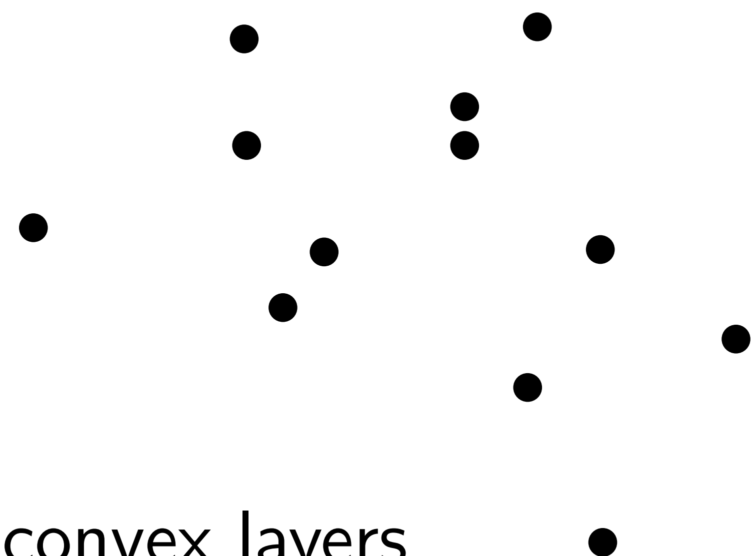
Grid Peeling and the Affine Curve-Shortening Flow (ACSF)

Günter Rote, Moritz Rüber, and Morteza Saghafian
Freie Universität Berlin / ISTA



convex layers
onion layers

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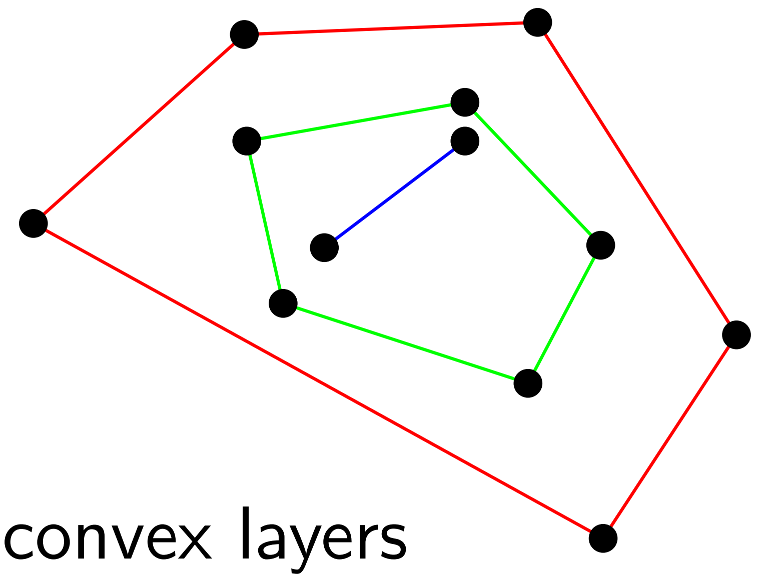

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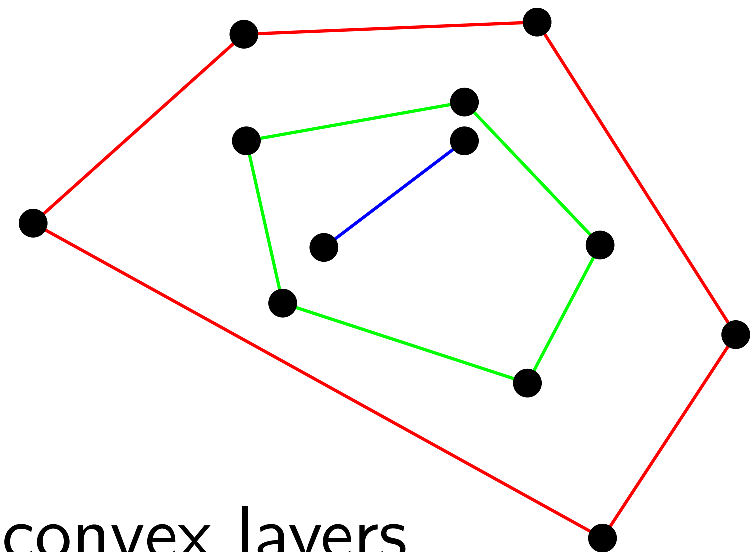

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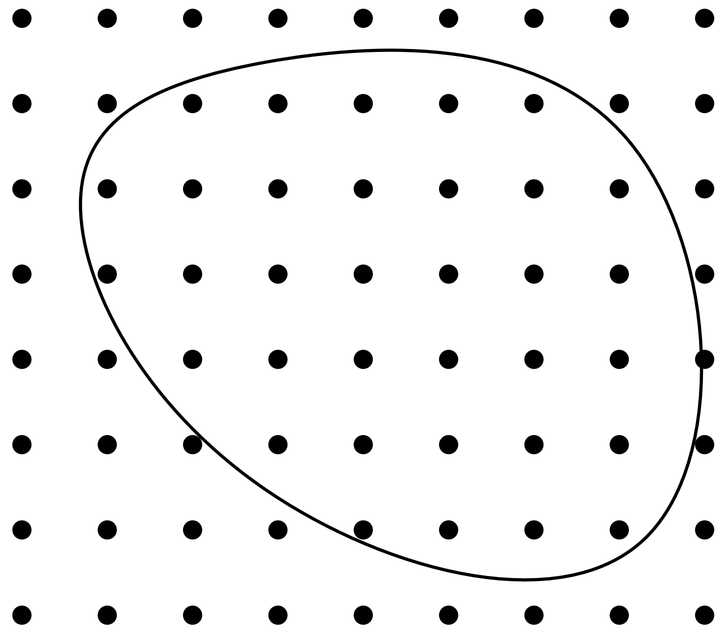

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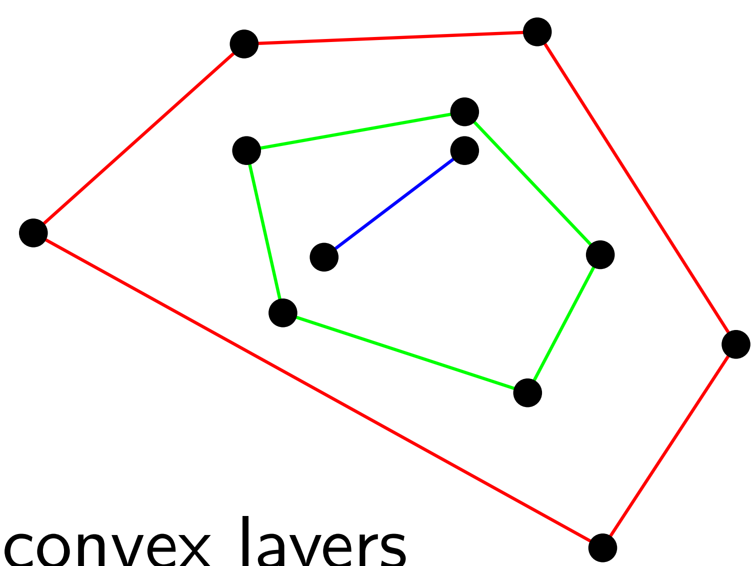

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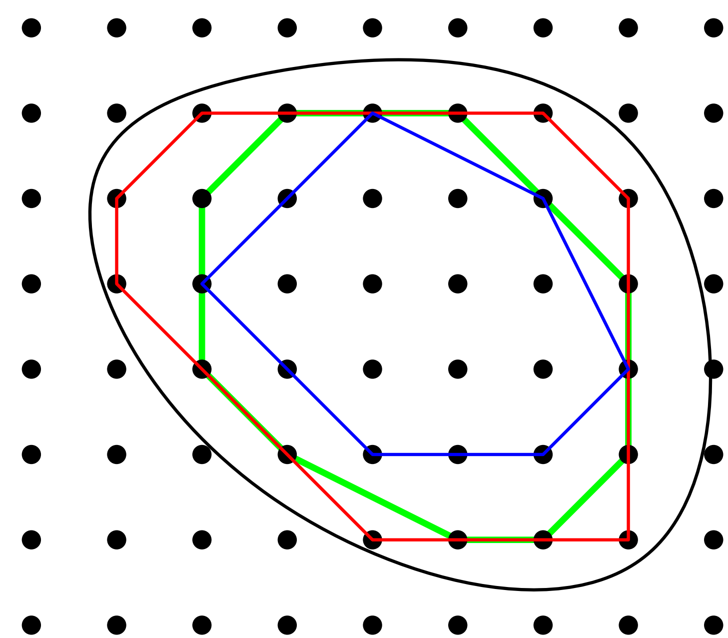
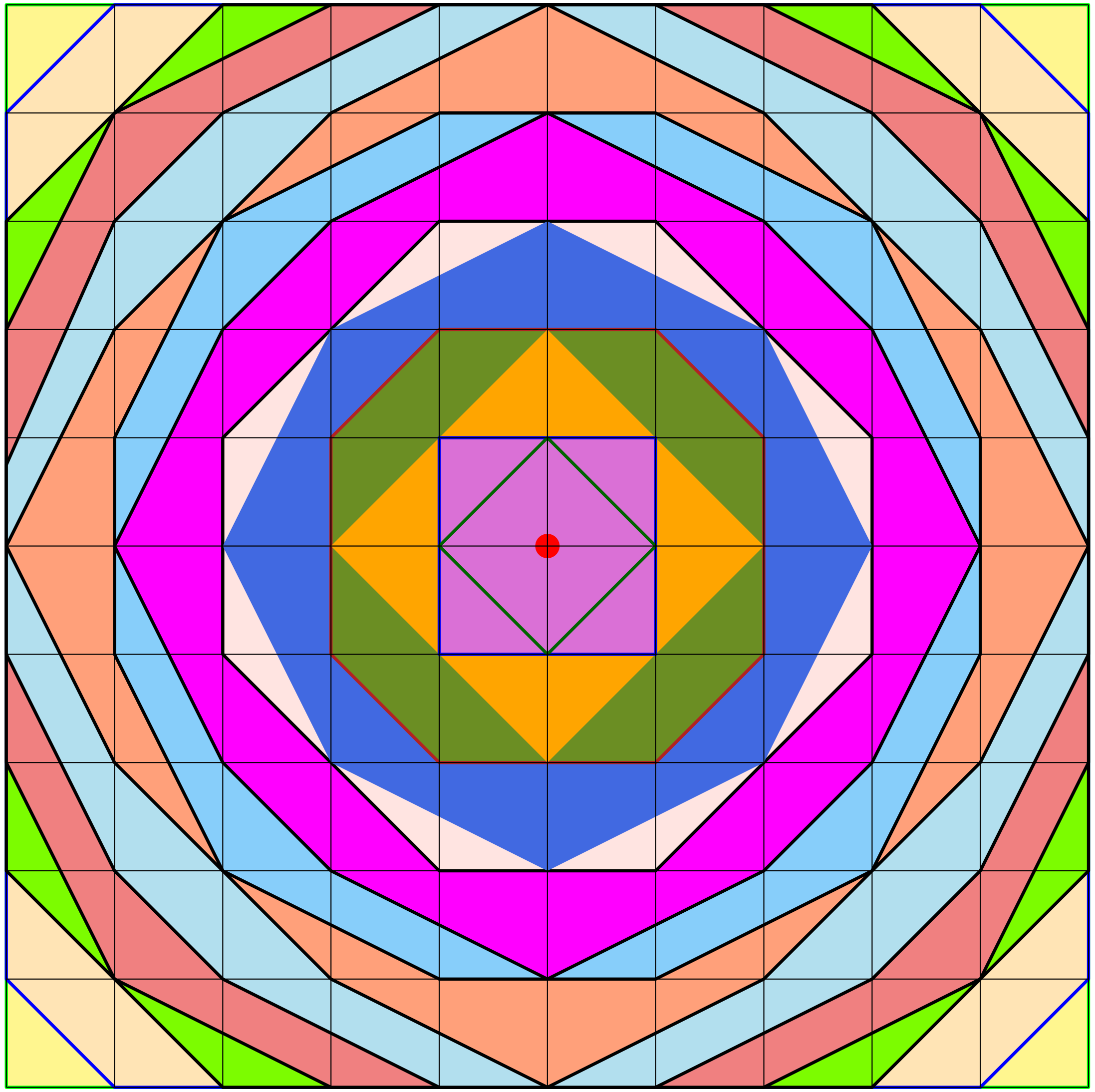


grid peeling

Grid Peeling of the Square

Freie Universität Berlin

[Sariel Har-Peled and Bernard Lidický 2013]



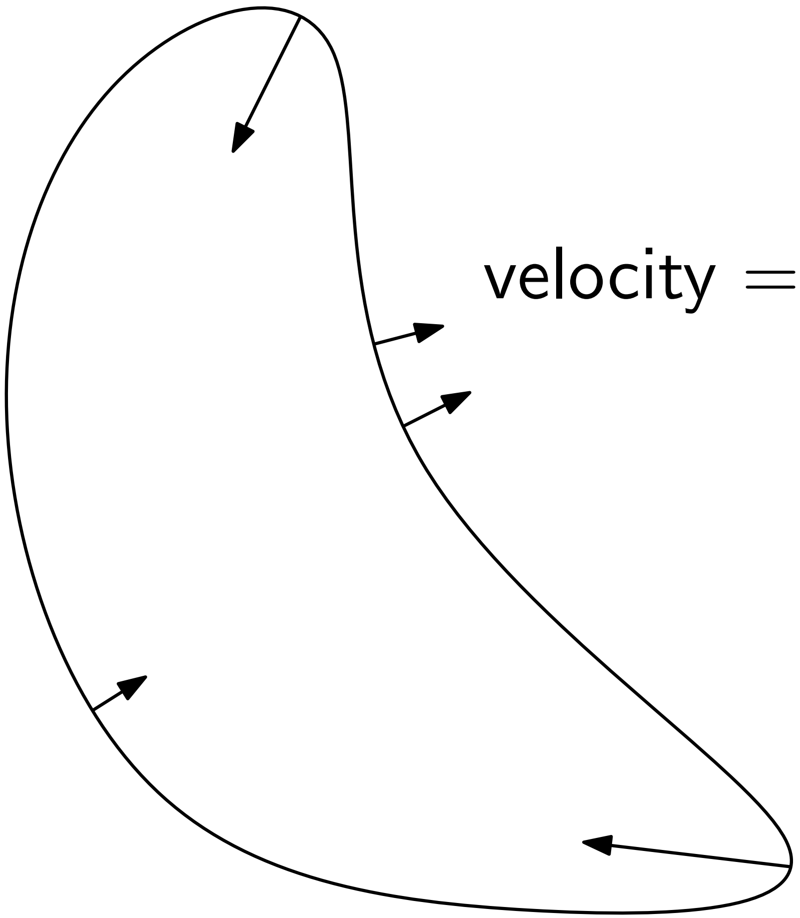
The $n \times n$ grid has $\Theta(n^{4/3})$ convex layers.

Affine Curve-Shortening Flow (ACSF)

Freie Universität Berlin

[L. Alvarez, F. Guichard, P.-L. Lions, J.-M. Morel:
"Axioms and fundamental equations of image processing" 1993]

[G. Sapiro and A. Tannenbaum:
"Affine invariant scale-space." Int. J. Computer Vision 1993]



velocity $= \kappa^{1/3}$ ($\kappa =$ curvature)

invariant under area-preserving affine transformations!

Peeling and the ACSF

Freie Universität Berlin

Conjecture:

David Eppstein, Sariel Har-Peled, and Gabriel Nivasch. Grid peeling and the affine curve shortening flow. Experimental Mathematics **29** (2020), 306–316

As the grid is more and more refined, grid peeling approaches the ACSF.

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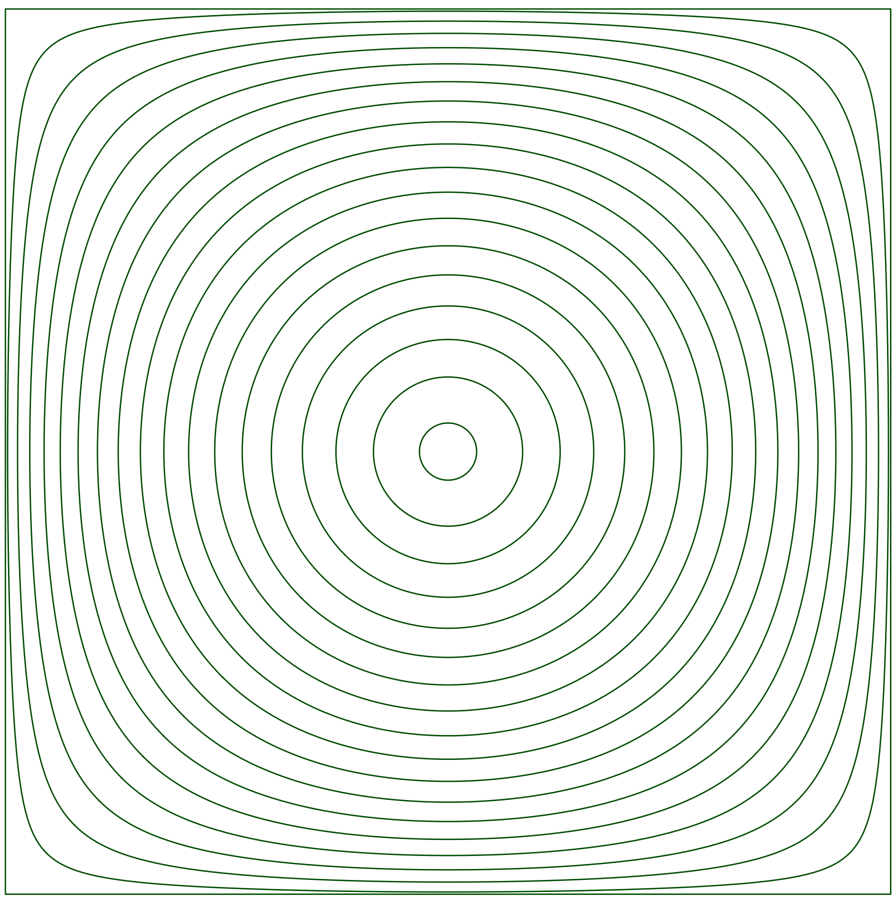

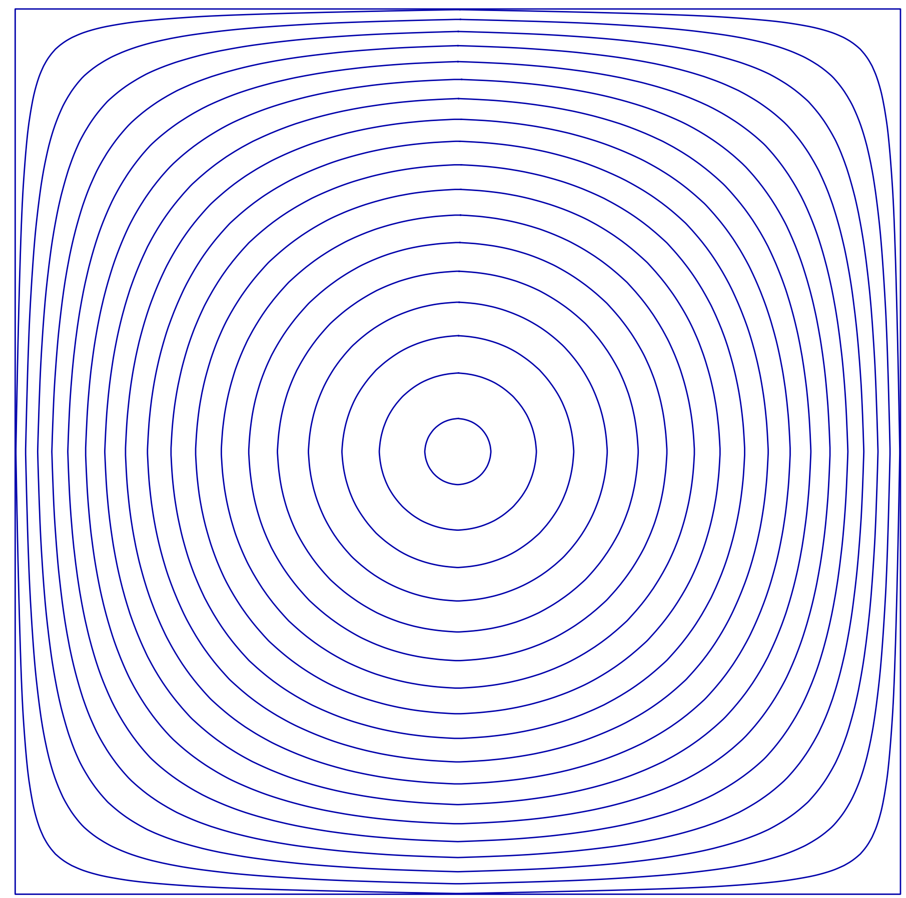

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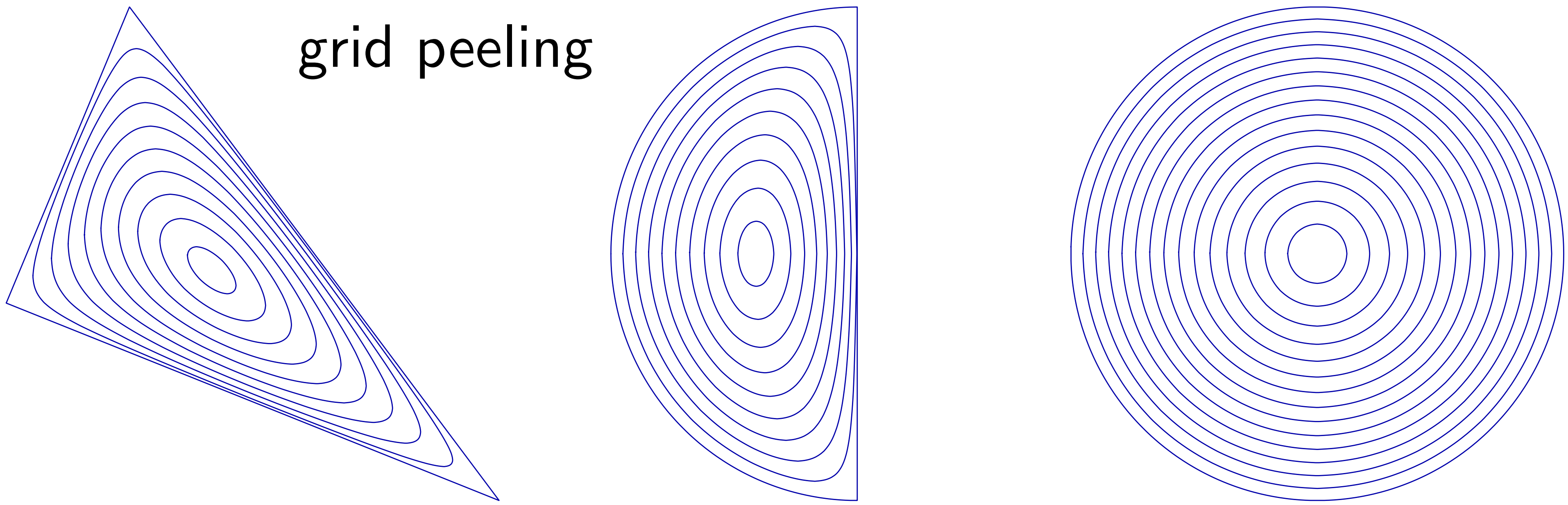

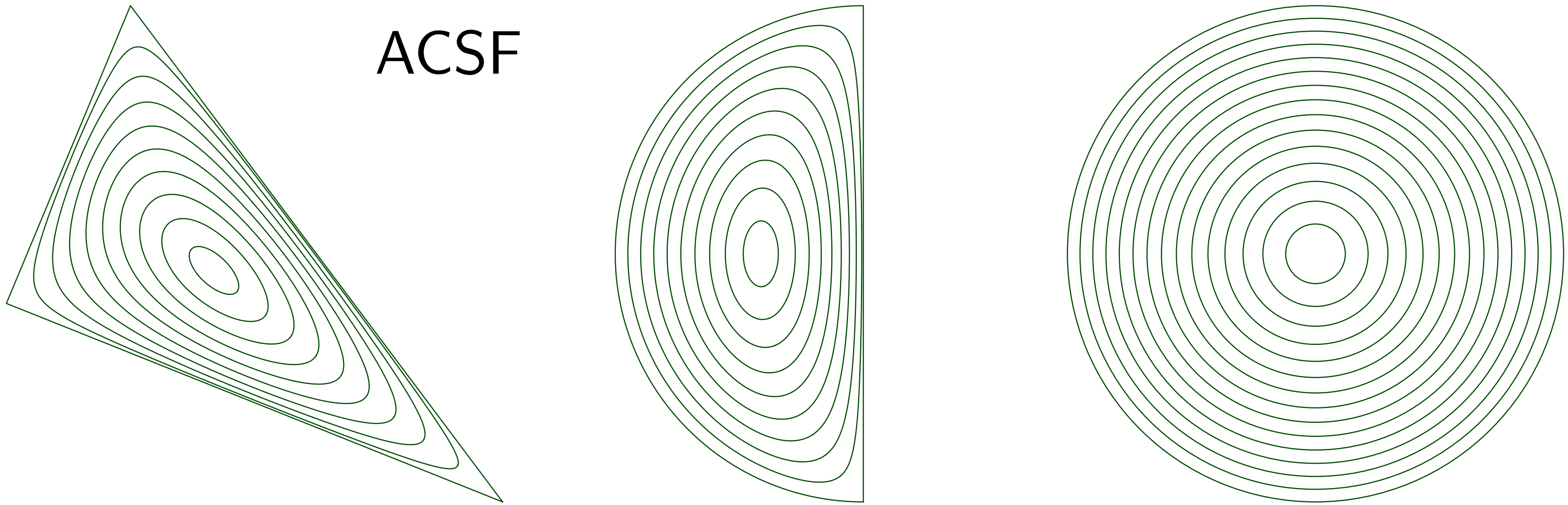

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ACSF at time $T \approx$ Grid peeling on $\frac{1}{n}$-grid after $C_g T n^{4/3}$ steps.

The value of the constant: (Moritz Rüber and Günter Rote)

$$C_g = \sqrt[3]{\frac{\pi^2}{2\zeta(3)}} \approx 1.60120980542577$$

Peeling and the ACSF

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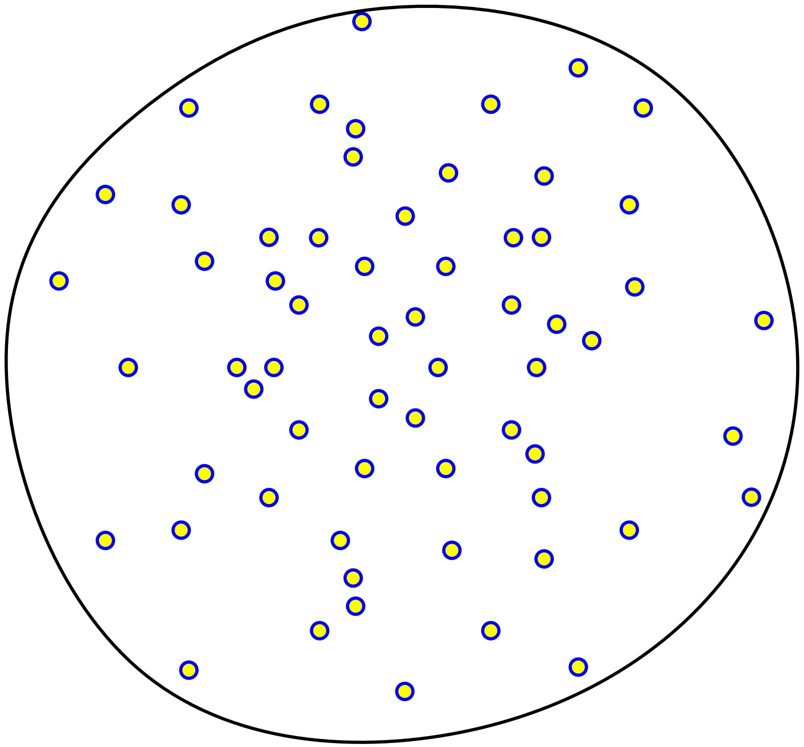
→ Jeff Calder and Charles K Smart. The limit shape of convex hull peeling. Duke Math. J. (2020)

random points



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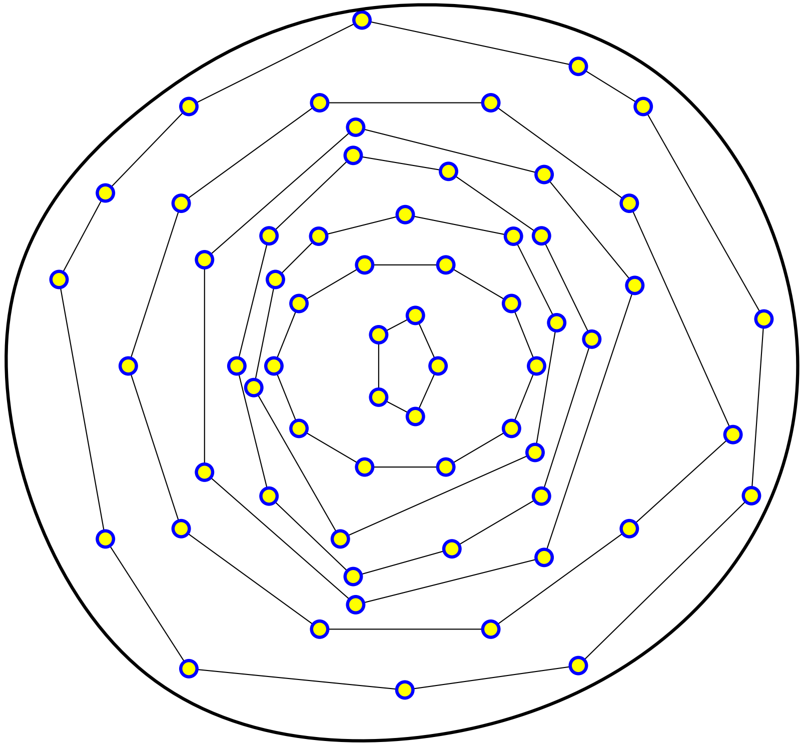

Peeling and the ACSF

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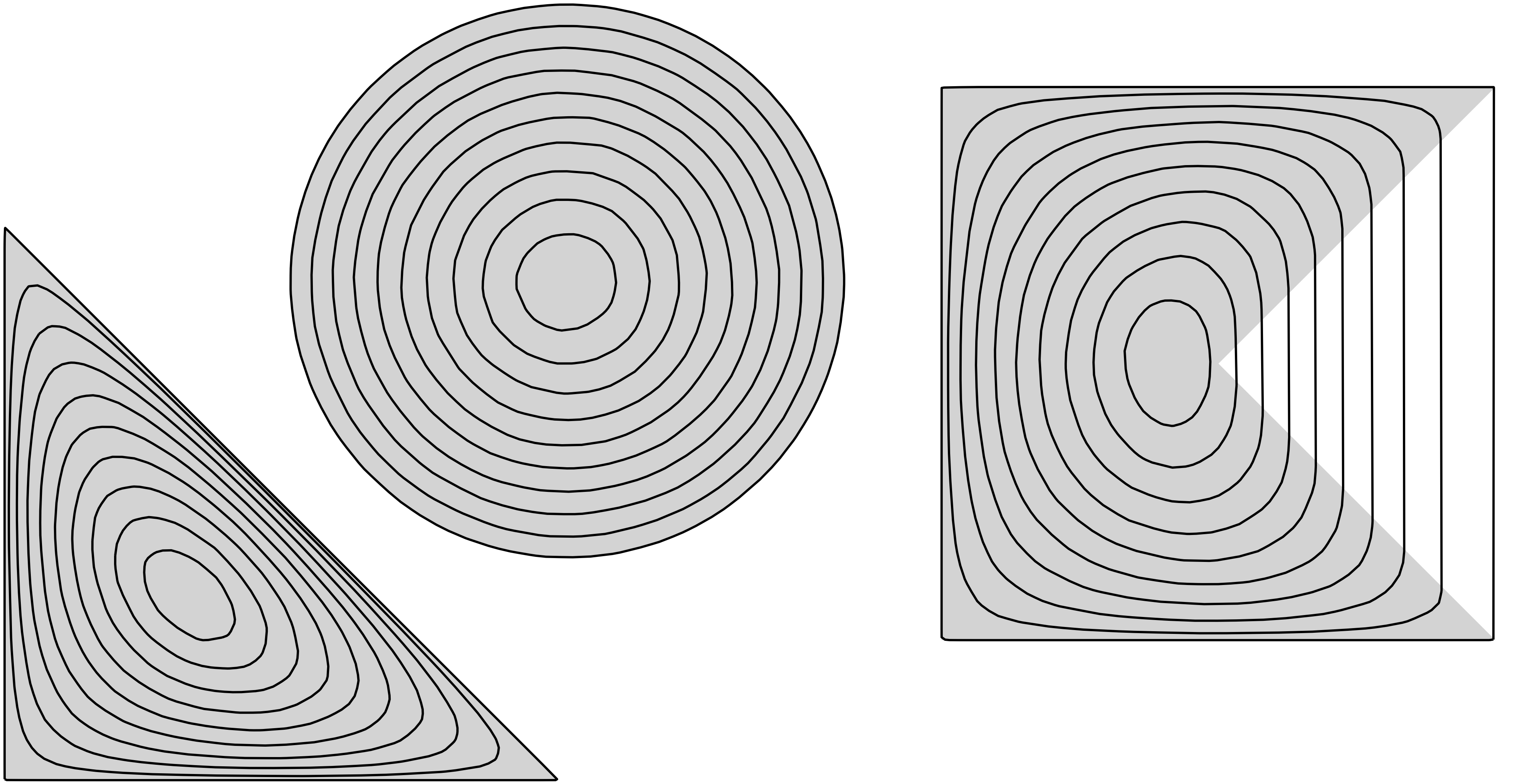
Jeff Calder and Charles K. Smart. The limit shape of convex hull peeling. 2020

10000 random points in the shaded region



Peeling and the ACSF

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Jeff Calder and Charles K Smart. The limit shape of convex hull peeling. Duke Math. J. **169** (2020)

Theorem:

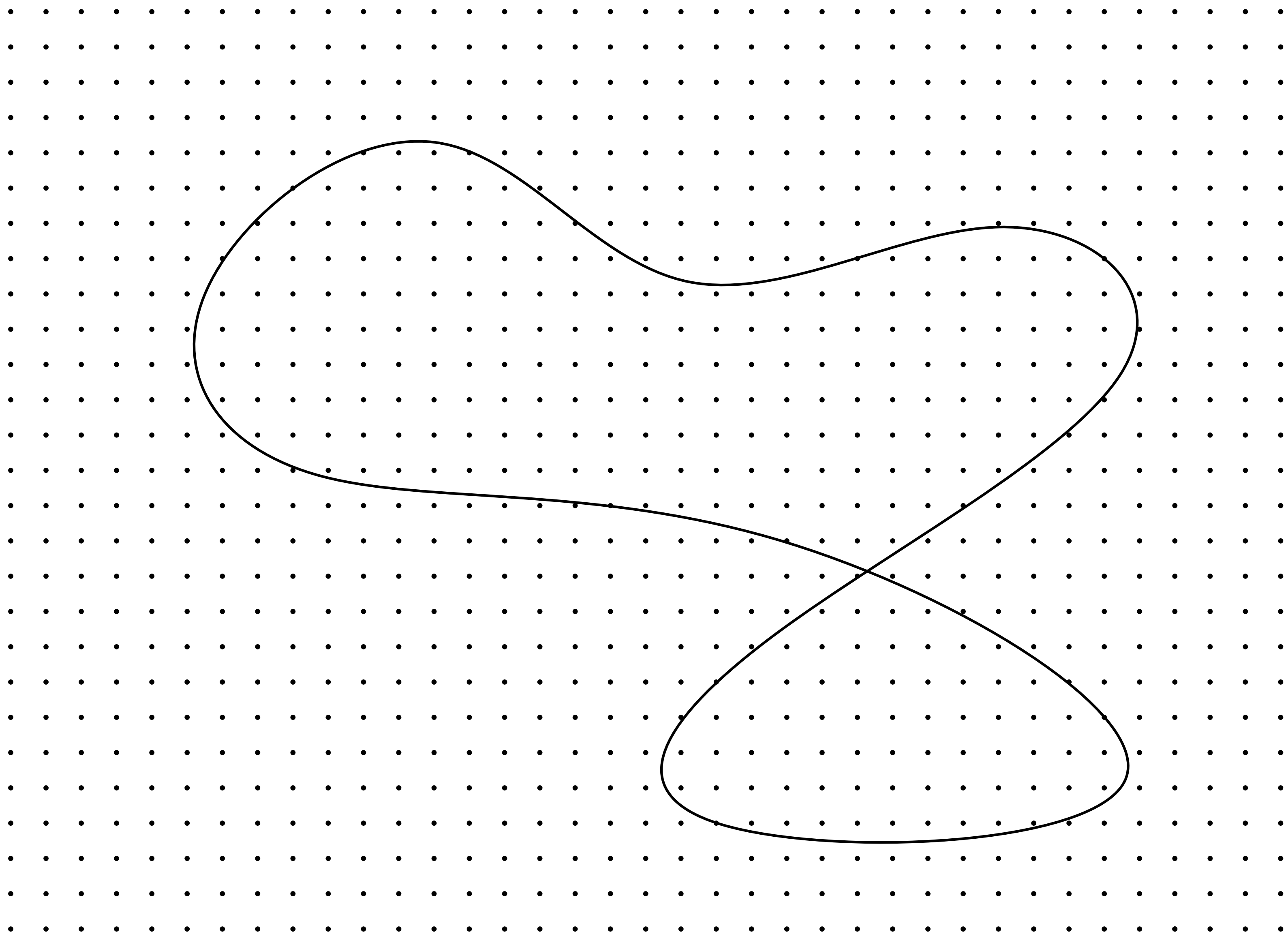
ACSF at time $T \approx$ Peeling on density-n^2 set after $C_r T n^{4/3}$ steps.

$$C_g \approx 1.6, \;\; C_r \approx 1.3$$

- Invariant under affine transformations?

Homotopic peeling

[Sergey Avvakumov and Gabriel Nivasch 2019]



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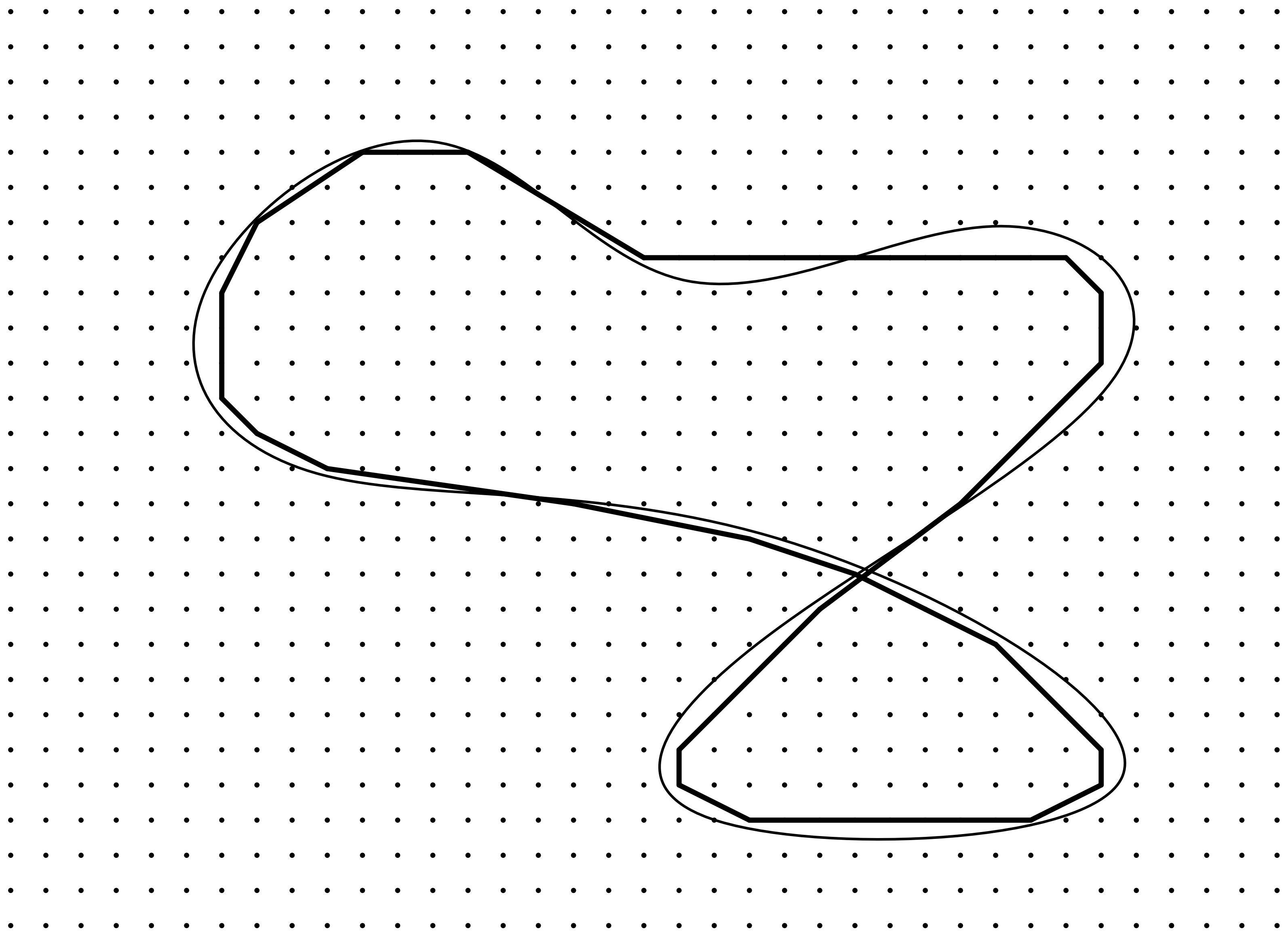

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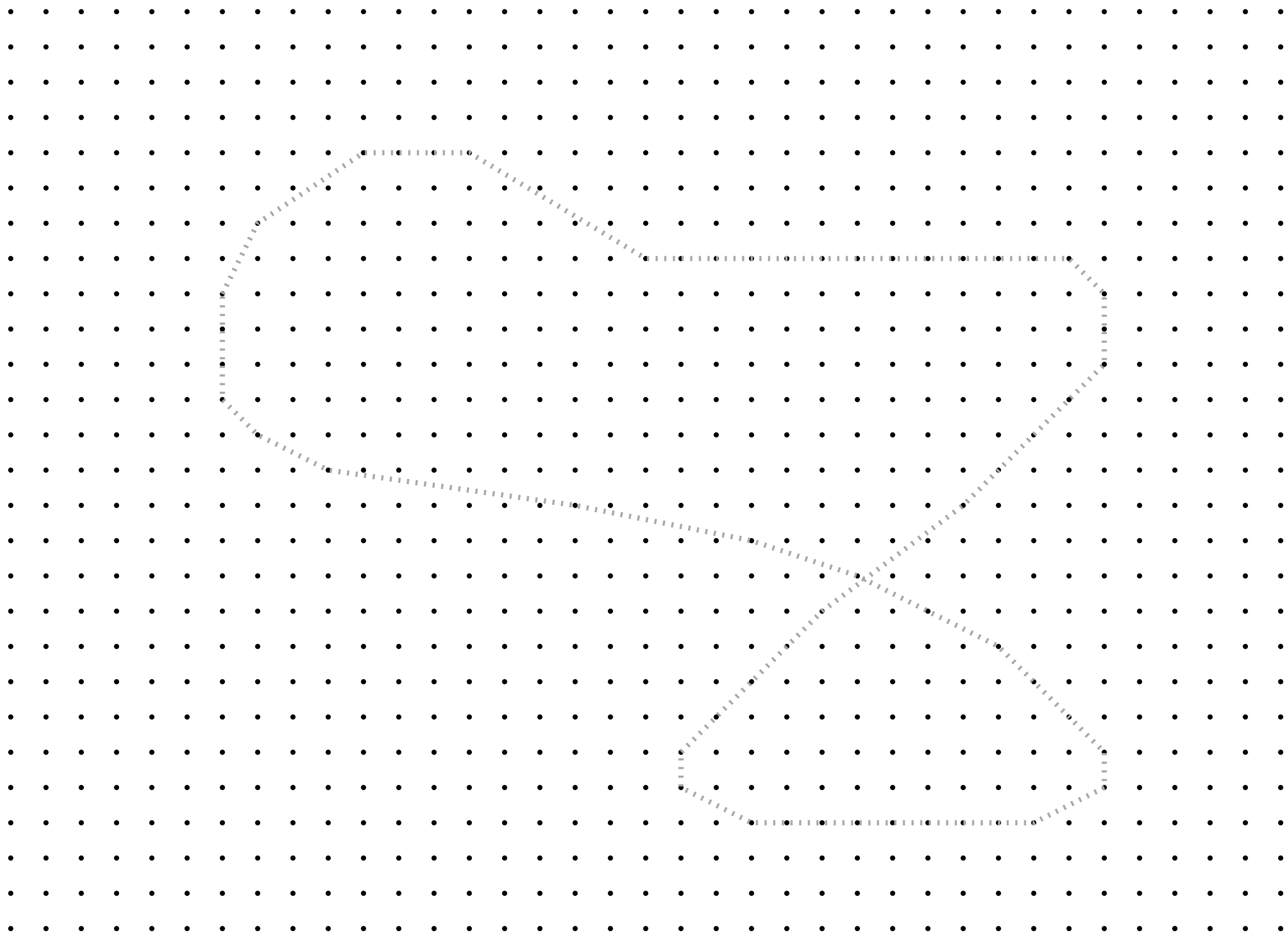

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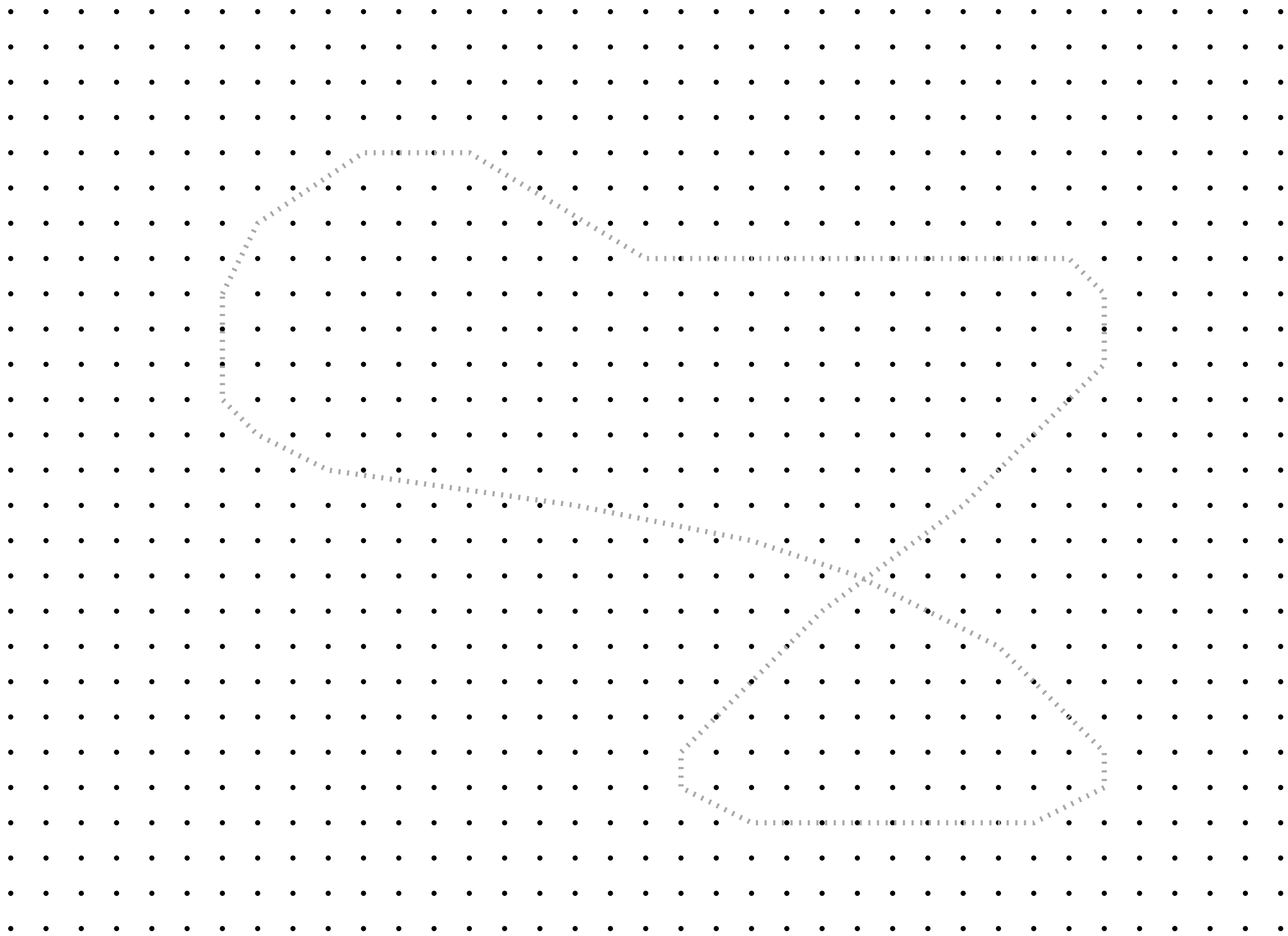

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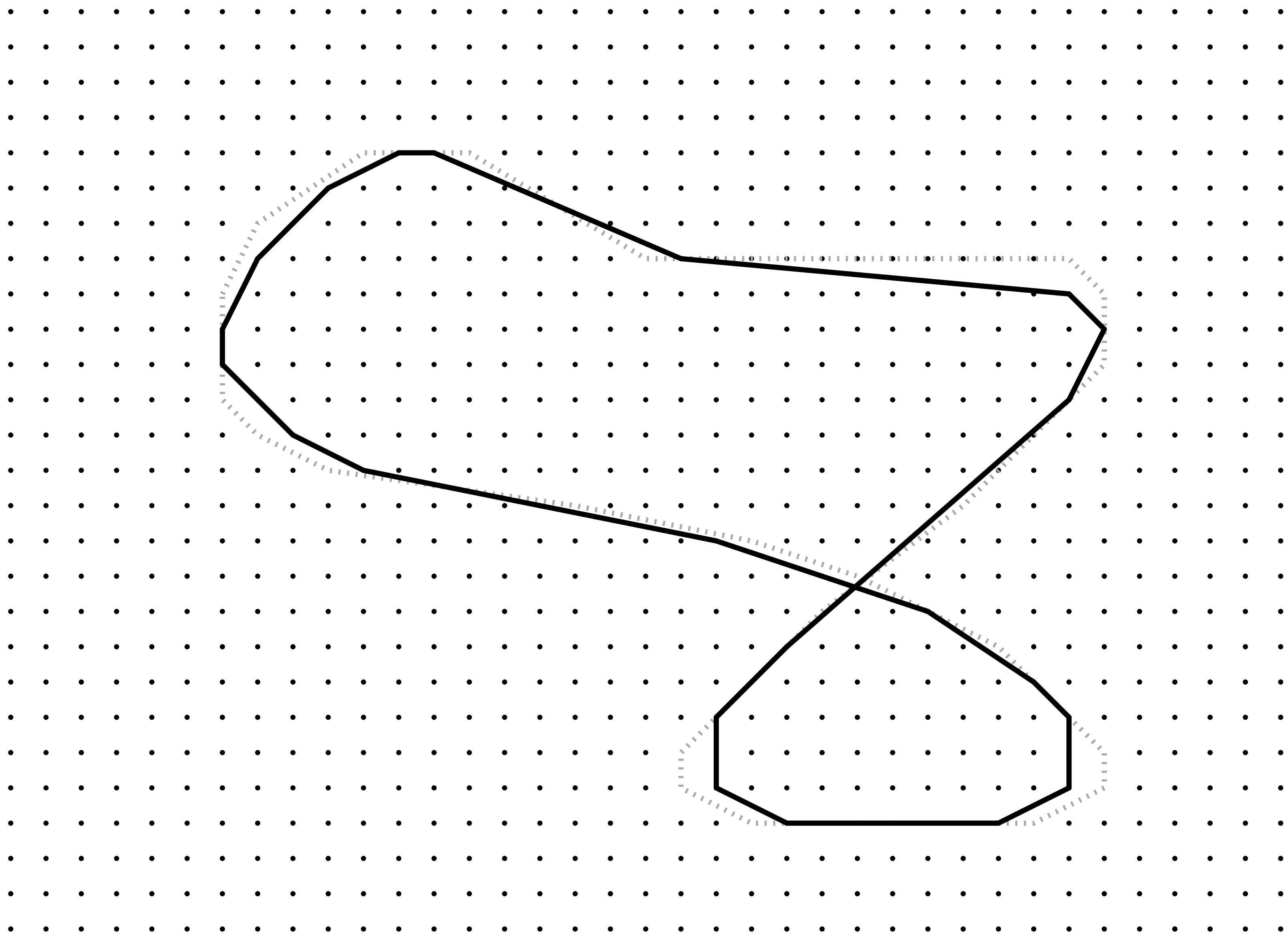

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Freie Universität Berlin

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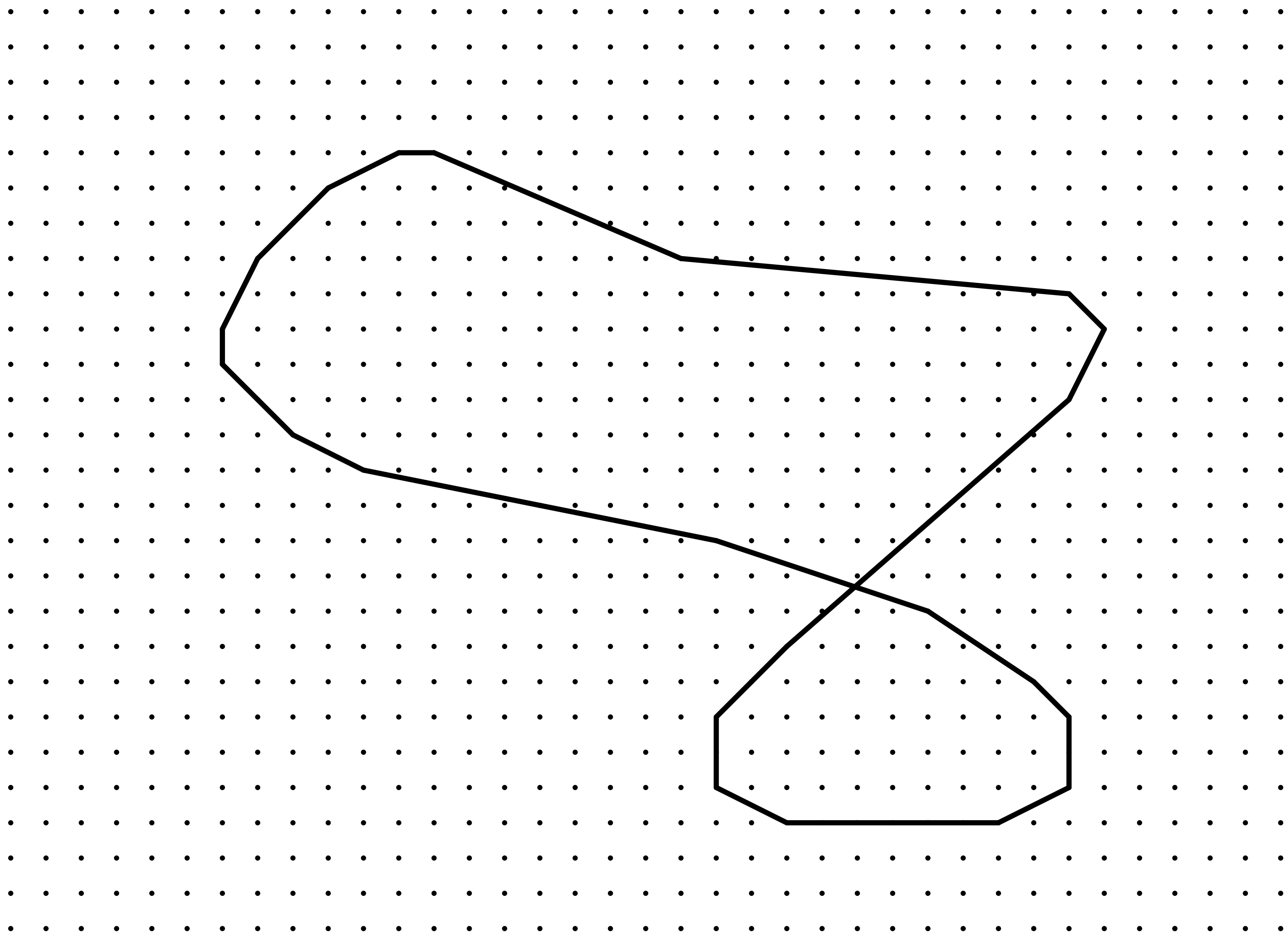

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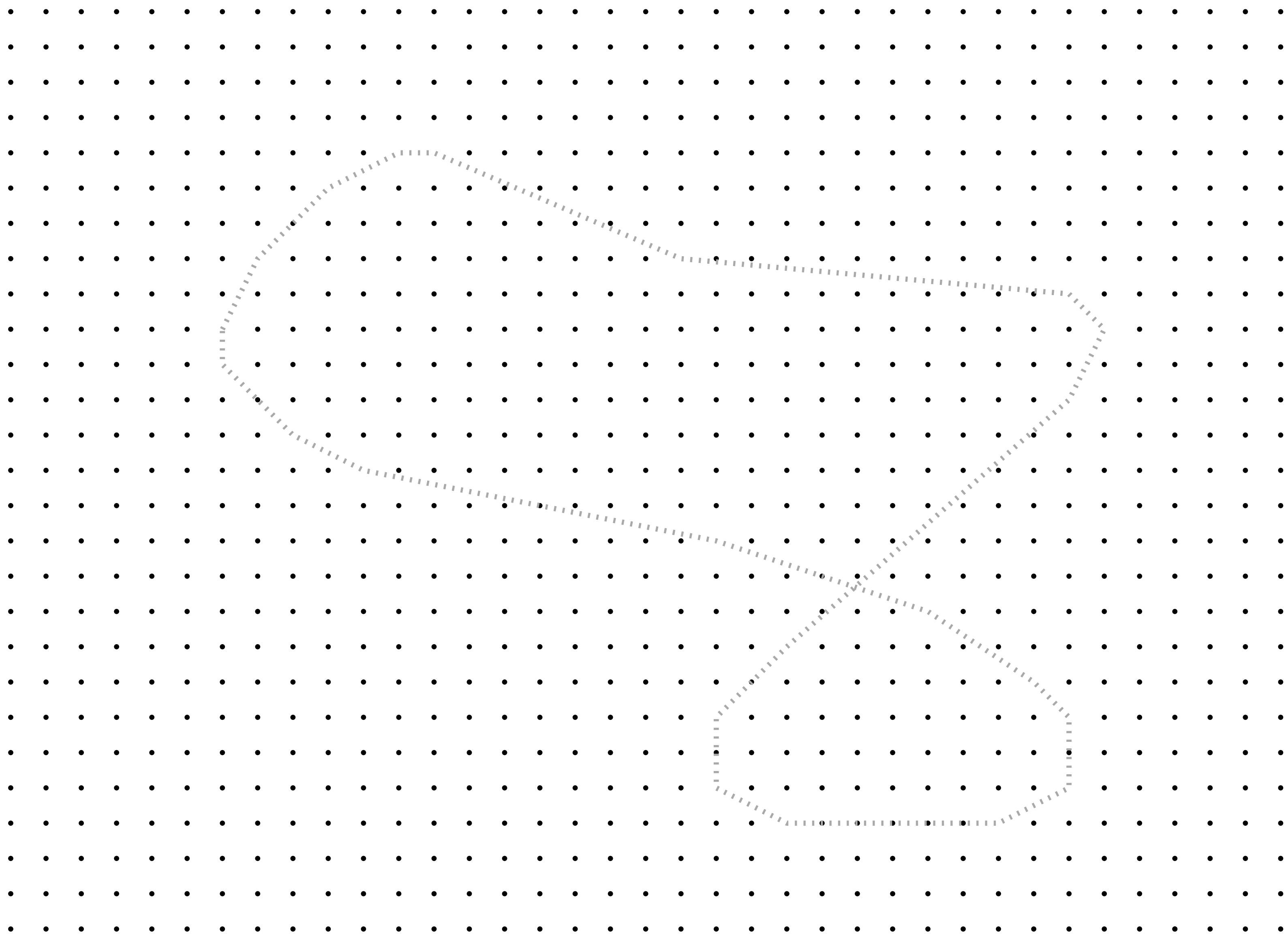

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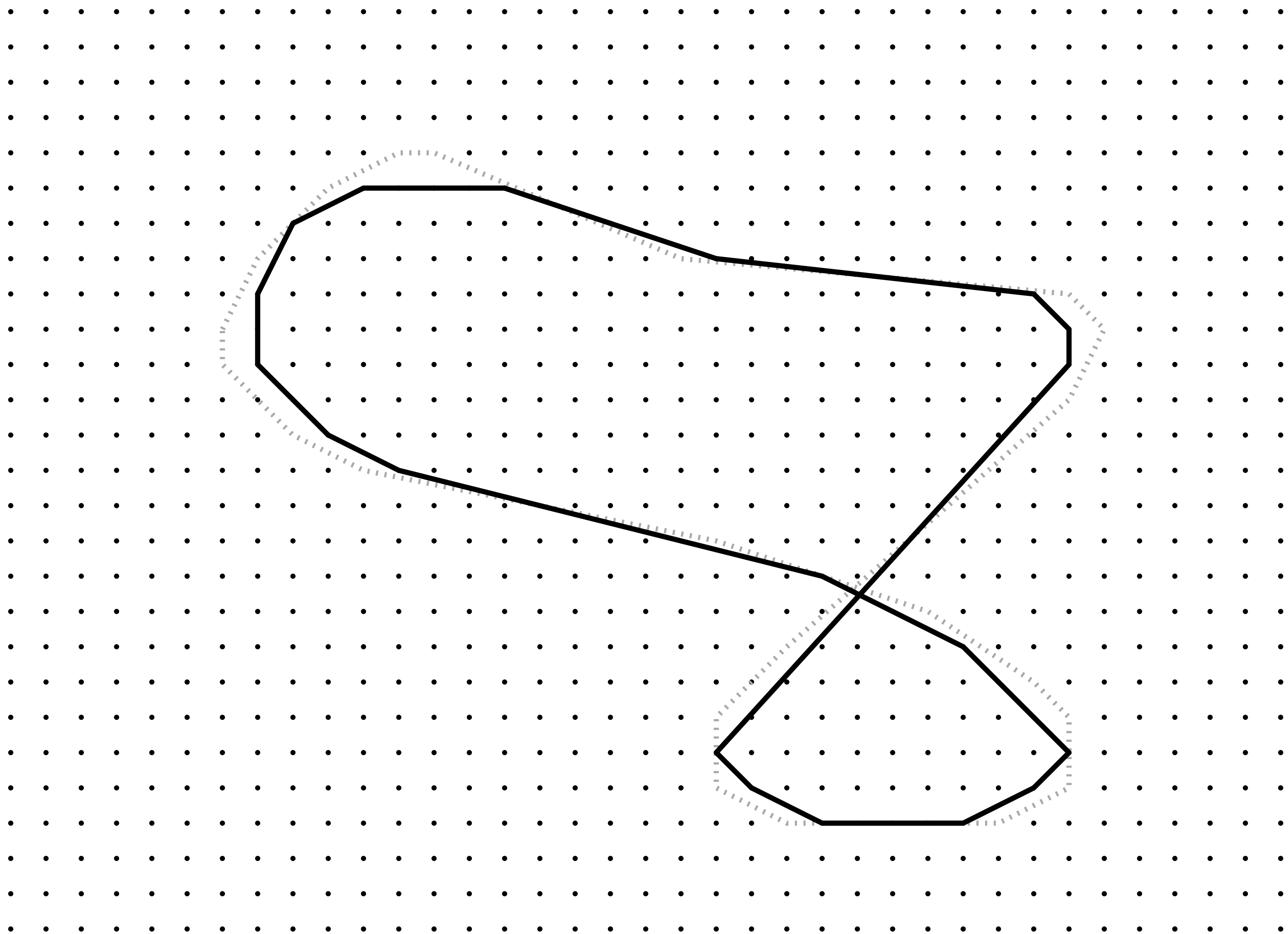

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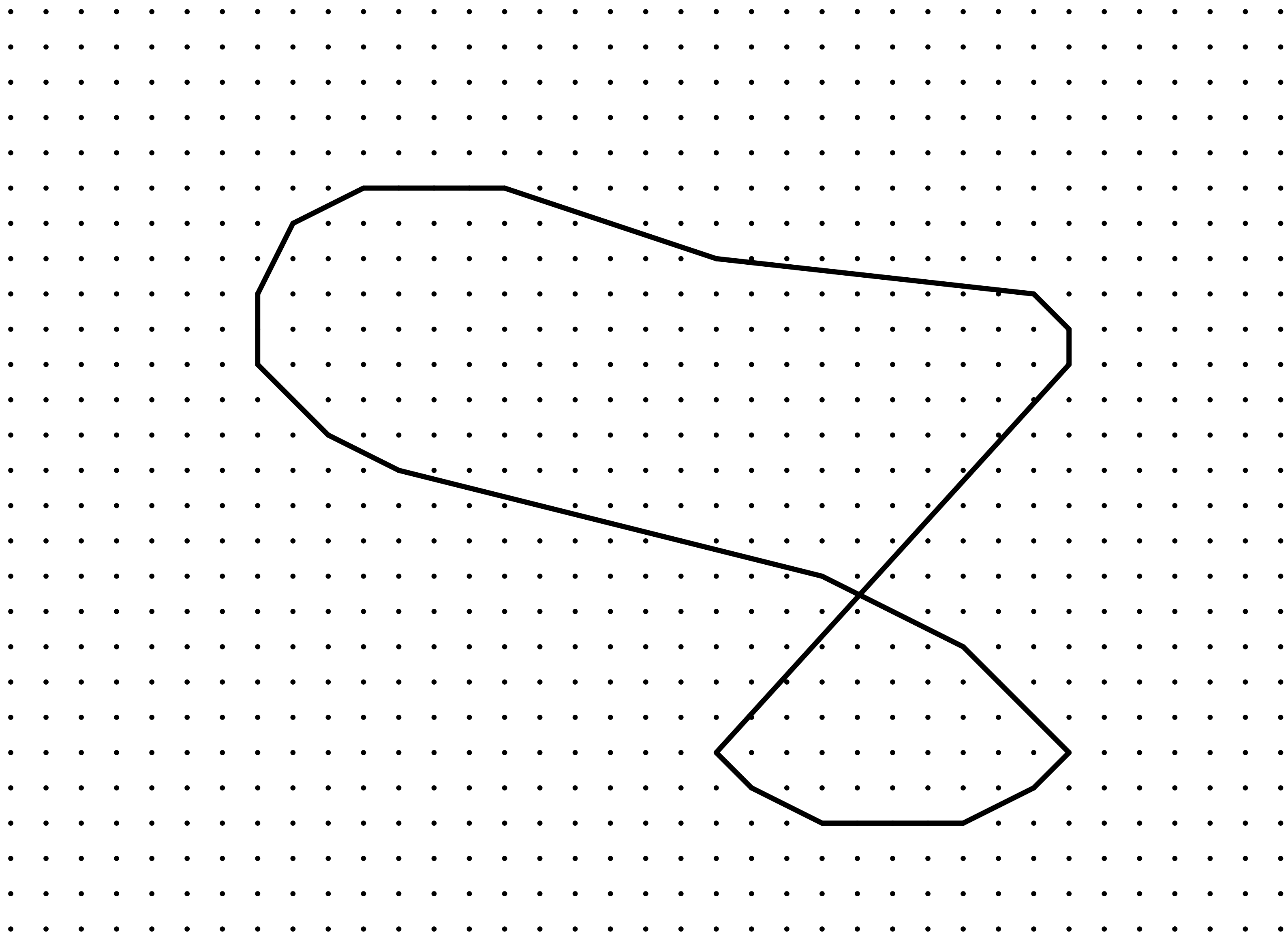

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Freie Universität Berlin

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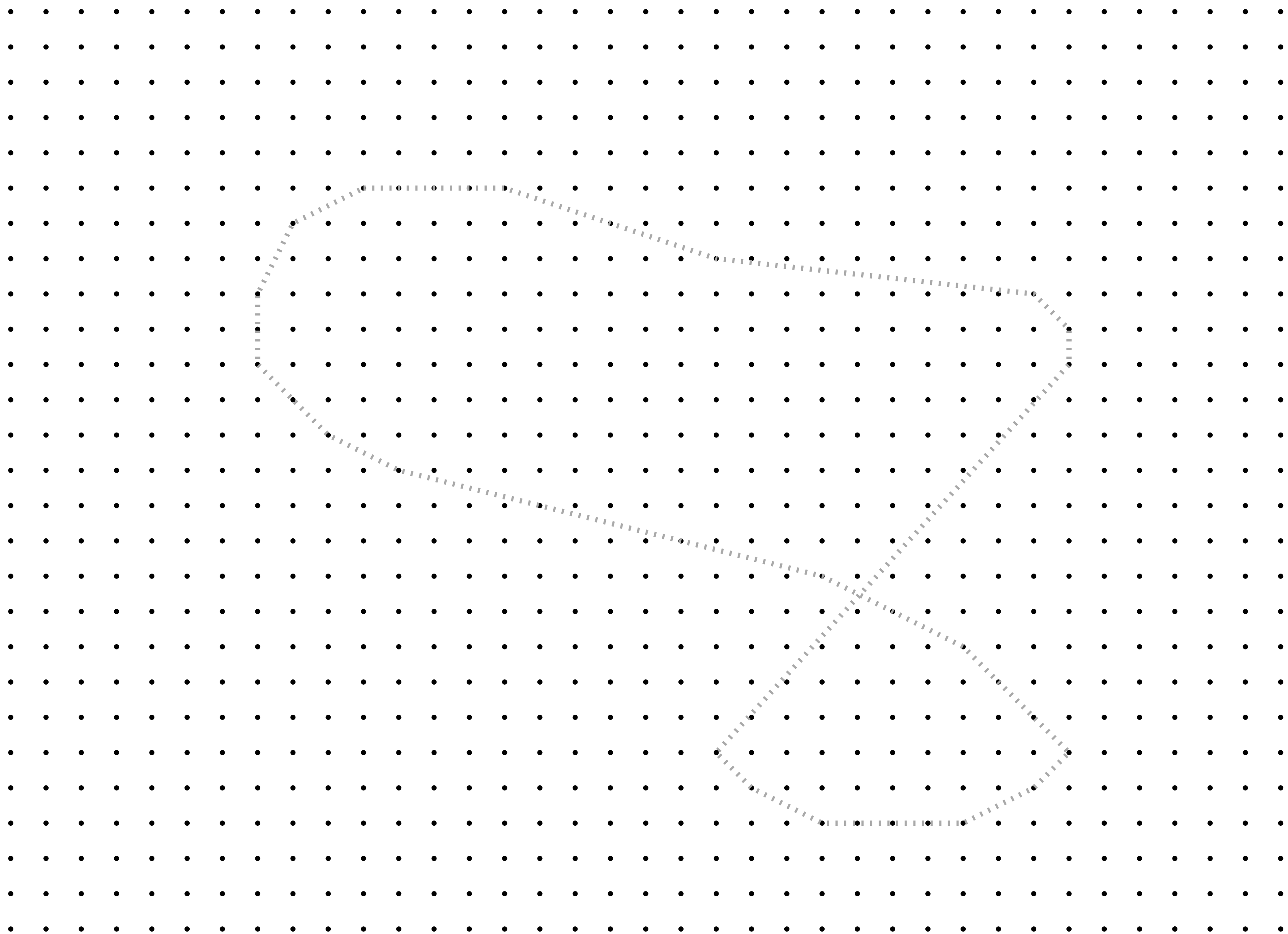

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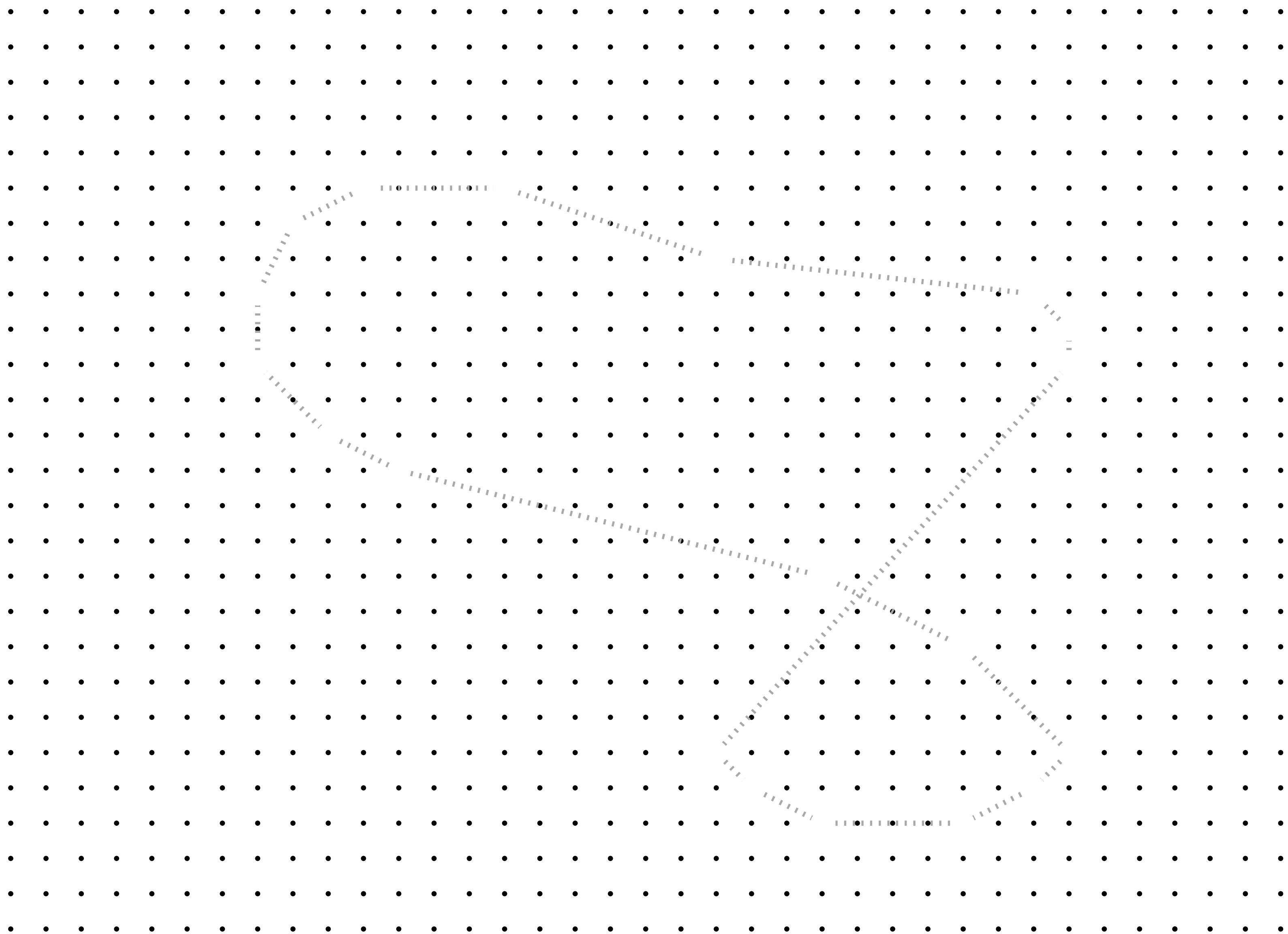

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Freie Universität Berlin

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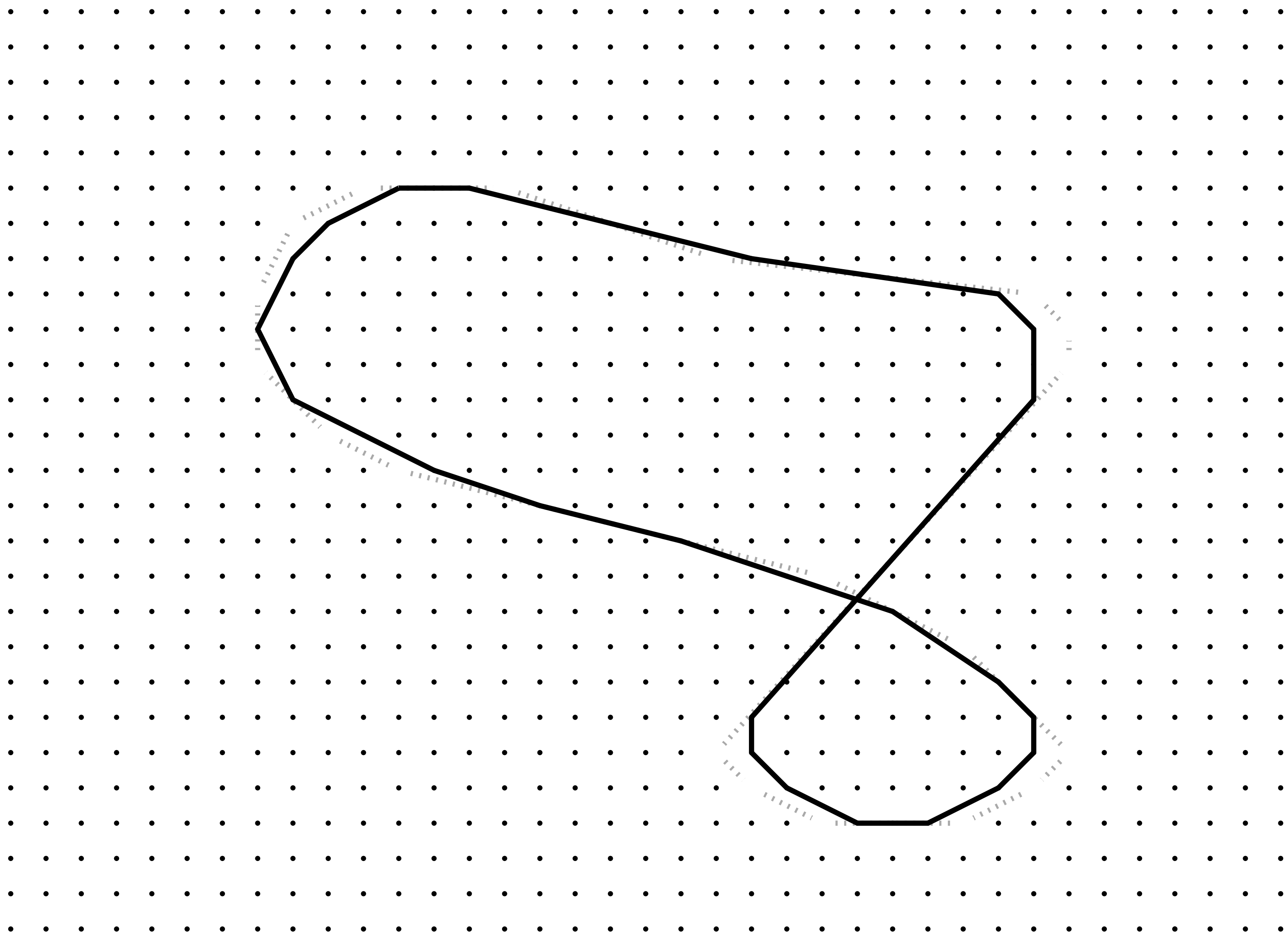

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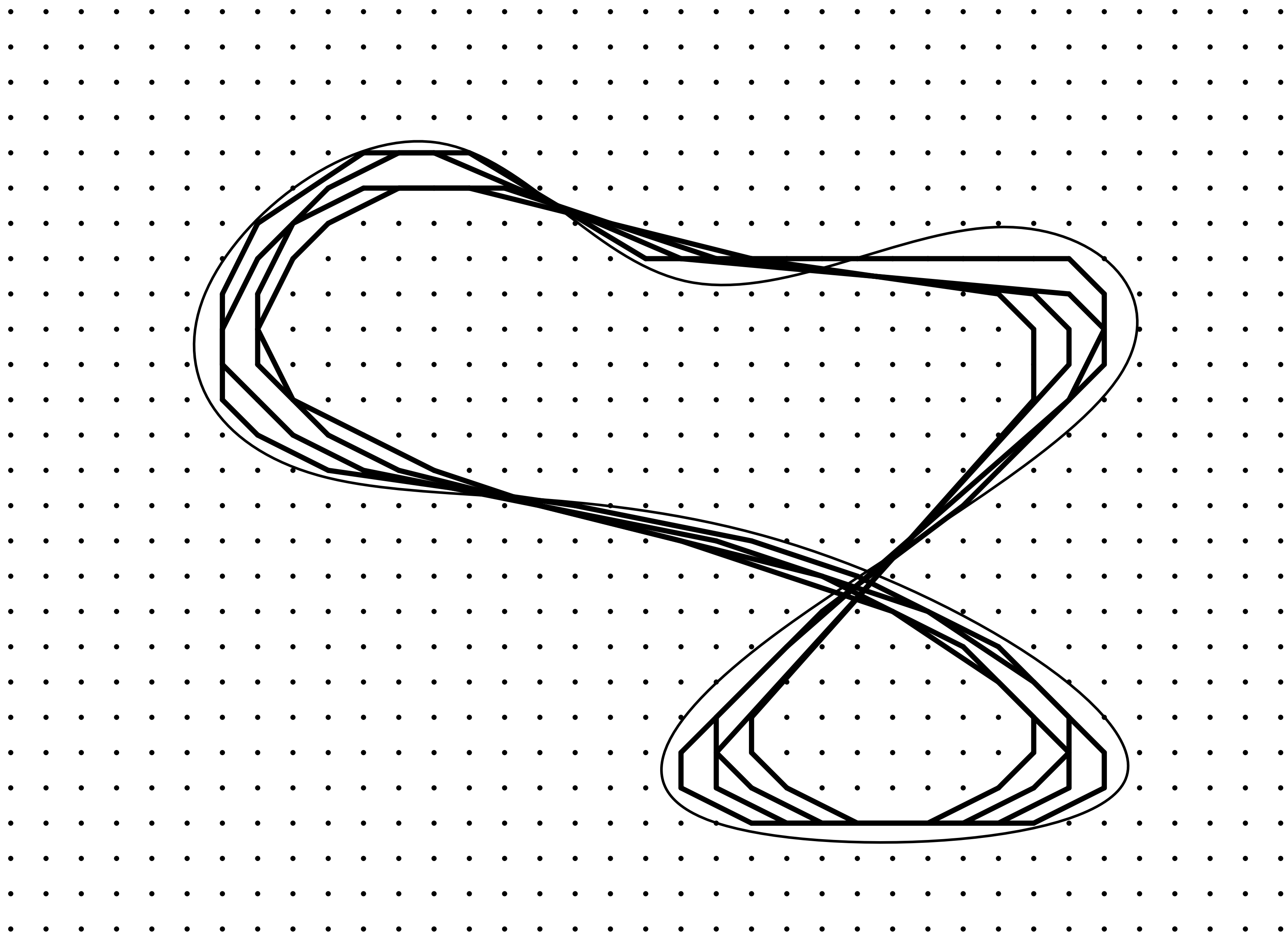
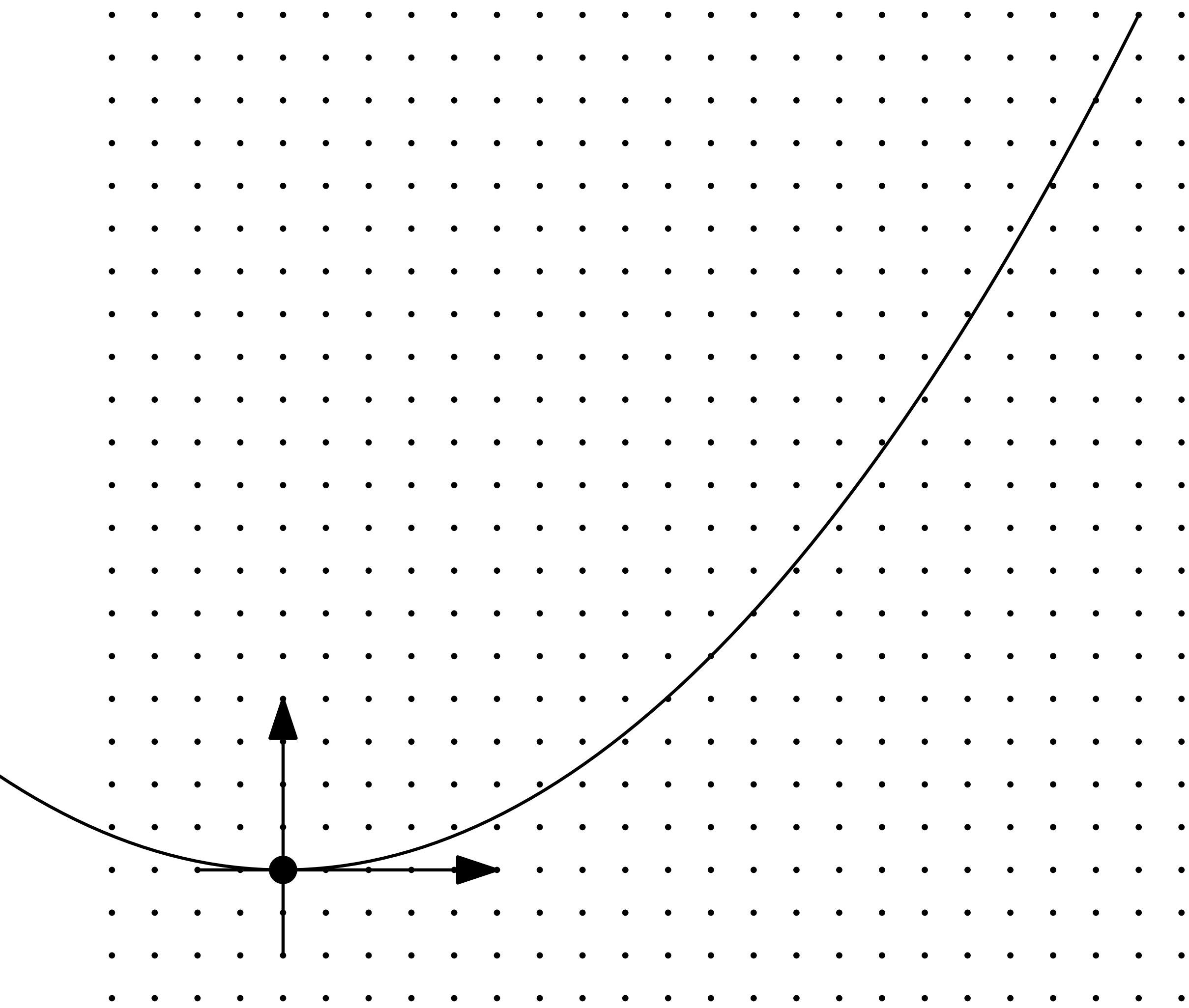


The parabola!



$$y = \frac{1}{20}x^2$$

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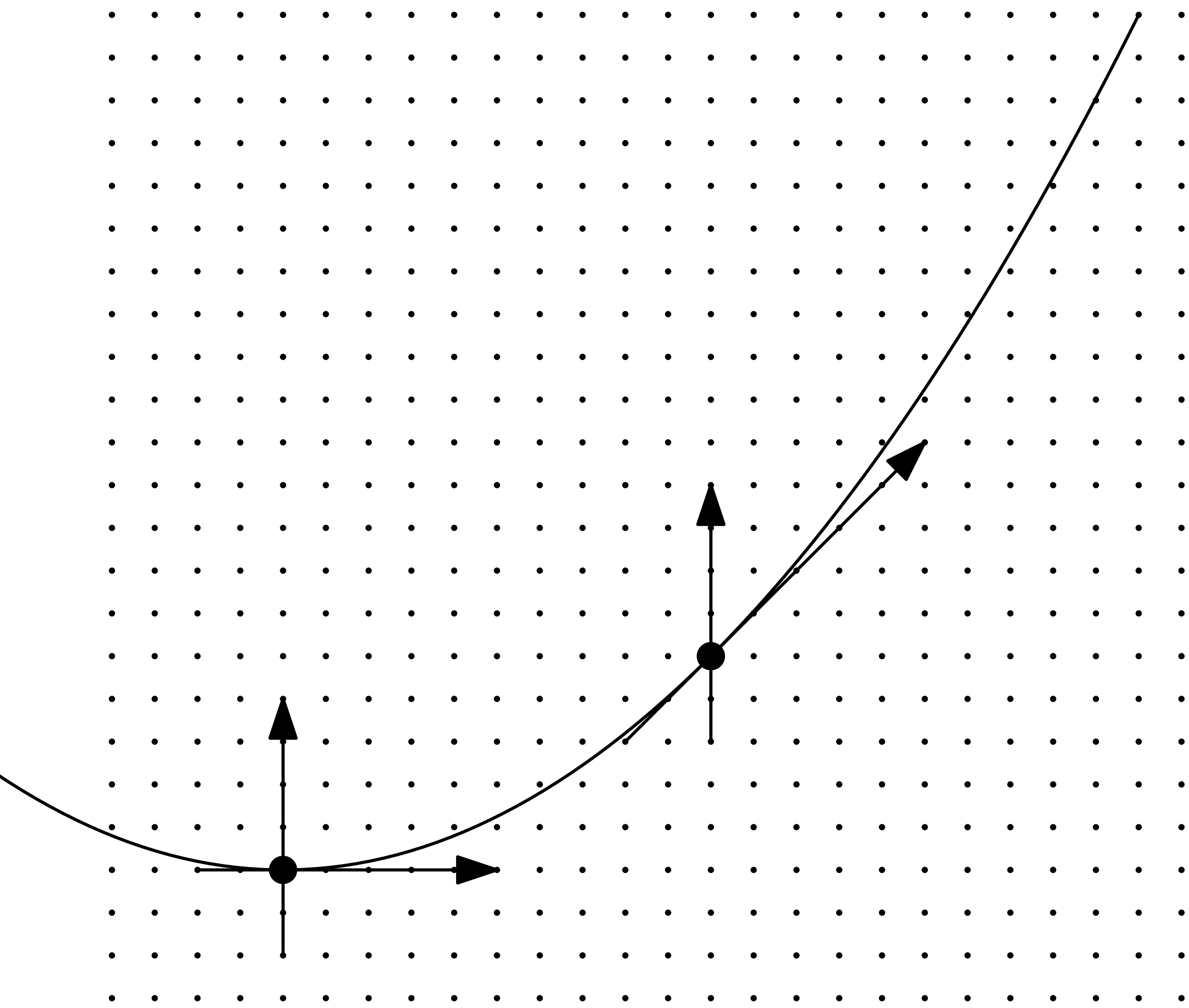

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affine lattice-preserving shearing transformations

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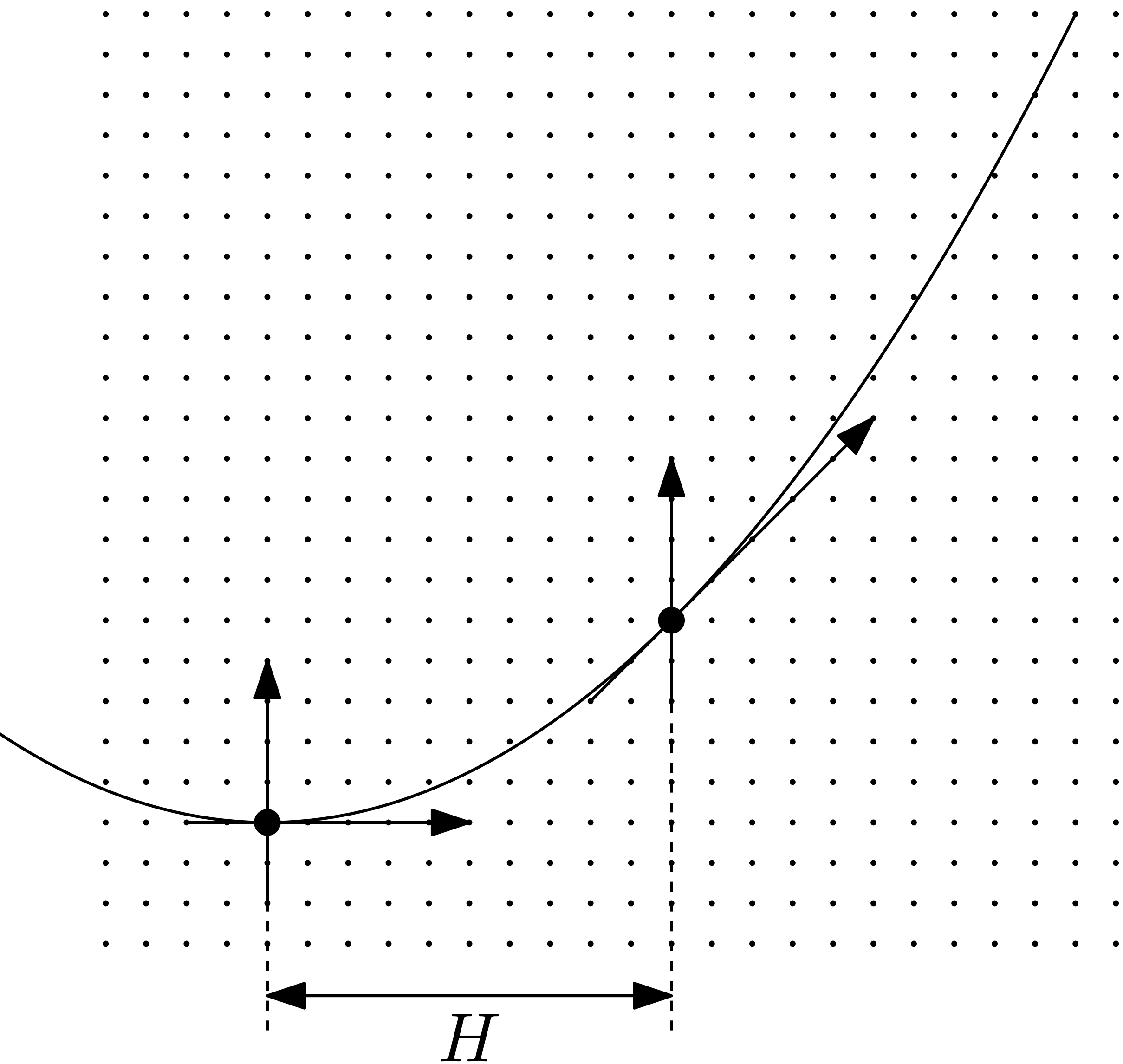

$y = \frac{1}{20}x^2$

affine lattice-preserving shearing transformations

$y = \frac{a_N}{a_D}x^2 + \frac{b_N}{b_D}x + c$

Lemma:

Horizontal period $H = \operatorname{lcm}(a_D, b_D)$ or $H = \operatorname{lcm}(a_D, b_D)/2$

Conics

Freie Universität Berlin

Conics maintain their shape under ACSF.

- Ellipses (and circles) *shrink* (and collapse to the center).
- Parabolas are *translated*.
- Hyperbolas *expand*.

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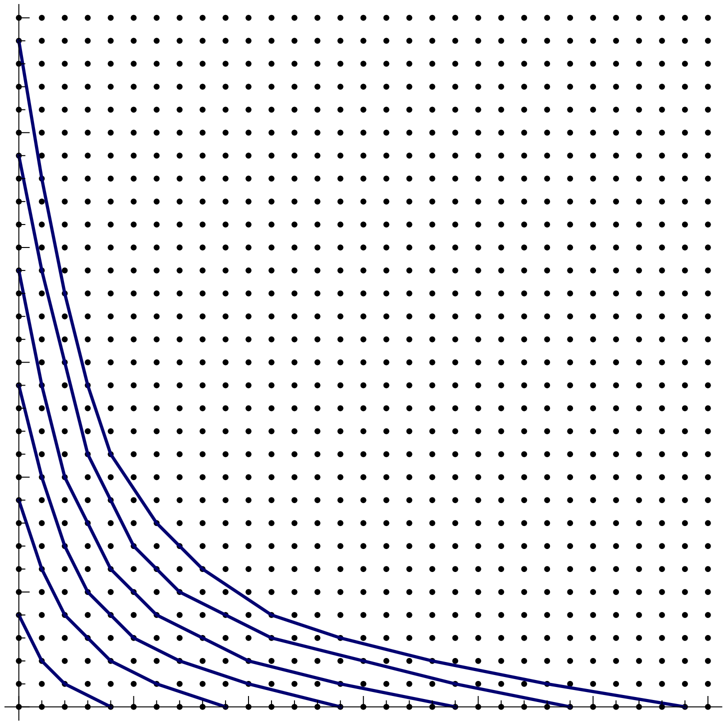
David Eppstein, Sariel Har-Peled, and Gabriel Nivasch 2020:



layers 5, 10, . . . , 30 of $\mathbb{N}^2$

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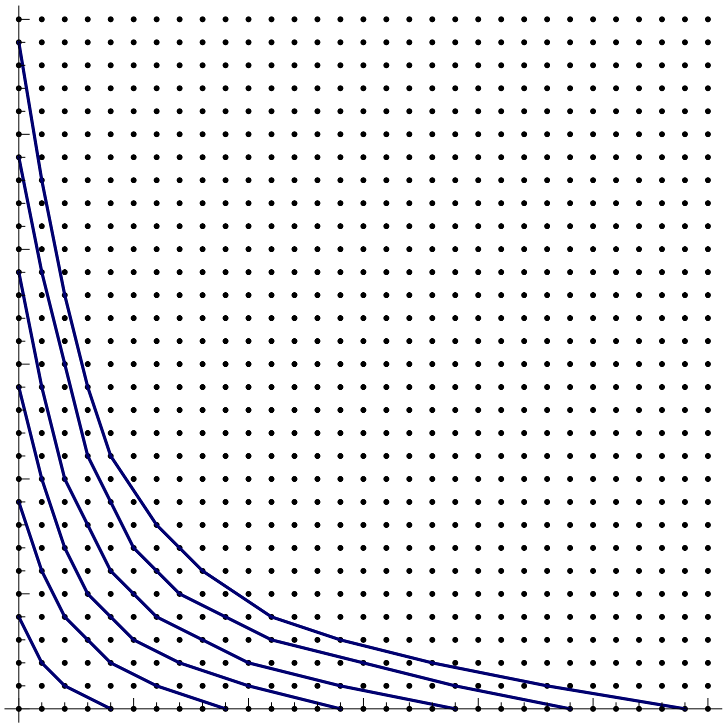

layers 5, 10, ..., 30 of $\mathbb{N}^2$

THEOREM:
The n-th layer of $\mathbb{N}^2$ is sandwiched between two hyperbolas:

$$c_1 n^{3/2} \le xy \le c_2 n^{3/2}$$

(except within $\sqrt{n}\log^2 n$ of the axes)

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THEOREM. Parabola $y = ax^2/2 + bx + c$. Time $T > 0$.
(A) ACSF $=$ a vertical translation by $a^{1/3}T$.
(B) Grid peeling with spacing $1/n$ for $m = \lfloor C_g T n^{4/3} \rfloor$ steps:
$\implies$ vertical distance between (A) and (B) is

$$O\left(\frac{Ta^{2/3}\log\frac{n}{a}}{n^{1/3}}\right). \qquad (\to 0 \text{ for } n \to \infty)$$

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Unimodular transformation:
vertical axis $\to$ axis with arbitrary rational slope

The "grid parabola" P_t

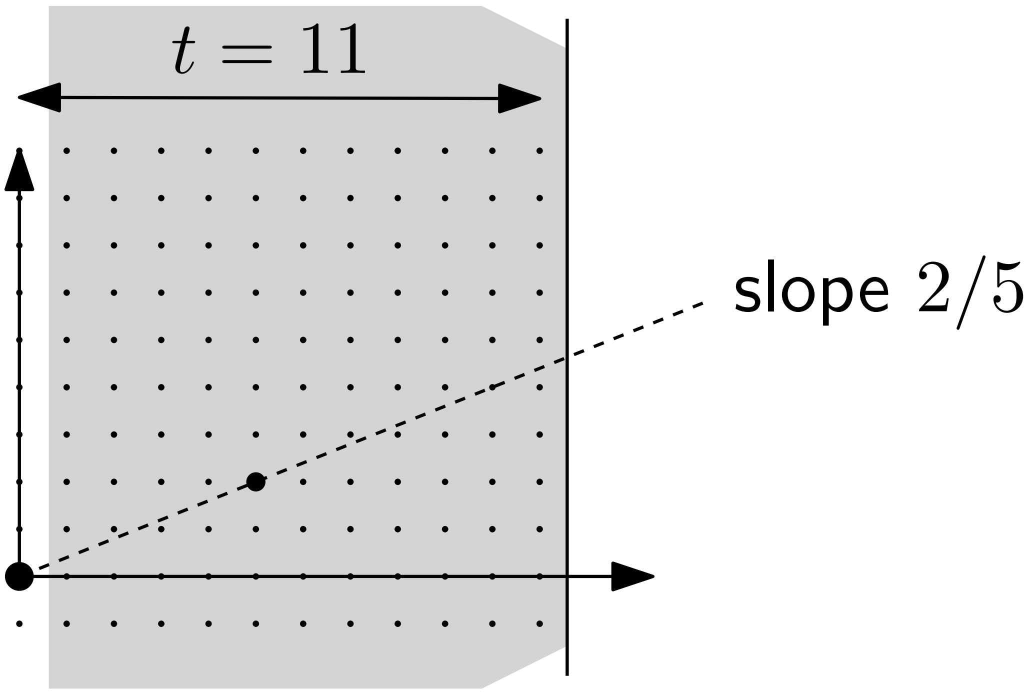
Freie Universität Berlin

- integer parameter $t \geq 1$
- $S_t :=$ { all slopes a/b with $0 < b \leq t$ }
- for each slope $a/b \in S_t$, take the longest integer vector

$$\binom{x}{y} = k\binom{b}{a} \quad (k \in \mathbb{Z})$$

with $0 < x \leq t$

Example



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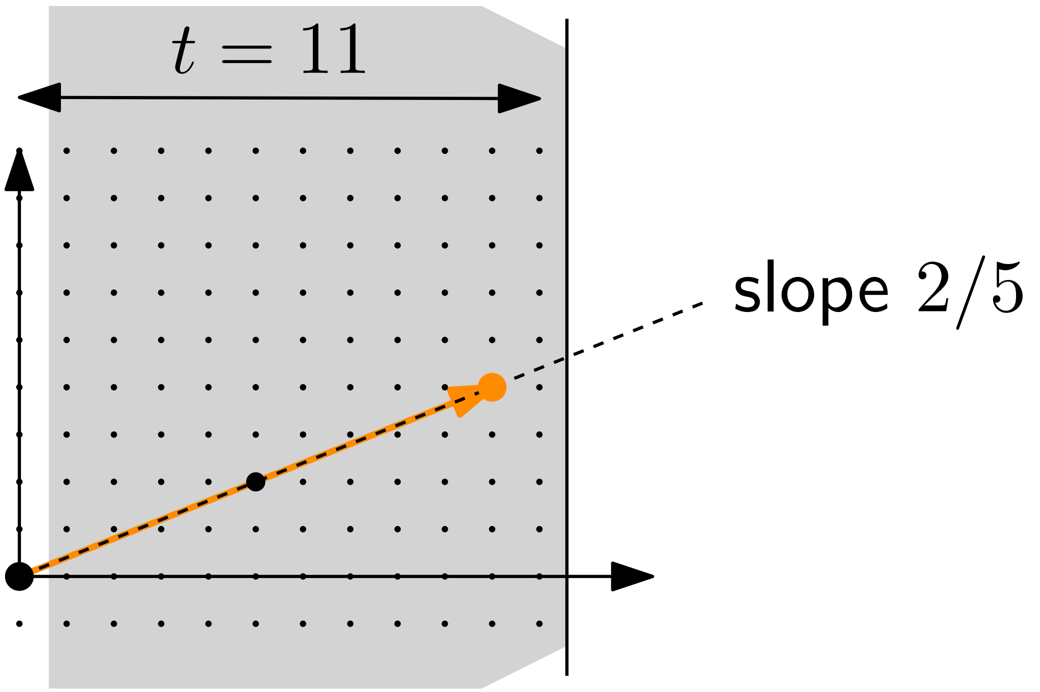

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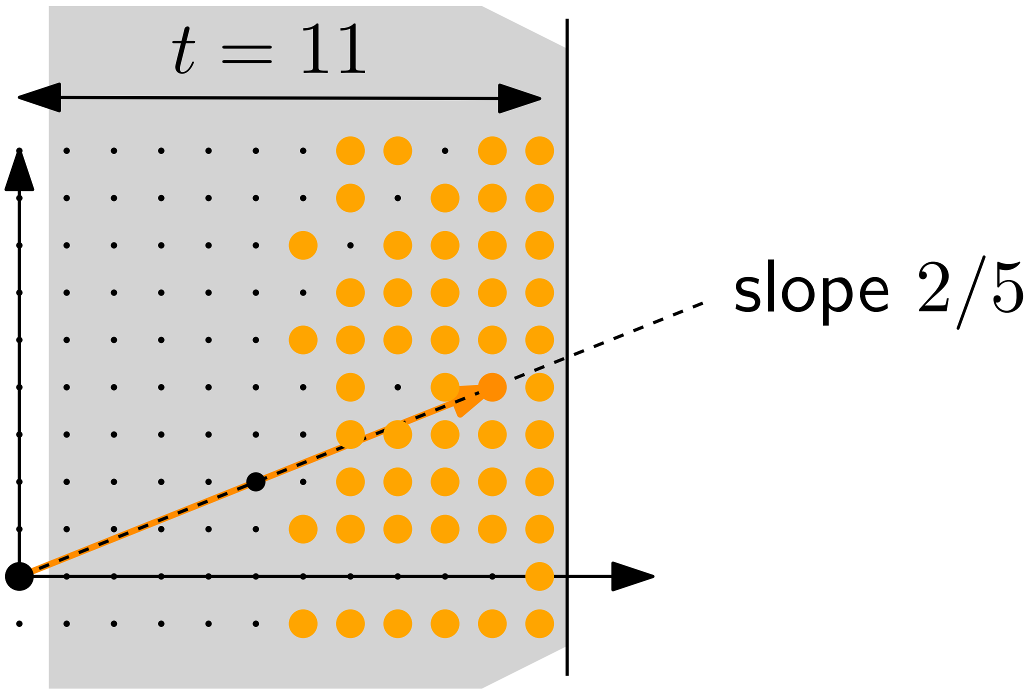
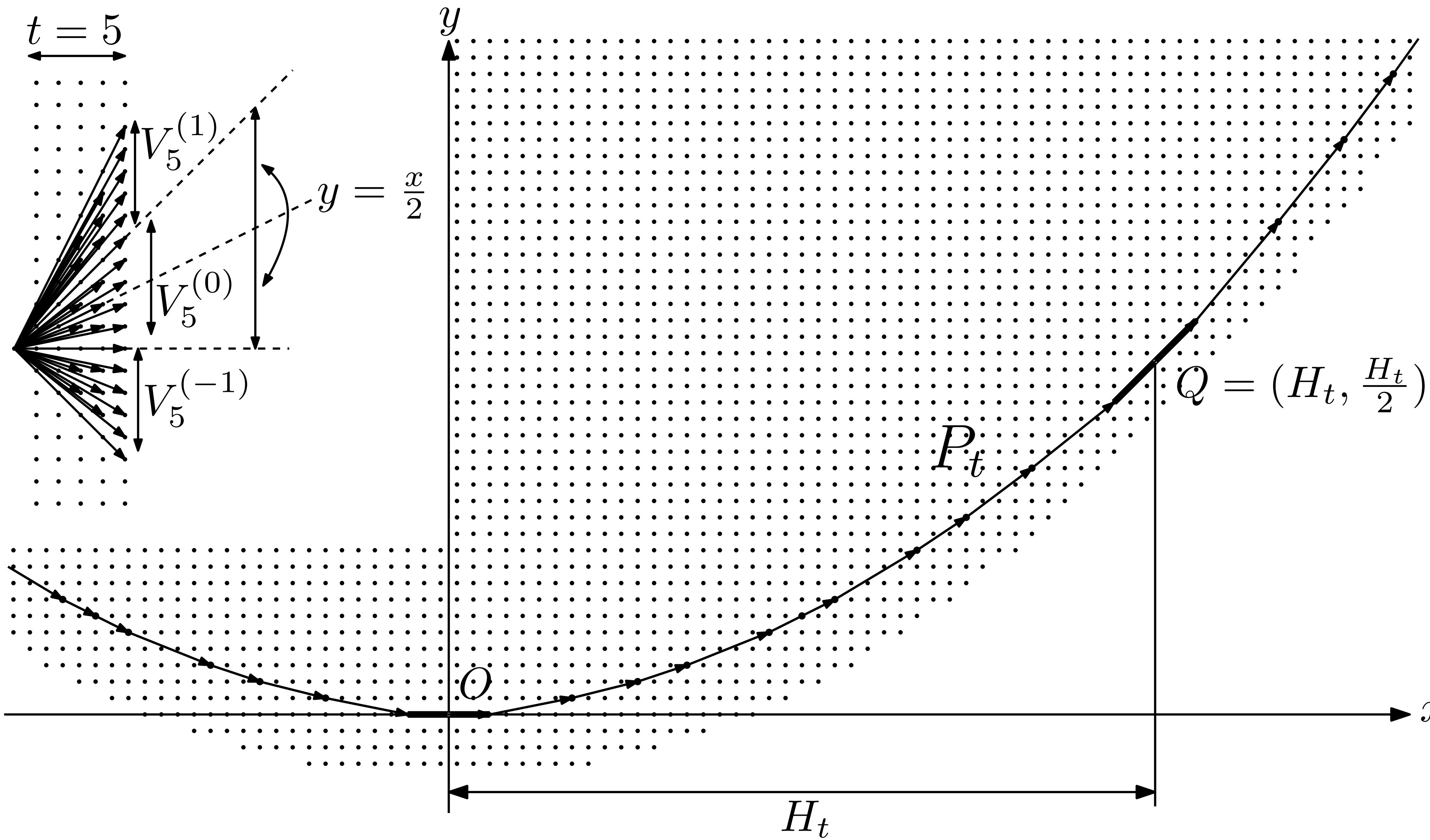


The "grid parabola" P_5



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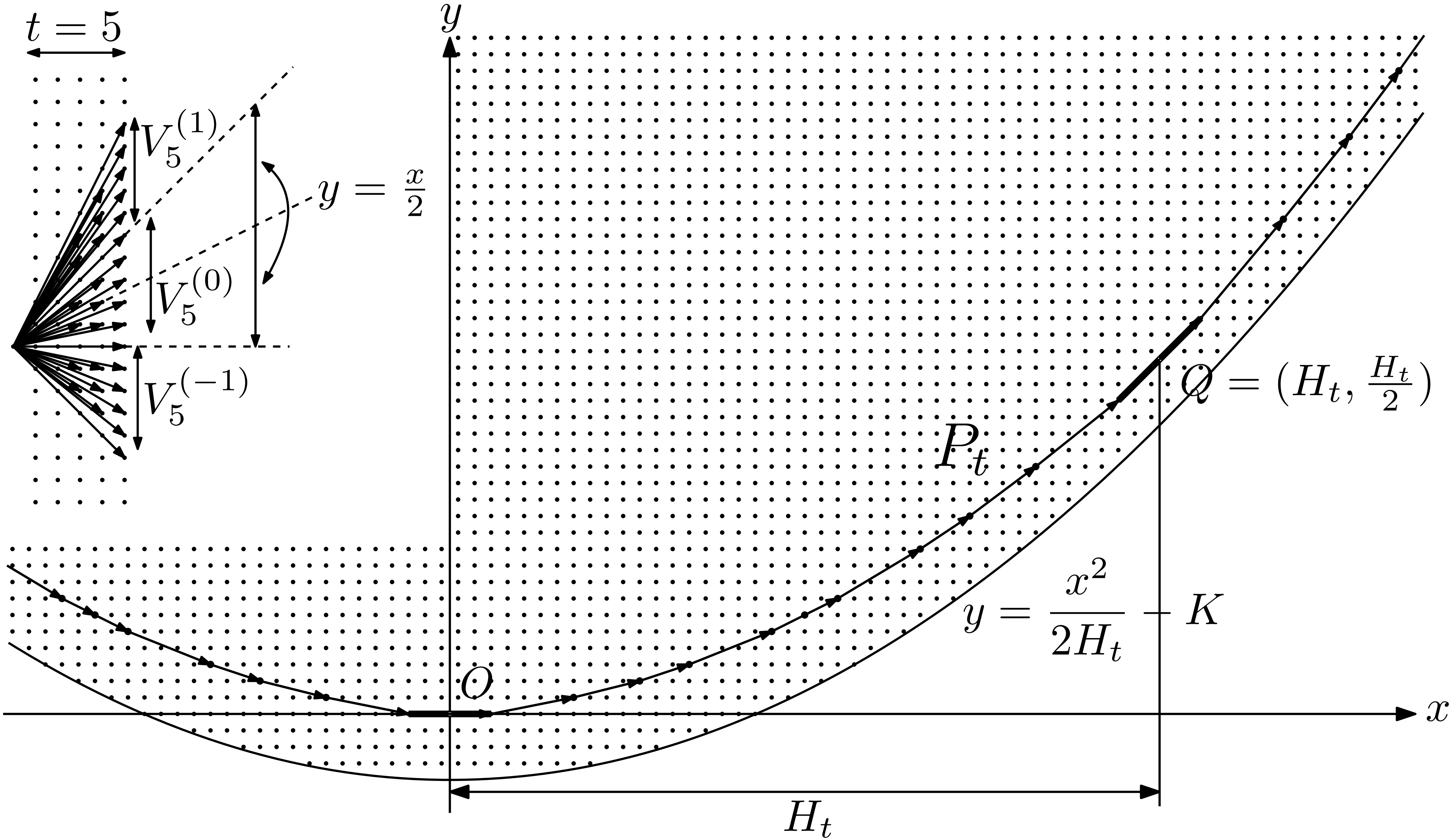

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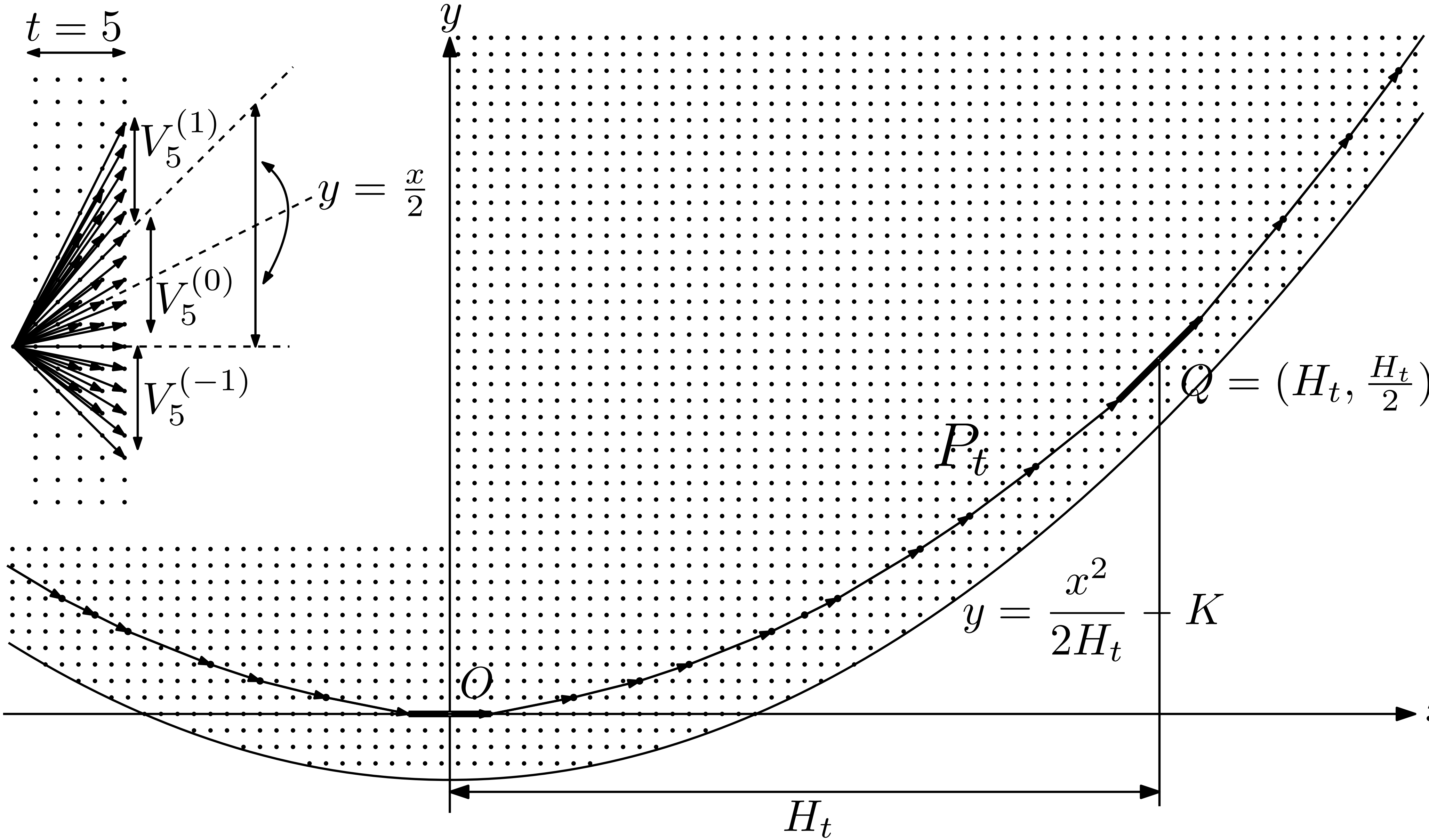

$$H_1, H_2, \ldots = 1, 4, 11, 22, 43, 64, 107, 150, 211, 274, 385, \ldots$$

The "grid parabola" P_5

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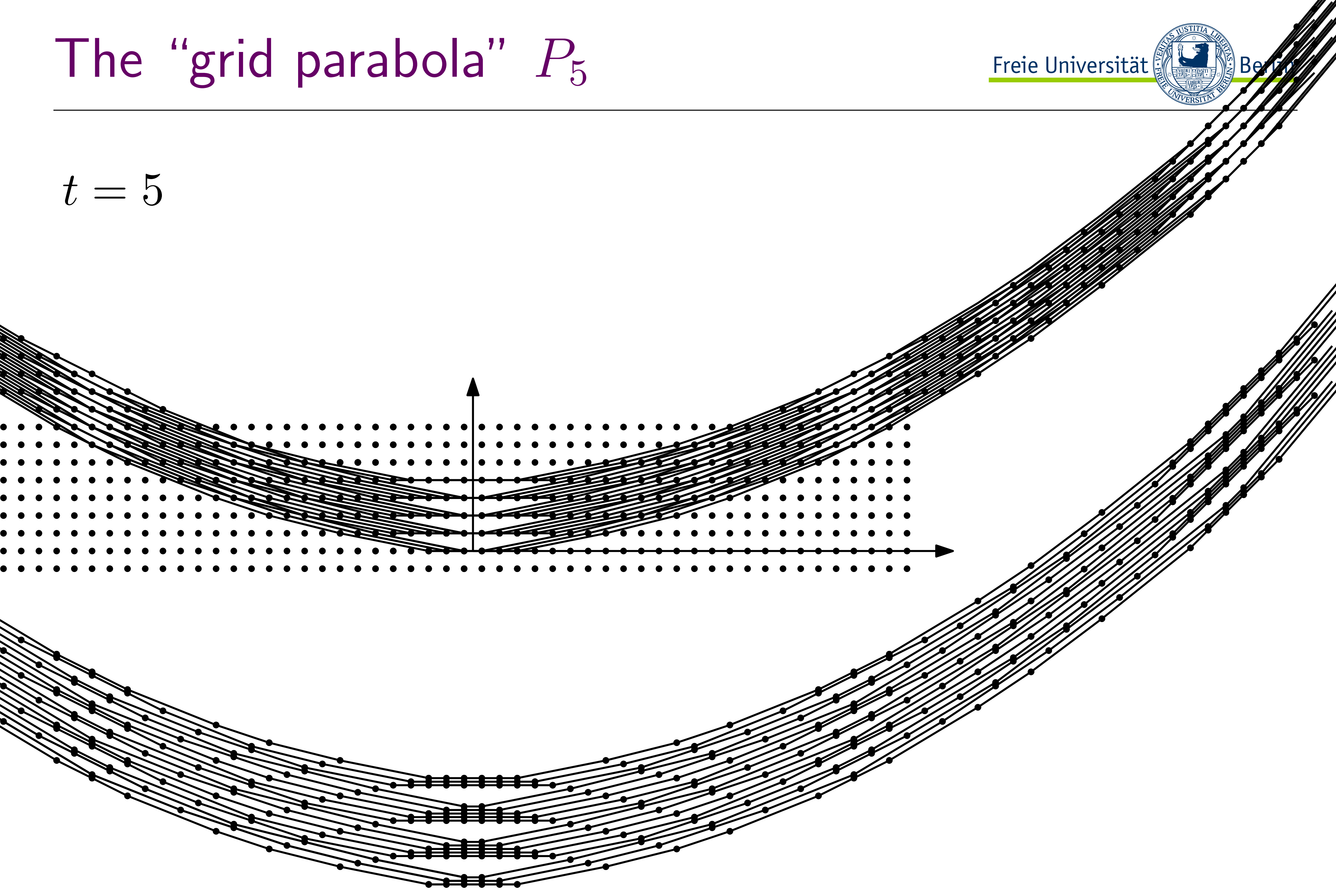
$t = 5$



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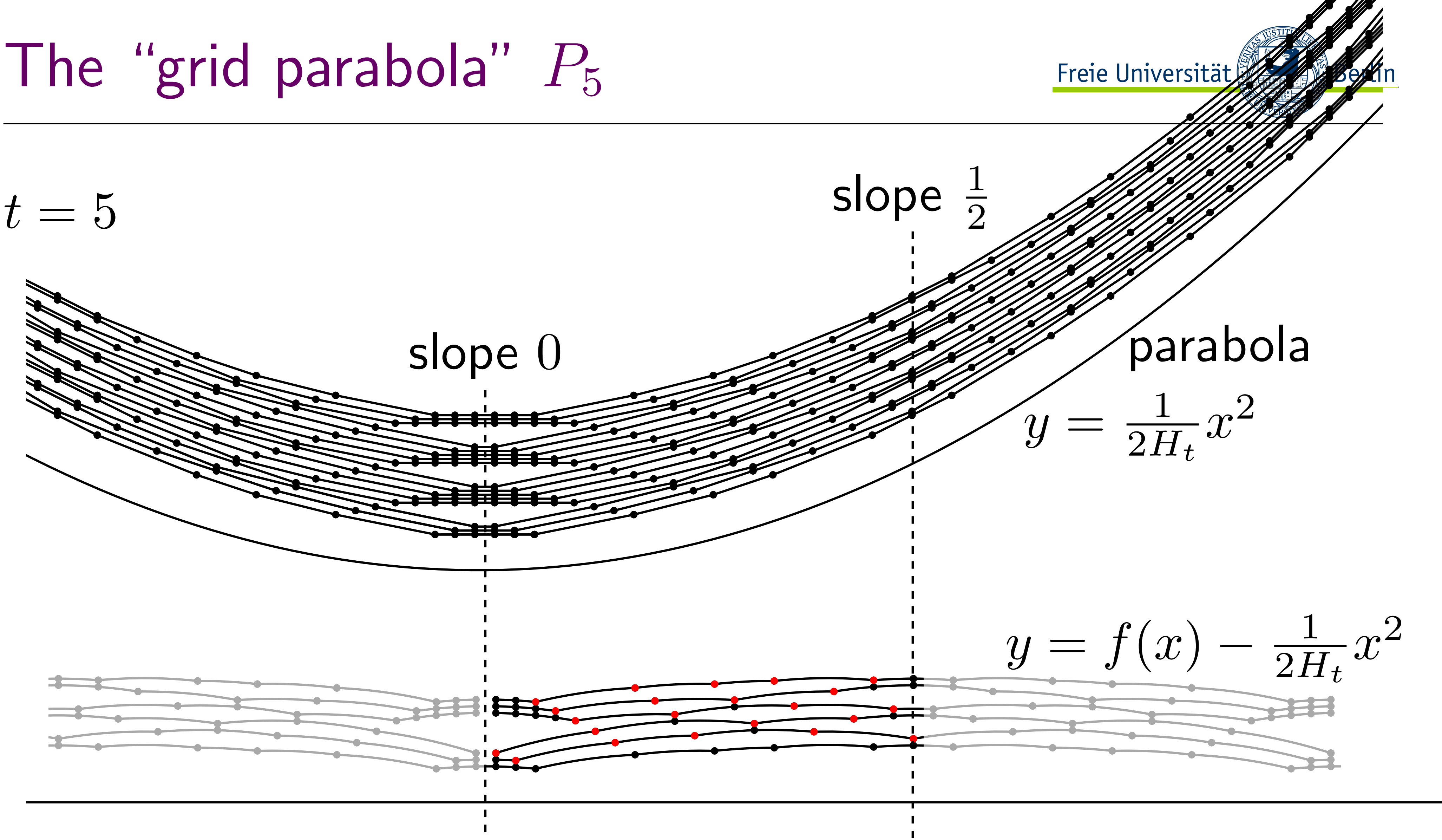
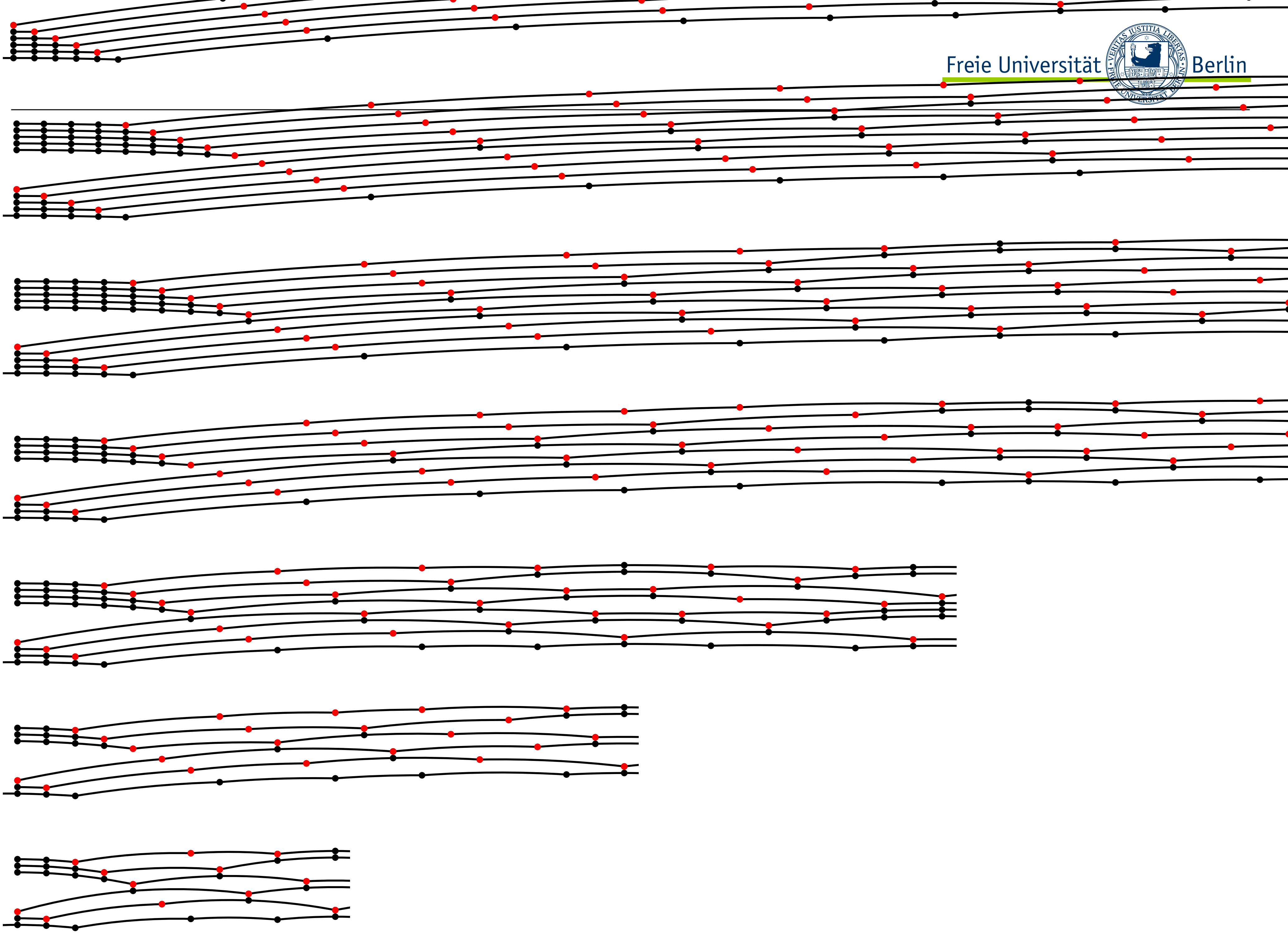


Theorem:
t odd: The polygon P_t repeats after t steps, one level higher.
(t even: after $t + 1$ steps.)



$t = 4, 5, 6, \ldots$

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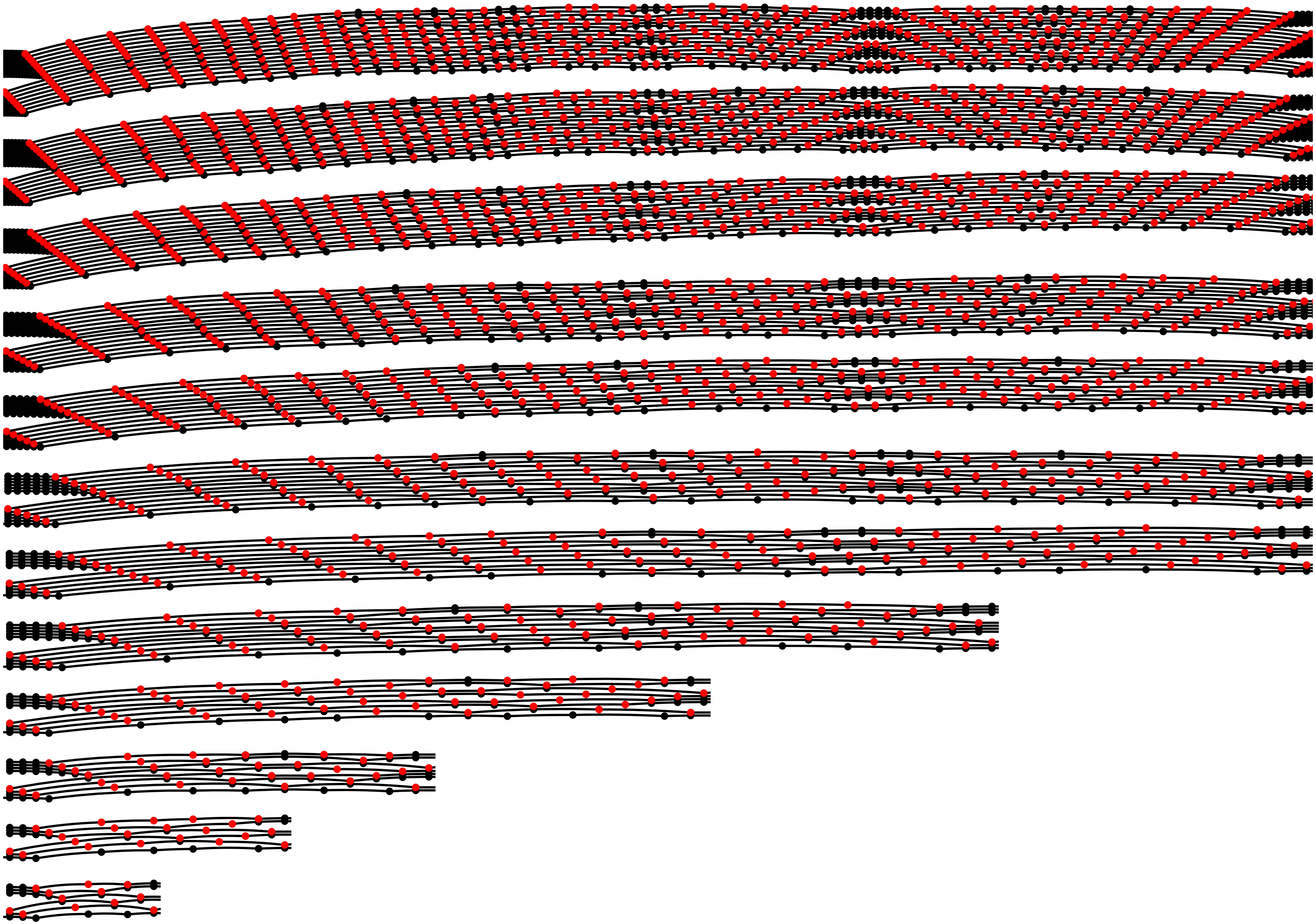
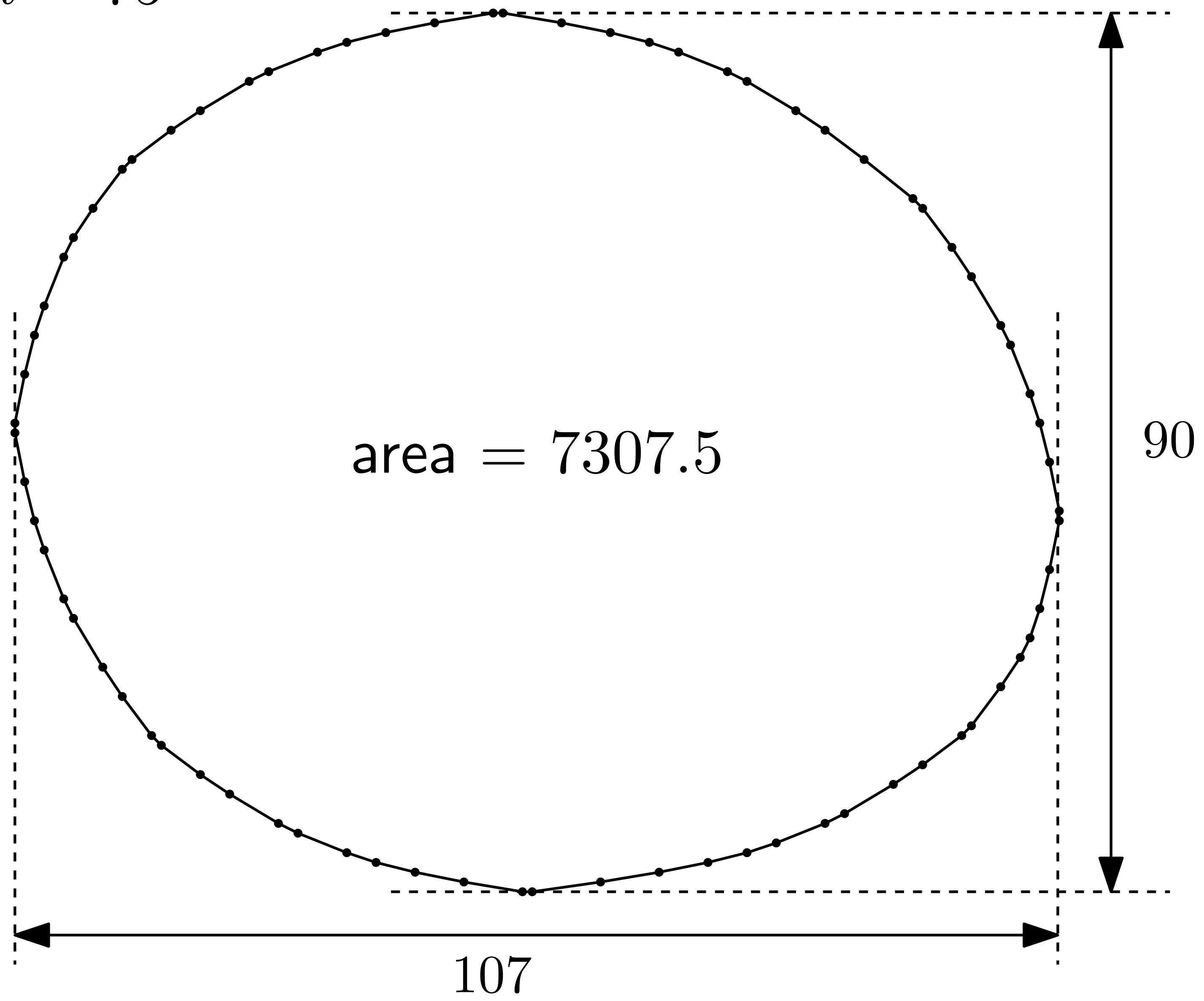


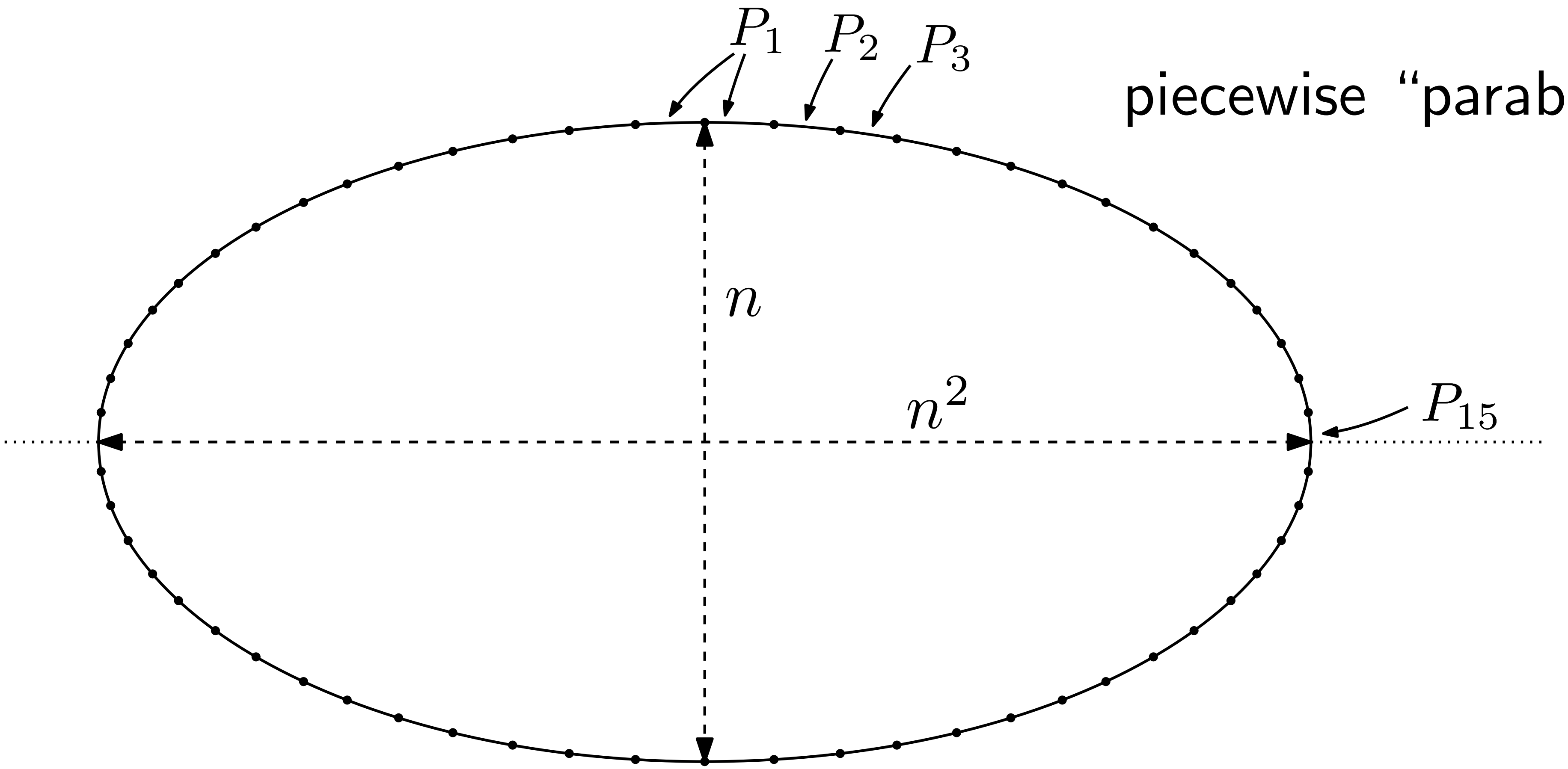
The minimum-area lattice n-gon

$n = 75$



The minimum-area lattice n-gon

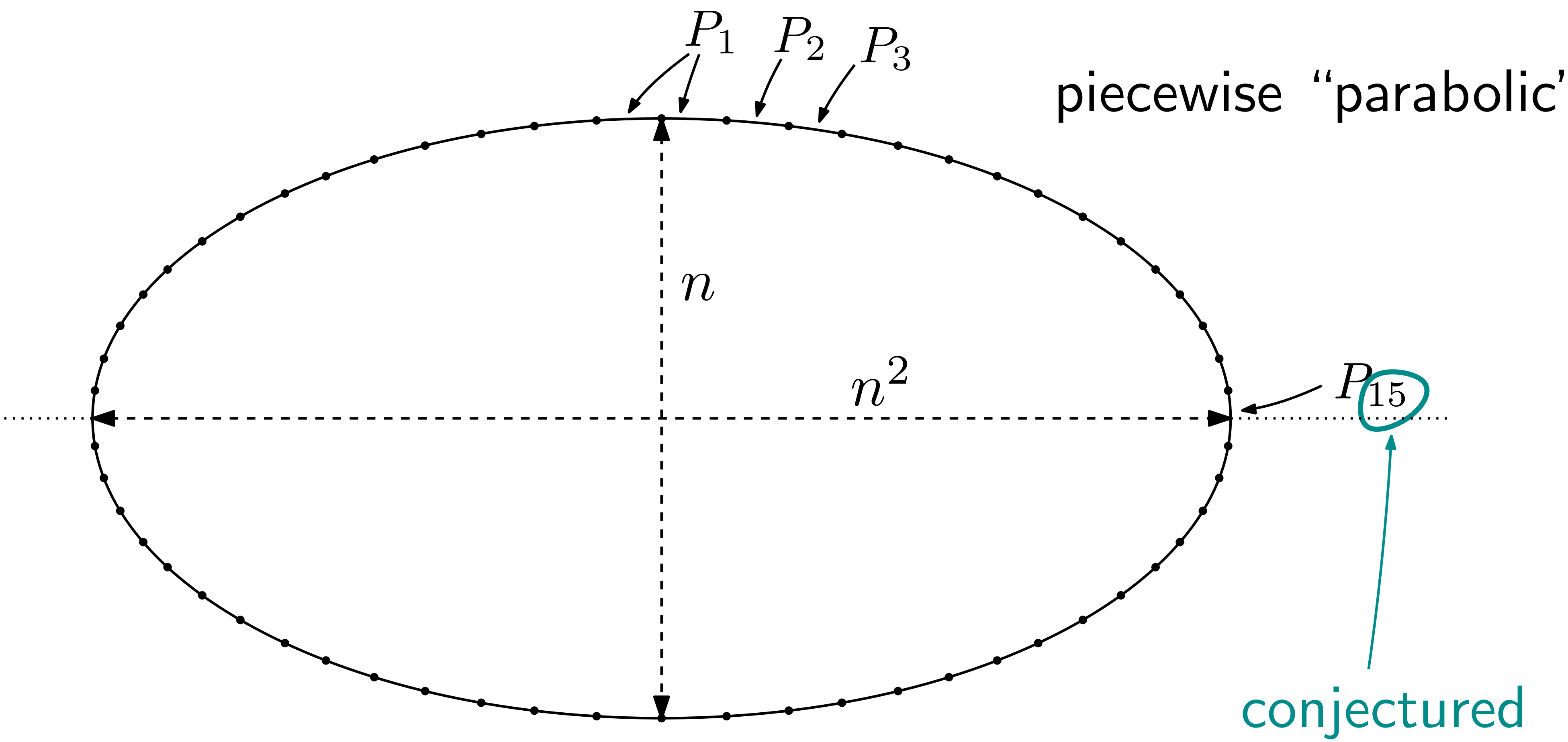
[Bárány and Tokushige, 2003] (n large)



piecewise "parabolic"

The minimum-area lattice n-gon

[Bárány and Tokushige, 2003]   (n large)



Asymptotic period

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$$H_1, H_2, \ldots = 1, 4, 11, 22, 43, 64, 107, 150, 211, 274, 385, \ldots$$

[OEIS A174405]

$$H_t := \sum_{\substack{0<y\le x\le t \\ \gcd(x,y)=1}} \left\lfloor \frac{t}{x} \right\rfloor x = \sum_{1\le i\le t} \sum_{d|i} d\varphi(d)$$

$$H_t = \frac{2\zeta(3)}{\pi^2} t^3 + O(t^2 \log t)$$

with $\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \approx 1.2020569$

[Sándor and Kramer 1999]

General curves

Idea: Approximate by parabolas from outside/inside

General curves

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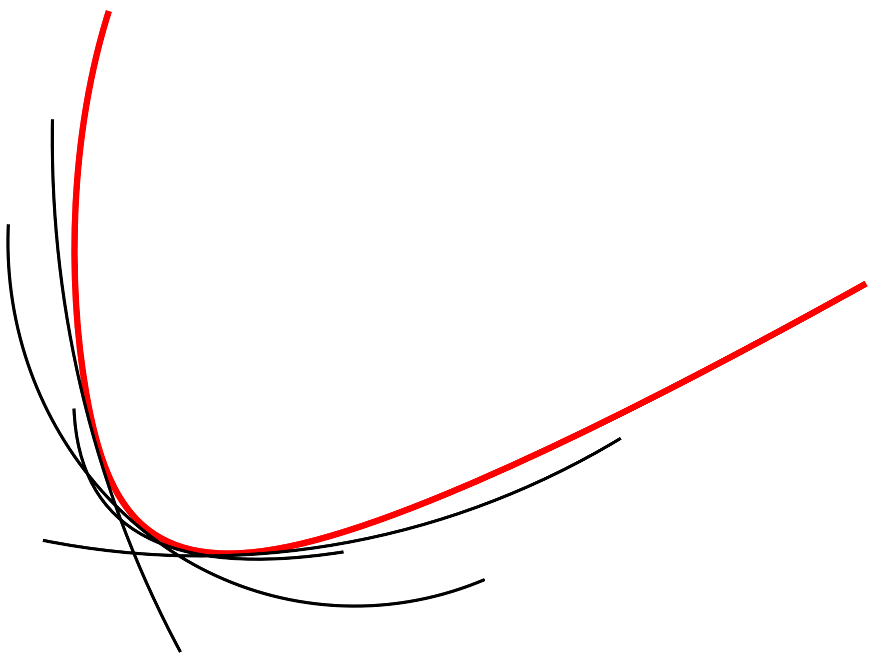

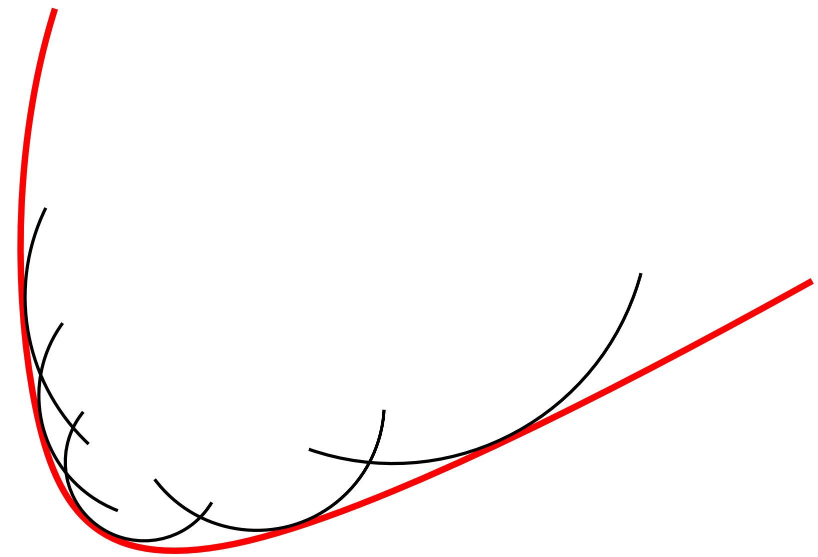

General curves

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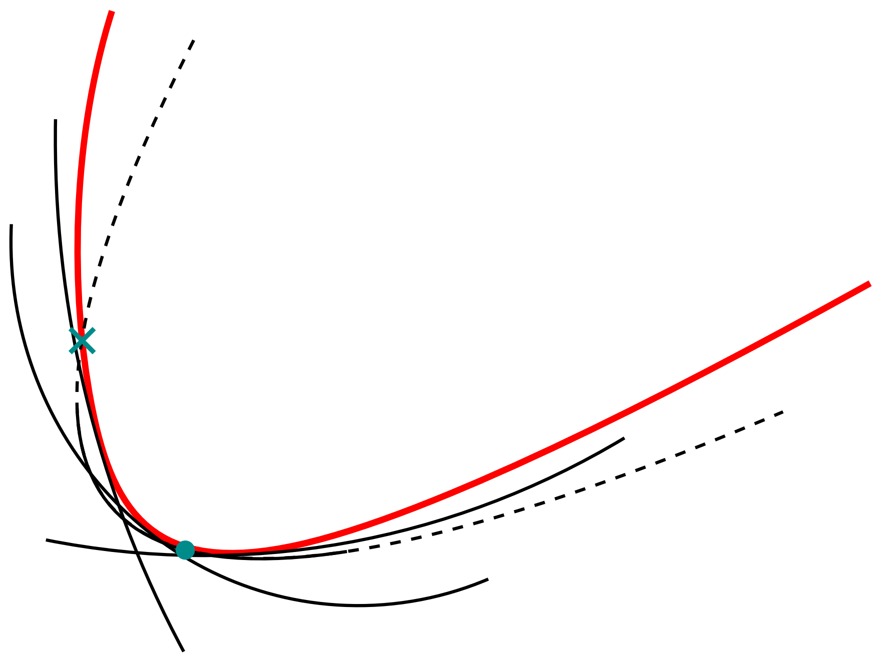

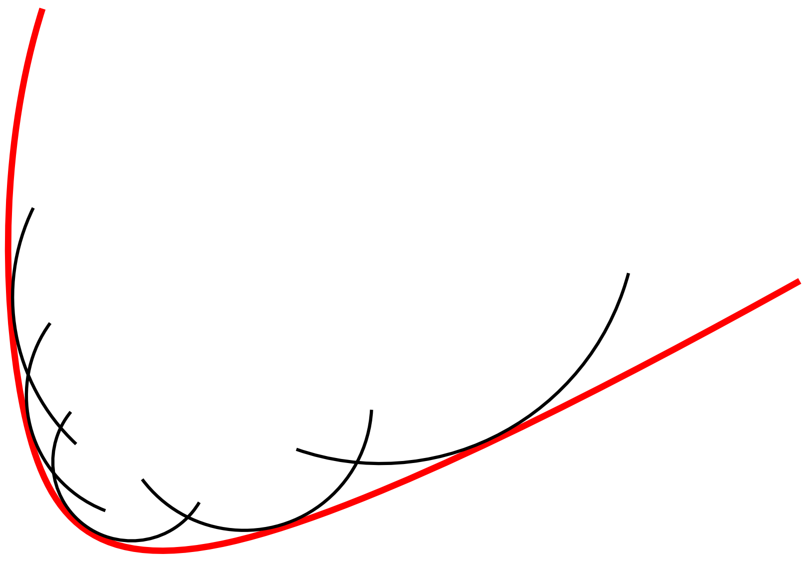

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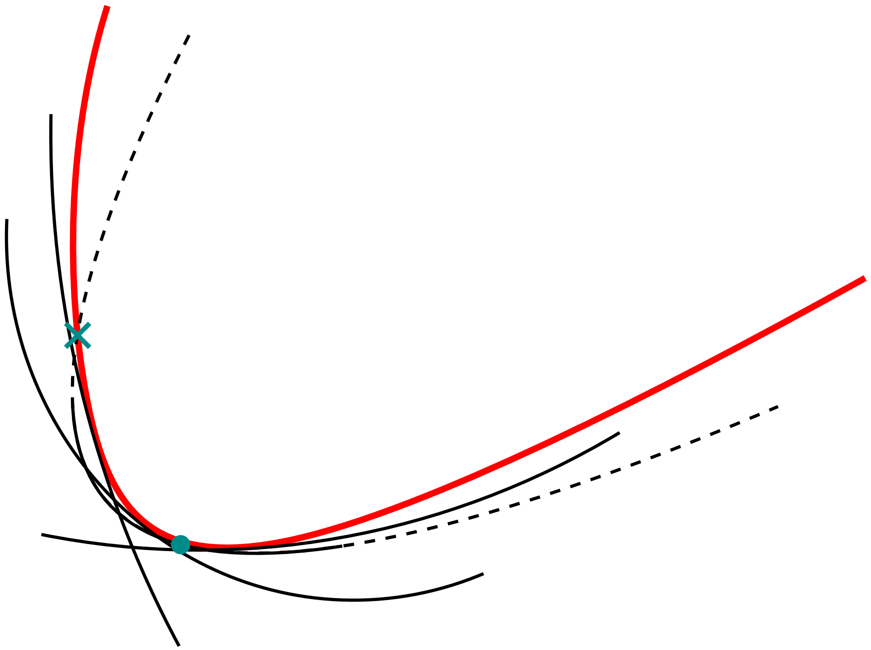
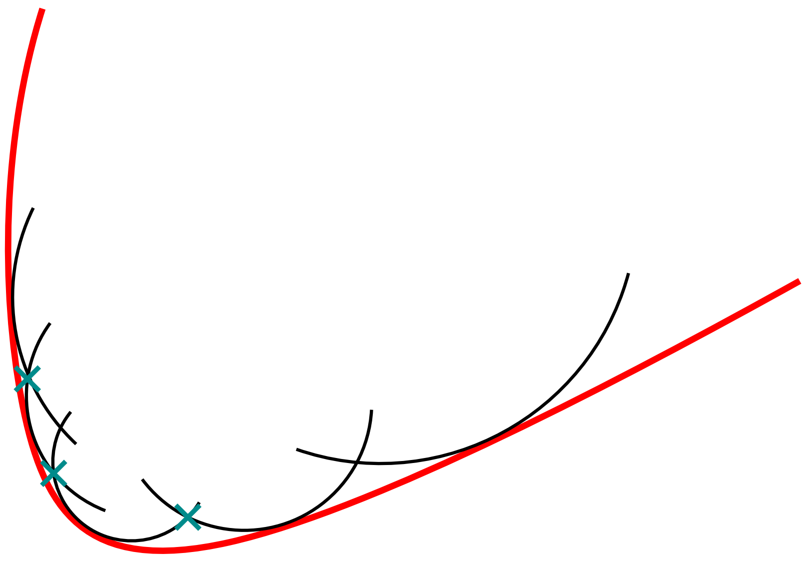
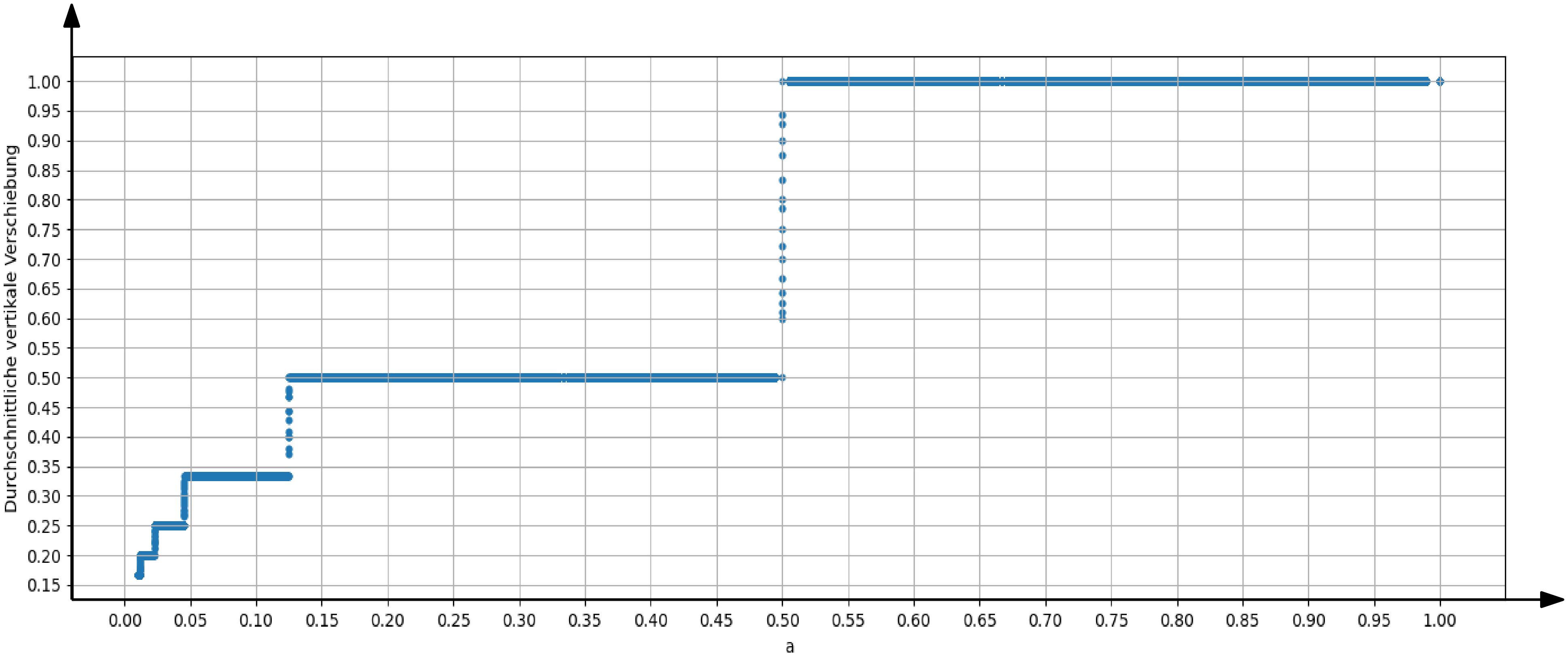




Time period for various parabolas

$$y = ax^2 + bx$$

average vertical speed depending on a (various values of b)



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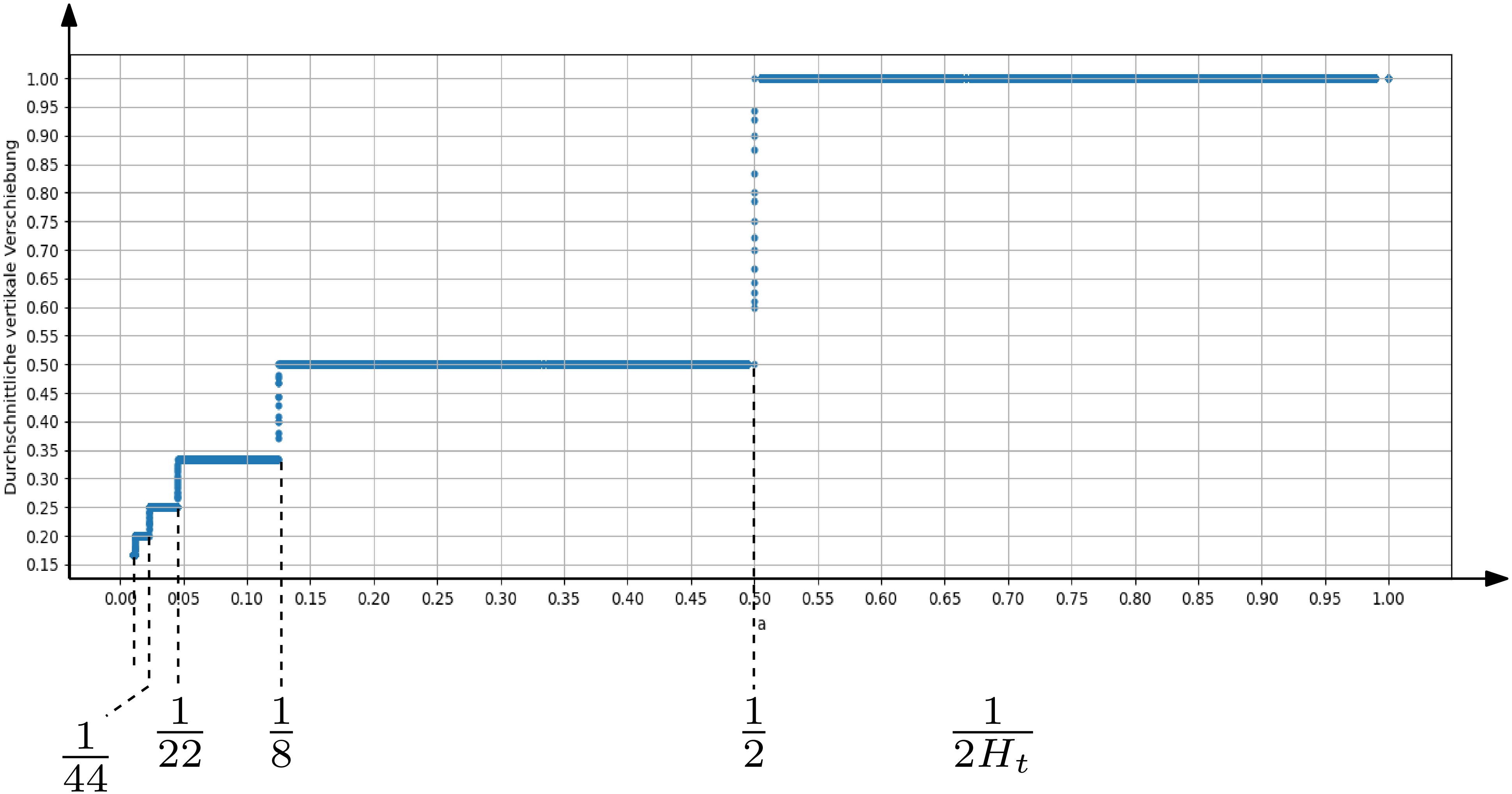
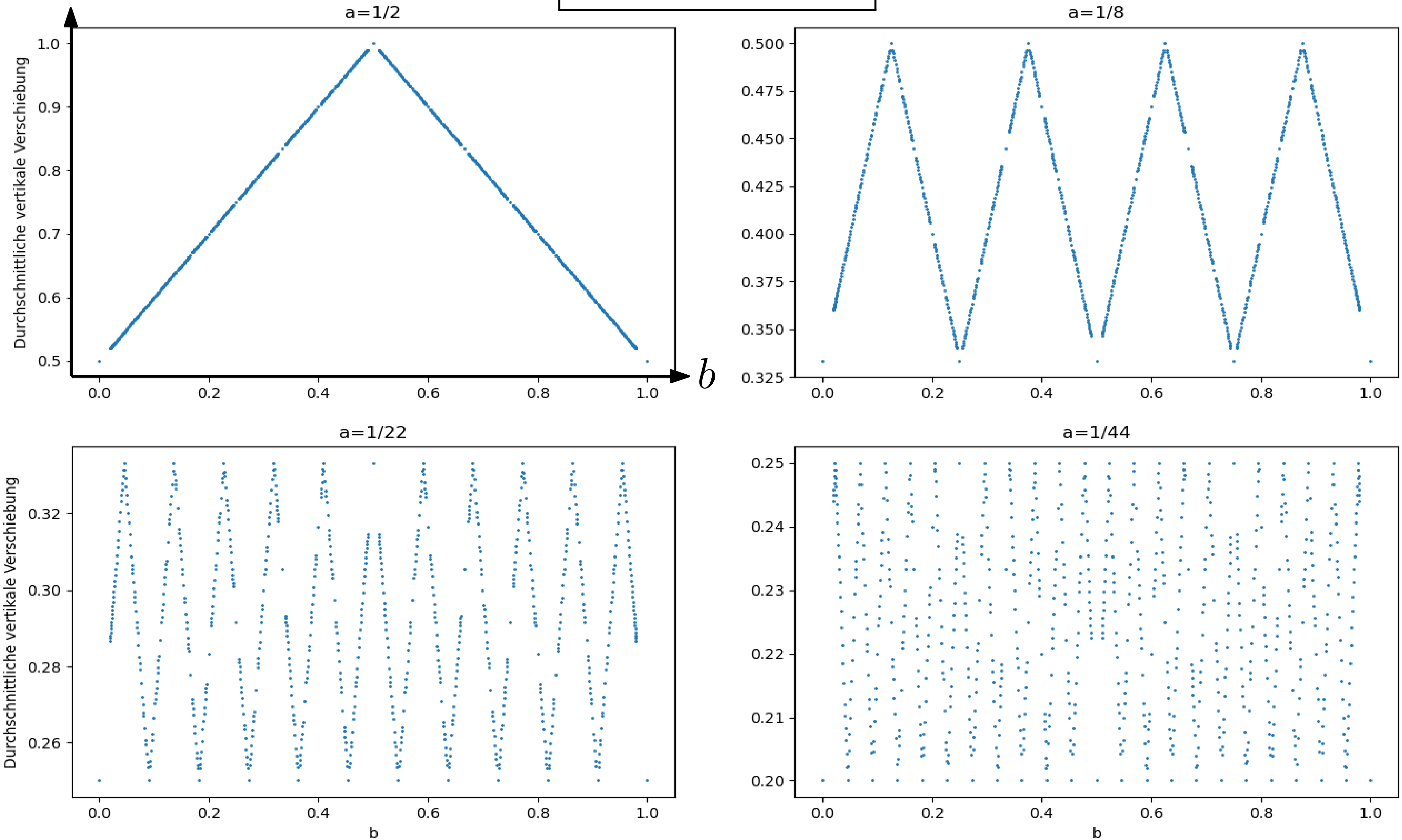


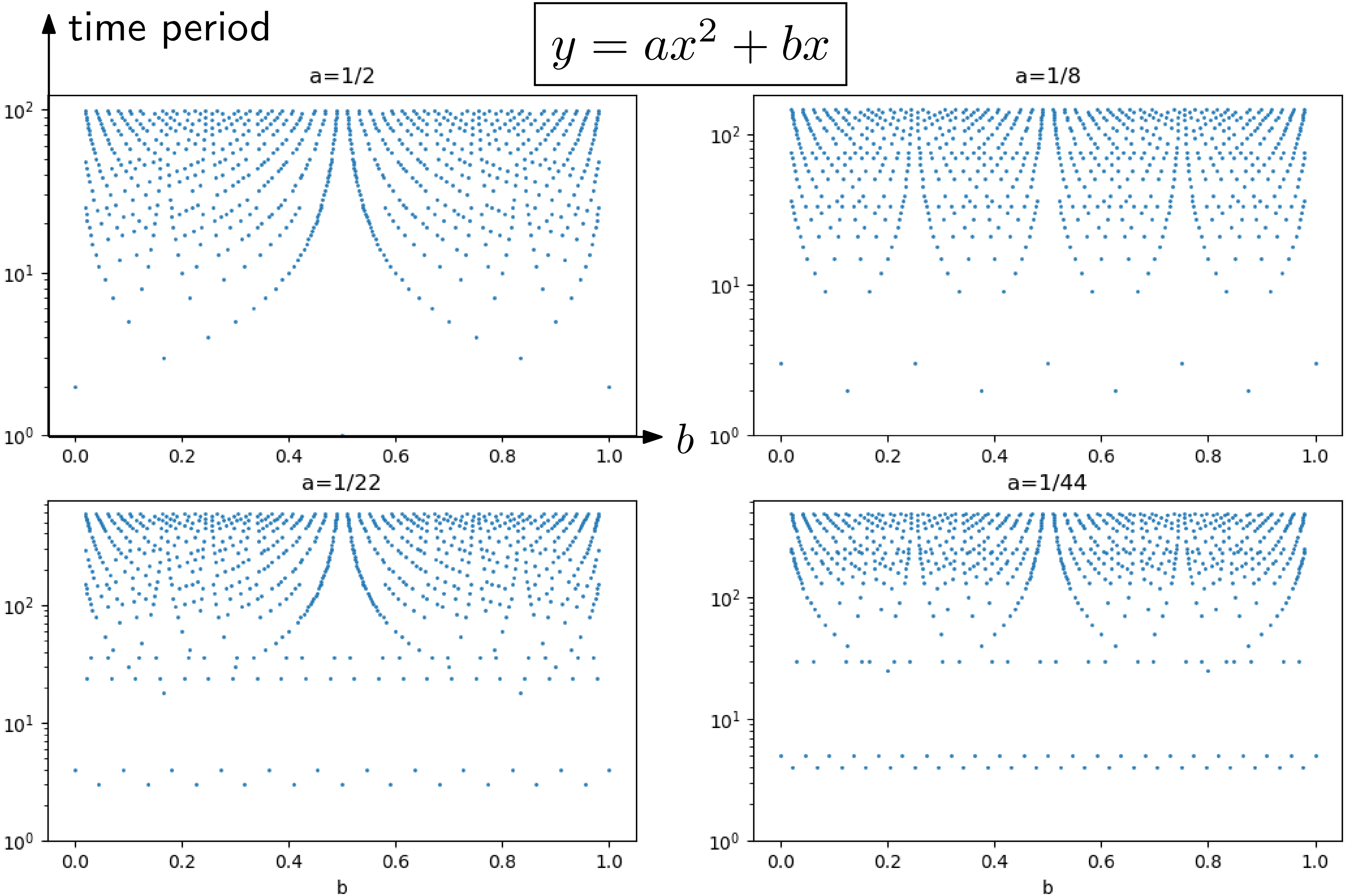
Time period for various parabolas

vertical speed

$$y = ax^2 + bx$$



Time period for various parabolas



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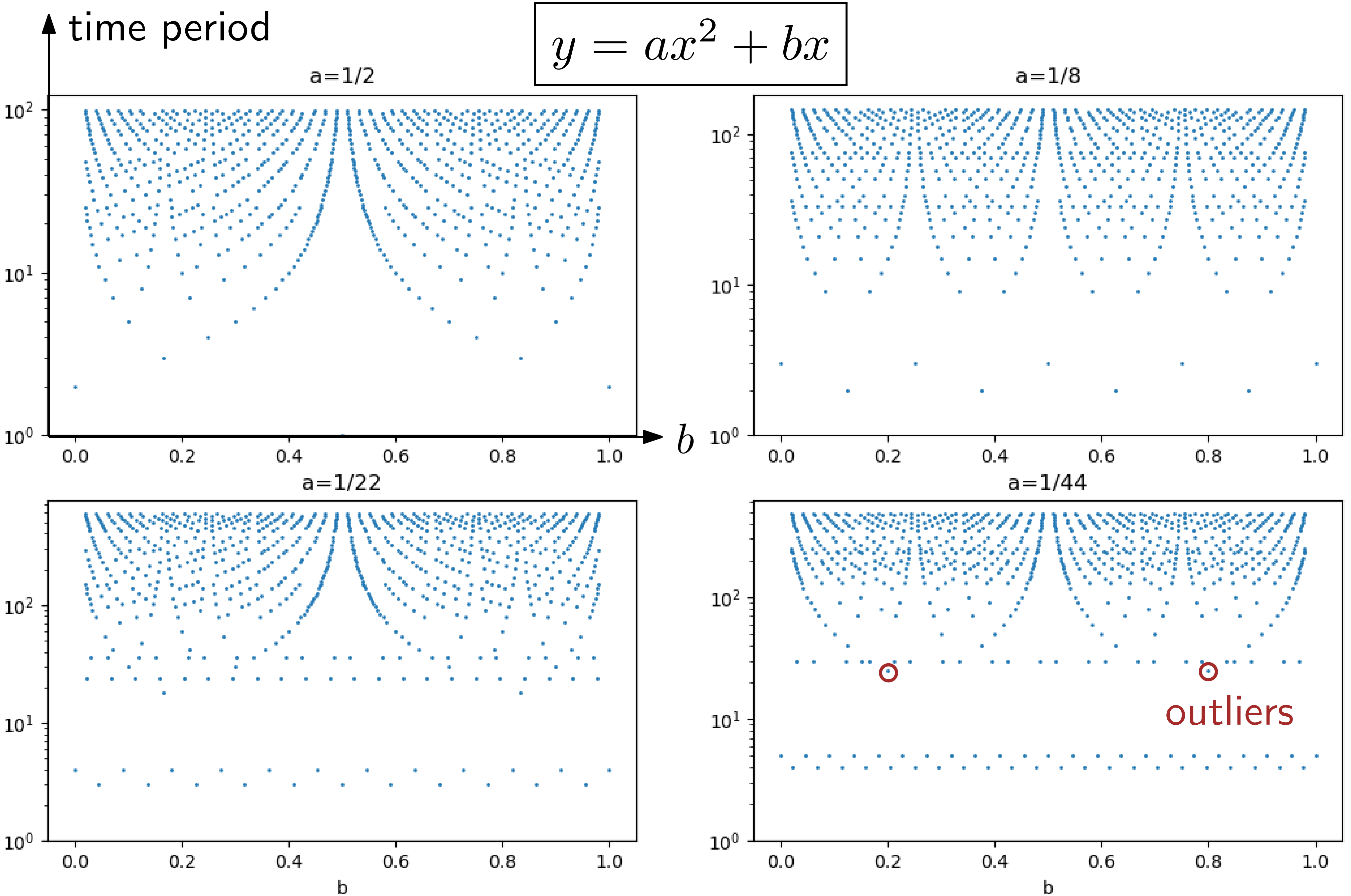
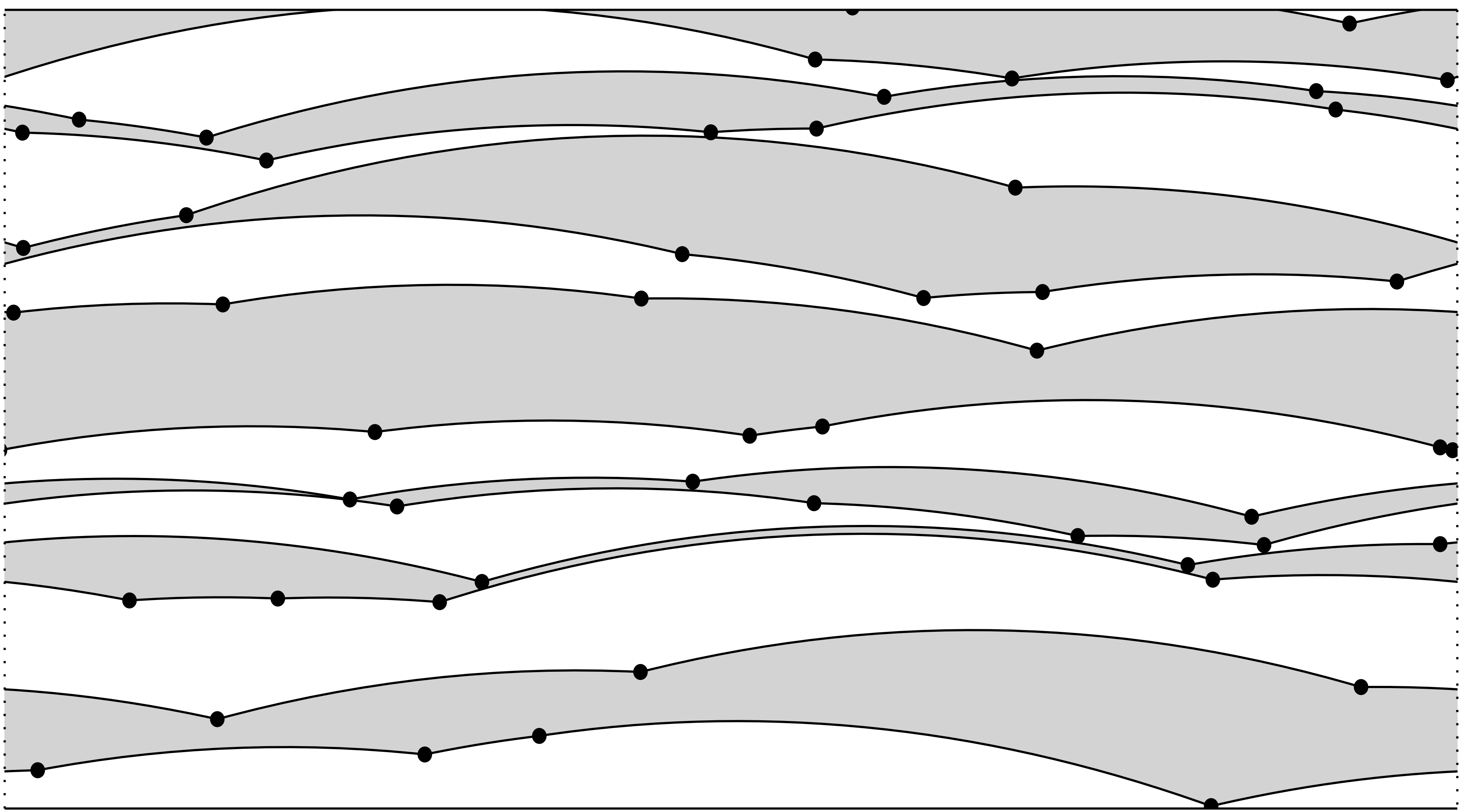


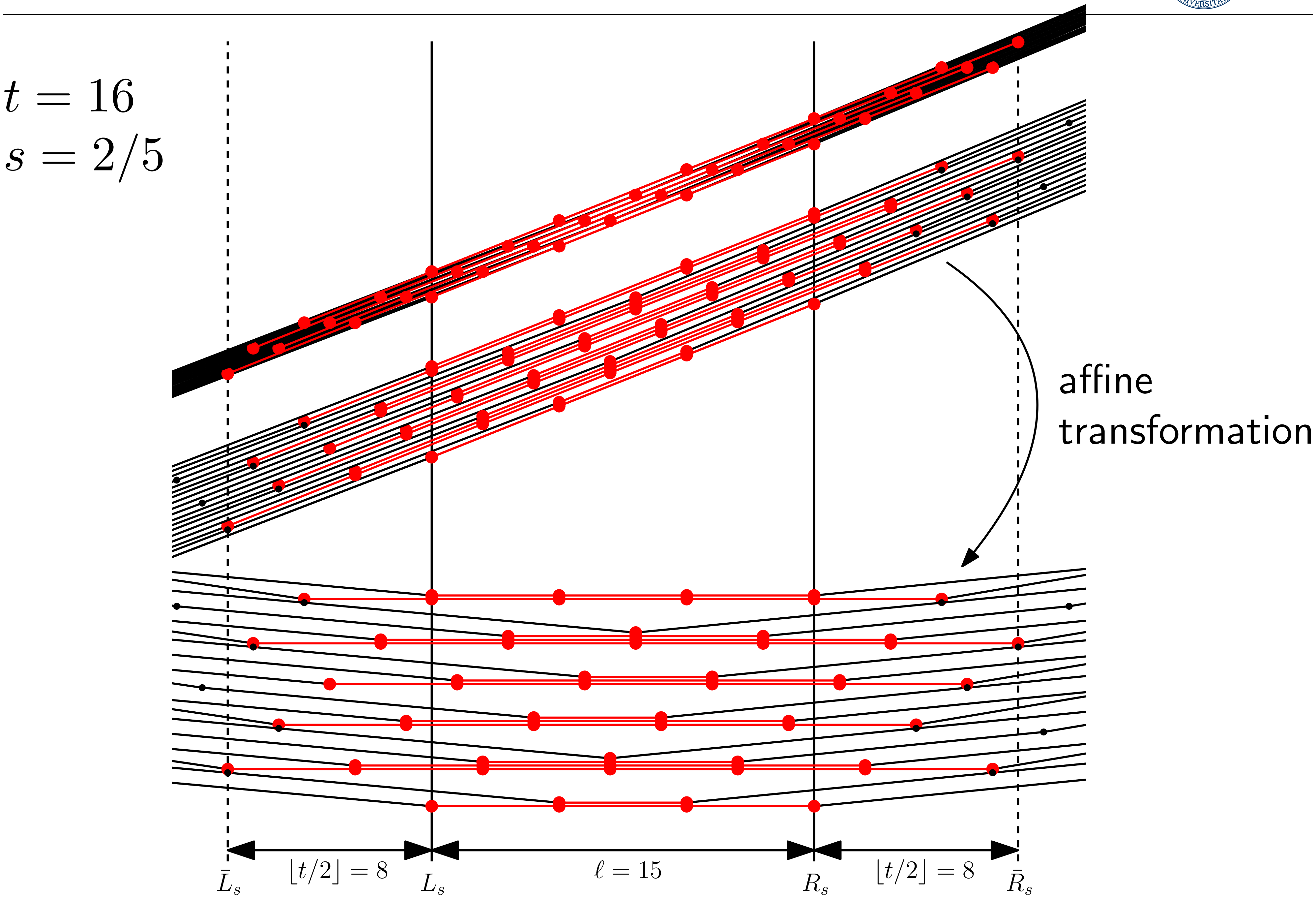
Random-set peeling

Jeff Calder and Charles K. Smart. The limit shape of convex hull peeling. 2020



semiconvex peeling, on a cylinder

Focus on one slope

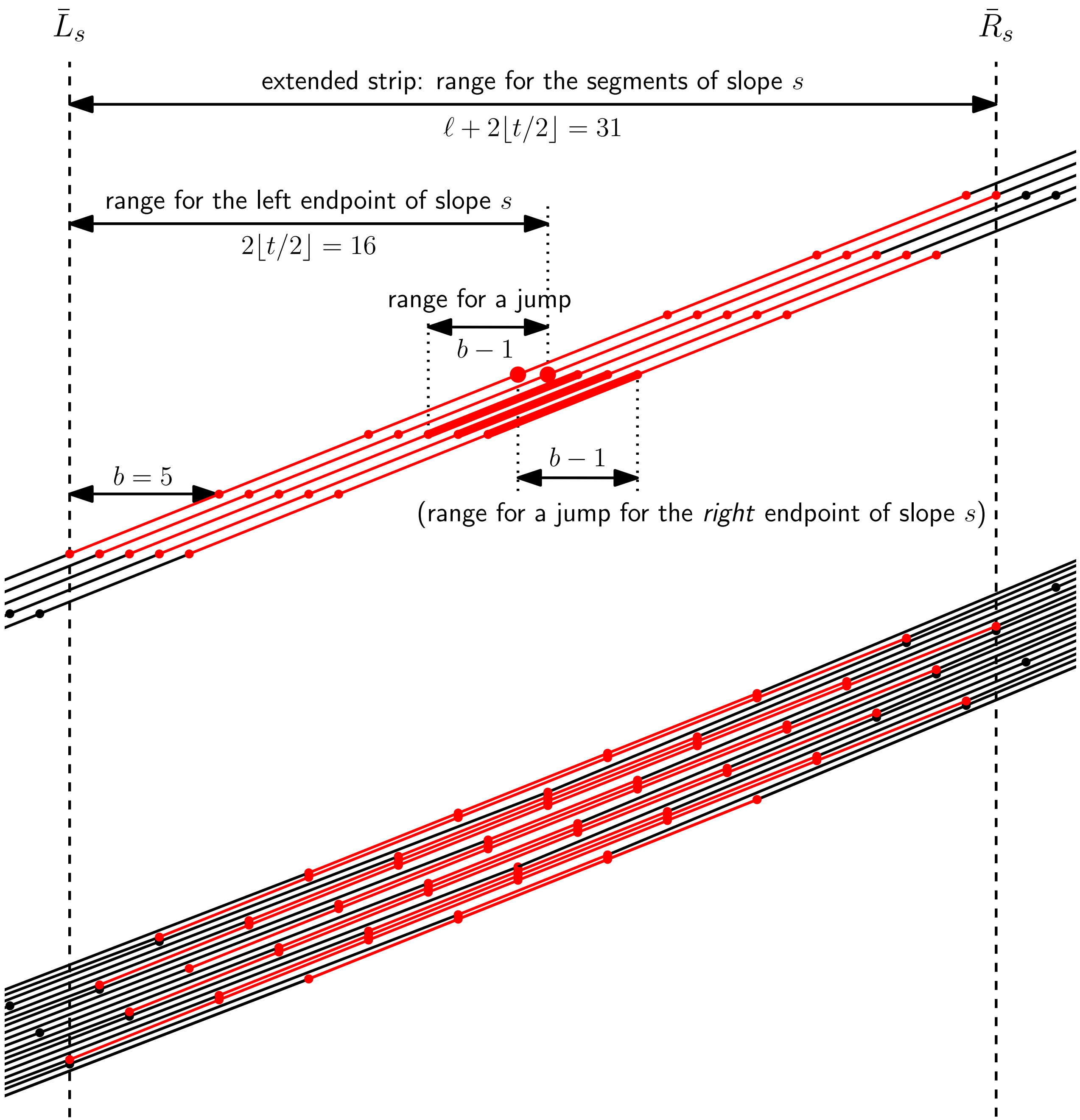


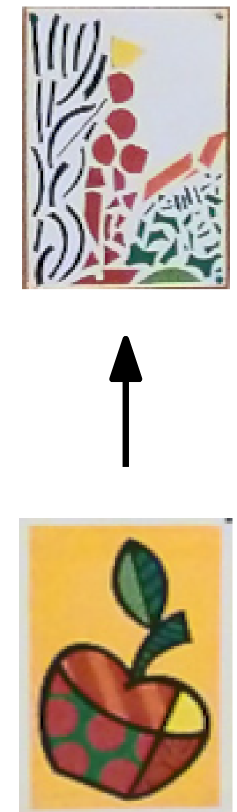
What happens at a jump?

JUMP RULES:

- jump to the *next* grid line of slope s
- fill the extended strip $[\bar{L}_s, \bar{R}_s]$ as much as possible

All possible grid lines of slope $s = 2/5$





cf. Ursus Wehrli:
Kunst aufräumen

Two adjacent slopes s, s'

Freie Universität Berlin