There is no triangulation of the torus with vertex degrees $5, 6, \dots, 6, 7$ and related results: Geometric proofs for combinatorial theorems

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Abstract

There is no 5,7-triangulation of the torus, that is, no triangulation with exactly two exceptional vertices, of degree 5 and 7. Similarly, there is no 3,5-quadrangulation. The vertices of a 2,4-hexangulation of the torus cannot be bicolored. Similar statements hold for 4,8-triangulations and 2,6-quadrangulations. We prove these results, of which the first two are known and the others seem to be new, as corollaries of a theorem on the holonomy group of a euclidean cone metric on the torus with just two cone points. We provide two proofs of this theorem: One argument is metric in nature, the other relies on the induced conformal structure and proceeds by invoking the residue theorem. Similar methods can be used to prove a theorem of Dress on infinite triangulations of the plane with exactly two irregular vertices. The non-existence results for torus decompositions provide infinite families of graphs which cannot be embedded in the torus.

1 Introduction

In any triangulation of the torus, the average vertex degree is 6, so vertices of degree $d \neq 6$ can be considered *exceptional.* It is easy to find *regular* triangulations with no exceptional vertices, as in Figure 1. Applying a single edge flip to such a triangulation produces a triangulation with four exceptional vertices (assuming the four vertices in question are distinct): two of degree 5 and two of degree 7, as in Figure 2(a). We call this a $5^2 7^2$-triangulation. Similarly, we can produce examples of triangulations with just two exceptional vertices, assuming these have degrees other than 5 and 7. Figure 3 shows 4,8-, 3,9-, 2,10- and 1,11-triangulations of the torus. However:

Theorem 1 (Jendroľ & Jucovič [14]). *The torus has no* 5,7*-triangulation, that is, no triangulation with exactly two exceptional vertices, of degree* 5 *and* 7.

We can also consider quadrangulations of the torus. In this case, the average vertex degree is 4, and an analogous theorem holds:

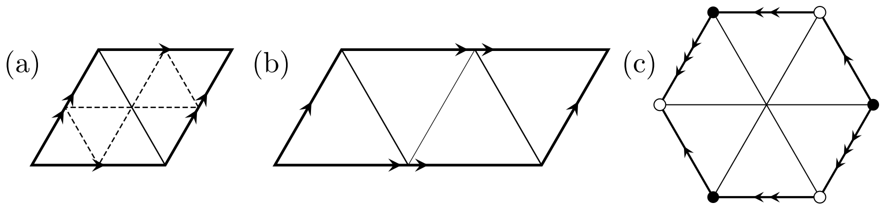

Figure 1: The simplest regular triangulations of the torus. (a) The solid lines show a triangulation with a single vertex of degree 6. Any regular triangulation is a cover of this one. After each triangle is split into four (dashed lines) there are four vertices. (b) The unique regular triangulation with two vertices. (c) There are two regular triangulations with three vertices, one analogous to (b), and this more symmetric one.

Theorem 2 (Barnette, Jucovič & Trenkler [3])**.** *The torus has no* 3,5*-quadrangulation, that is, no quadrangulation with exactly two exceptional vertices, of degree* 3 *and* 5*.*

On the other hand, 2,4- and $3^2 5^2$-quadrangulations do exist, as shown in Figure 4. Finally, one can consider hexangulations of the torus, with average vertex degree 3. In this case, the corresponding result takes a different form. Hexangulations with two irregular vertices of degree 2 and 4 exist, as shown in Figure 5. But any such hexangulation has an odd edge-cycle:

Theorem 3. *The vertices of a* 2,4*-hexangulation of the torus cannot be bicolored (that is, the* 1*-skeleton is not bipartite).*

Similarly, 4,8-triangulations and 2,6-quadrangulations are subject to combinatorial restrictions:

Theorem 4. *The faces of a* 4,8*-triangulation of the torus cannot be* 2*-colored (that is, the dual graph is not bipartite).*

Theorem 5. *The edges of a* 2,6*-quadrangulation of the torus cannot be bicolored with colors alternating around each face (or equivalently, around each vertex).*

We prove Theorems 1–5 by converting the statements about combinatorics to statements about geometry (Sections 3 and 4). Namely, any triangulated torus has a natural *equilateral* metric obtained by declaring each edge to have length 1 and each triangle

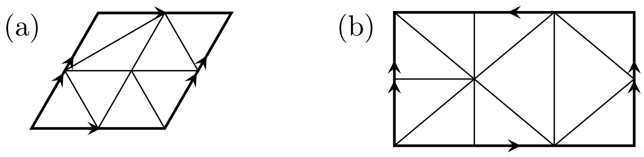

Figure 2: (a) Flipping one edge in the refined triangulation of Figure 1(a) gives a $5^2 7^2$-triangulation which has only the four exceptional vertices. (b) The Klein bottle does have 5,7-triangulations, for example this one with five vertices.

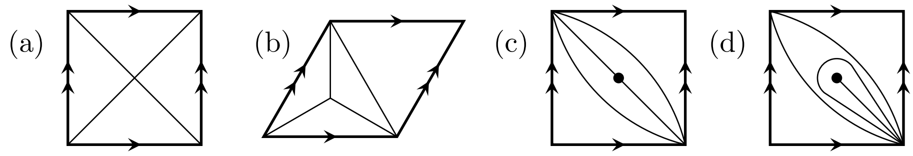

Figure 3: The irregular triangulations of the torus with exactly two vertices: (a) a 4,8-triangulation, (b) a 3,9-triangulation, (c) a 2,10-triangulation and (d) a 1,11-triangulation.

to be a euclidean equilateral triangle. This approach was also used by Thurston [18] to classify triangulations of the sphere with vertex degrees at most 6. In our case, if there were a 5,7-triangulation of the torus, its equilateral metric would be a euclidean metric with exactly two cone singularities (see Section 3). We obtain a contradiction by studying the possible holonomy groups of such metrics (see Section 4). Specifically, we prove the following theorem.

Theorem 6. *Suppose the torus is equipped with a euclidean cone metric with exactly two cone points $p_\pm$ of curvature $\pm 2\pi/n$, for some integer $n \geq 2$. Then the holonomy group H contains the cyclic group C_n of order n as a proper subgroup: $C_n \lneqq H$.*

We provide two proofs for Theorem 6. One argument is metric in nature (Section 5), the other relies on the induced conformal structure and proceeds by invoking the residue theorem (Section 6).

Jendroľ & Jucovič [14] actually prove a stronger statement than Theorem 1. They also show that every distribution of irregular degrees except $5, 7$ does indeed occur in some triangulation as long as the average is 6. (Their notion of triangulation is more restrictive. See Section 2 for the precise statements.) Barnette, Jucovič & Trenkler [3] prove an analogous existence result as well. These existence proofs involve more or less explicit constructions and are rather different in nature from the non-existence proofs. In this article, we do not deal with such existence statements at all. Neither are we concerned with the realization of tori as polyhedral surfaces in $\mathbb{R}^3$ (as considered, for instance, in [13] and the references therein).

We have phrased Theorems 1–3 in terms of triangulations, quadrangulations and hexangulations with some exceptional vertices. One could reformulate these results in dual terms: For example, dual to a triangulation is a map where the vertices all have degree three. Hexagon faces would then be considered regular, while pentagons and

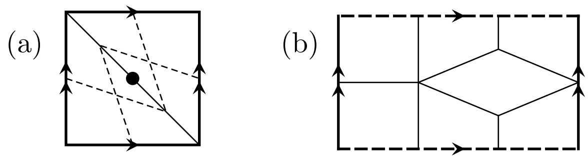

Figure 4: (a) A 2,6-quadrangulation of the torus with two vertices, and its refinement with eight vertices. (b) A $3^2 5^2$-quadrangulation of the torus with seven vertices. Erasing the dashed edges would give a smaller example containing only the four exceptional vertices.

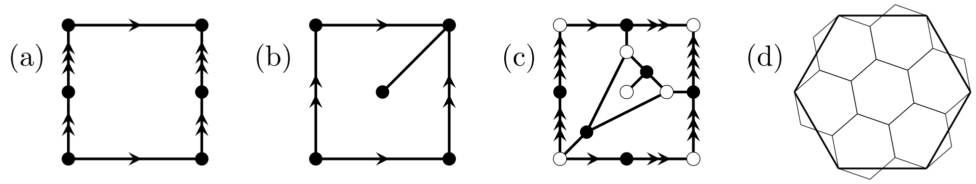

Figure 5: (a) A 2,4-hexangulation of the torus. The 1-skeleton fails to be bipartite because it has a loop edge. (b) A 1,5-hexangulation. (c) A 1,5-hexangulation with bipartite 1-skeleton. (d) A subdivision scheme for hexangulations, which can be used to generate bigger examples.

heptagons would be exceptional. In fact, Theorem 1 was originally stated in that dual form. We prefer our formulation because it is more closely connected to the euclidean cone metric we use.

Acknowledgments

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2 Maps on surfaces

Suppose M is a closed connected surface. For us, a *map* on M will mean an embedding of a finite graph G in M such that each face is topologically an open disk. Here, a *face* is a component of the complement $M \setminus G$, and a *graph* is a one-dimensional cell complex. In particular, a graph can have loops and multiple edges. This notion of map corresponds to the construction of surfaces by starting with a collection of (topological) polygons (including 1- and 2-gons) and gluing their edges in pairs. Any map has a combinatorial dual, with a vertex for every face and vice versa. A map on M is called a *triangulation* (*quadrangulation*, *hexangulation*) of M if all faces are triangles (quadrilaterals, hexagons).

Some authors put more restrictions on the combinatorics of a map. Suppose (a) the graph G has no loops or multiple edges and (b) the boundary of each face is an embedded circle and (c) any two faces meet either along a single edge or at a single vertex or not at all. Then we will call the map *polyhedral.* In a polyhedral map, each vertex has degree $d \geq 3$ and each face has $k \geq 3$ sides. A polyhedral triangulation is a simplicial complex. For a map on the sphere, the following statements are equivalent [12, Section 13.1]: (i) The map is polyhedral. (ii) The map can be realized by a convex polyhedron. (iii) The graph is 3-connected. (iv) The dual map is polyhedral.

Consider a map with V vertices, E edges, and F faces on a surface with Euler characteristic χ. Let v_k be the number of vertices with degree k and let p_k be the number of k-gons among the faces. The Euler equation $\chi = V - E + F$ (where of course $V = \sum_k v_k$ and $F = \sum_k p_k$) and the double counting formula $\sum_k kv_k = \sum_k kp_k = 2E$ imply relations between the numbers v_k and p_k. If all faces are n-gons, one obtains

$$\sum_k (\bar{n} - k)v_k = \bar{n}\chi, \tag{1}$$

where $\bar{n} := 2n/(n-2)$. In the case $\chi = 0$, we see the average vertex degree is exactly $\bar{n}$. (For any χ, the average vertex degree for large maps with n-gonal faces approaches $\bar{n}$.)

Note that $\bar{n}$ is an integer for $n = 3, 4, 6$; then $v_{\bar{n}}$ does not appear in equation (1). That is, the Euler relation imposes no restictions on $v_{\bar{n}}$ in these cases. For polyhedral triangulations ($n = 3$, $\bar{n} = 6$) of the sphere, equation (1) is the only relation among the v_k for $k \neq 6$:

Theorem 7 (Eberhard [8]). *Suppose* $(v_3, v_4, v_5; v_7, v_8, \ldots)$ *is a sequence of nonnegative integers with finitely many nonzero terms satisfying* $\sum(6-k)v_k = 12$*. Then for some* $v_6 \geq 0$ *there exists a polyhedral triangulation of the sphere with* v_k *vertices of degree* k *for all* k*.*

The theorem is usually phrased in dual terms as a theorem about polyhedra with vertices of degree 3 and prescribed number of k-gons for $k \neq 6$; a proof can be found in [12, Section 13.3]. In some cases, the possible values of v_6 are known exactly. (See [12, Section 13.4] and also [18, Section 6].) For instance, a 5^{12}-triangulation of the sphere exists exactly for $v_6 \neq 1$; a 3^4-triangulation exists exactly for v_6 even and not equal to 2.

Jendroľ & Jucovič proved an analogous theorem for the torus and found that one exceptional case has to be excluded:

Theorem 8 (Jendroľ & Jucovič [14]). *Suppose* $p = (p_3, p_4, p_5; p_7, p_8, \ldots)$ *is a sequence of nonnegative integers with finitely many nonzero terms satisfying* $\sum(6-k)p_k = 0$*. Then there exists a nonnegative integer* p_6 *and a map on the torus with* 3*-connected* 3*-valent graph having* p_k k*-gons if and only if* $p \neq (0, 0, 1; 1, 0, \ldots)$*.*

We formulated the non-existence statement for the exceptional case in dual form as Theorem 1. Jendroľ & Jucovič only consider 3-connected graphs, but they do not seem to use this assumption in their non-existence proof. In an earlier paper, Grünbaum had not noticed the exceptional case and made a wrong claim [10, Theorem 3].

Without the assumption that all faces have the same number of sides, one obtains the following equation, which is symmetric in p and v:

$$\sum_k (4-k)p_k + \sum_k (4-k)v_k = 4\chi. \tag{2}$$

Note that p_4 and v_4 do not occur. The double counting formula implies $\sum_{k\neq 4} kp_k = 2E - 4p_4$ and $\sum_{k\neq 4} kv_k = 2E - 4v_4$, so that

$$\sum_{k\neq 4} kp_k \quad \text{and} \quad \sum_{k\neq 4} kv_k \quad \text{are even.} \tag{3}$$

Question. Given p_k and v_k for $k \neq 4$ satisfying (2) and (3), is there corresponding map?

For the case of the sphere (and considering only polyhedral maps), Grünbaum [11] showed that the answer is yes. In the case of the torus, only two dual exceptional cases must be excluded:

Theorem 9 (Barnette, Jucovič & Trenkler [3])**.** *Suppose* $p = (p_3; p_5, p_6, p_7 \ldots)$ *and* $v = (v_3; v_5, v_6, v_7 \ldots)$ *are sequences of nonnegative integers with finitely many nonzero terms satisfying* (2) *and* (3)*. Then there exist* $p_4, v_4 \geq 0$ *and a map on the torus with* 3*-connected graph having* v_k *vertices of degree* k *and* p_k k*-gons for each* $k \geq 3$*, if and only if it is not the case that* $p = (1; 1, 0, 0 \ldots)$ *and* $v = (0; 0, \ldots)$ *or dually that* $p = (0; 0, \ldots)$ *and* $v = (1; 1, 0, 0 \ldots)$*.*

We formulated the non-existence statement as Theorem 2; the assumption of 3-connectedness is not necessary. (Another proof of the existence statement in the case $v = (0; 0, \ldots)$ is due to Zaks [19].)

Jucovič & Trenkler [15] also answered the above question for closed orientable surfaces of genus $g \geq 2$ (assuming that all faces and vertices have degree at least 3). The answer is yes, with no exceptional cases.

3 Euclidean cone metrics

Let $\omega > 0$ be different from 2π. A *euclidean cone* of angle ω is the metric space resulting from gluing together the two edges of a planar wedge of angle ω (when $\omega > 2\pi$, we are gluing together several wedges of total angle ω). A *euclidean cone metric* on a surface M is one in which every point has a neighborhood isometric either to an open subset of the euclidean plane or to a neighborhood of the apex of a euclidean cone. Points mapped to an apex are called *cone points* and form a discrete subset of M. Each cone point has a *curvature* $\kappa := 2\pi - \omega$, where ω is the angle of the corresponding euclidean cone. The complement of the set of cone points is denoted by M^o and is clearly locally euclidean.

The metric induced on any polyhedral surface in $\mathbb{R}^n$ is such a euclidean cone metric. Indeed, points along the edges and in the interior of faces have flat neighborhoods, but a vertex v is a cone point (unless the angle sum around v equals 2π).

Suppose we are given a map on a surface M without 1- or 2-sided faces. This map induces a euclidean cone metric on M, called the *equilateral metric*, as follows: each edge is a segment of length 1, and each k-sided face is isometric to a euclidean regular k-gon. If all faces are n-gons, a vertex of degree k has curvature $2\pi(1 - k/\bar{n})$, where $\bar{n} = 2n/(n-2)$ as in equation (1). Thus equation (1) is exactly the Gauss–Bonnet theorem for the equilateral metric.

In particular, given a triangulation, quadrangulation or hexangulation of a torus, it is only the *exceptional* vertices (those not of degree $\bar{n}$) that become cone points in the equilateral metric. This observation allows us to give an easy proof of the classification of (degree-)regular tilings of the torus. This has been treated by many authors [2, 16, 17, 5, 4], although most of the results are restricted to the class of polyhedral maps.

Theorem 10. *Any triangulation, quadrangulation or hexangulation of the torus with no exceptional vertices is a quotient of the corresponding infinite regular tiling of the plane. Equivalently, it is a finite cover of the* 1*-vertex triangulation (Figure 1(a)), the* 1*-vertex quadrangulation, or the* 2*-vertex hexangulation (Figure 1(c)), respectively.*

Proof. Since there are no exceptional vertices, the equilateral metric is a euclidean (flat) metric on the torus M. Thus its universal cover is the euclidean plane $\mathbb{R}^2$, so M is the quotient by some lattice Λ of translations. The map on M pulls back to the cover, giving the infinite tiling T by regular k-gons. The translational symmetries of T form a lattice Λ_0, and T/Λ_0 is the corresponding minimal map on the torus. Since Λ preserves T, we have $\Lambda < \Lambda_0$. The covering of $\mathbb{R}^2/\Lambda_0$ by $M = \mathbb{R}^2/\Lambda$ restricts to a covering of the minimal map T/Λ_0 by the original map T/Λ. $\square$

4 Holonomy: Proofs of Theorems 1–5

Given a euclidean cone metric on an oriented surface M, the holonomy group $H(M)$ is defined as follows. Pick a basepoint $x \in M^o$, and consider a loop γ in M^o based at x. Parallel transport along γ induces a rotation $h(\gamma)$ of the tangent space T_xM, that is, an element of SO_2 called the holonomy of γ. Because the metric on M^o is flat, this holonomy is unchanged if we replace γ by a homotopic loop. Thus we get a map $\pi_1(M^o, x) \to SO_2$ which is independent of the choice of basepoint x since SO_2 is abelian. Its image is the *holonomy group* $H(M) < SO_2$,

Note that $\pi_1(M^o, x)$ is generated by (generators of) $\pi_1(M, x)$ together with loops around each of the cone points. A loop around a single cone point of curvature κ has $e^{i\kappa}$. We will be particularly interested in the case where the holonomy group is finite. The finite subgroups of SO_2 are of course the cyclic groups $C_n := \langle e^{2\pi i/n} \rangle$.

Lemma 11. *Suppose we are given a triangulation, quadrangulation or hexangulation of a surface M. Then the holonomy group of the associated equilateral metric is a subgroup of C_6 (for triangulations or hexangulations) or a subgroup of C_4 (for quadrangulations).*

Proof. Take a basepoint $x \in M^o$ lying on an edge of the map. Consider parallel translation of a vector v along a loop γ, which we may assume is transverse to the edges of the map. Each time γ crosses an edge, look at the angle between v and a vector along that edge. Successive angles will differ by the angle between two edges of some face of the map. Since that face is a euclidean regular k-gon, the angles differ by a multiple of $\pi/3$ in the case of triangulations and hexangulations, and by a multiple of $\pi/2$ in the case of quadrangulations. Summing these changes, the same is true for the angle between v and its parallel translate along γ. $\square$

This argument could alternatively be phrased in terms of the developing map from the universal cover of M^o to the euclidean plane, whose image is the corresponding regular tiling.

Stronger statements about the holonomy can be made if the map has additional combinatorial properties.

Lemma 12. *For a hexangulation, the holonomy group is a subgroup of C_3 if and only if the vertices can be 2-colored (that is, the 1-skeleton is bipartite). For a triangulation, $H < C_3$ if and only if the triangles can be 2-colored (that is, the dual graph is bipartite). For a quadrangulation, $H < C_2$ if and only if the edges can be 2-colored (with colors alternating around each face or, equivalently, around each vertex).*

Proof. In a hexangulation with 2-colored vertices, orient each edge from black to white; in a triangulations with 2-colored faces, orient each edge with black to its left. Now repeat the parallel translation argument of Lemma 11 but keeping track of the angle between v and the *oriented* edges. Since this changes only by multiples of $2\pi/3$, we get $H < C_3$.

For a quadrangulation with 2-colored edges, we claim that the angles from v to edges of the same color are congruent modulo π, and that angles from v to edges of different colors differ by an odd multiple of $\pi/2$. This is easily proved by looking at the moments when γ enters and leaves a face. Since γ starts and ends on the same edge, v comes back either unchanged or rotated by π.

In all cases, the converse follows by propagating a 2-coloring along arbitrary paths; the holonomy condition shows we will never encounter a contradiction. □

Lemma 12 can also be proved by looking at the developing map from the universal cover of M^o to the euclidean plane. Note that the regular tilings can be colored in the ways described and the rotational symmetries of the colored tilings are C_3 or C_2.

Recall that Theorem 6, our main tool, restricts the holonomy group of a torus with just two cone points. Before proving it in the next sections, we apply it to prove the results we listed as Theorems 1–5.

Proofs of Theorems 1–5. In the situation of Theorems 1–5, Lemmas 11 and 12 say that the holonomy group is a subgroup of C_n with $n = 6, 4, 3, 3, 2$, respectively. On the other hand, the assumptions on the exceptional vertices imply that the equilateral metric is a euclidean cone metric with two cone points, of curvature $\pm 2\pi/n$. Then, by Theorem 6, C_n is a proper subgroup of the holonomy group. This contradiction proves the theorems. □

Because there are no intermediate subgroups between C_3 or C_2 and C_6 – or between C_2 and C_4 – Lemma 11 and Theorem 6 immediately give the following:

Corollary 13. *The equilateral metric on any 4,8- or 3,9-triangulation of the torus has holonomy exactly $H = C_6$. Similarly, any 2,6-quadrangulation has $H = C_4$ and any 2,4-hexangulation has $H = C_6$.*

5 Proof of Theorem 6 using metric geometry

Recall that we are given a torus with exactly two cone points $p_\pm$, of curvature $\pm 2\pi/n$. Since the holonomy around $p_\pm$ is $e^{\pm 2\pi i/n}$, we know C_n is a subgroup of H and we must prove they are not equal.

Let γ be any shortest non-contractible loop on M. Away from the cone points, it must be a geodesic in the usual sense. If it passes through a cone point p with angle ω, the angles to the left and right of γ at p sum to ω. But each of these angles must be at least π, for otherwise we could shorten γ by moving it off p to that side. Thus we see that γ cannot pass through the cone point of positive curvature.

On the other hand, we are free to assume that γ does pass through p_-. If not, then near any point it looks like a straight segment, so γ has a neighborhood isometric to a euclidean cylinder. Thus γ can be translated sideways, while remaining a shortest geodesic. This translation can be continued until we hit a singularity, which must be p_-.

So let $\pi + \alpha$ and $\pi + \beta$ be the angles formed by γ at p_-, with $\alpha \geq \beta \geq 0$ and $\alpha+\beta = 2\pi/n$. If we consider curves parallel to γ just to either side, they have holonomy α and β. Thus unless $\beta = 0$, we have $H \neq C_n$, as desired.

If $\beta = 0$ then γ has to one side a euclidean cylinder neighborhood, and as above, it can be translated sideways through this neighborhood. Again, it will never hit a cone point of positive curvature. This time we stop when the translated γ' first touches γ. This first contact must happen (only) at p_-, for if it happened at a regular point, γ and γ' would coincide, and we would have traced out the whole torus without seeing the positive cone point.

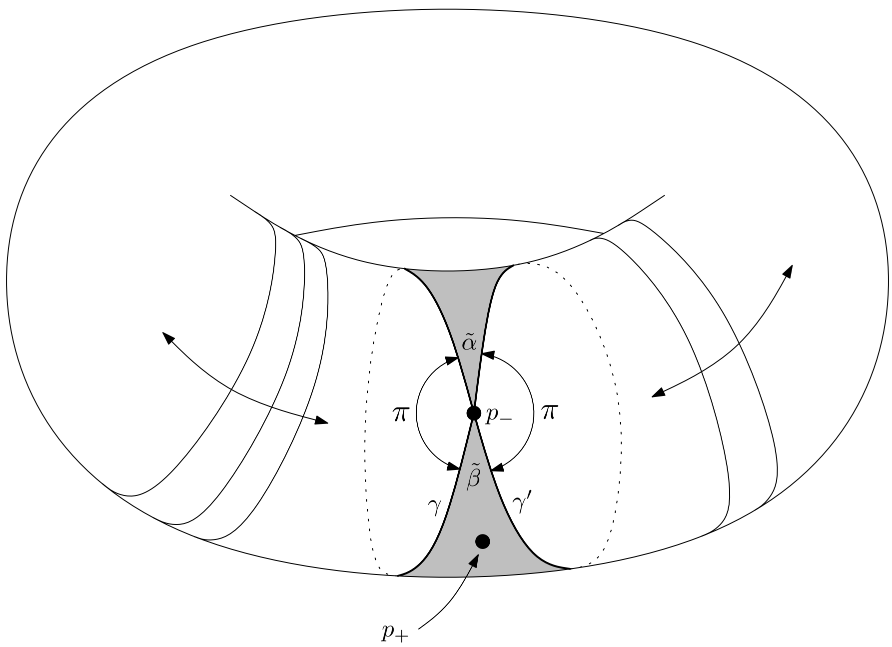

Figure 6: A schematic drawing of the parallel translates of the geodesic γ on the torus. The point p_+ lies in the shaded digon with angles $\tilde\alpha$ and $\tilde\beta$; the unshaded region is a euclidean cylinder.

It follows that γ and γ' bound a digon (both of whose vertices are at p_-) containing the point p_+, as shown in Figure 6. Let $\tilde\alpha \geq \tilde\beta > 0$ be the angles of the digon, with $\tilde\alpha + \tilde\beta = 2\pi/n$. The complement of the digon is a euclidean cylinder, also meeting itself at p_-, foliated by translates of γ. Consider a closed path based at p_- running once along the length of this cylinder. If it is perturbed off p_- into the angle $\tilde{\alpha}$ then its holonomy is $\tilde{\alpha} \in (0, 2\pi/n)$. This proves, as desired, that H is bigger than C_n.

Remark. There are many different equivalent ways to phrase this argument. For instance, given a 5,7-triangulation, one could cut open the torus along geodesic arcs connecting the cone points to give a planar polygon. Its vertices would lie in the triangular lattice in the plane. Our statements about curvature and holonomy would become conditions on the angles of such a polygon.

Figure 7 shows a simple example of a euclidean cone metric on the torus with two cone points.

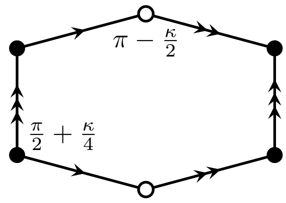

Figure 7: Example of a torus equipped with a euclidean cone metric with two cone singularities of curvature $\pm\kappa$. The holonomy group is generated by a rotation through angle $\kappa/2$.

6 Proof of Theorem 6 using conformal geometry

Suppose the torus M is equipped with a euclidean cone metric whose holonomy group is $C_n = \langle e^{2\pi i/n} \rangle$. Then each cone point has curvature an integral multiple of $2\pi/n$. We have to rule out the possibility that there are only two cone points, with curvature $\pm 2\pi/n$. To this end, we will prove the following lemma.

Lemma 14. *Suppose the torus M is equipped with the conformal structure induced by a euclidean cone metric with holonomy group $H = C_n = \langle e^{2\pi i/n} \rangle$ and m cone points p_j with curvature $\kappa_j = 2\pi k_j/n$. Then the divisor $\sum_{j=1}^{m} k_j p_j$ is a principal divisor.*

By the definition of principal divisors, this means that there exists a meromorphic function on the torus (an *elliptic function*) that has a zero of order k_j at each cone point p_j of positive curvature, a pole of order $-k_j$ at each cone point p_j of negative curvature, and no other zeros or poles. (We will prove the lemma by constructing such an elliptic function.) By the residue theorem, there are no elliptic functions with only one simple pole [1, Theorem 4, p. 271]. Thus the lemma implies Theorem 6.

Proof of Lemma 14. The universal cover of $M^o = M \setminus \{p_1, \dots, p_m\}$ extends, via metric completion, to a branched cover $\hat{M} \to M$, ramified over the singularities p_j. Let $w : \hat{M} \to \mathbb{C}$ be the developing map (where $\mathbb{C}$ is equipped with the standard euclidean metric). We will view the function w also as a branched multivalued function on M.

Recall that the conformal structure of M is defined by the following atlas. In a sufficiently small neighborhood of a nonsingular point, any branch of w may serve as a coordinate. In a sufficiently small neighborhood of a singular point p_j, choose any connected set of branches of w and let $w_j \in \mathbb{C}$ be their common value at p_j. Since the cone angle at p_j is $2\pi(n-k_j)/n$, a coordinate function around p_j can be defined by

$$u_j = (w - w_j)^{n/(n-k_j)}. \tag{4}$$

(The expression on the right hand side is multivalued on M, but it is unramified at p_j; any choice of branch will do.)

Claim. *The form $(dw)^n$ is a well-defined meromorphic differential of degree n on T, that is, a meromorphic section of K^n, where K is the canonical bundle. It has poles of order k_j at the singularities p_j with $\kappa_j > 0$, and zeros of order $-k_j$ at the singularities p_j with $\kappa_j < 0$.*

Once we have proved the claim, Lemma 14 follows: Given any nonvanishing holomorphic differential $\omega \in K$ on the torus, $\omega^n/(dw)^n$ is a meromorphic function with divisor $\sum_{j=1}^m k_j p_j$.

To prove the claim, first note that for any deck transformation $\phi : \hat{M} \to \hat{M}$, there exist $a \in H$ and $b \in \mathbb{C}$ such that

$$w \circ \phi = aw + b.$$

This implies $\phi^* \, dw = a \, dw$. By assumption, H is generated by $e^{2\pi i/n}$, so $a^n = 1$ and $(dw)^n$ is a well-defined meromorphic differential of degree n on M. Finally, near a singular point p_j, equation (4) implies

$$dw = \tfrac{n-k}{n} \, u_j^{-k_j/n} \, du_j,$$

giving

$$(dw)^n = \left(\tfrac{n-k}{n}\right)^n u_j^{-k_j} \, (du_j)^n,$$

from which the statement about the zeros and poles of $(dw)^n$ follows. □

7 Burgers vectors and a theorem of Dress

Since we have shown there is no there is no 5,7-triangulation of the torus, it may be surprising that it is possible to find a 5,7-triangulation of the infinite plane. Figure 8 shows an example nicely laid out in the euclidean plane by Ken Stephenson with his *CirclePack* program.

We note, however, that this triangulation is not isomorphic near infinity to the regular one. Physicists and crystallographers measure the difference—a dislocation in the lattice—by the so-called Burgers vector. As shown in Figure 8, if a closed path enclosing both exceptional vertices is transferred onto the regular triangular lattice, it

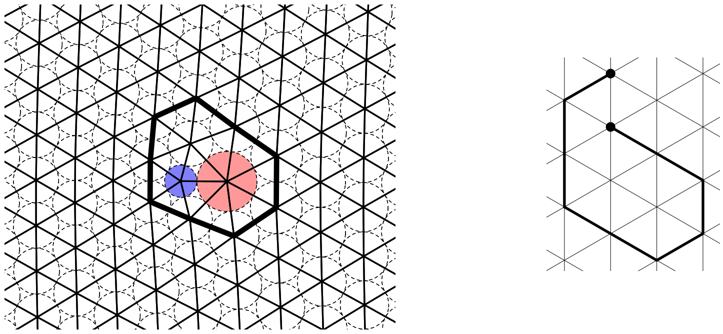

Figure 8: This figure (left), computed by Ken Stephenson with *CirclePack*, shows a 5,7-triangulation of the plane, in which the two exceptional vertices are adjacent, forming a dislocation in the hexagonal lattice. The heavy lines show a loop around the dislocation, from which the Burgers vector can be calculated: A path in the regular hexagonal lattice (right) with the same 9 steps and 6 left turns would fail to close; the Burgers vector is the resulting gap, here one vertical step. The nonzero Burgers vector shows this triangulation is not isomorphic to the regular one near infinity—no matter how far out we go, we still measure the same Burgers vector for this dislocation.

will fail to close. The gap is the Burgers vector, and is independent of the path chosen; the dislocation can be measured near its source or arbitrarily far away.

Andreas Dress used similar ideas in [7] to study triangulations of the plane that are isomorphic to the regular one outside some bounded region. Such a triangulation has a finite number of exceptional vertices. Dress sketched a proof that the number of exceptional vertices cannot be one or two. The first step is to consider a large rectangle enclosing the exceptional vertices, whose boundary is within the regular part of the triangulation. Its opposite sides can thus be glued to form a torus. The Euler characteristic then shows immediately that there cannot be a single exceptional vertex. The importance of Dress's theorem is that it also rules out the case of two cone points. Thus, for instance, a 5,7-triangulation of the plane cannot be isomorphic to the regular triangulation near infinity. Of course, applying edge flips to the regular triangulation produces examples which are still regular near infinity, including a $5^2 7^2$-triangulation.

Before sketching Dress's argument further, we note that our Theorem 6 gives an alternative proof of the three most important cases. (These are the only cases arising in simplicial triangulations—and indeed the only cases considered by Dress [7], although we will see below that his original argument applies equally well to the cases of 2,10- and 1,11-triangulations.)

Corollary 15. *A* 5,7-*,* 4,8- *or* 3,9-*triangulation of the plane cannot be isomorphic to the regular triangulation near infinity.*

Proof. As above, given a triangulation which is regular near infinity, we can glue opposite sides of a large rectangle to produce a triangulation of the torus, with the same two exceptional vertices. The 5,7 case is then ruled out immediately by Theorem 1.

For a 4,8- or 3,9-triangulation, consider the holonomy of the equilateral metric on the triangulated torus. The sides of the rectangle generate the fundamental group of the (unpunctured) torus, but since they lie in the regular background triangulation, they have no rotational holonomy. Thus the holonomy group is generated just by loops around the cone points, contradicting Theorem 6. □

The proof sketched by Dress can be understood as an argument that a triangulation with two exceptional vertices has nonzero Burgers vector, while a triangulation regular near infinity must have zero Burgers vector. To make this precise, we now propose a mathematical interpretation of the Burgers vector in terms of the developing map.

Definition. Given any euclidean cone metric on an oriented surface M, a *developing map* is an oriented local isometry mapping the universal cover of M^o to the euclidean plane. Pre-composing a developing map with a deck transformation of the covering gives a new developing map. But any two developing maps for the same cone metric differ by (post-composition with) a euclidean motion in the plane. Identifying the group of deck transformations with the fundamental group $\pi_1(M^o, x)$, where x is any basepoint, we thus get a homomorphism

$$\hat{h} : \pi_1(M^o, x) \to E_2^+,$$

where $E_2^+ = SO_2 \ltimes \mathbb{R}^2$ is the euclidean group. We call $\hat{h}$ the *holonomy* of the developing map. Its rotational part (that is, its composition with the projection $E_2^+ \to SO_2$) is the metric holonomy h discussed above.

For the equilateral metric arising from a triangulation of M, the developing map sends each triangle linearly to one in the regular triangulation of the plane. Thus its holonomy lies in the symmetry group of the triangular lattice.

The image of a closed path γ in M^o is a locally isometric path in the plane, with the same geodesic curvature. In particular, if γ follows edges of the triangulation through regular vertices, then its image follows edges of the triangular lattice in the plane, making the same turn at each corresponding vertex. This mimics the usual crystallographic definition of the Burgers vector. In particular, suppose γ is a loop in M^o bounding a disk in M in which the cone points have total curvature zero. Then $h(\gamma) = 0$, so $\hat{h}(\gamma)$ is a translation, called the *Burgers vector*.

We can now flesh out Dress's argument.

Theorem 16 (Dress [7]). *A triangulation of the plane with exactly two exceptional vertices cannot be isomorphic to the regular triangulation near infinity.*

Proof. Given a triangulation of the plane isomorphic to the regular one near infinity, let γ be a loop around all the exceptional vertices. Taking a representative of the homotopy class near infinity, we see $\hat{h}(\gamma) = 0$. If there are just two opposite cone points $p_\pm$, then γ can also be written as a loop around p_+ followed by one around p_-. The holonomy of each individual loop is a rotation around the image of the cone point under the developing map. Of course each cone point has many images, but the images of $p_\pm$ we care about are are distinct points in the plane – indeed separated by the same distance as the original points $p_\pm$. Then we are done, since the two opposite rotations around distinct centers compose to give a nonzero translation $\hat{h}(\gamma)$, the Burgers vector. □

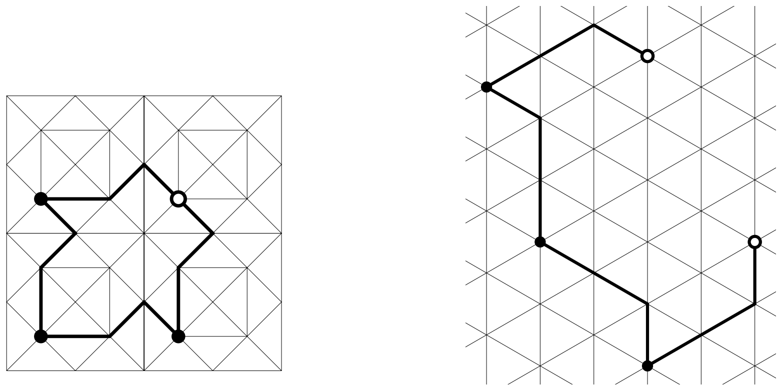

Figure 9: The 2-fold refinement (left) of the 4,8-triangulation of Figure 3(a) has a fundamental domain whose boundary curve γ (shown as the heavier line) only passes through regular vertices. If we trace the same steps in a regular hexagonal lattice, we get a nonclosed path (right); the difference of its endpoints is the nonzero Burgers vector of γ. Note that opposite arcs in the boundary of the fundamental domain (which get glued to each other to form the torus) are no longer parallel to each other when developed into the regular lattice; they are rotated by $\pi/3$, corresponding to the holonomy $H = C_6$ guaranteed by Corollary 13. The intuition that a periodic structure should have zero Burgers vector thus fails. Indeed the rotational holonomy shows that $\hat{h}$ is not a translation for the nontrivial loops α and β on the torus, so their Burgers vectors aren't even defined.

We have seen above that Theorem 6 gives an alternative proof of most cases of Dress's theorem. Physicists have often speculated that, conversely, arguments like those of Dress could be used to prove Theorem 1, that is, to rule out of 5,7-triangulation of a torus. More precisely, the argument is supposed to be as follows: Given a triangulation of the torus with two exceptional vertices, choose standard generators α and β for the fundamental group of the torus. The loop

$$\gamma := \alpha \cdot \beta \cdot \alpha^{-1} \cdot \beta^{-1}$$

is a loop around both cone points, so as in Dress's argument, its holonomy must be a nonzero translation, the product of two opposite rotations about different centers. The contradiction is now supposed to arise from the fact that $\gamma = [\alpha, \beta]$ is a commutator and thus should give zero Burgers vector. This argument would work if the holonomy of α and β were purely translational. Instead, we see that this argument proves at least one of them has nontrivial rotational holonomy so that the commutator in E_2^+ is nontrivial, as in Figure 9. This is essentially a weaker variant of our Theorem 6, which also says something about the amount of rotation; it does not seem that Theorems 1–5 would follow from this weaker variant.

8 Examples of non-toroidal graphs

A graph is called *toroidal* if it can be embedded in the torus. As implied by the Robertson–Seymour theorem, toroidal graphs are characterized by a finite set of forbidden minors. (See [9] for more background information.)

Theorems 1–3 provide an infinite family of non-toroidal graphs. Recall that the *girth* of a graph is the length of the shortest cycle in it. The girth is at least 3 if and only if the graph has no loops or multiple edges

Corollary 17. *If a graph G satisfies any one of the following sets of assumptions, then it cannot be embedded in the torus:*

(a) All vertices of G have degree 6, except for one of degree 5 and one of degree 7, and G has girth at least 3.

(b) All vertices of G have degree 4, except for one of degree 3 and one of degree 5, and G has girth at least 4.

(c) All vertices of G have degree 3, except for one of degree 2 and one of degree 4, and G is bipartite with girth at least 6.

Proof. Our assumptions mean that the average vertex degree k is 6, 4 or 3, respectively, while the girth is at least $\bar{k} := 2k/(k-2)$. If G is embedded in a torus, then each face has at least $\bar{k}$ sides. (Here, a face is a connected component of the complement of G. A priori, it need not be a topological disk, nor does its boundary have to be connected.) By double counting the edges in two ways as before, we obtain the inequality

$$0 = \chi = V - E + F \leq 2E\Big(\frac{1}{k} - \frac{1}{2} + \frac{1}{\bar{k}}\Big).$$

On the other hand, we have $V - E + F \geq 0$, with equality if and only if all faces are topological disks. It follows that all faces are topological disks with $\bar{k}$ sides. But then we have a triangulation, quadrangulation or bipartite hexangulation of the torus. Such maps, with the given vertex degrees, have been ruled out by Theorems 1–3. □

The following procedure produces graphs to which Corollary 17 is applicable. Let G be any 6-regular graph without loops or multiple edges and with at least 8 vertices. Choose an edge ij of G, and let k be any vertex not adjacent to i. Remove the edge ij and insert an edge ik. Vertices j and k now have degrees 5 and 7, respectively, while all the other vertices still have degree 6.

The same can be done with 4- and 3-regular graphs. We start with a graph of the required girth (again, the regular tessellations of the torus provide examples) and then simply choose vertex k in the above procedure far enough from i. Figures 10 and 11 show graphs constructed by this procedure; they are not embeddable in the torus.

Any 5,7-triangulation of the Klein bottle provides another example of such a graph, as long as it has girth 3. The one in Figure 2(b) is too small—its 1-skeleton has girth only 2—but a refinement of it would work.

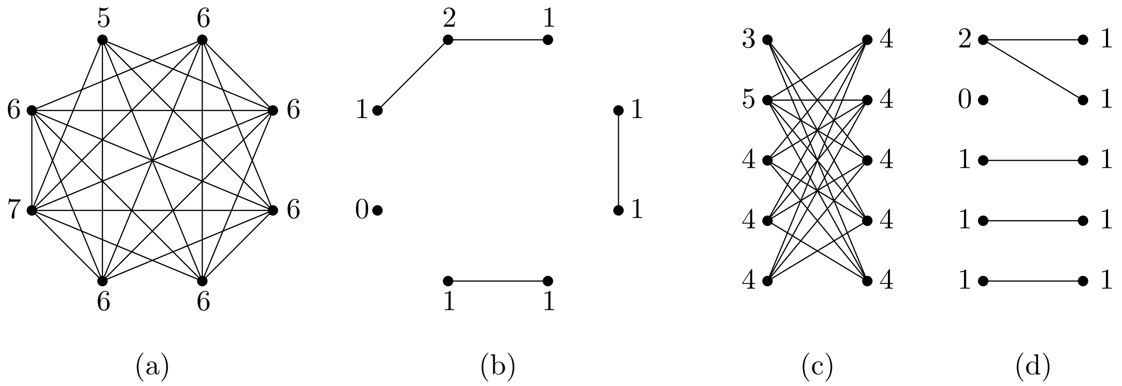

Figure 10: (a) A graph with vertex degrees $5\,6^6 7$, and (b) its complement; (c) a graph of girth 4 with vertex degrees $3\,4^8 5$, and (d) its complement. These are the smallest examples of their respective classes of graphs.

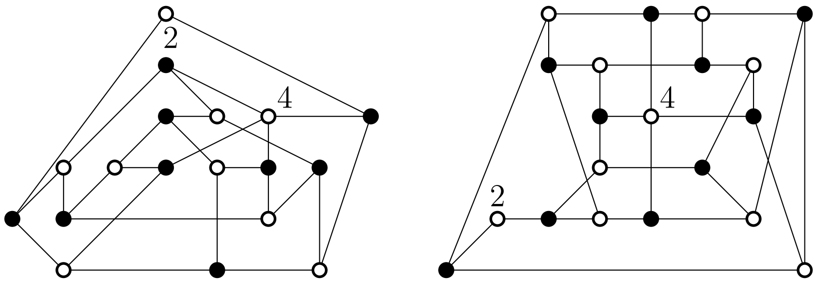

Figure 11: Bipartite graphs with girth 6 and vertex degrees $2\,3^k 4$.

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