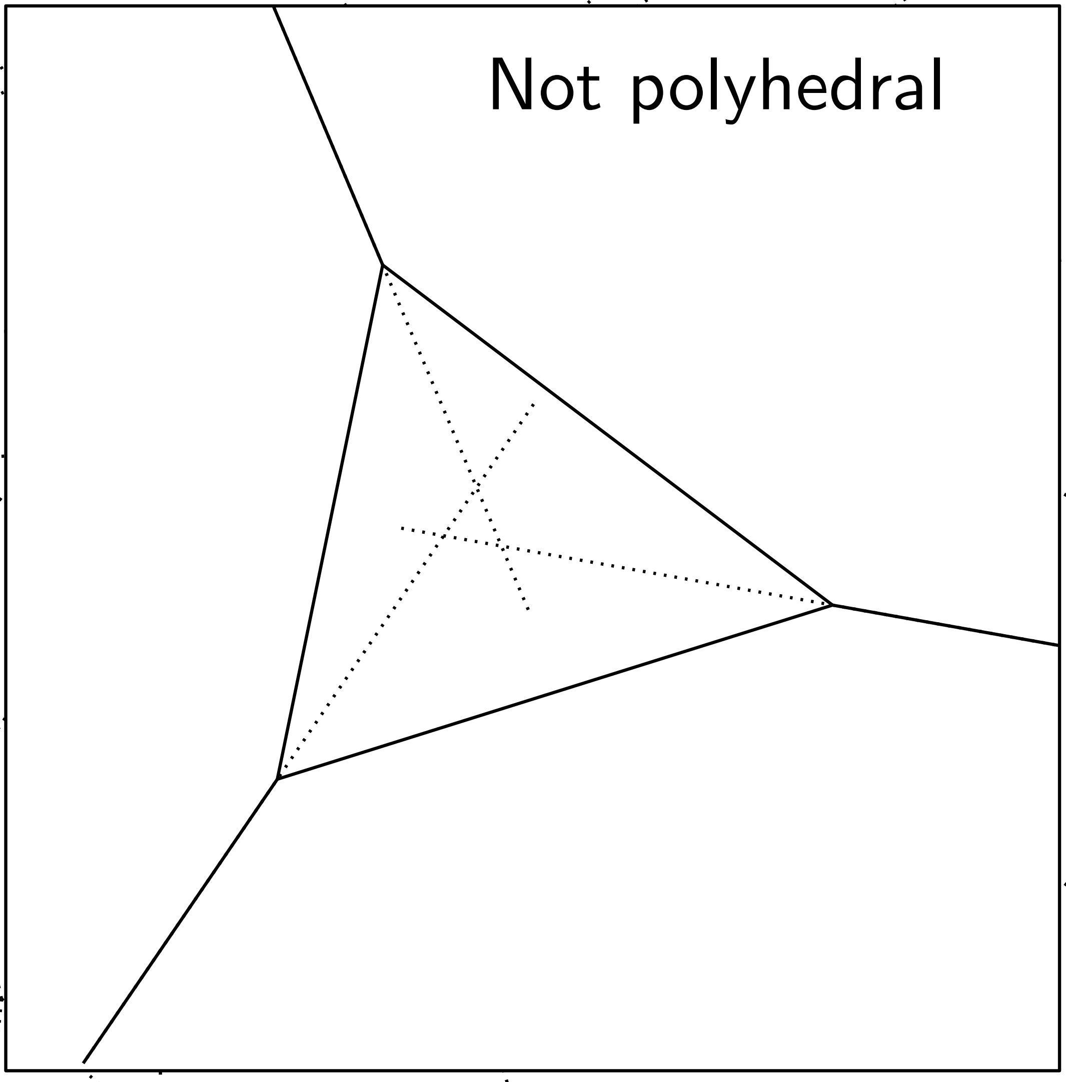

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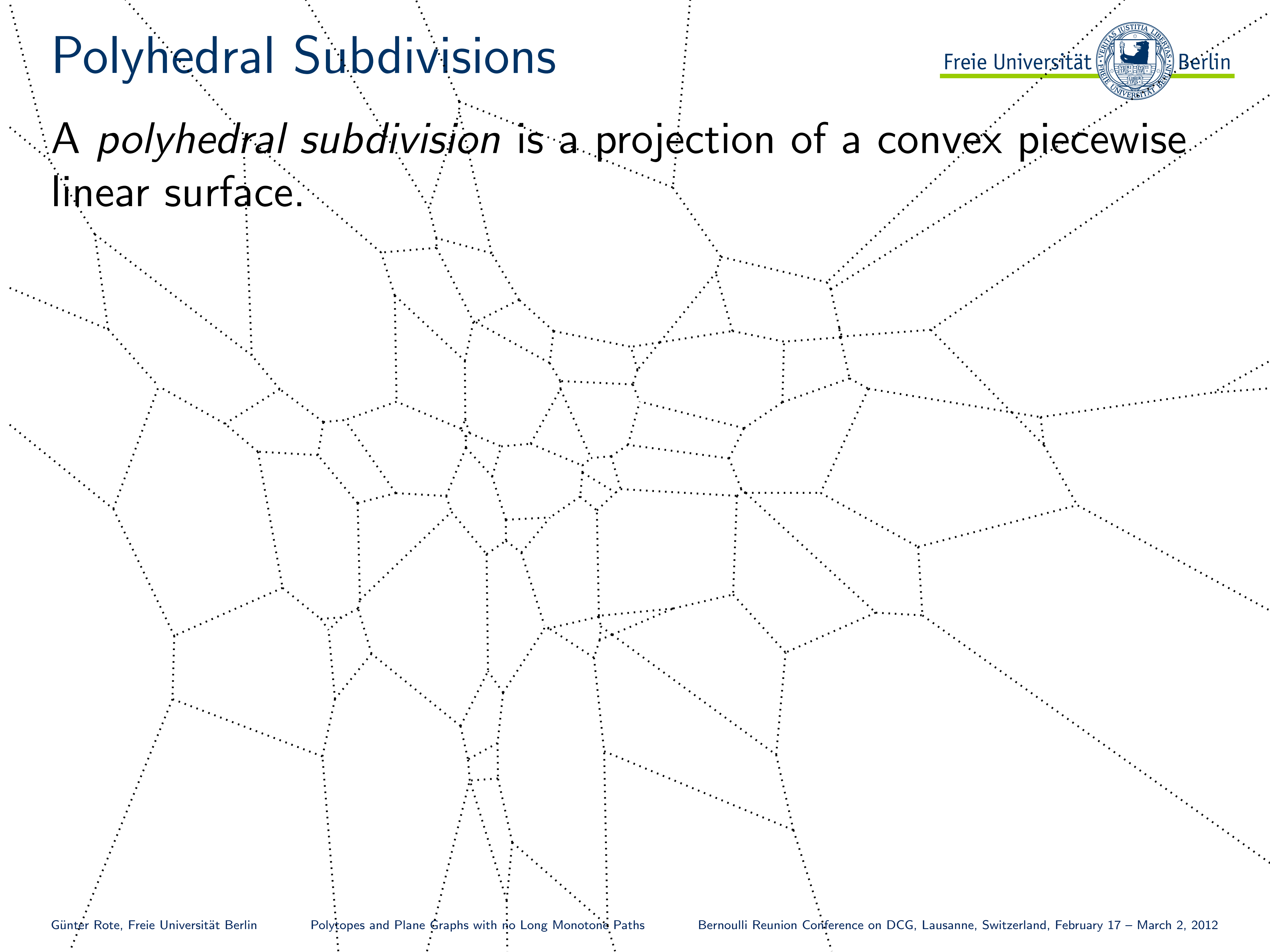

# Monotone Paths in Convex Subdivisions

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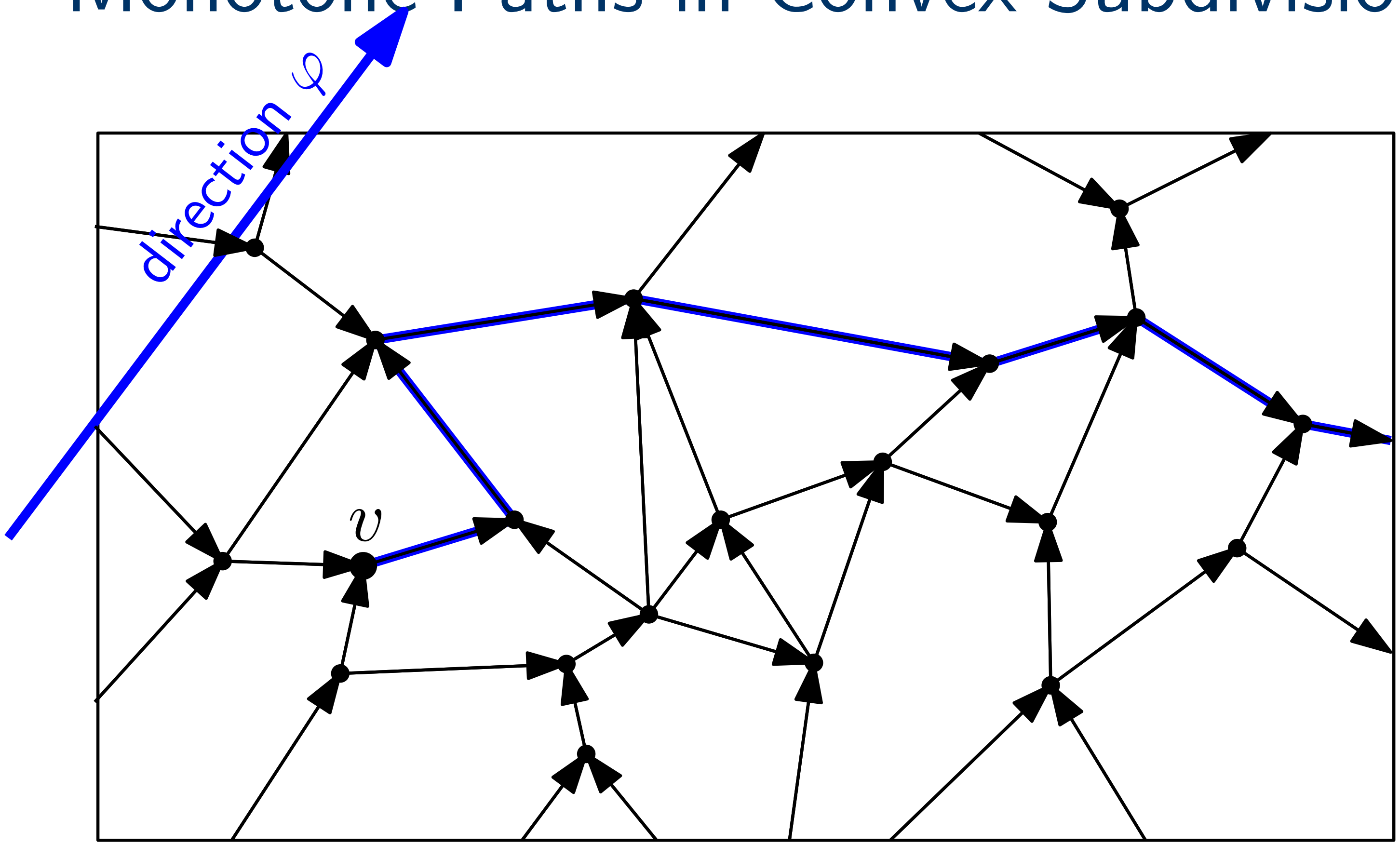


$n$ vertices
$\Theta(n)$ edges
$\Theta(n)$ faces

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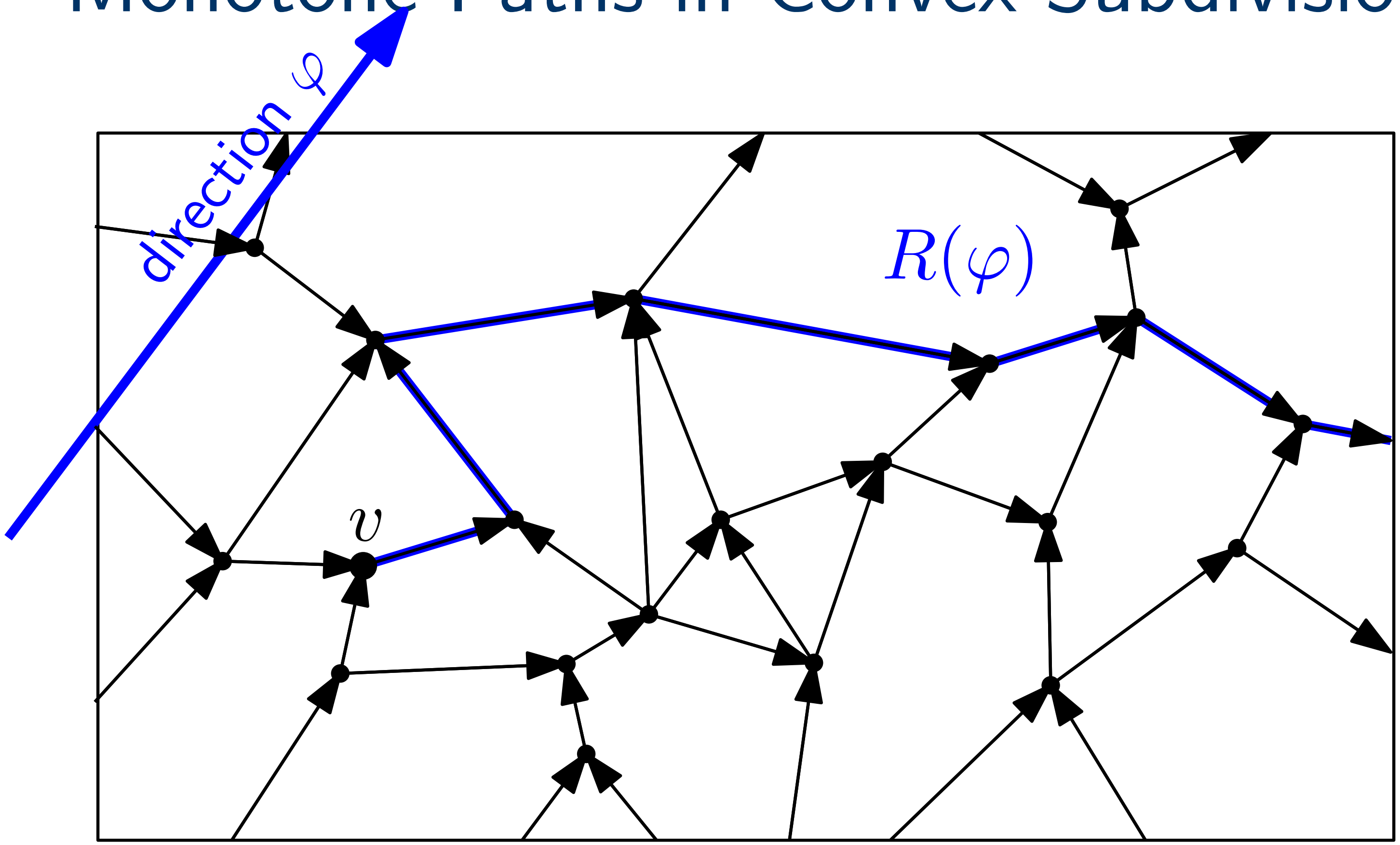


$R(\varphi) :=$
the *rightmost*
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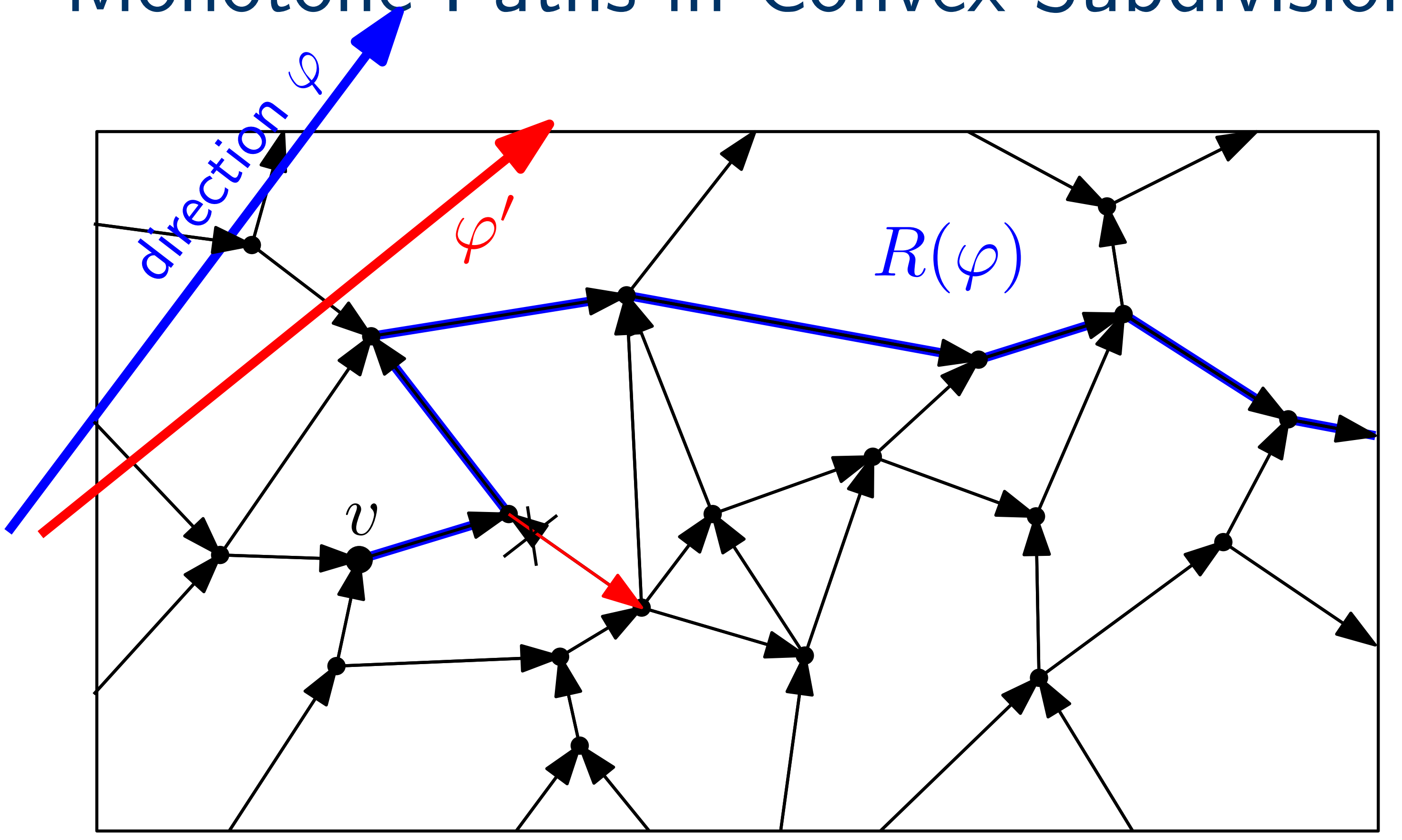


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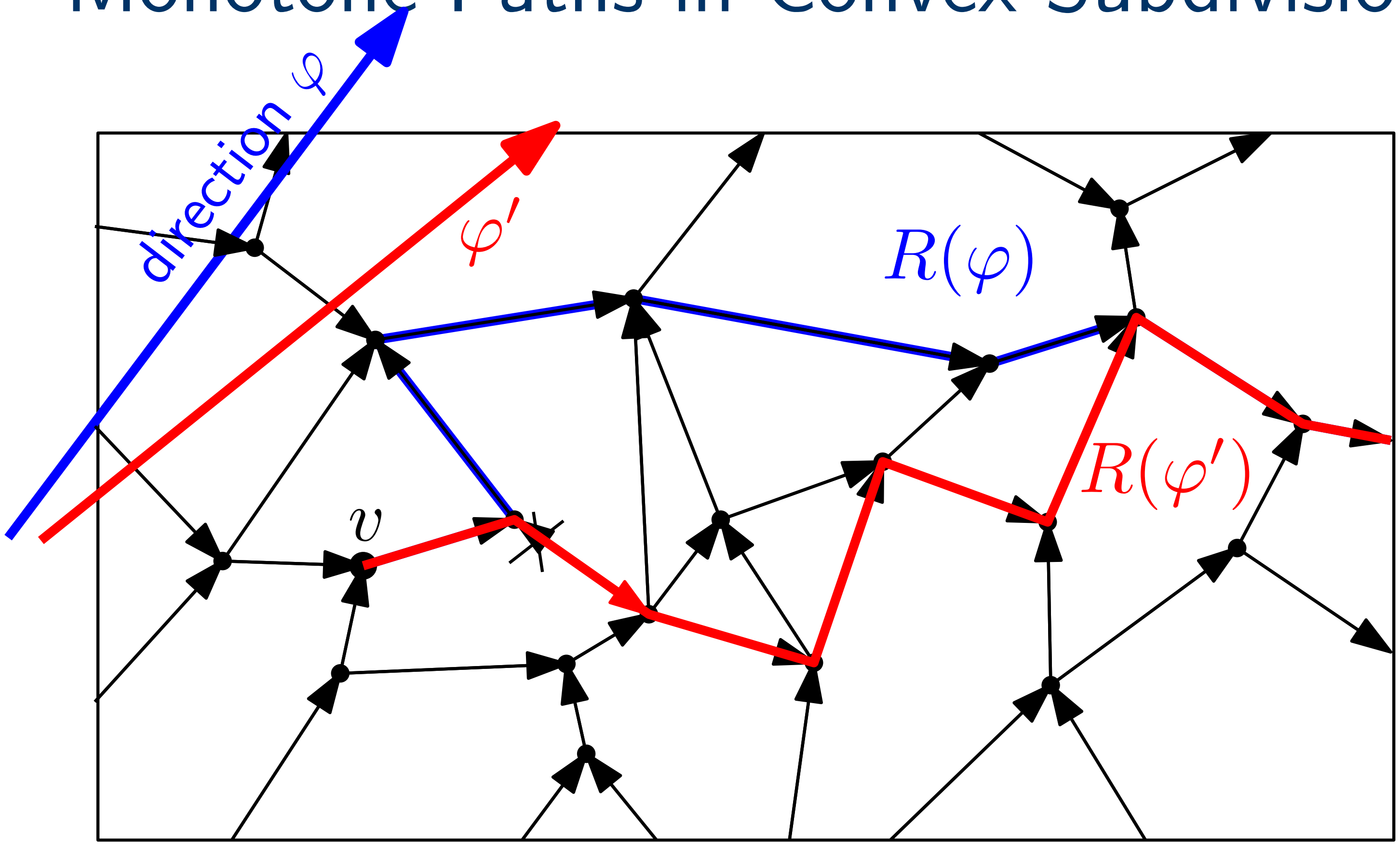


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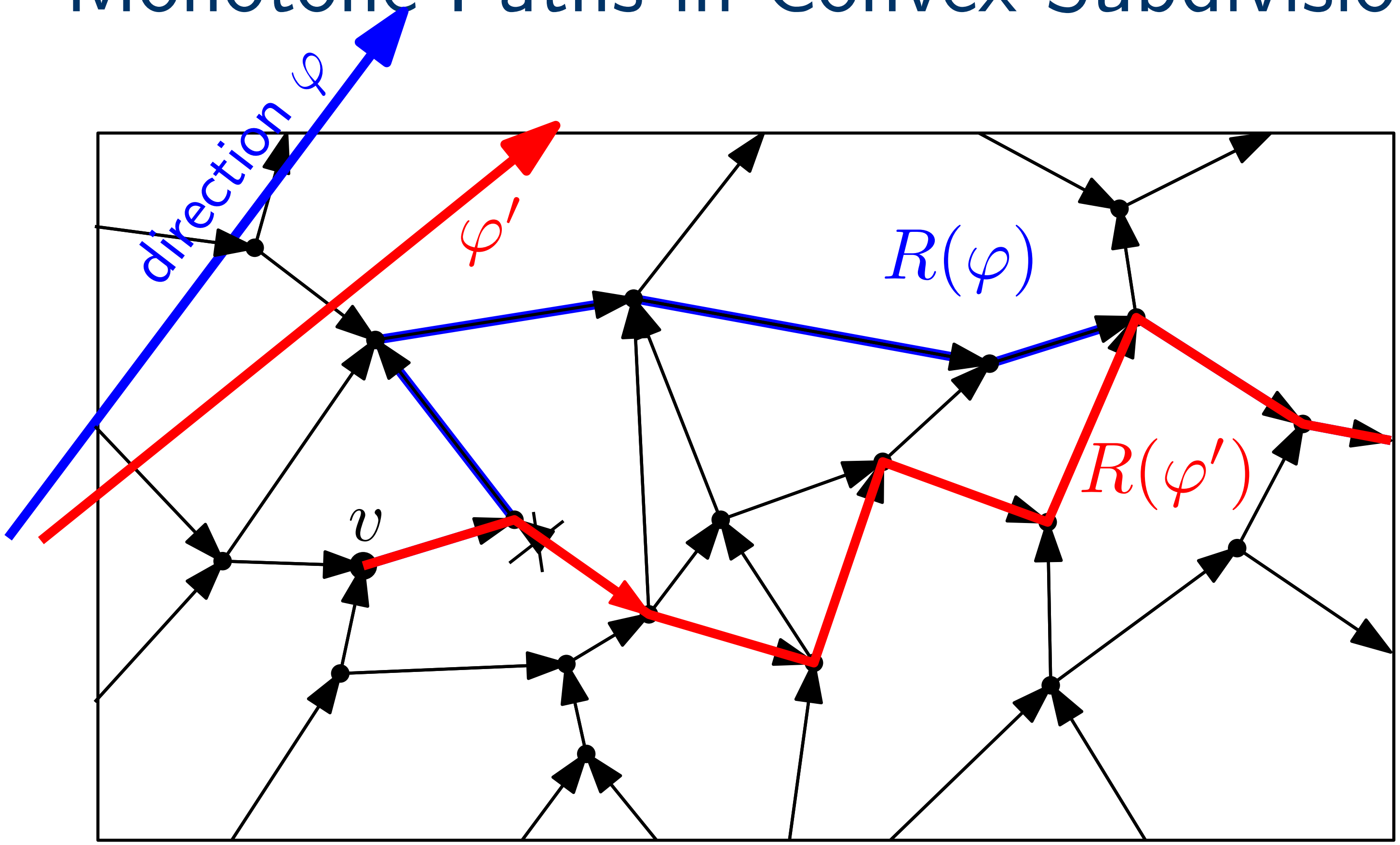


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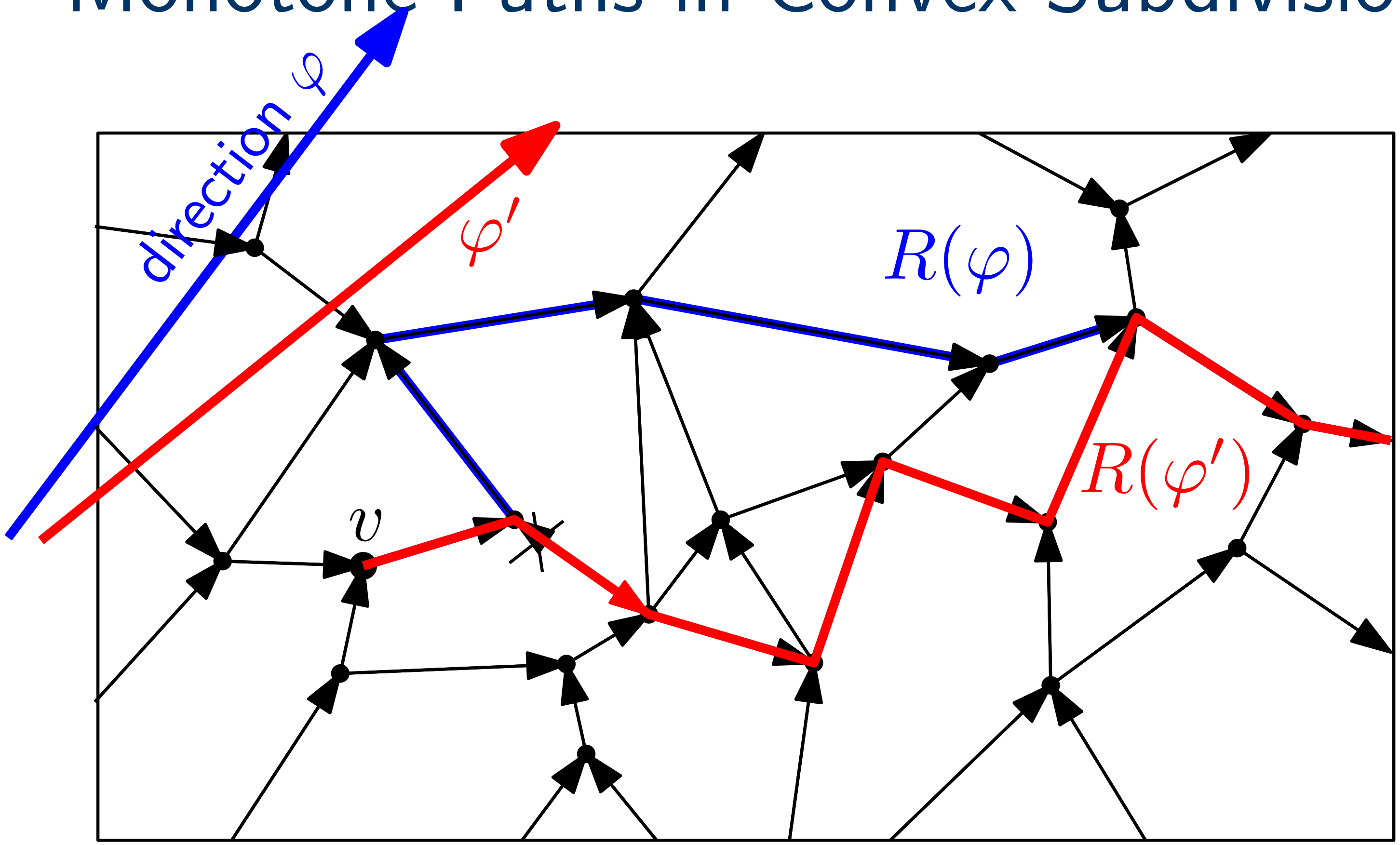


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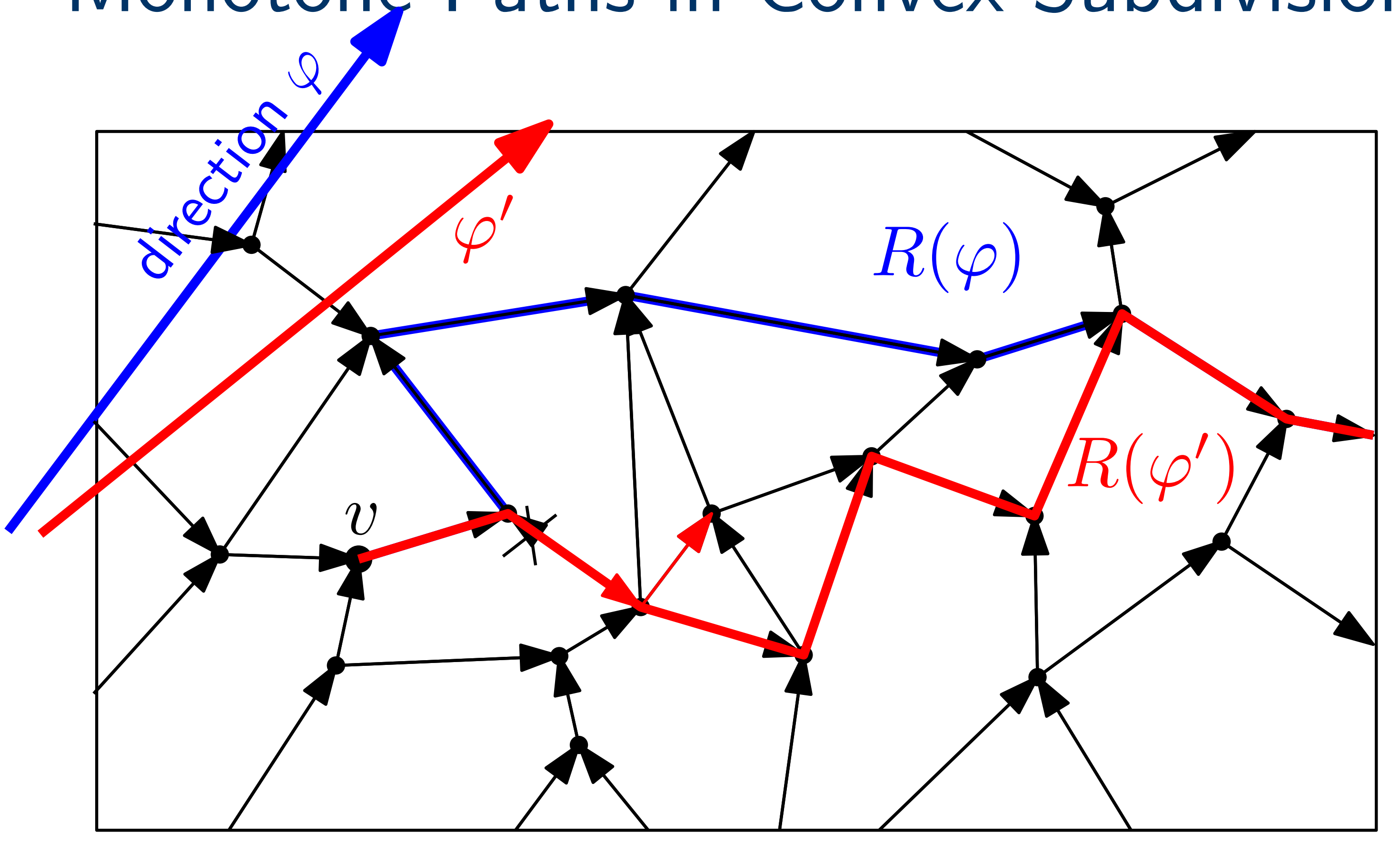


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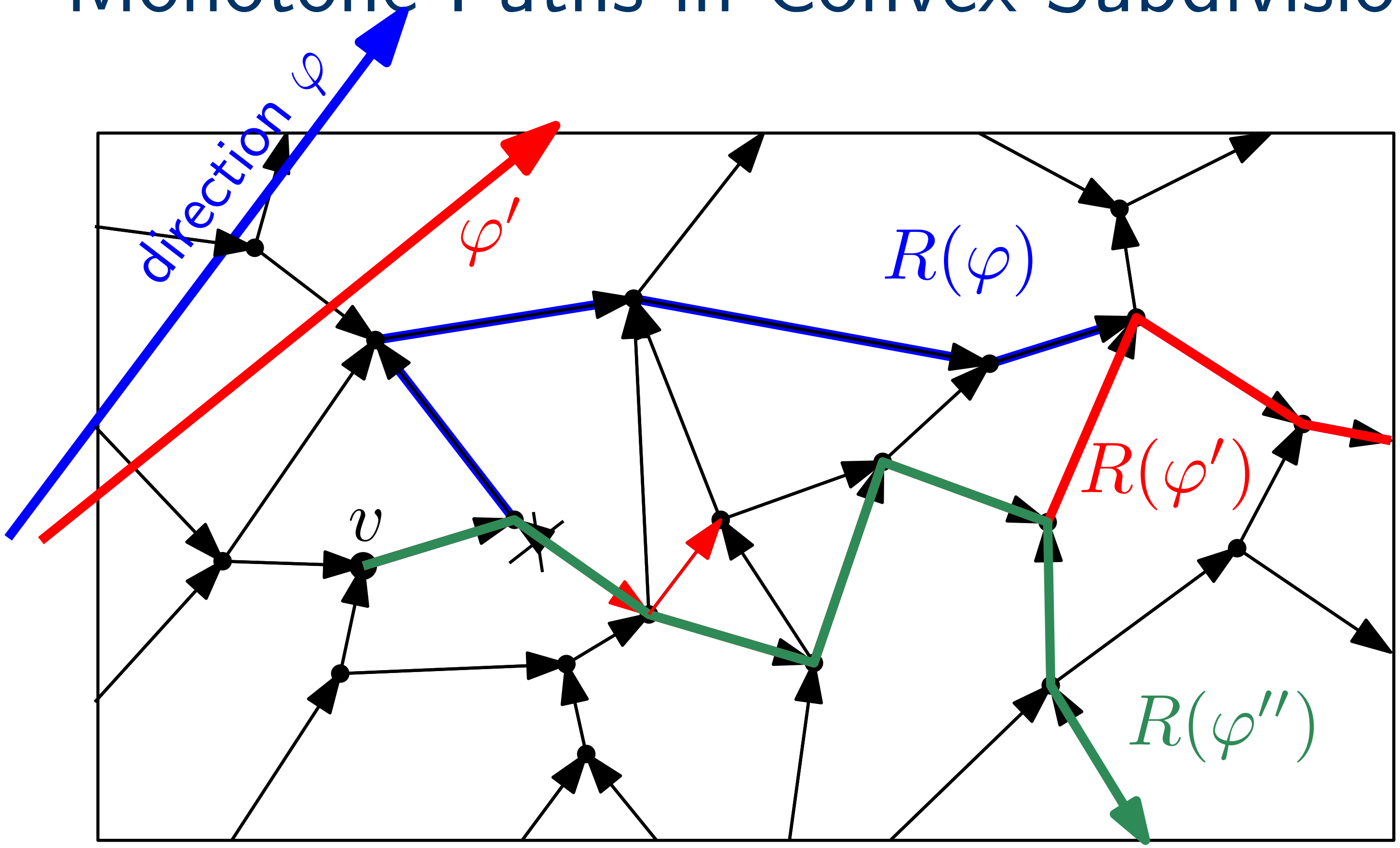


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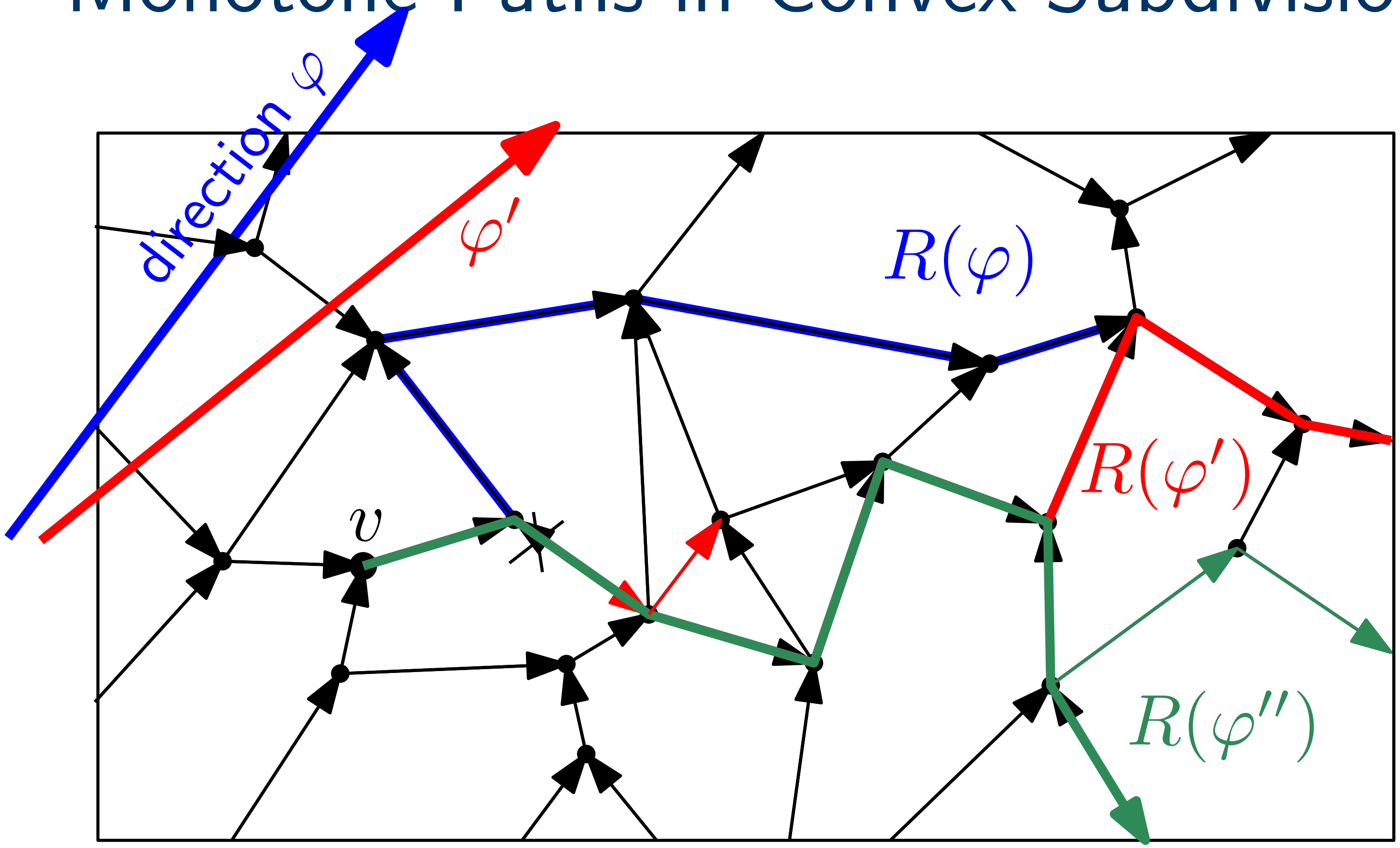


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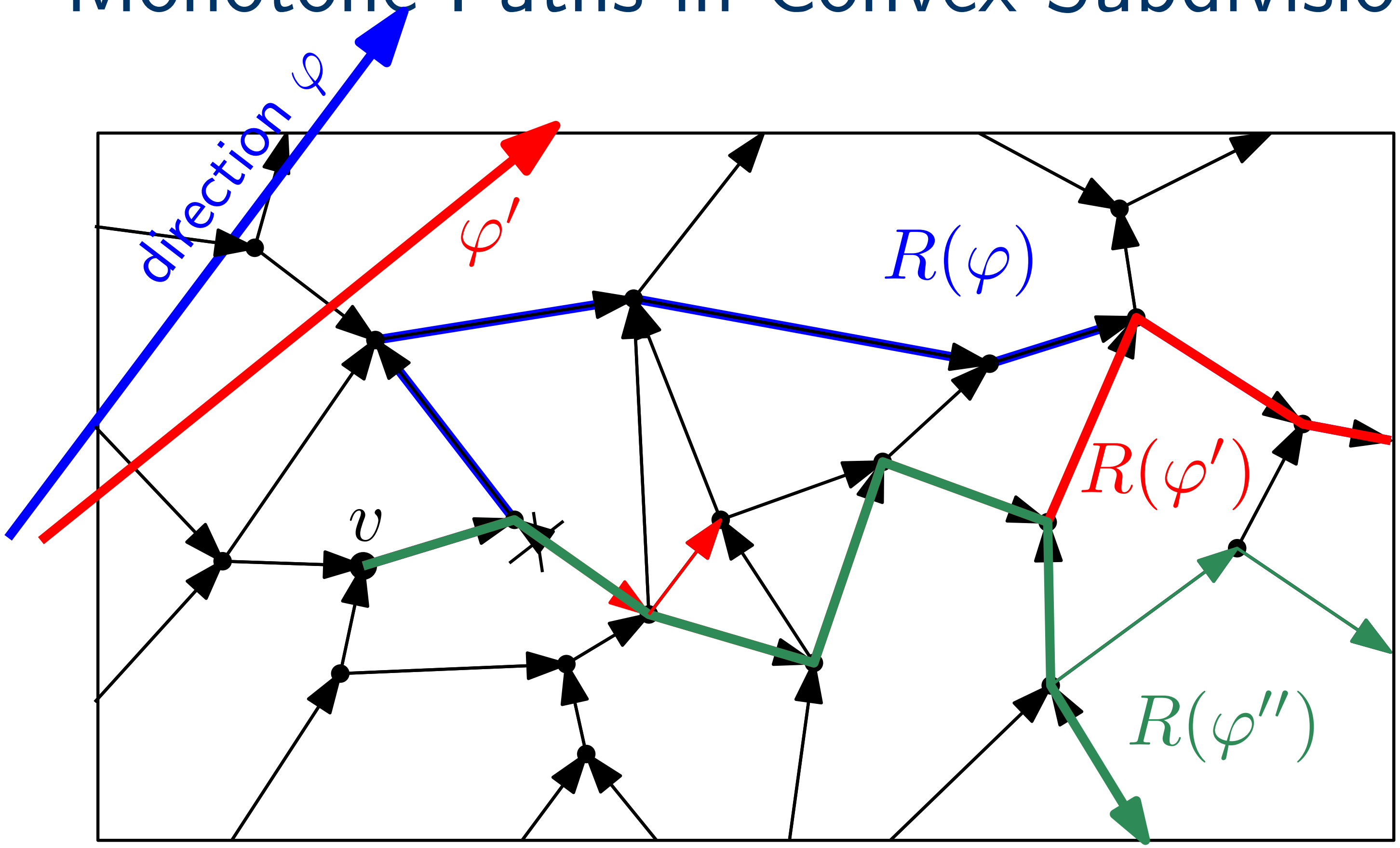


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$\rightarrow$ a directed graph in which $v$ can reach every vertex by a monotone path.

# Monotone Paths in Convex Subdivisions

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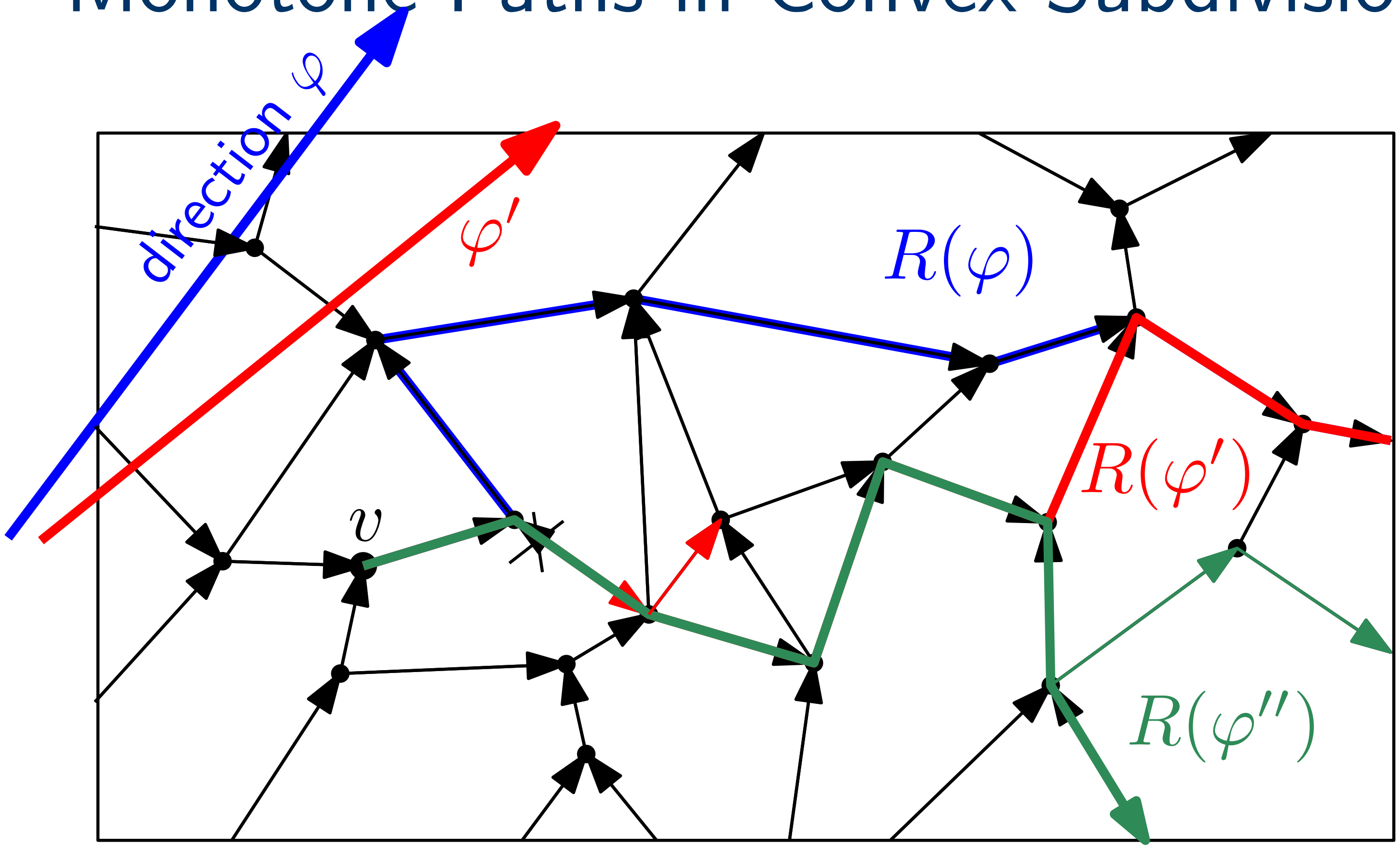


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- degree $\leq d \implies$ longest path $\geq \log_d n$. QED

# Degenerate Situations

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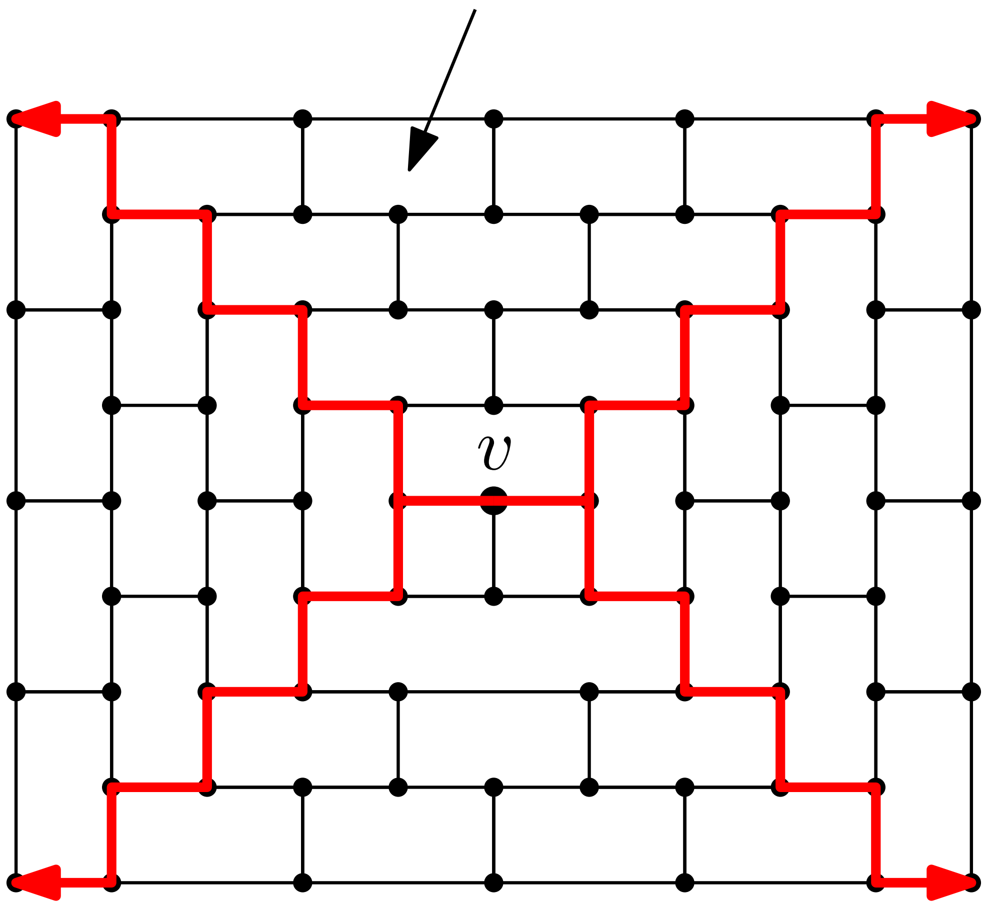
Faces are not strictly convex.

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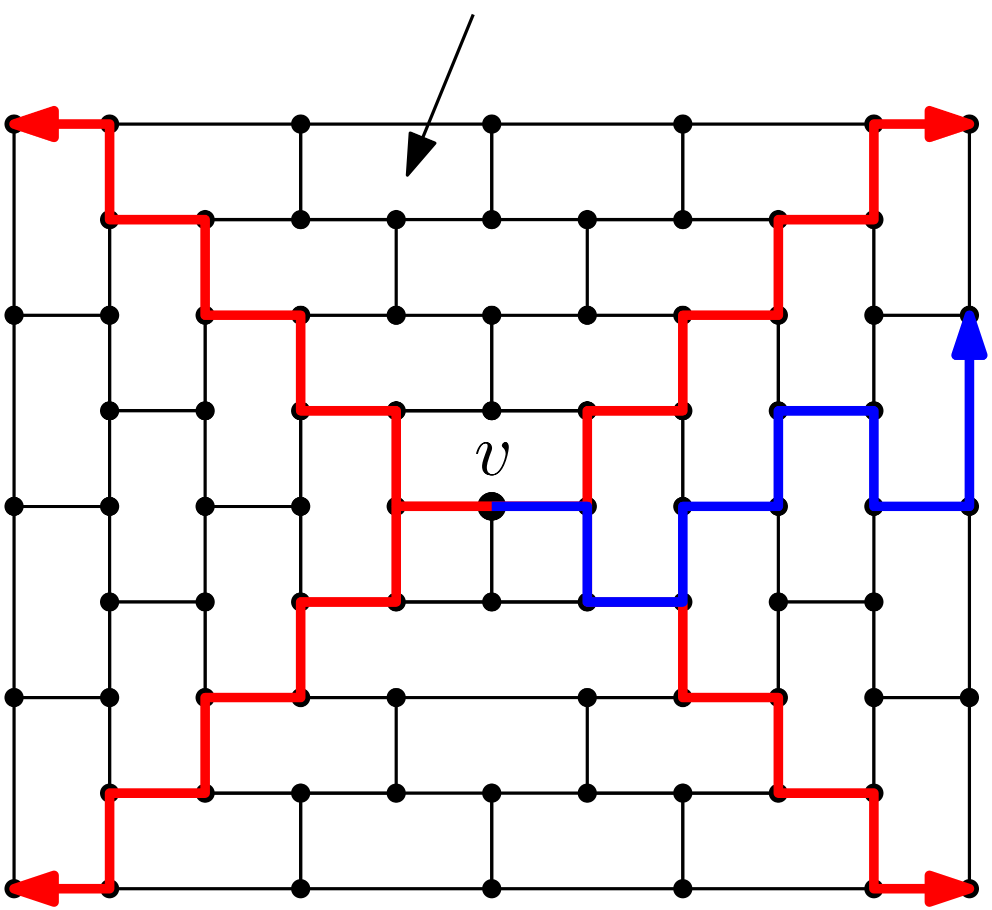


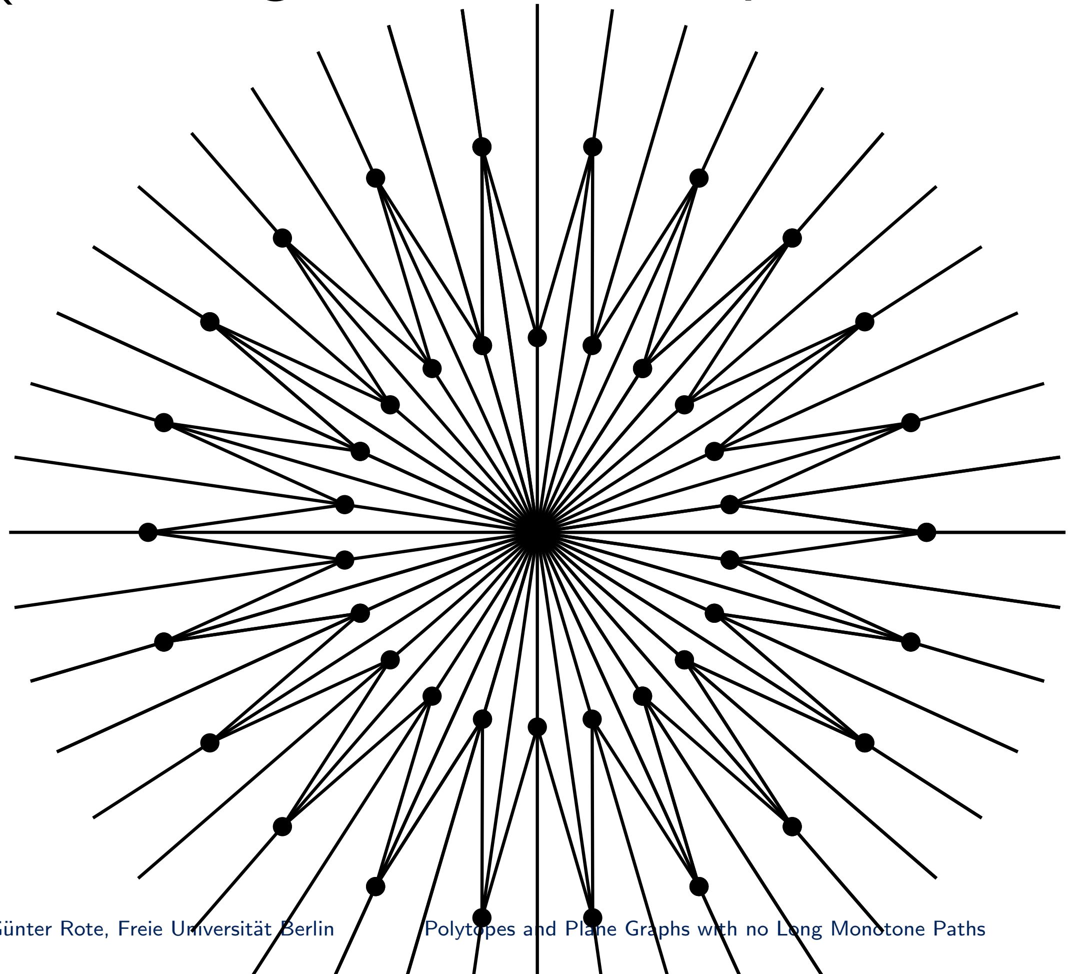

Not every vertex can be reached by a strictly monotone path.

Weakly monotone paths work.

# Tightness

THEOREM. Every convex subdivision of the plane with $n$ vertices and degree $\leq d$ contains a monotone path with $\geq \Omega(\log_d n)$ edges.

For $d \approx n$, this is tight, even for triangulations.
(The longest monotone path is bounded by a constant.)



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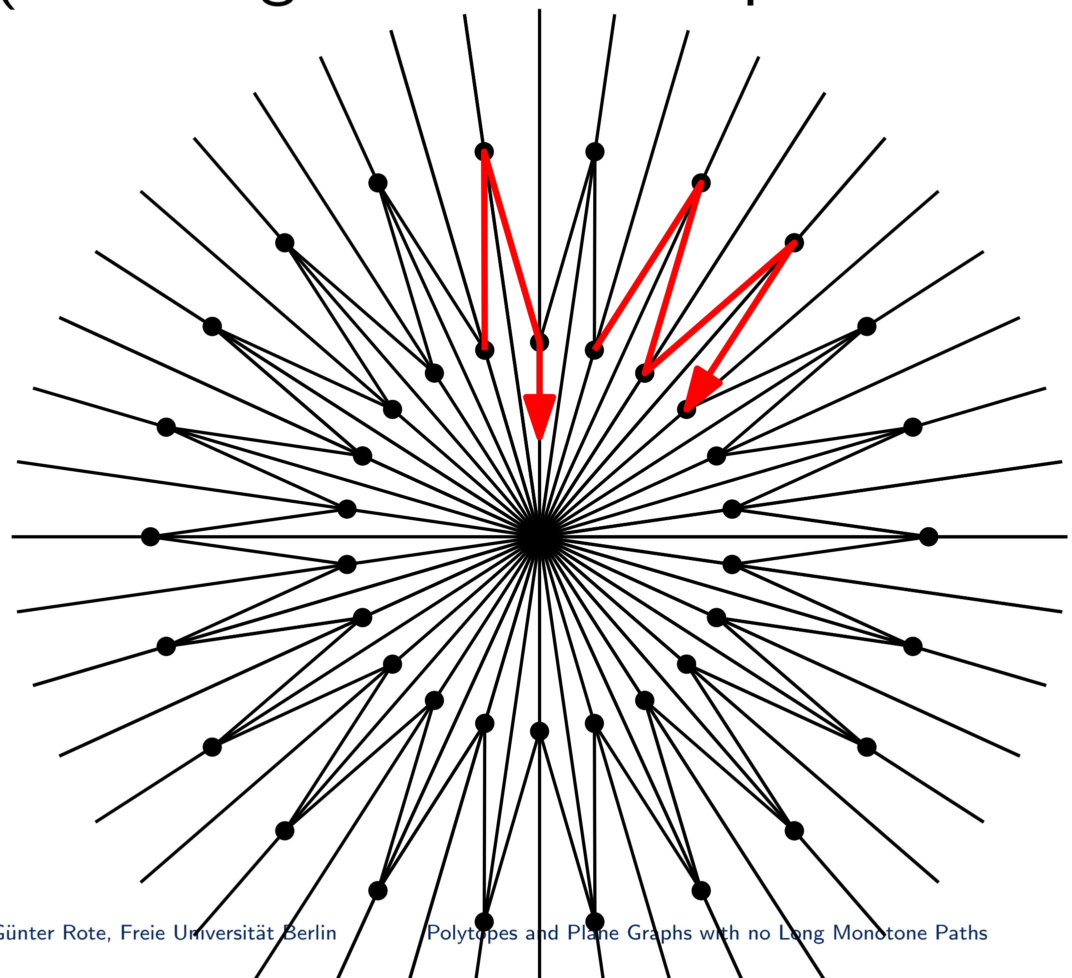

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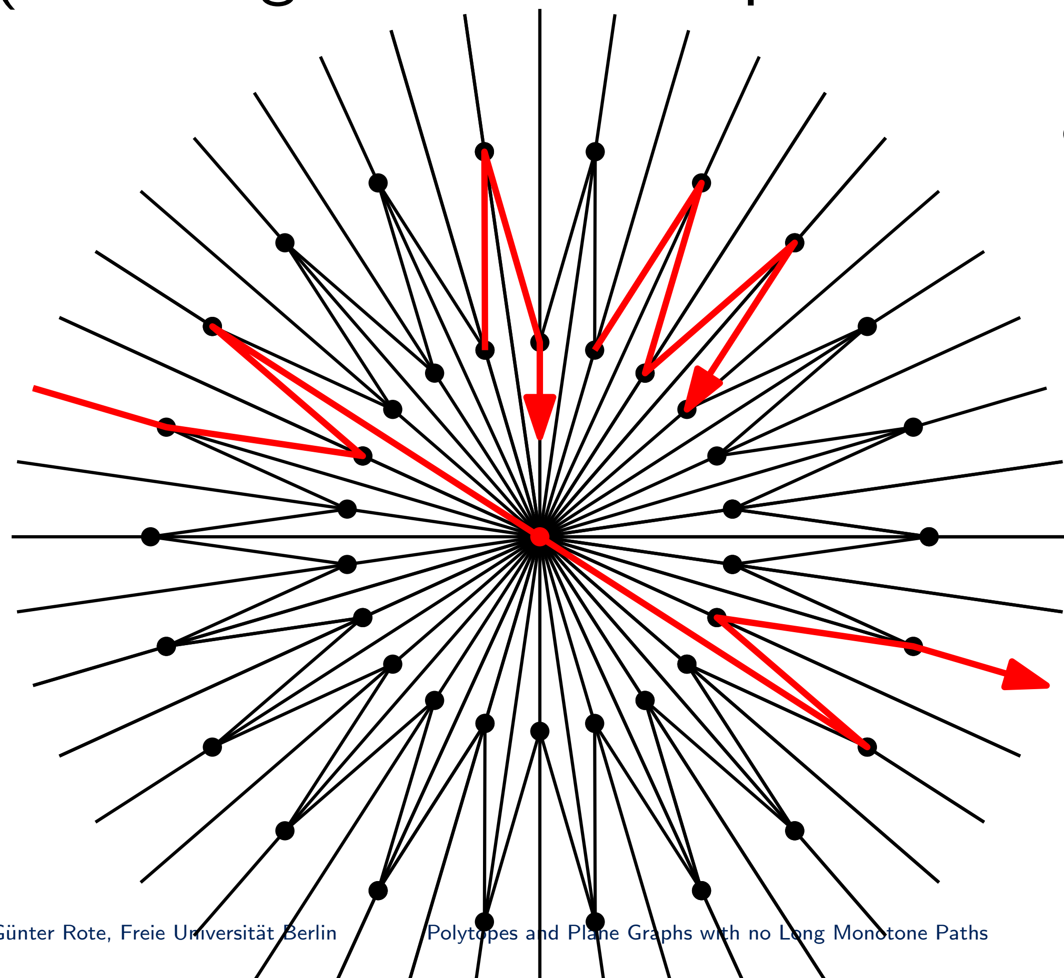

8 edges.

What happens if the number of unbounded edges is bounded by a constant (say, 3)?

# Few Unbounded Faces

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This bound is best possible.

# Few Unbounded Faces

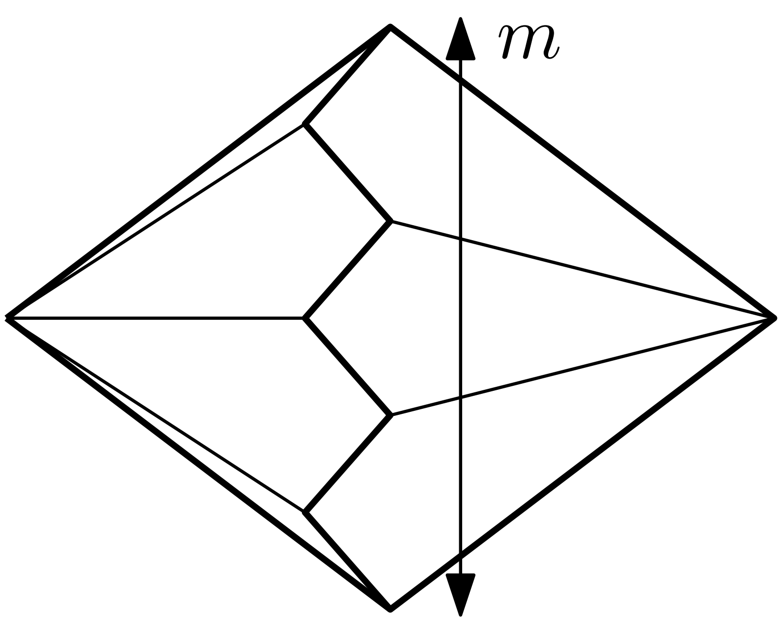


THEOREM. Let $G$ be a convex subdivision of the plane with $n$ vertices and $k$ unbounded faces. Then $G$ contains a path with $\geq \Omega(\log \frac{n}{k} / \log\log \frac{n}{k})$ edges that is monotone in *some* direction.

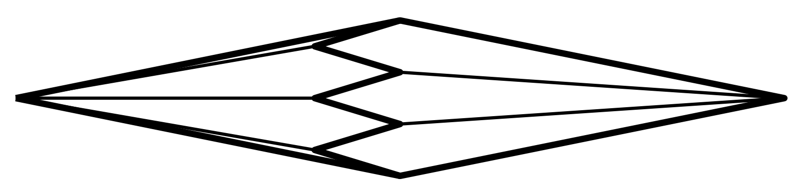
This bound is best possible.

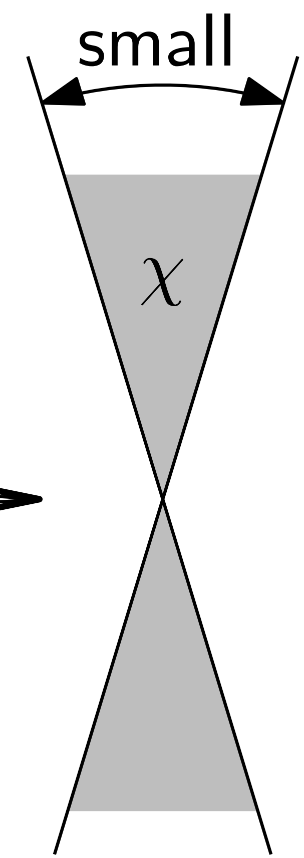
Upper-bound construction for $k$ constant.

$m := 2\log n/\log\log n$, $m^m > n$.




Characteristic region $\chi$:
can follow the zigzag





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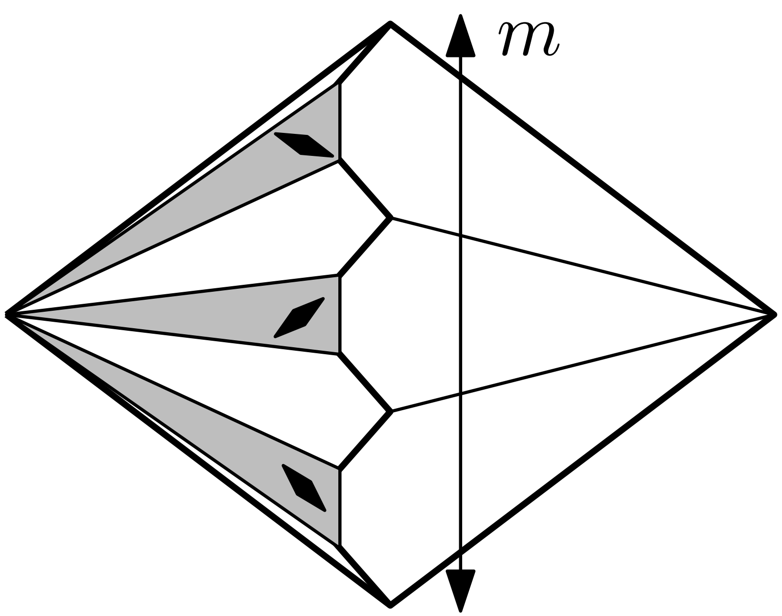


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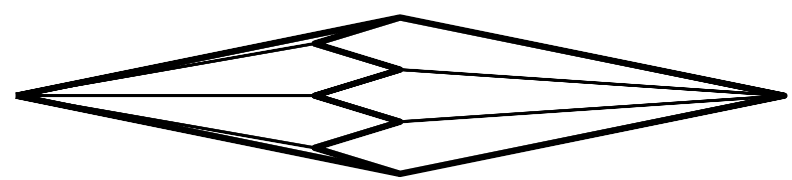
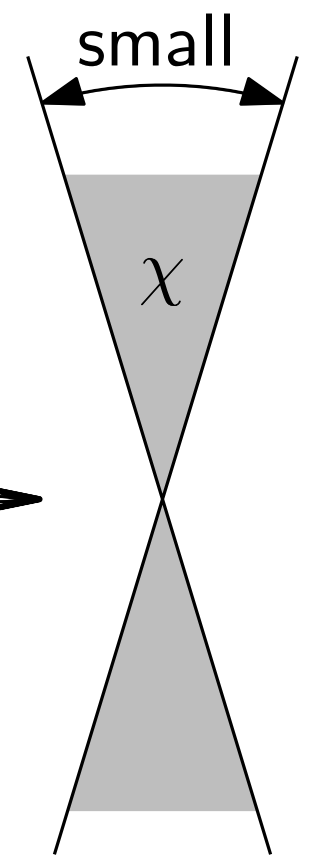


$m$ levels of fanout $m$.

Longest path $\leq m + m$

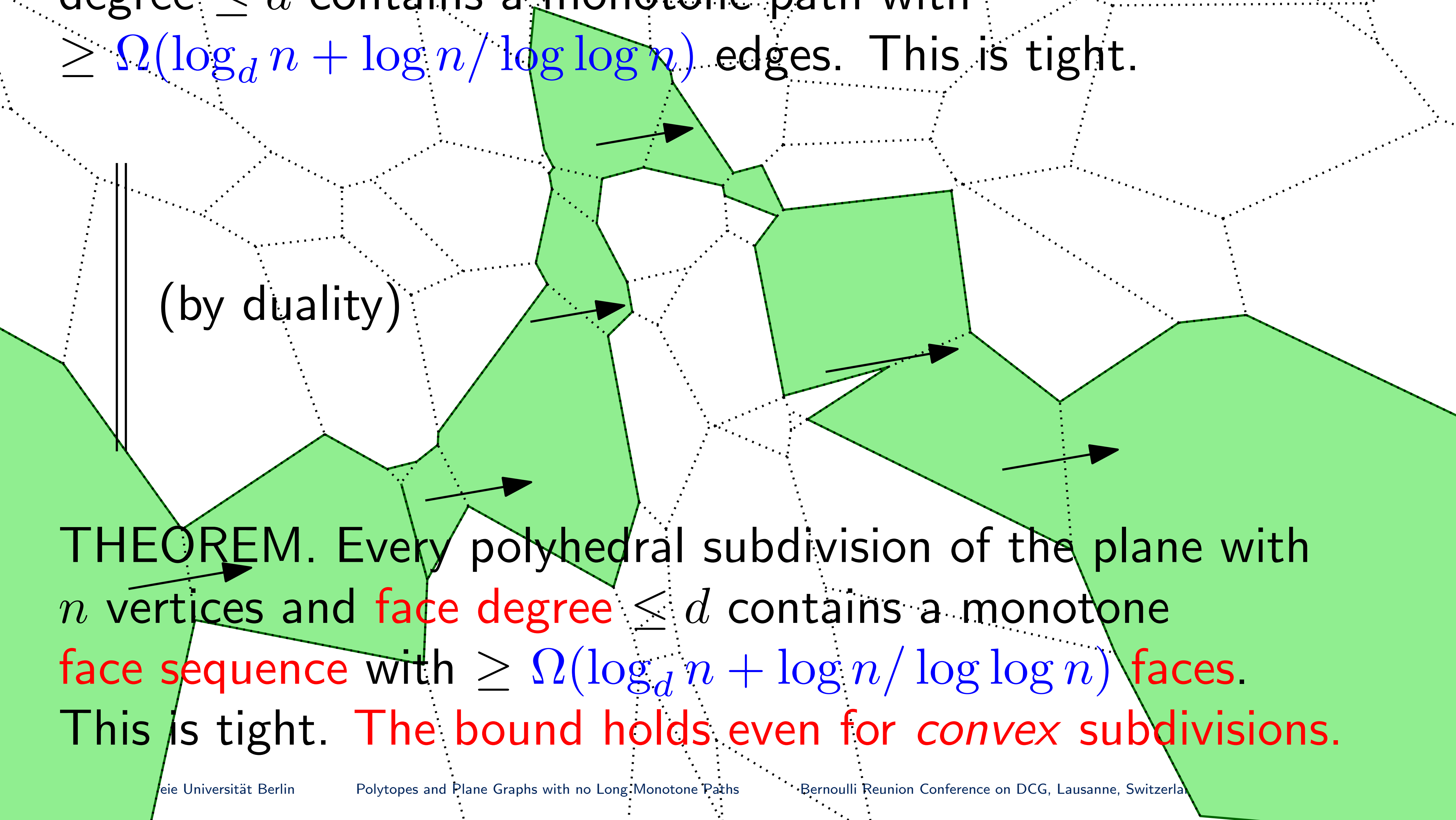


# Monotone Face Chains

Freie Universität Berlin

THEOREM (Chazelle, Edelsbrunner, Guibas 1989).
Every *polyhedral* subdivision of the plane with $n$ vertices and degree $\leq d$ contains a monotone path with $\geq \Omega(\log_d n + \log n / \log \log n)$ edges. This is tight.

$\Vert$ (by duality)



THEOREM. Every polyhedral subdivision of the plane with $n$ vertices and face degree $\leq d$ contains a monotone face sequence with $\geq \Omega(\log_d n + \log n / \log \log n)$ faces.
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