Adaptive Intersection of Bézier Splines by the SUPER-COMPOSITION Method

Günter Rote
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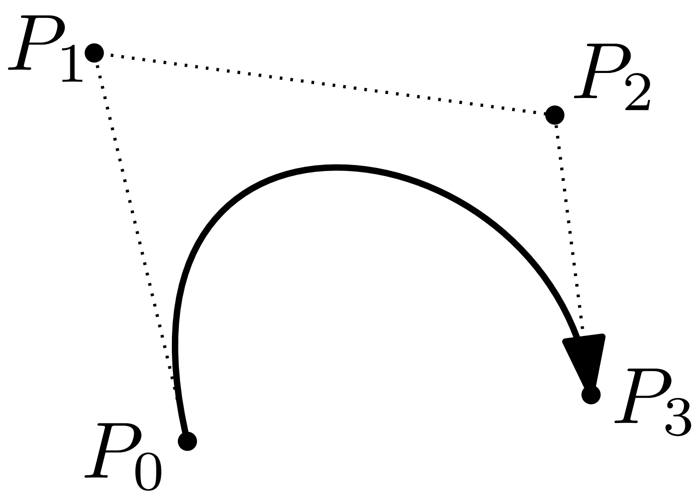
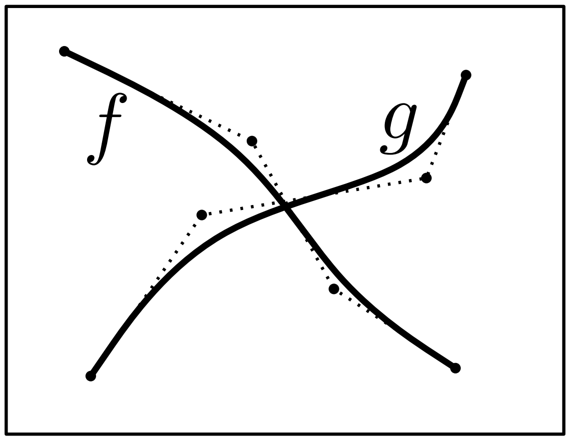
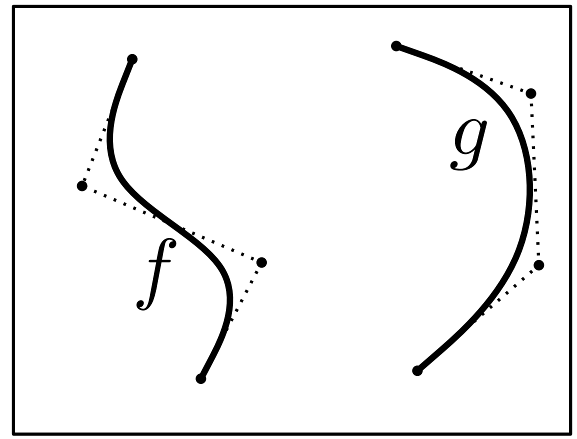


$$f(t) = \underbrace{\sum_{i=0}^{d} \binom{d}{i} t^i (1-t)^{d-i}}_{\text{Bernstein polynomials}} \cdot P_i$$

Intersecting two Bézier splines



easy



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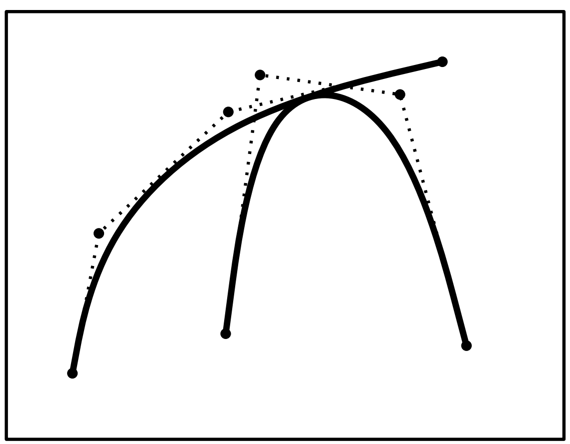
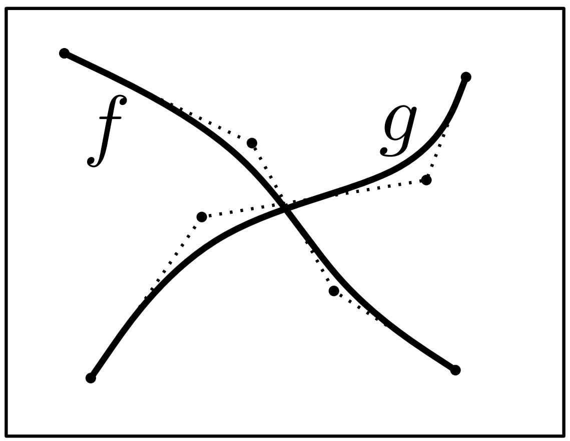
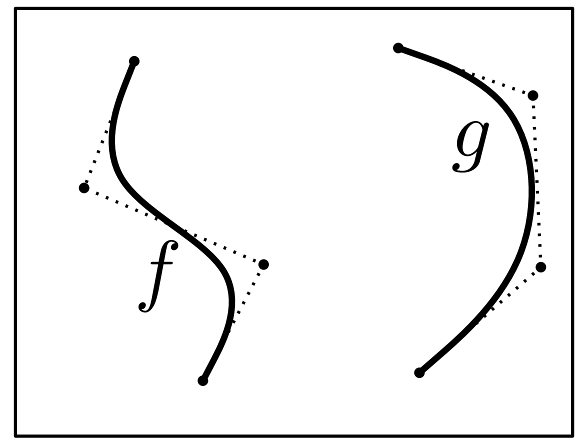


hard

Intersecting two Bézier splines



easy



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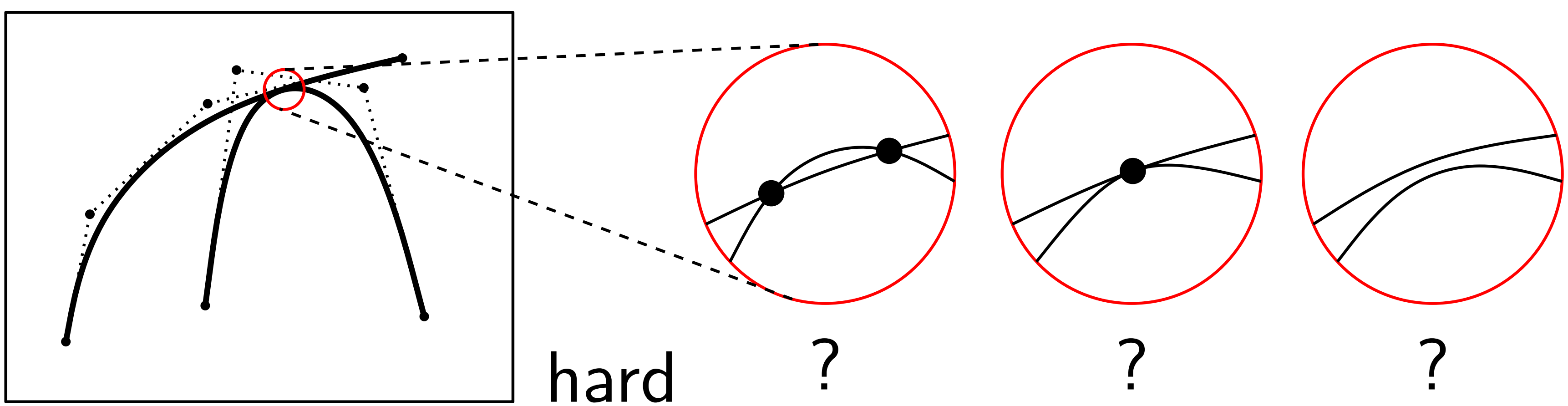
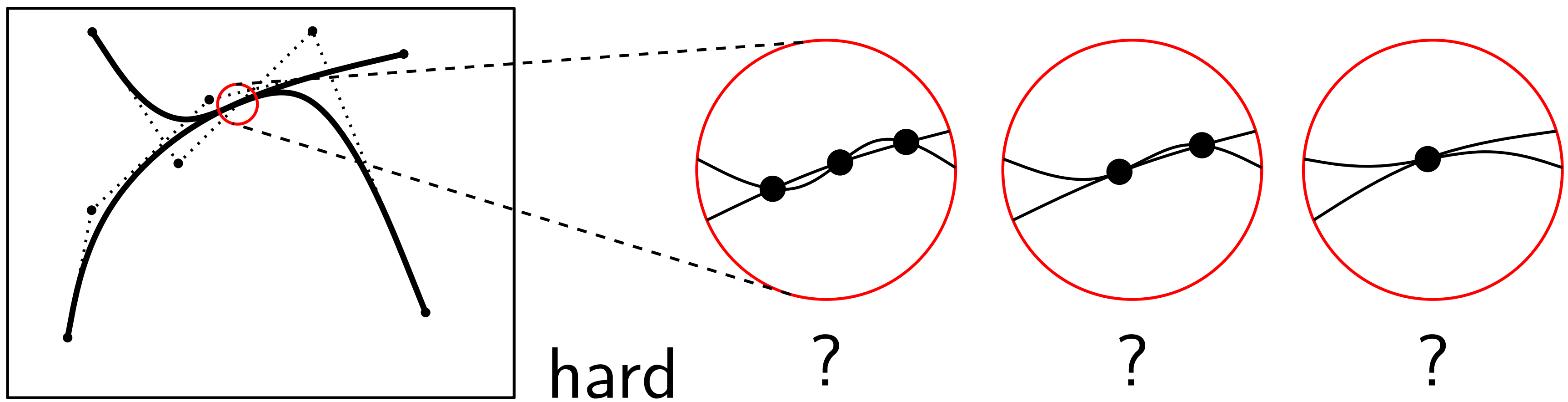
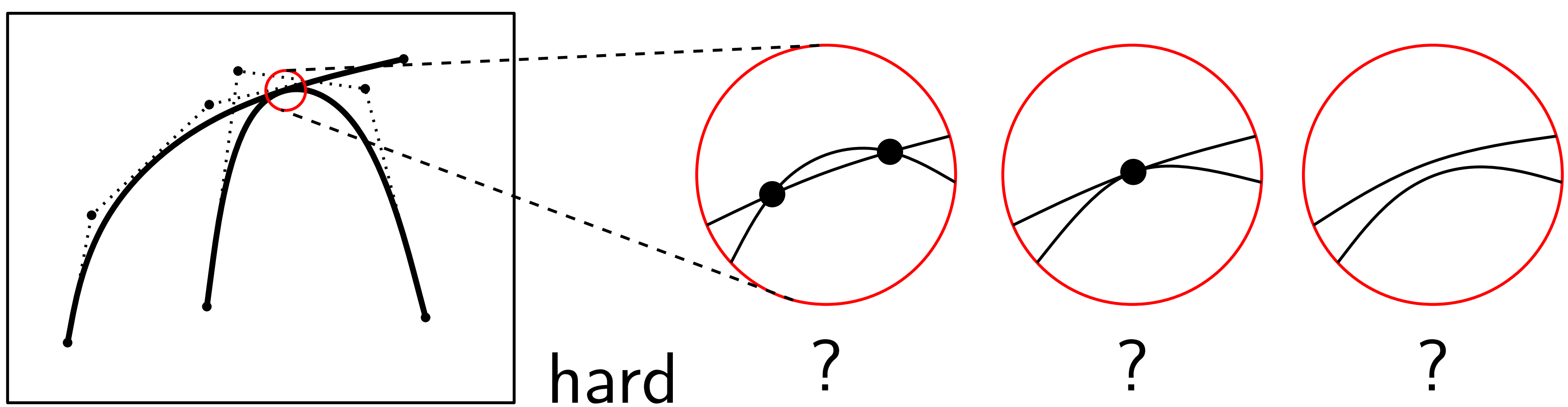


hard

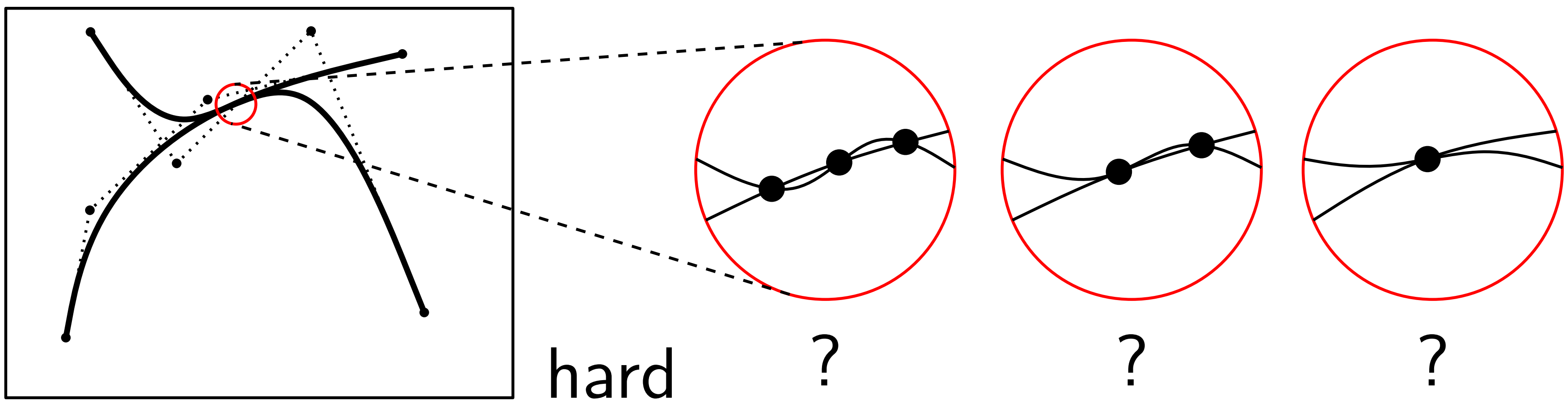
? ? ?

other hard cases

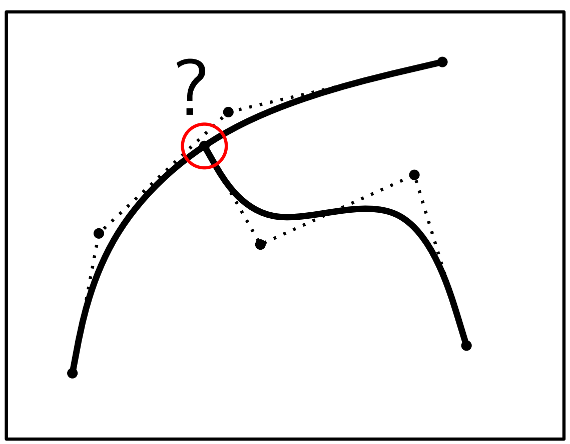




other hard cases



hard



hard

Intersecting two Bézier splines

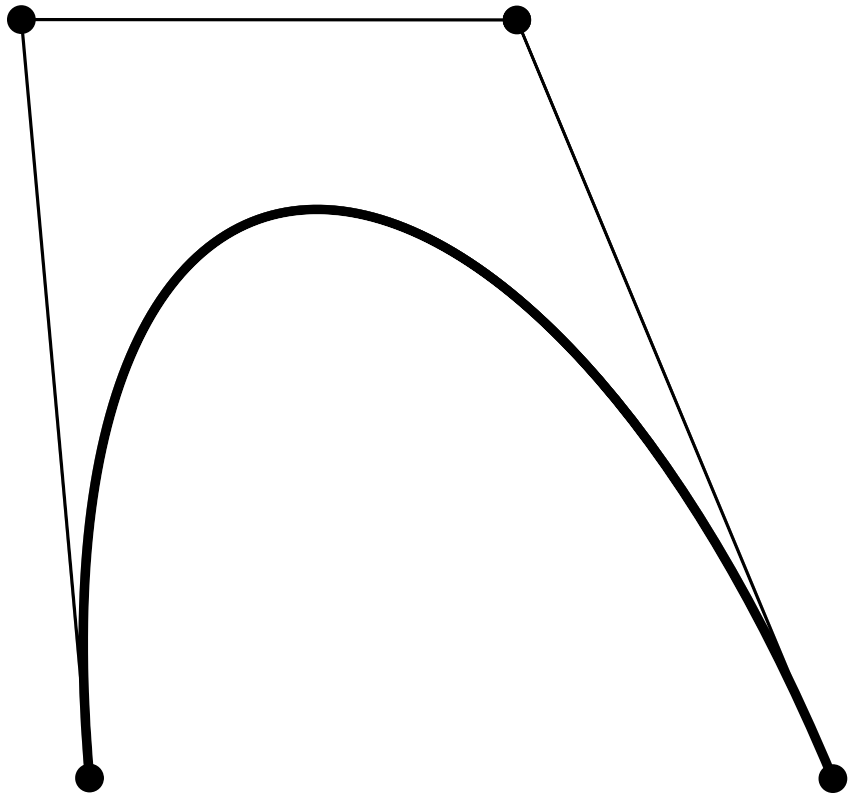
Easy cases:

- *transverse* intersection (large angle)
- *large* distance between curves

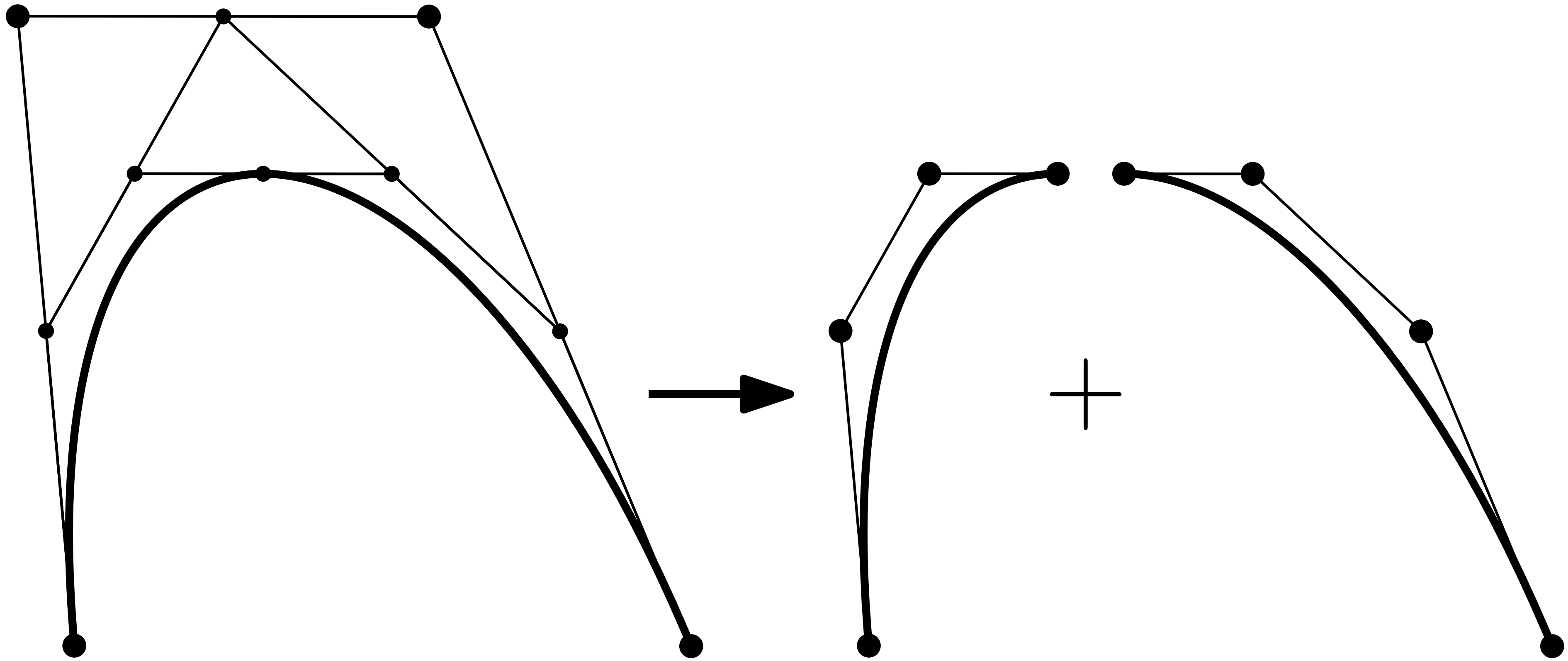
Hard cases:

- intersection with small angle or even *tangency*
- curves come *close* without intersecting
- endpoint of one curve near the other curve

Bézier curve subdivision



Bézier curve subdivision



Intersection algorithm by subdivision

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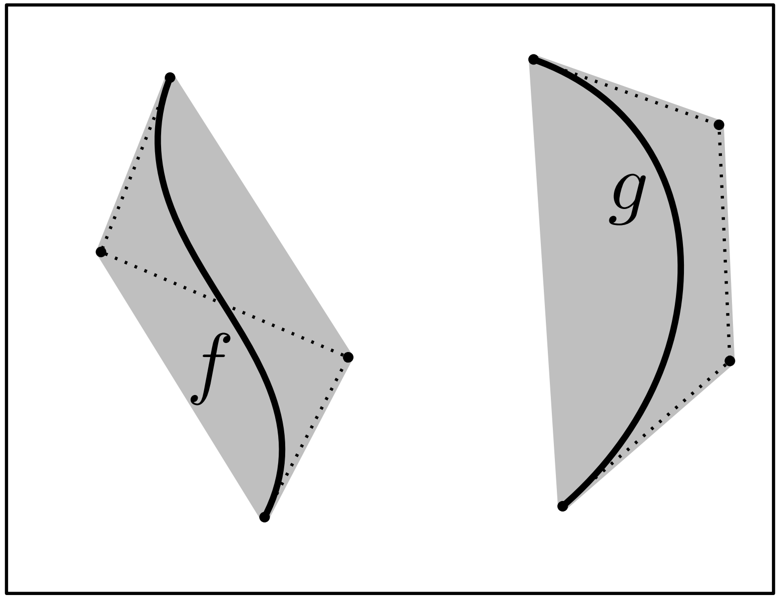
PUSH (f, g) on stack
while stack is not empty:
  POP (f, g) from stack
  if f, g are guaranteed to have no intersection:
    discard f, g
  elsif f, g are guaranteed to have a unique intersection
   and the precision of intersection is good enough:
    report intersection
  else:
    subdivide the larger curve (say, f) into f_1 and f_2.
    PUSH (f_1, g) on stack
    PUSH (f_2, g) on stack

Sufficient condition for disjointness



(easy)

The curve is inside the convex hull of the control polygon.

convex hulls disjoint $\Rightarrow$ no intersection.

Sufficient condition for unique crossing

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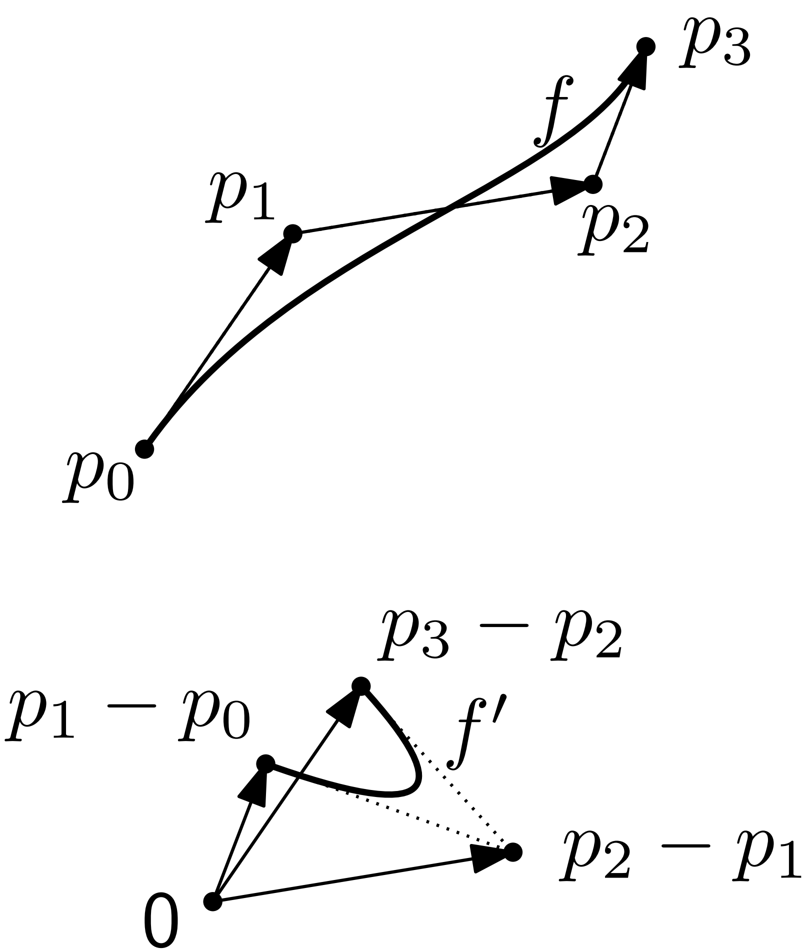
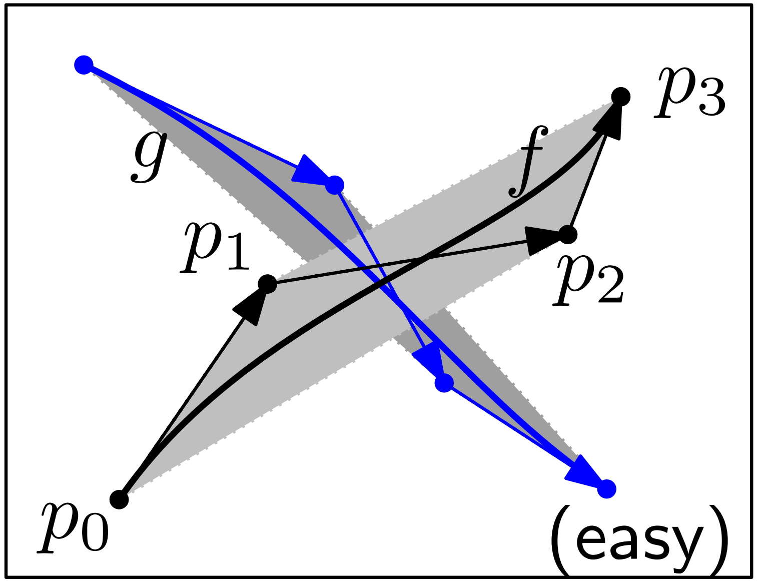
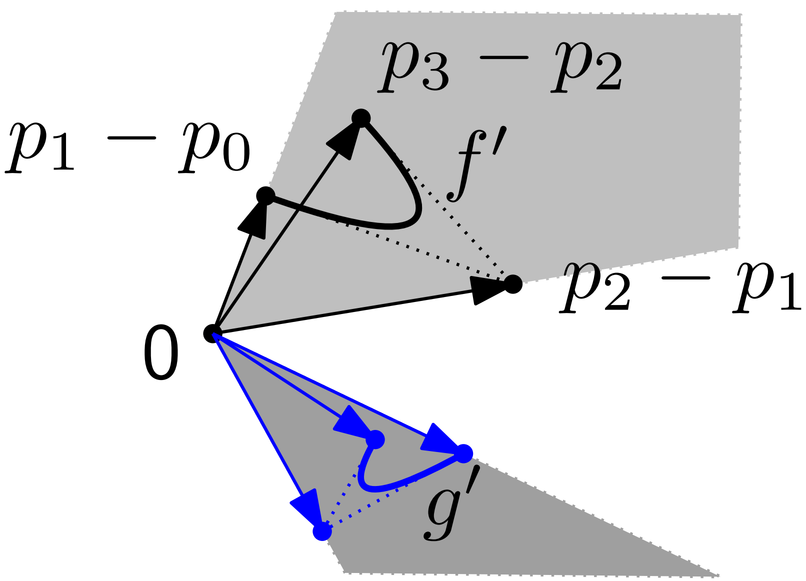


The derivative of a Bézier curve f is a Bézier curve f', of degree one less.

The control polygon of f' is formed from the differences $p_{i+1} - p_i$ of the original control polygon, times a constant factor.

Sufficient condition for unique crossing





If

- the convex hulls of f and g "cross" (the endpoints stick out),
- and the cones of directions of f' and g' are disjoint,

then f and g intersect in a single point.

Intersection algorithm by subdivision

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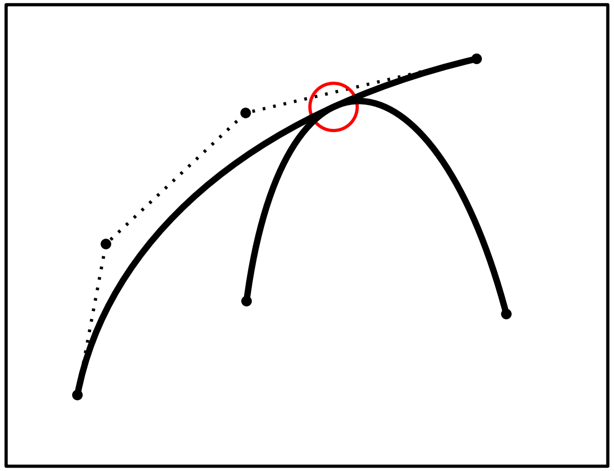
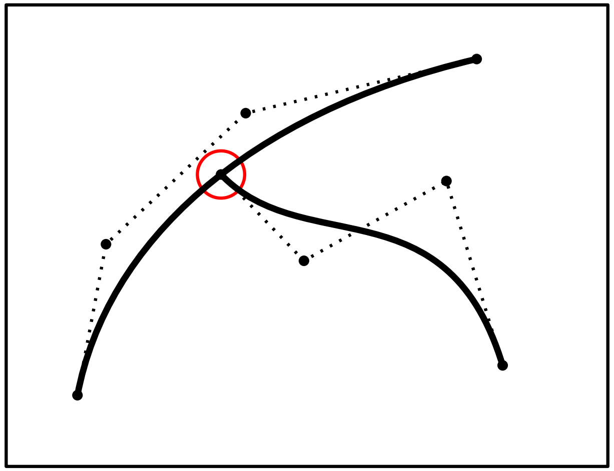
Problems with termination

The subdivision algorithm
will *never* terminate



- when curves are *tangent*, or

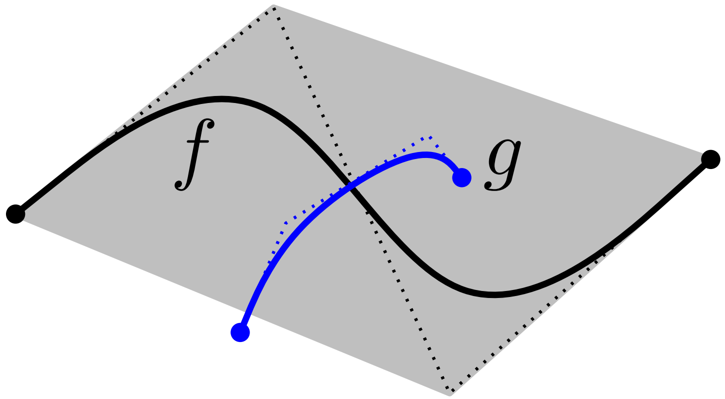


- when an endpoint lies *on* a curve.

(hard cases)

Problems with termination

The algorithm may also fail in easy cases:



no decision.

→ subdivide $f = f_1 + f_2$

Problems with termination

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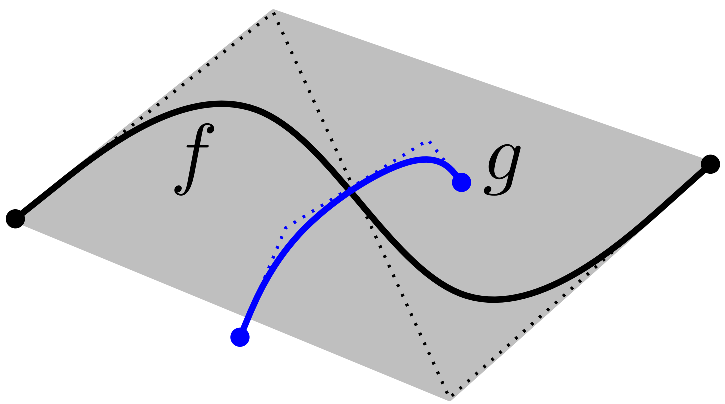
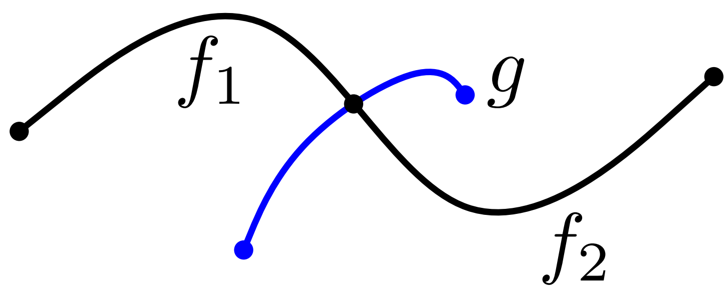


no decision.
$\rightarrow$ subdivide $f = f_1 + f_2$



The subdivision point $f(1/2)$ happens to fall on g.
$\rightarrow$ infinite loop for (f_1, g) and for (f_2, g)

Problems with termination

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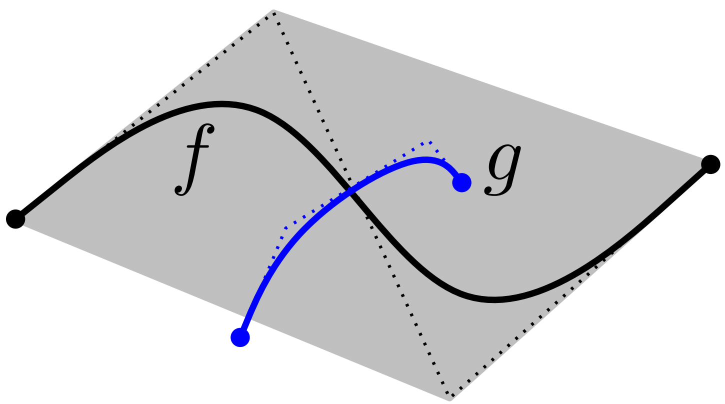

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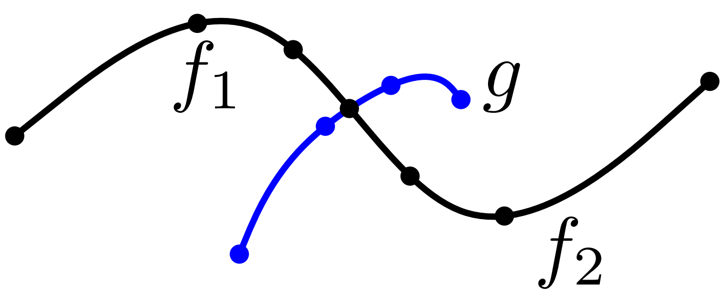

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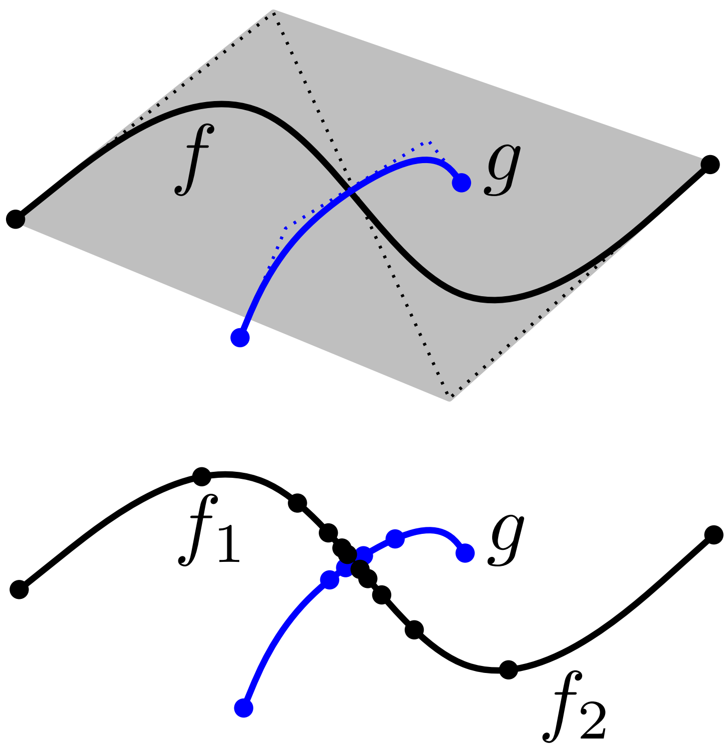

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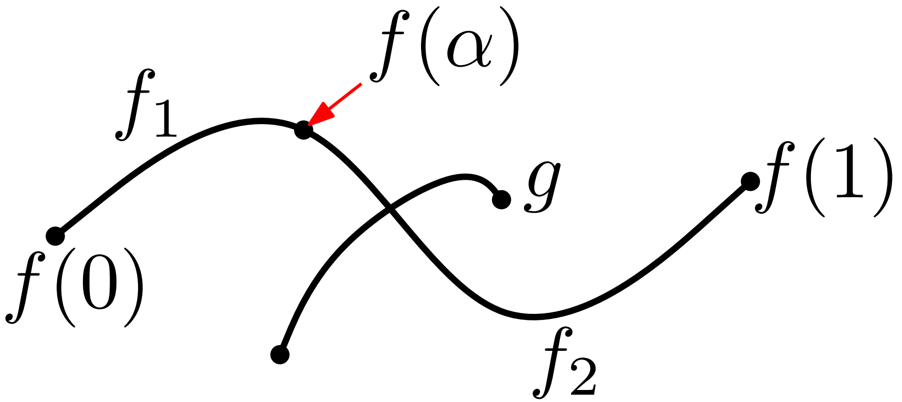
The subdivision point $f(1/2)$ happens to fall on g.
$\rightarrow$ infinite loop for (f_1, g) and for (f_2, g)

Even it the algorithm terminates, it may make unnecessarily many subdivision steps.

Randomization

One possible solution:
Don't subdivide at $1/2$, but at a *random* point $0 < \alpha < 1$.



$\rightarrow$ The problematic case is avoided with high probability.

[A. Eigenwillig, L. Kettner, W. Krandick, K. Mehlhorn, S. Schmitt, N. Wolpert: A Descartes algorithm for polynomials with bit-stream coefficients (CASC 2005)]

Drawback:
Subdivision at points other than $1/2$ is costly in terms of bit complexity.

Zeros of a polynomial

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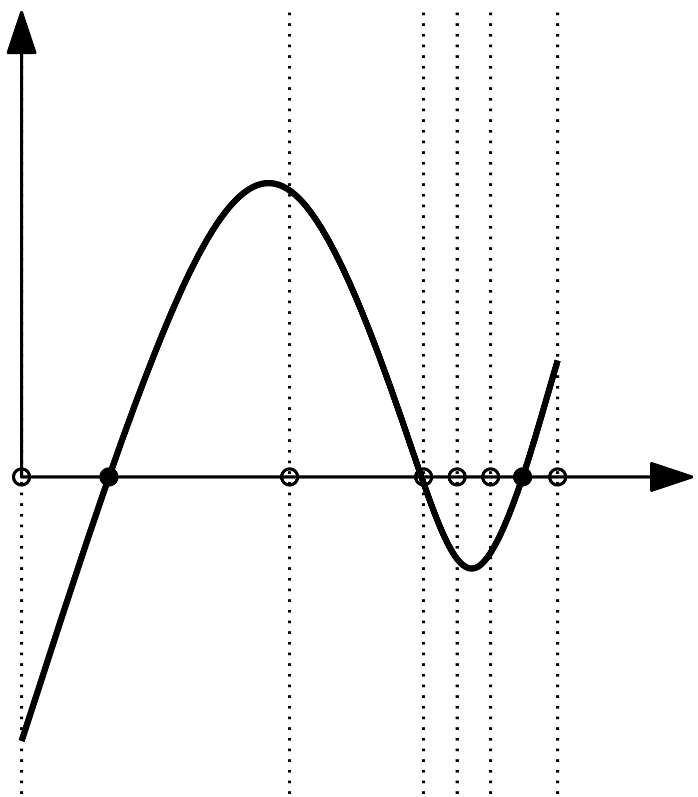
Eigenwillig, Kettner, Krandick, Mehlhorn, Schmitt, Wolpert (2005)
A Descartes algorithm for polynomials with bit-stream coefficients



Task: root isolation

Assumption: no multiple roots

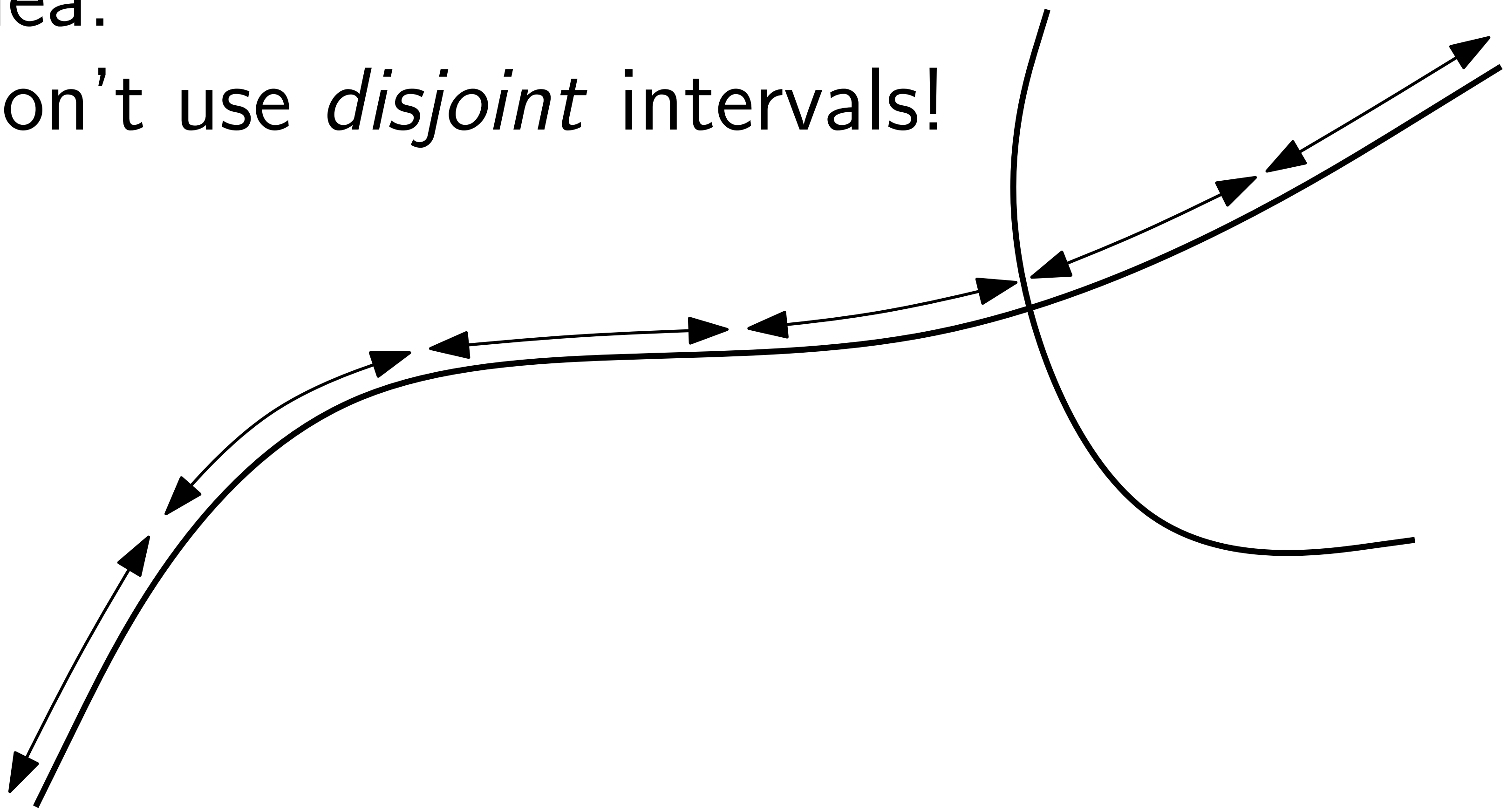
Descartes Rule of Signs may identify an interval as contain 0 roots or exactly 1 root.

If not, bisect and repeat.

SUPER-Composition

Idea:

Don't use *disjoint* intervals!



SUPER-Composition

Idea:
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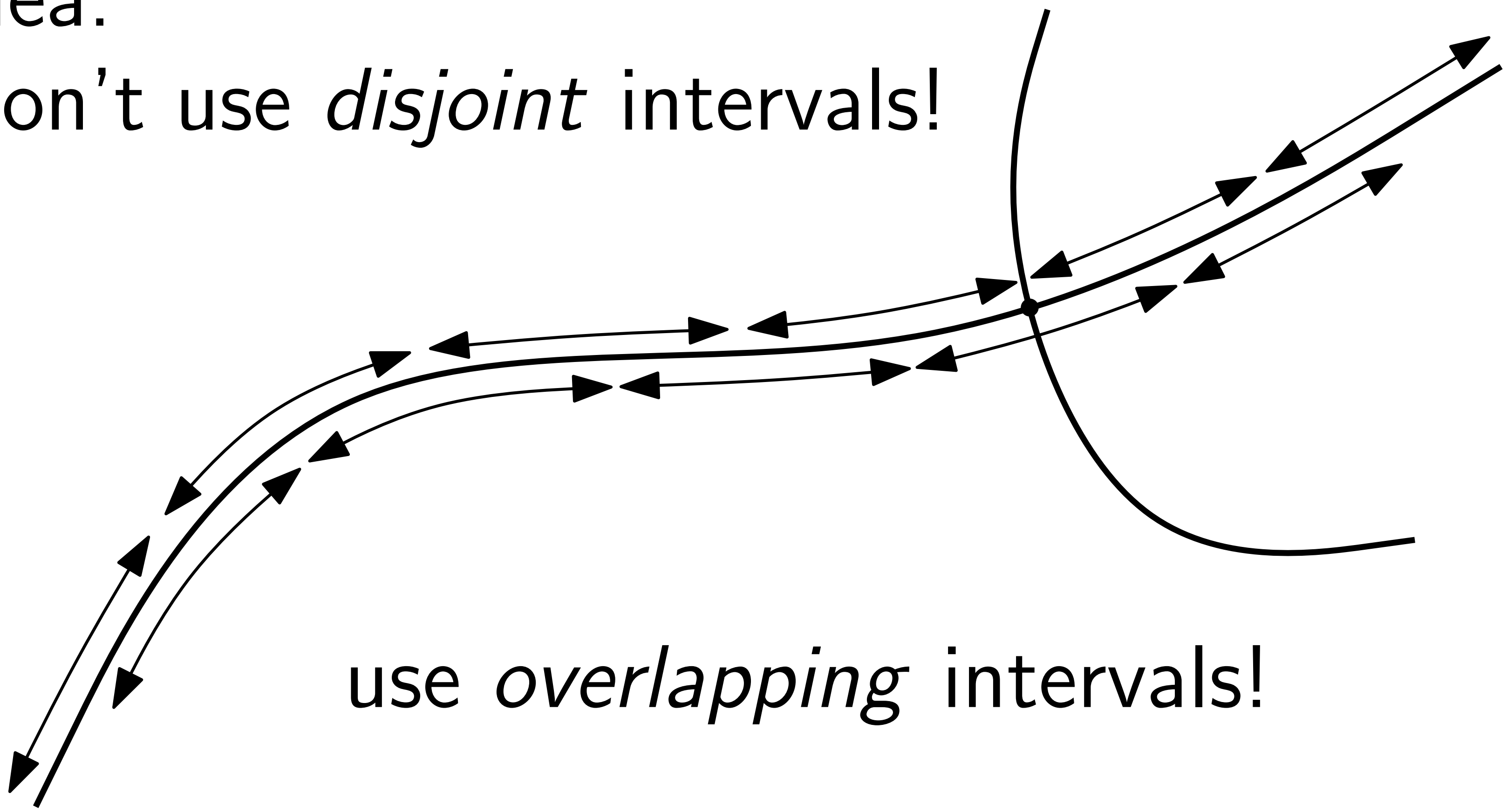
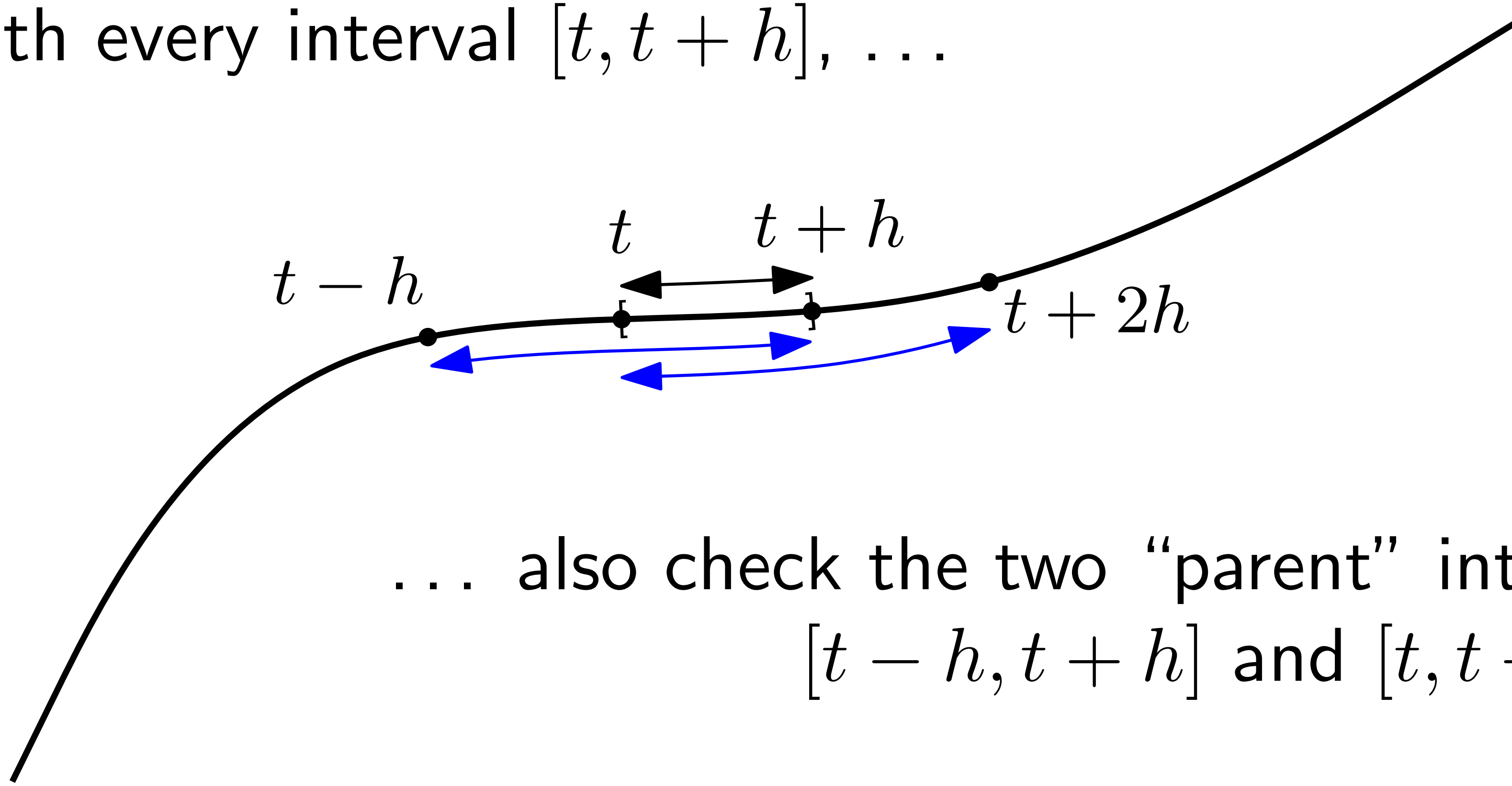


use *overlapping* intervals!

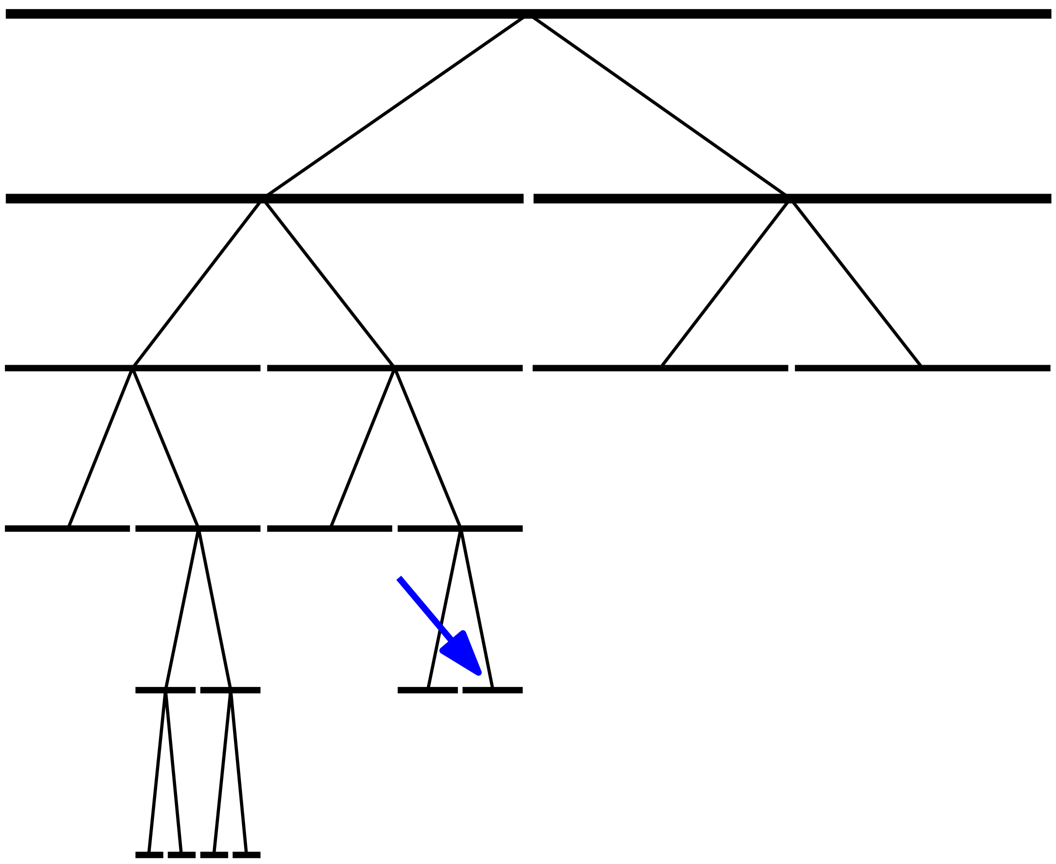
SUPER-Composition

With every interval $[t, t+h]$, . . .



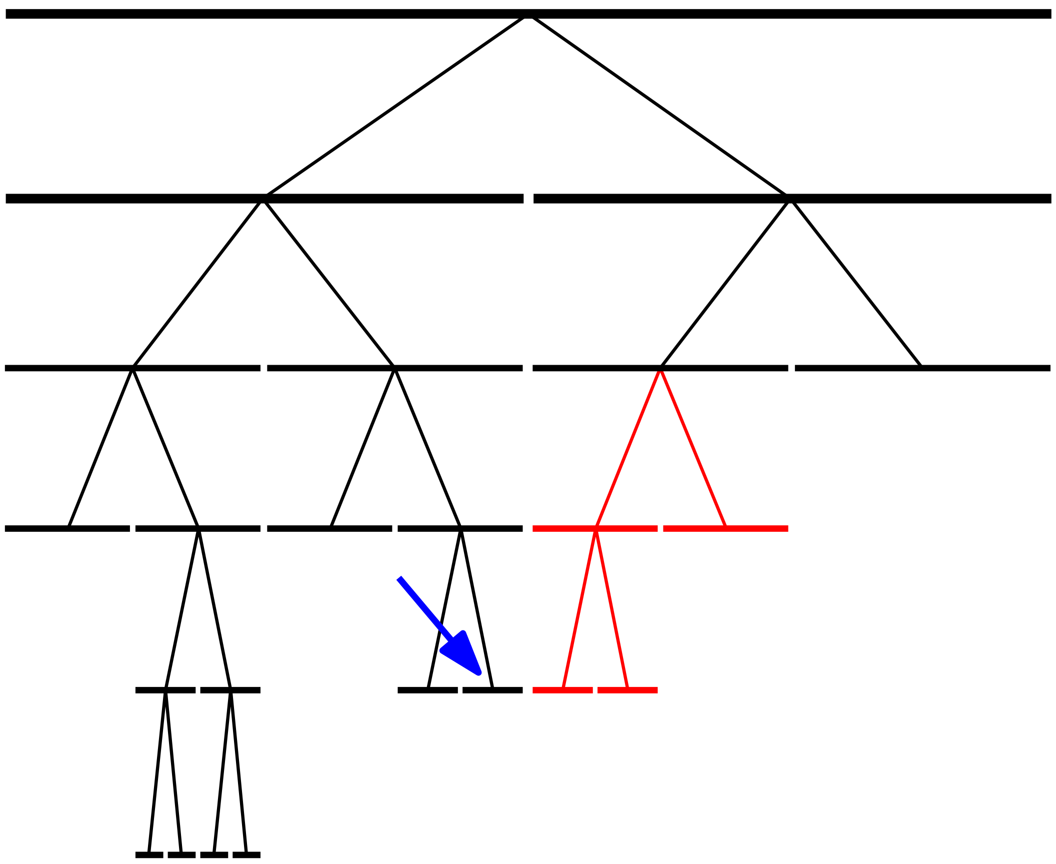
. . . also check the two “parent” intervals $[t-h, t+h]$ and $[t, t+2h]$.

SUPER-Composition



binary
decomposition tree

SUPER-Composition



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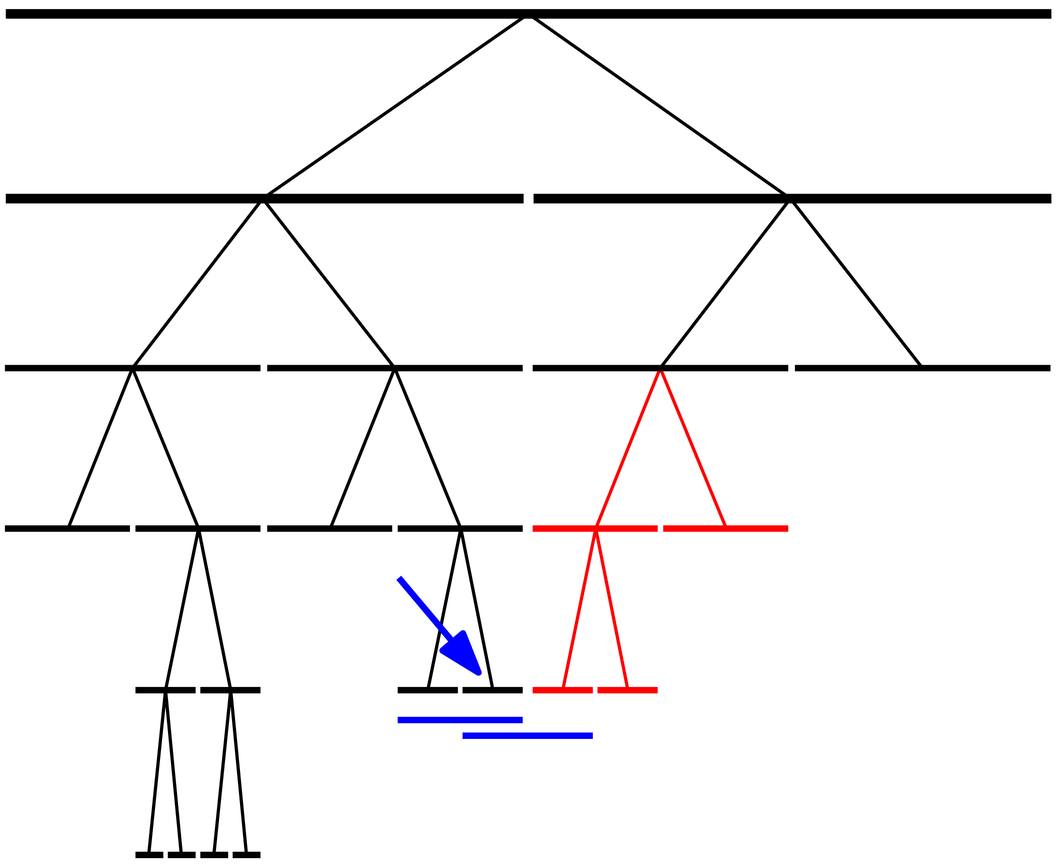
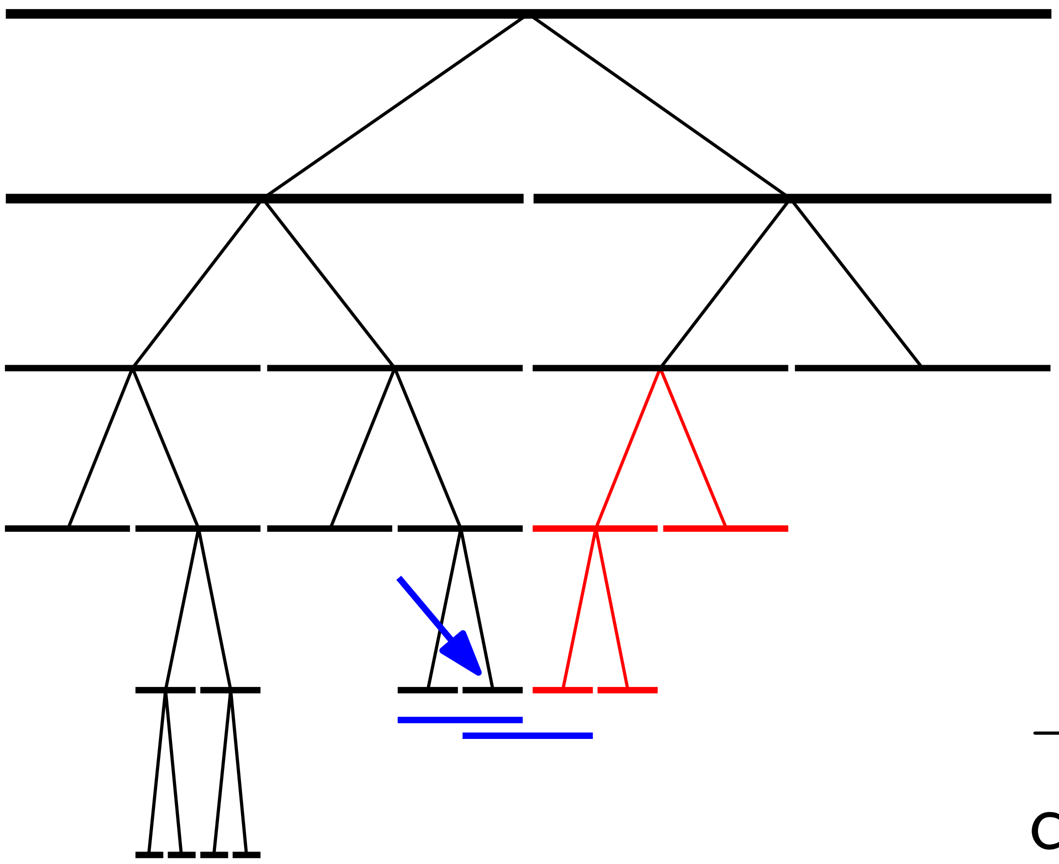
SUPER-Composition



binary decomposition tree

SUPER-Composition



binary decomposition tree

algorithm must cross subtree boundaries.

$\longrightarrow$

constant-factor overhead (cf. balanced quadtree)

Result

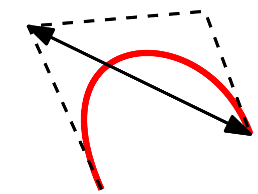
Theorem 1 *If*

- *the derivative of* f *and* g *is nowhere zero, and*
- *at every intersection point, the curves cross at a positive angle, and*
- *no endpoint of* f *or* g *lies on the other curve,*

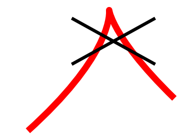
then the subdivision-supercomposition algorithm terminates.

Result

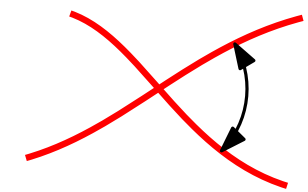
Theorem 2 • *If the diameter of the control polygon of f and g is at most D,*



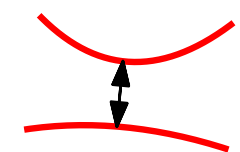
- $\|f'(t)\| \geq v_{\min}$ *and* $\|g'(t)\| \geq v_{\min}$ *everywhere,*



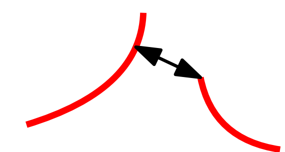
- *every intersection angle is at least α, and*



- *the distance between f and g is at least ε, at every local minimum and at every endpoint,*

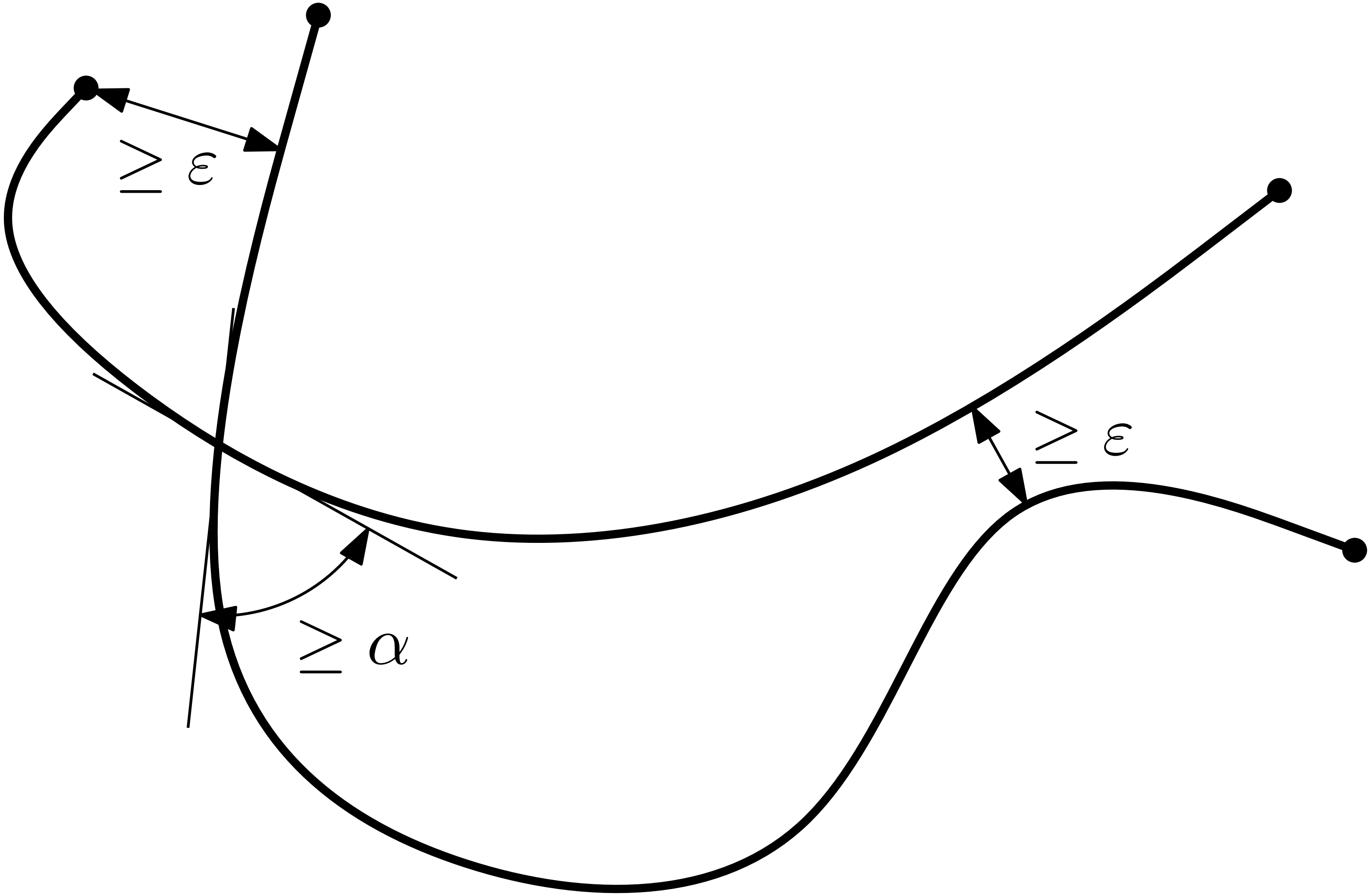


then the number of subdivision levels is at most



$$\max\left\{\log_2 \frac{D}{v_{\min} \cdot \alpha}, \log_4 \frac{D}{\varepsilon}\right\} + 2\log_2 d + 4.$$

The conditions of the theorem



Proof of the theorem

Assumptions:
f and g are Bézier curves of degree d.
Initial parameter interval $= [0, 1]$.

diameter of control polygon at most D
$\Rightarrow$

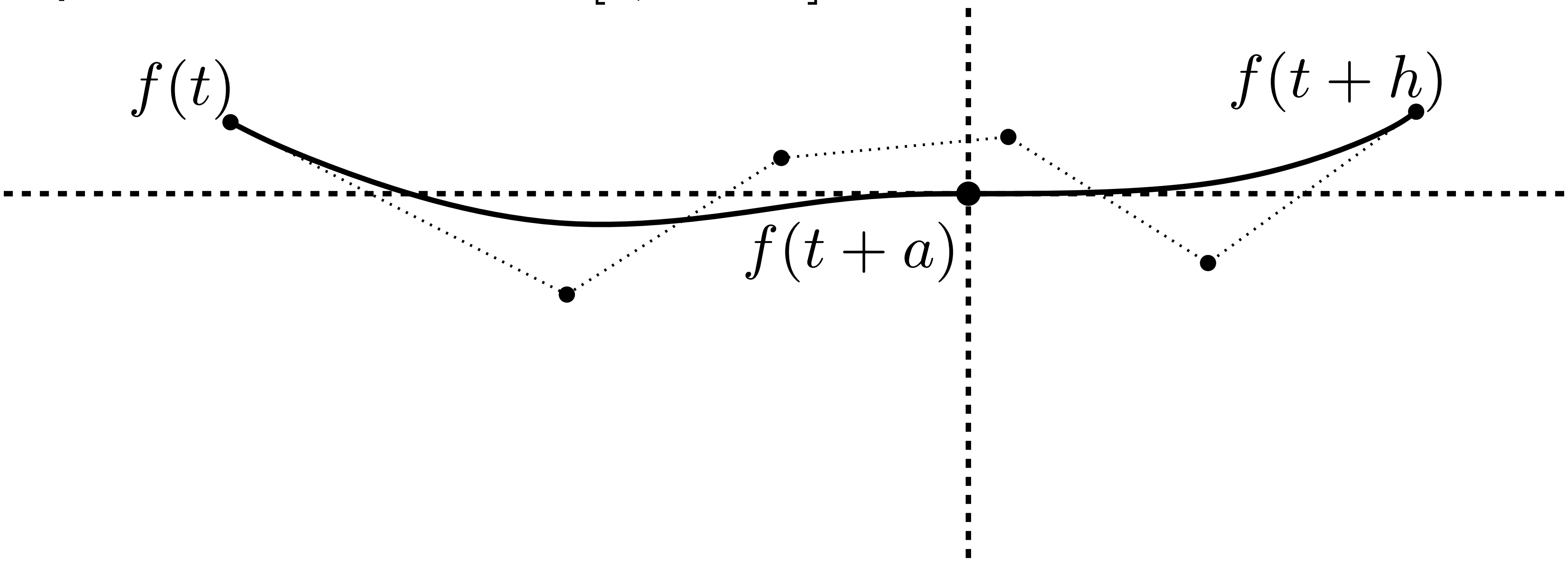
$$\|f'(t)\|, \|g'(t)\| \le 2dD$$

$$\|f''(t)\|, \|g''(t)\| \le S := 4d(d-1)D$$

$\|f'(t)\|, \|g'(t)\| \ge v_{\min}$
$\Rightarrow$ curvature of f and g is at least $S/v_{\min}$

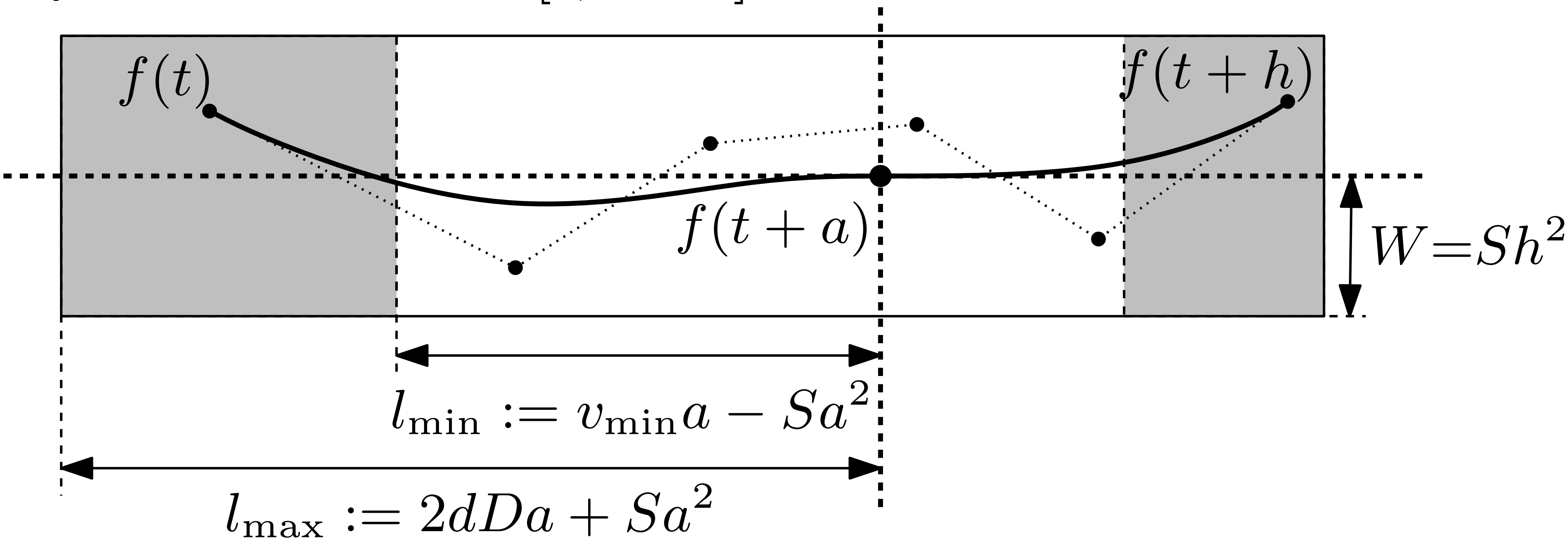
Proof of the theorem

parameter interval $[t, t+h]$



Proof of the theorem

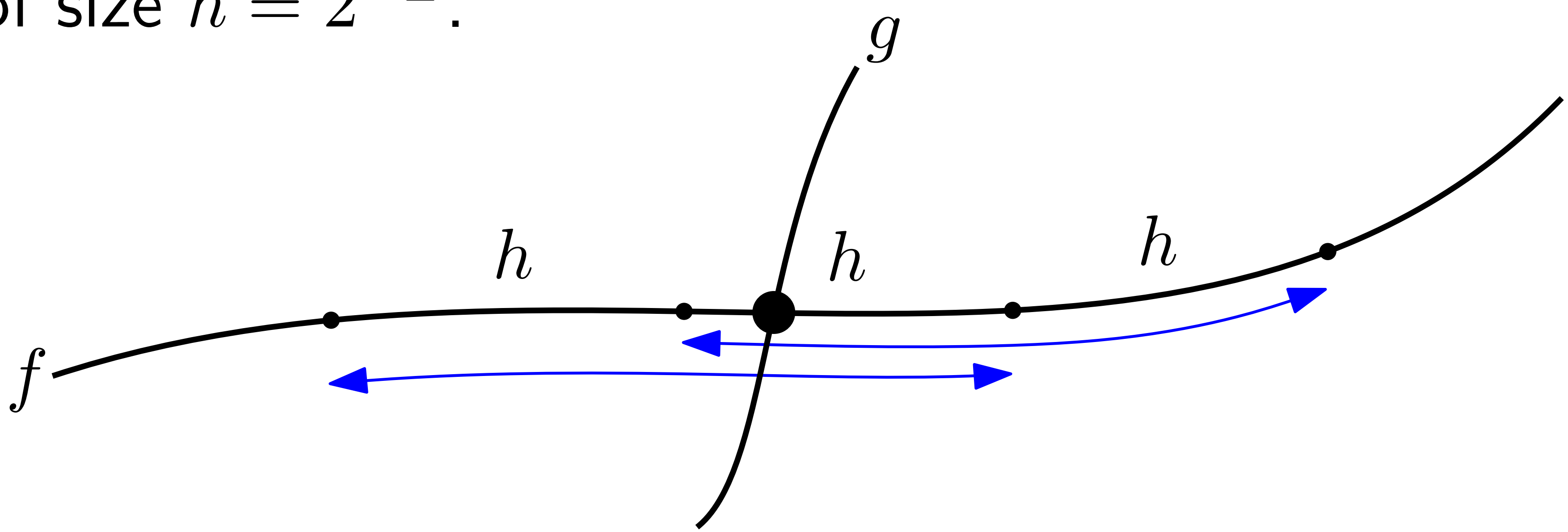
parameter interval $[t, t+h]$



The curve and the control polygon is contained in a rectangular strip of width $W = S \cdot h^2$ and length $\Theta(h)$.

Intersection points

Assume complete subdivision for L levels into intervals of size $h = 2^{-L}$.



Then there is an interval where the intersection point is at least $h/2$ away from both ends.

Intersection points

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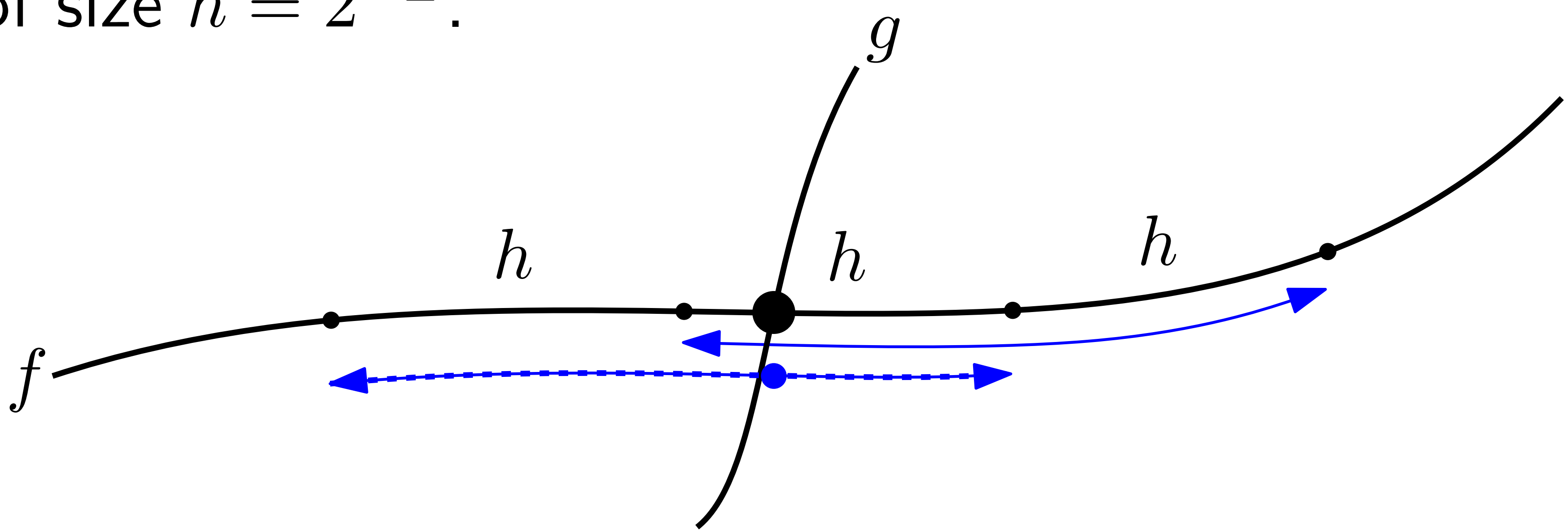

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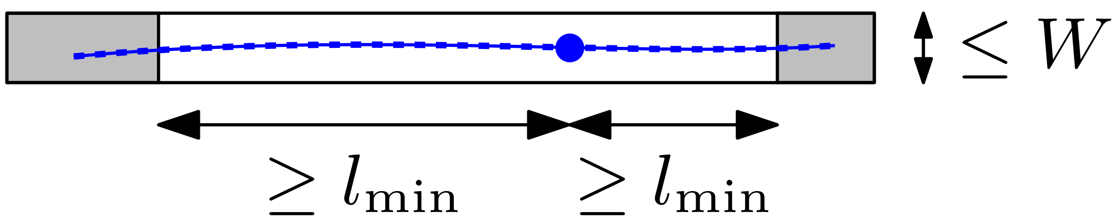

Intersection points

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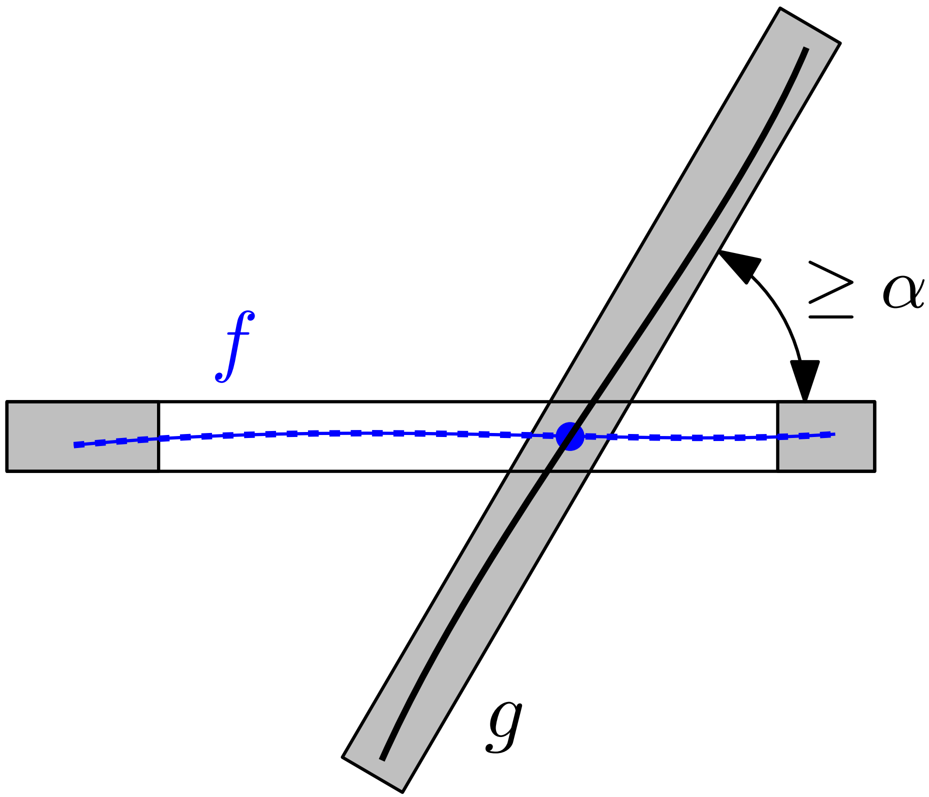
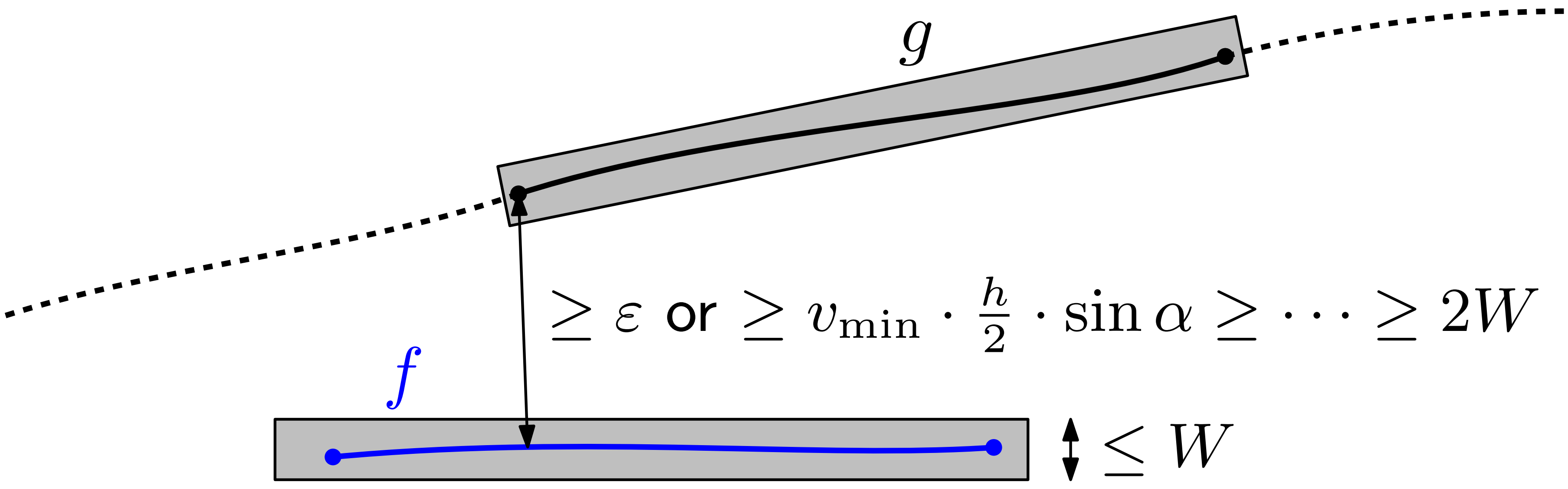


$\Rightarrow$

The endpoints of f stick out of the control polygon of g.

No intersections



Runtime analysis

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Theorem 2 *The number of subdivision levels is at most*

$$L := \max\left\{\log_2 \frac{D}{v_{\min}\cdot\alpha}, \frac{1}{2}\cdot\log_2\frac{D}{\varepsilon}\right\} + O(1).$$

Corollary 1 *The running time is at most*

$$2^L \times 2^L = O\left(\frac{D^2}{(v_{\min}\cdot\alpha)^2} + \frac{D}{\varepsilon}\right).$$

Tightness of the bound

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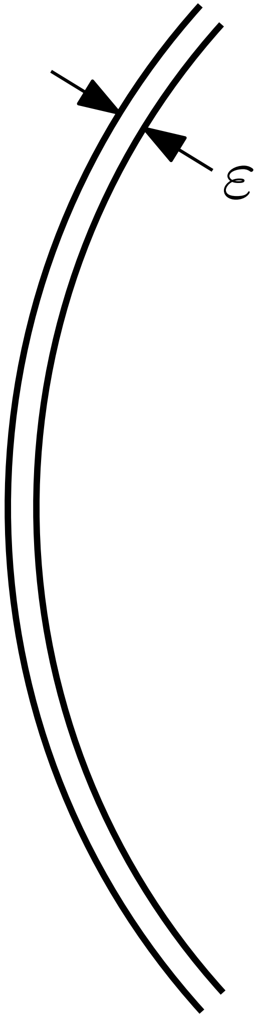
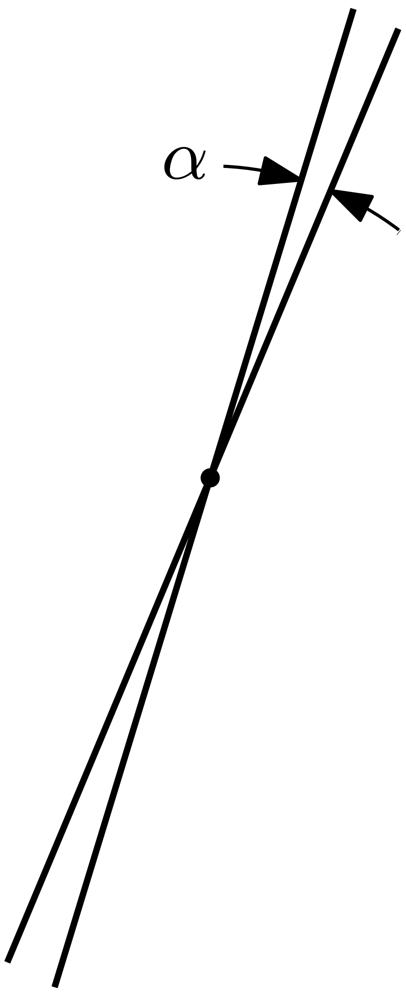


Each curve is subdivided into $O(\sqrt{1/\varepsilon})$ pieces.

Running time is $O(\sqrt{1/\varepsilon})$.

NOT $O(\sqrt{1/\varepsilon} \times \sqrt{1/\varepsilon})$, as given by the corollary!

Tightness of the bound



Each curve is subdivided into $O(\log \frac{1}{\alpha})$ pieces.

Running time is $O(\log \frac{1}{\alpha})^2$.

NOT $O(1/\alpha^2)$, as given by the corollary!

Future work — Open questions

- Better analysis of the runtime

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- higher dimensions
  (e. g., intersecting a curve with a surface patch)

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- Better analysis of the runtime
- higher dimensions
  (e. g., intersecting a curve with a surface patch)
- What is the right name
  for $\begin{cases} \text{Sturm-Habitch sequences?} \\ \text{Sturm-Habicht sequences?} \end{cases}$