Strictly Convex Drawings of Planar Graphs

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July 11, 2004

Abstract

Every three-connected planar graph with n vertices has a drawing on an $O(n^{7/3}) \times O(n^{7/3})$ grid in which all faces are strictly convex polygons.

1 Introduction

A drawing of a planar graph in which all faces, including the outer face, are strictly convex polygons, is called a *strictly convex* drawing.

Theorem 1. *A three-connected planar graph with n vertices has a strictly convex drawing*

(i) *on an $O(n^{7/3}) \times O(n^{7/3})$ grid,*

(ii) *or more generally, on an $O(n^2 s^2) \times O(n^{5/2}/s)$ grid, for any choice of a parameter $1 \le s \le \sqrt[6]{n}$,*

(iii) *or on an $O(n) \times O(n^3)$ grid.*

The main idea is to start with a (non-strictly) convex embedding and to perturb the vertices to obtain strict convexity. We will use an embedding with special properties that is provided by the so-called *Schnyder embeddings*, which are introduced in Section 2.

The analysis of the perturbation leads to a number-theoretic question from the geometry of numbers whose solution would yield grid drawings on an $O(n^2) \times O(n^2)$ grid, see Section 4.3.

Historic context. The problem of drawing graphs with straight lines has a long history. It is related to realizing three-connected planar graphs as three-dimensional polyhedra. By a suitable projection on a plane, one obtains from a polyhedron a straight-line drawing, a so-called *Schlegel diagram.* The faces in such a drawing are automatically strictly convex. However, the problem of realizing a graph as a polytope is more restricted: not every drawing with strictly convex faces is the projection of a polytope. In fact, there is an exponential gap between the known grid size for strictly convex planar drawings and for polytopes in space.

The approaches for realizing a graph as a polytope or for drawing it in the plane come in several flavors. The classical methods of Steinitz (for polytopes) and Fáry (for graphs) work incrementally, making local modifications to the graph and adapting the geometric structure accordingly. Tutte [13, 14] gave a "one-shot" approach for drawing graphs that sets up a system of equations. This method yields also a polytope via the Maxwell-Cremona correspondence [10]. All these methods give embeddings that can be drawn on an integer grid but require an exponential grid size (or even larger, if one is not careful).

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The first methods for straight-line drawings of graphs on an $O(n) \times O(n)$ grid were proposed for triangulated graphs, independently by de Fraysseix, Pach and Pollack [6] and by Schnyder [11]. The method of de Fraysseix, Pach and Pollack [6] is incremental: it inserts vertices in a special order, and modifies a partial grid drawing to accommodate new vertices. In contrast, Schnyder's method is another "one-shot" method: it constructs some combinatorial structure in the graph, from which the coordinates of the embedding can be readily determined afterwards. Both methods work in linear time. $O(n) \times O(n)$ is still the best known asymptotic bound for planar grid drawings.

If graphs are not triangulated, the challenge is to get faces which are convex (but not necessarily strictly convex). (Without the convexity requirement one can just remove edges from a triangulated graph.) Many algorithms are now known that achieve this with $O(n) \times O(n)$ size, for example by Chrobak and Kant [4] and Chrobak, Goodrich and Tamassia [5] (à la Fraysseix, Pach and Pollack); or Schnyder and Trotter [12] and Felsner [7] (à la Schnyder). Our algorithm builds on the output of Felsner's algorithm, which is described in the next section.

The idea of getting a strictly convex drawing by perturbing a convex drawing was pioneered by Chrobak, Goodrich and Tamassia [5], who claimed a strictly convex embeddings on an $O(n^3) \times O(n^3)$ grid, without giving full details, however.

2 Preliminaries: Schnyder Embeddings of Three-Connected Plane Graphs

Felsner [7] (see also [8]) has extended the straight-line drawing algorithm of Schnyder, which works for triangulated planar graphs, to arbitrary three-connected graphs. The edges of the graph are covered by three directed trees which are rooted at three selected vertices a, b, c on the boundary, forming a *Schnyder wood.* The three trees define for each vertex v three paths, which partition the graph into three regions. Counting the faces in each region gives three numbers x, y, z which can be used as barycentric coordinates for the point v with respect to the points a, b, and c. Selecting abc as an equilateral triangle of side length $f-1$ (the number of interior faces of the graph) yields vertices which lie on a hexagonal grid formed by equilateral triangles of side length 1, see Figure 1a.

This straight-line embedding has the following important property: Every vertex except the corners a, b, c has at least one incident edge in each of the three $60°$ wedges shown in Figure 2a. From this it follows immediately that there can be no angle larger than $180°$, and hence all faces are convex. Moreover, the faces F have the following special shape, see Figure 4a: Consider the line $x = \text{const}$ through the point with maximum x-coordinate, and similarly for the other three coordinate directions. These three lines form a triangle T_F which encloses F. The special property is that all vertices of F must lie on the boundary of T_F.

The Schnyder wood and the coordinates of the points can be calculated in linear time.

3 Rough Perturbation

Before making all faces strictly convex, we perform an initial perturbation to a refined grid which is smaller by only a constant factor. This preparatory step will ensure that the main perturbation step can treat each face independently.

We overlay a triangular grid which is scaled by a factor of $1/7$, see Figure 3. As shown in the figure, a point may be moved to one of three alternate positions. The precise rules are as follows: A vertex v on an interior face F is moved if and only if (i) the interior angle of F at v is

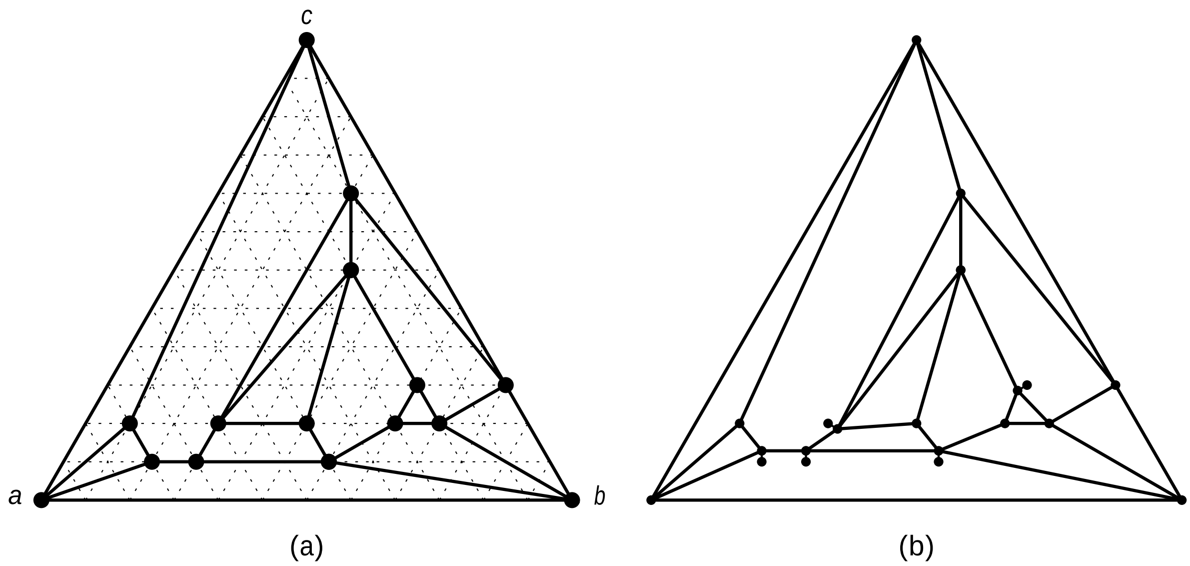

Figure 1: (a) A Schnyder embedding on a hexagonal grid and (b) on the refined grid after the initial (rough) perturbation

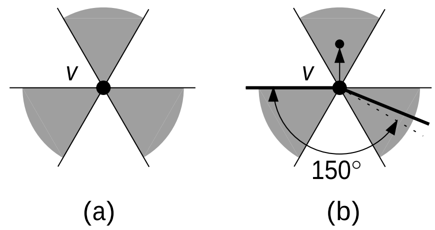

Figure 2: (a) Each shaded wedge contains at least one edge incident to v. (b) A typical situation at a vertex which is perturbed.

larger than 150°; and (ii) it is incident to an edge of F which lies on the bounding triangle T_F. See Figure 2b for a typical case. Such a vertex is then pushed "out", perpendicular to the edge of T_F. We call F the critical face of v. For a boundary vertex different from a, b, c, the exterior face is the critical face, but we perform no perturbation for these vertices.

Examples can be seen in Figure 4b–c and Figure 5. The result of perturbing the example in Figure 1a is shown in Figure 1b.

There can be no conflict in applying the rules by regarding a vertex v as part of different faces: the bound of 150° on the angle, together with the existence of an edge in every wedge ensures that there is at most one critical face for every vertex, see Figure 2b.

Moreover, the result has the following properties, which are easy to see:

Lemma 1. *1. The resulting embedding is still convex.*

2. If each vertex is additionally perturbed by at most $1/30$, the only reflex angle that might arise at a vertex v is in the critical face of v.

This lemma is important because it means that we only have to take care of *one* incident face when we decide the final perturbation of v. We can thus work on each face independently to make it strictly convex.

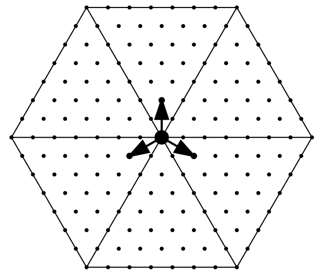

Figure 3: The possible positions for a single point in the rough perturbation.

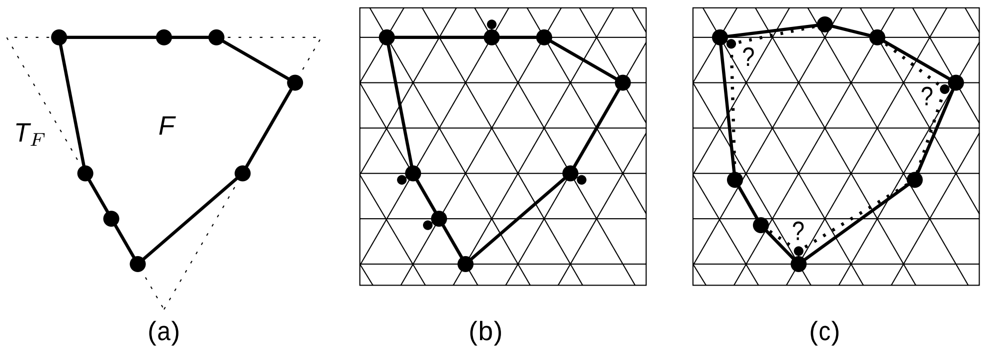

Figure 4: (a) A typical face F constructed by the convex embedding algorithm. (b) The new positions of the vertices of F which are pushed out are indicated. (c) The result of the initial perturbation. The perturbation of the vertices with question marks depends on the other faces incident to these vertices.

4 Fine Perturbation

4.1 The Setting after the Rough Perturbation

We are now in the following situation. Consider a horizontal chain of vertices on the upper edge of a face F, as in Figure 6a. We first discuss the case when the chain is horizontal. According to Lemma 1 we have to ensure that the angles on this chain whose critical face is F are smaller than 180° after the perturbation. In Figure 6a, these are the vertices v_2, v_3, and v_4. Let us call these vertices *critical vertices*.

We place a disk of radius 1/30 around every perturbed point on this edge, including all neighbors of critical vertices and all intermediate grid points on this edge, see Figure 6b. The centers of these circles are placed as if all points were perturbed (from their original position) in the same direction as the critical points, so that they form a regular row of circles.

This will permit a more a uniform treatment in the next step: we find a strictly convex chain which selects one vertex out of each little disk, as shown in Figure 6c.

Finally, we use these perturbed positions for our critical points, see Figure 6d. We simply ignore the perturbed positions for intermediate points which were inserted, and also for the two extreme points, which are only neighbors of critical points: their position is determined independently. It is clear that the omission of the intermediate points does not destroy convexity. We still have to check that the angle at the left-most and right-most critical vertex (v_2 and v_4 in this case) is also convex.

Vertices v_2 and v_4 in this example represent the possible cases that have to be considered.

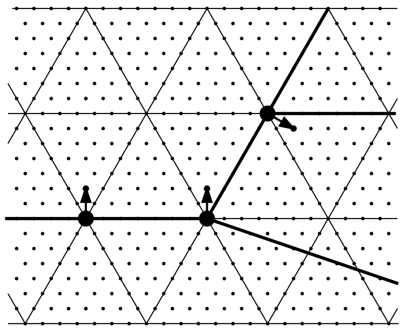

Figure 5: Example of the initial perturbation.

For visual clarity, the circles in Figure 6 have been drawn with a much larger radius than $1/30$. Since the circles are actually small enough, the angle at v_4 will be convex no matter where the point v_5 is placed in its own circle. (This position is determined when the critical face of v_5 is considered.) A similar statement holds at v_2, where the perturbed position of v_1 in Figure 6c is replaced by the original position of v_1; this will always turn the edge v_2v_1 counterclockwise and thus preserve convexity at v_2.

4.2 Convex Chains in the Grid

We have $O(n)$ vertices arrayed on a line which must be perturbed into convex position. This is a very standardized situation in which the only variability is in the number of points. It is more convenient to work with a rectangular grid. So we extend the hexagonal grid to a rectangular grid as shown in Figure 7a–b. This grid will be refined sufficiently in order to allow a strictly convex chain to be drawn inside a sequence of circles. Figure 8 gives a schematic picture of the situation. (This drawing not to scale.) For a change, we are now constructing an *upward* convex chain, but this makes no real difference. Inside each disk (of radius $1/30$) we fit a square of side length $1/50$, which is subdivided into a subgrid of width $2w$ and height $2h$. More precisely, we are looking for a sequence of points $p_i = (x_i, y_i)$ in these circles, where the integer coordinates $-w \le x_i \le w$ and $-h \le y_i \le h$ measure the distance from the center of each circle in units of little grid cells. Eventually, when the whole subgrid is scaled to the standard grid $\mathbb{Z} \times \mathbb{Z}$, x_i and y_i will become true distances again. The total size of the resulting integer grid will be $O(nw) \times O(nh)$.

The condition of convexity can be translated into a condition on the angle between successive *difference vectors*

$$\Delta p_i := p_{i+1} + \binom{S}{0} - p_i,$$

where $S = 100w$ is the distance between successive circle centers (which has length 1) measured in terms of subgrid units. The sequence of vectors Δp_i should turn left. We can draw the vectors $q_i := p_{i+1} - p_i$ in the grid $-2w \le u \le 2w$ and $-2h \le \Delta y \le 2h$ as shown in the right part of Figure 8b.

The convex chain $\ldots, p_i, p_{i+1}, \ldots$ has a descending part up to a point with minimum y coordinate and an ascending part. We will construct the two parts symmetrically. We therefore look only at the ascending part. We renumber this part it to $p_1, p_2, \ldots, p_N$ and we fix the point $p_1 = \binom{0}{-h}$. The number of points N is reduced to half, which is still $O(n)$.

We now have to carry out the following task: Select a sequence of points $q_i = \binom{u_i}{\Delta y_i}$ with $-2w \le u_i \le 2w$ and $\Delta y_i \ge 1$, moving counterclockwise around the *base point* $B := \binom{-S}{0}$,

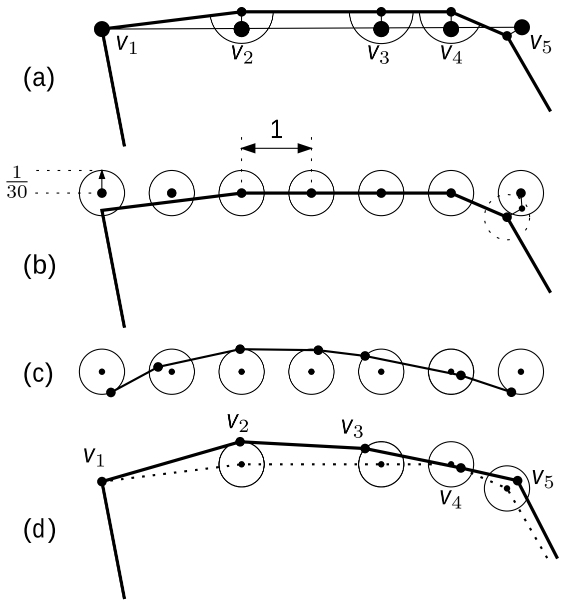

Figure 6: The setting of the fine perturbation process: (a) The initial situation after the rough perturbation. The angles in which it is necessary to ensure a convex angle are marked. (b) The circles in which the fine perturbation is found. The size of the circles is exaggerated to make the perturbation more conspicuous. (c) A strictly convex polygon inside the circles. (d) The final result.

such that the resulting sequence p_i defined by $p_1 = \binom{x_1}{y_1} = \binom{0}{-h}$ and

$$p_{i+1} = \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} = \begin{pmatrix} x_i + u_i \\ y_i + \Delta y_i \end{pmatrix}$$

satisfies $-w \leq x_i \leq w$. If the points p_i fall into this horizontal range, we call the sequence of points q_i *feasible*. The total amount of y-increments

$$H := \sum_{i=1}^{N-1} \Delta y_i$$

determines the necessary height of the grid: We must have $2h \geq H$. The integer grid in which the graph is finally embedden has a total width of $\Theta(nw)$. and a total height of $\Theta(nH)$. If

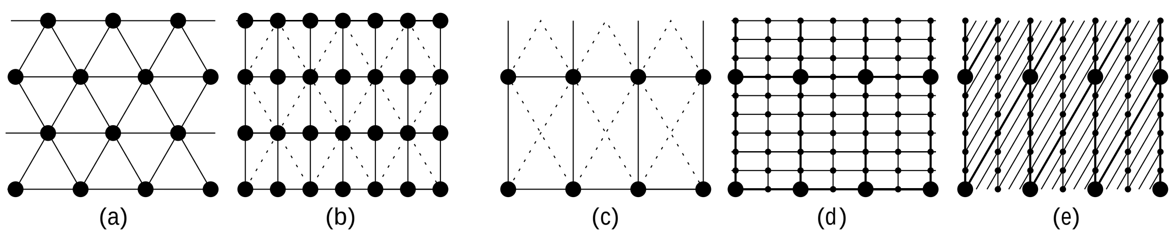

Figure 7: The hexagonal grid (a) is contained in a rectangular grid (b). A 2×6 refinement (d) of a rectangular grid (c), and a shearing of the refined grid (e) whose grid-points coincide with the untransformed grid.

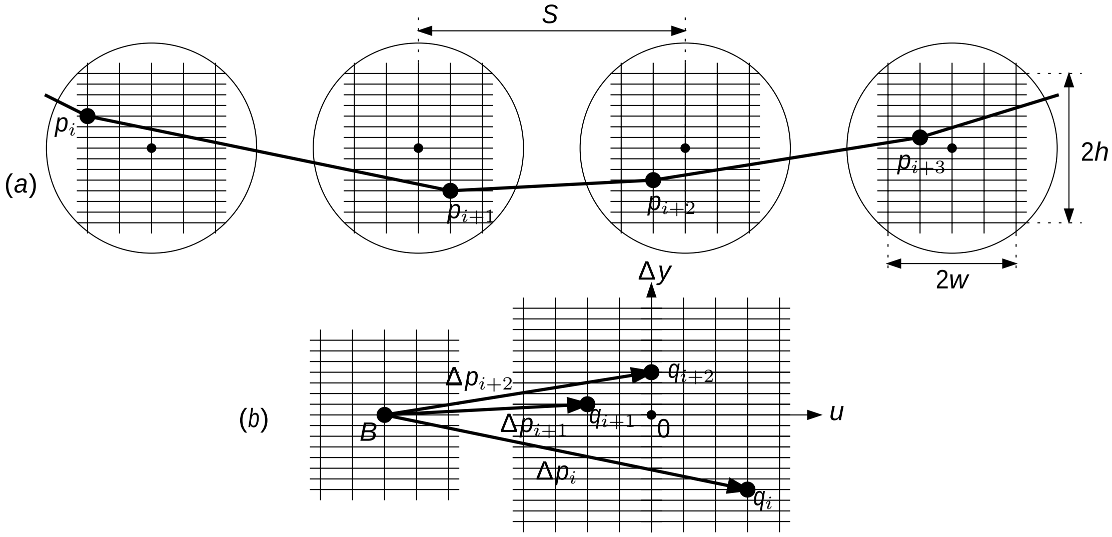

Figure 8: (a) A convex chain formed of one grid point in each circle. (Again, the radius of the circles is drawn much too large compared to their distance.) (b) The difference vectors.

the desired length of one of these dimensions is specified, we can determine w from it and try to minimize the other dimension.

We can thus set up the problem as follows:

> *The Convex-Chain Grid-Point Selection Problem:*
> For given N and w, and find a feasible sequence of $N-1$ points q_i which minimizes H.

One fesible solution is to set $u_i = 0$ and $\Delta y_i = i$. This leads to a total height $H = \sum_{i=1}^{N-1} i = O(n^2)$, and an $O(n) \times O(n^3)$ grid for embedding the whole graph, proving part (iii) of the theorem.

A different family of solutions is constructed as follows, see Figure 9. We choose $w \geq N$ and we assume for simplicity that $\sqrt{w}$ is an integer. Then we start in the row with $\Delta y = 1$ and successively pick $2\sqrt{w}+1$ points with $u_i = \sqrt{w}, \sqrt{w}-1, \ldots, 2, 1, 0, -1, -2 \ldots, -\sqrt{w}+1, -\sqrt{w}$. The points turn counterclockwise as seen from the base point $B = \binom{-S}{0}$ and the sum of the positive u_i values is $\sqrt{w}(\sqrt{w}-1)/2 \leq w$; thus, x_i does not leave the permitted horizontal range. Moreover, the negative u_i values will completely cancel the movement to the right, and in the end we are at the same x-coordinate as at the beginning: $x_{2\sqrt{w}+2} = 0$. We can therefore continue with $2\sqrt{w}+1$ points from the next row with $\Delta y = 2$ and the same sequence of u-coordinates, and so on. We need a total of $R = \lceil N/\sqrt{w}\,\rceil$ rows. The only thing that remains to be checked is that the points also turn counterclockwise around B when we proceed from row Δy to $\Delta y + 1$ (from q_7 to q_8 and from q_{14} to q_{15} in the example). In other words, we need to show

$$\frac{\Delta y + 1}{S + \sqrt{w}} > \frac{\Delta y}{S - \sqrt{w}},$$

which simplifies to $S > \sqrt{w}(2\Delta y + 1)$. This relation holds because Δy is bounded by $R \leq N/\sqrt{w}+1$, and therefore $\sqrt{w}(2\Delta y+1) \leq 2N + 3\sqrt{w} \leq 2w + 3\sqrt{w} \leq 5w$. This is much smaller than $S = 100w$.

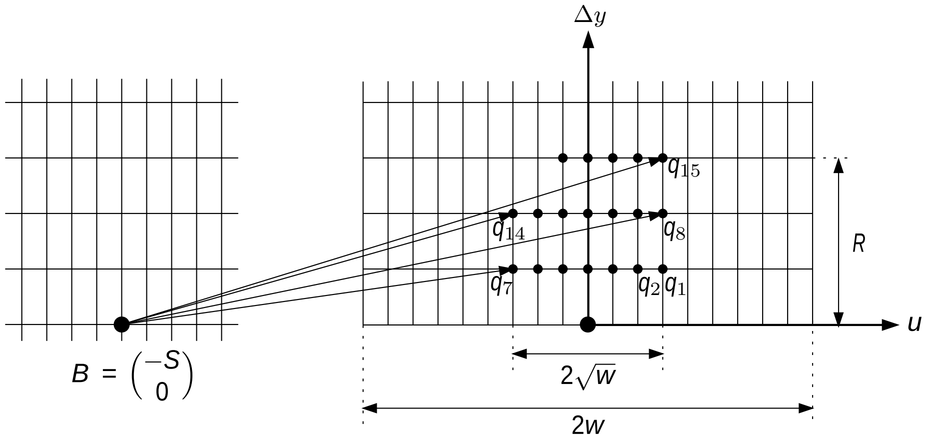

Figure 9: A solution to the convex-chain grid-point selection problem.

The total accumulated height is bounded by

$$H = \sum_{i=1}^{N-1} \Delta y_i \leq (N-1) \cdot R = O(N^2/\sqrt{w}) = O(n^2/\sqrt{w}).$$

We thus get a grid of total size $O(nw) \times O(nH) = O(nw) \times O(n^3/\sqrt{w})$. The choice $w = n^{4/3}$ yields a square grid of size $O(n^{7/3}) \times O(n^{7/3})$, proving part (i) of the theorem. The choice $w = ns^2$ proves part (ii) of the theorem.

So far, we have treated only a sequence of vertices on a horizontal straight line. The same scheme can be applied to lines of the two other directions by applying the shearing transformation $\binom{x}{y} \mapsto \binom{x}{y+\sqrt{3}/2\cdot x}$ or $\binom{x}{y} \mapsto \binom{x}{y-\sqrt{3}/2\cdot x}$ which moves points only in vertical direction. If h is a multiple of v, the transformation will produce a grid like in Figure 7e which is contained in the original grid of Figure 7d. One needs to reduce the size of the little square subgrid to ensure that the sheared square still fits inside the circle, and one has to adjust the constant S accordingly. For the interesting range of parameters discussed above, the height h of the subgrid is never smaller than the width w; thus, the choice of h as a multiple of v does not change the analysis.

On the exterior edges, the points must of course be perturbed to form an *outward* convex chain.

The whole procedure, as described above, is quite explicit and can be carried out in linear time. The only tricky part is the treatment of the intermediate grid vertices, since there might be $\Theta(n^2)$ of them in total. These vertices must not be handled explicitly, but one must jump over a sequence of intermediate vertices and accumulate their u_i and Δ_i contributions in one step.

4.3 Towards a Smaller Grid

The general solution of the convex-chain grid-point selection problem which we have presented in the previous section is very systematic and conservative. It is possible to find the *optimal* convex chain in polynomial time by dynamic programming. In practice, a simple greedy approach for selecting the points q_i one by one already gives a very good solution which is much better than the systematic approach: we sort all grid points q in the strip $[-2w, 2w] \times [1, \infty)$ by slope around the base point $B = \binom{-S}{0}$ and process them in this order. In other words, we rotate a line around B and pick up the points as we meet them. Two situations might prevent a point q from being taken as the next point q_i: (i) the vector might be collinear with the last vector which was taken; (ii) the vector might lead p_{i+1} out of the horizontal range $-w \leq x_{i+1} \leq w$.

The first criterion excludes only a constant fraction of the possible points: It is known that the proportion of *primitive vectors* (vectors whose components are relatively prime) among the integer vectors in some large enough area is approximately $6/\pi^2$ [9]. The second criterion should also be not too serious. Let us restrict our attention to the region $[-w, w] \times [1, \infty)$, as in Figure 9. A point $q = (u, \Delta y)$ is certainly acceptable to become the point q_i if u has the opposite sign from the current value x_i: this ensures that $x_i + q_i = x_{i+1}$ lies in the range $[-w, +w]$. As soon as the slope from $\binom{-S}{0}$ is large enough, the rotating line will intersect several horizontal rows in the region $[-w, w] \times [1, \infty)$, some of them in the area of negative u values, and some of them in the area of positive u values. (In Figure 9, this happens already very early; with the actual dimensions, S is much bigger, and some larger—but still constant—number of initial rows must be swept before the rotating line starts intersecting two successive rows.)

Under the heuristic assumption that the next point that is hit by the rotating sweep-line is equally likely to contribute a positive sign or a negative sign, independently of the current value of x_i, this means that, on average, at most one point is skipped before a point can be taken. The consequence would be that only a constant fraction of points have to be skipped, and one gets a much smaller value of the total vertical displacement H. Some numerical experiments have confirmed this hypothesis on the growth of H, which would lead to a grid of area $O(n^4)$, with dimensions anywhere in the range between $O(n) \times O(n^3)$ and $O(n^2) \times O(n^2)$. This would be the optimum bound that can be obtained by the approach of perturbing the vertices of an initial convex grid embedding inside small disks of constant size: suppose a cycle of n vertices is embedded as a triangle abc. Some side must contain $N \geq n/3$ vertices, leading to a convex-chain grid-point selection problem with N points. Even if we could select *all* the N vectors $q = (u, \Delta y)$ in the strip $[-2w, 2w] \times [1, \infty)$ with the lowest Δy values, we would get $H = \Theta(N^2/w)$, and thus a total area of $\Theta(n^4)$.

In practice, one would of course not insert the intermediate grid points as in Figure 6b. The approach indicated above can be modified to take into account perturbation disks which are not equidistant.

5 Conclusion

In practice, the algorithm behaves much better than indicated by the rough worst-case bounds that we have proved. We have not attempted to optimize the constants in our proof. For example, if we don't take a 7×7-subgrid but a 12×12 subgrid, the permissible amount of perturbation in Lemma 1 increases to $1/8$, but it would make the pictures harder to draw.

Lower Bounds. The only known lower bound comes from the fact that a single convex n-gon on the integer grid needs $\Omega(n^3)$ area, see Bárány and Tokushige [3], or Acketa and Žunić [1, 2] for the easier case of a *square* grid.

Extensions. The class of three-connected graphs is not the most general class of graphs which allow strictly convex embeddings. The simplest example of this is a single cycle. A planar graph, with a specified face cycle C as the outer boundary, has a strictly convex embedding if and only if it is three-connected to the boundary, i. e., if every interior vertex (not on C) has three vertex-disjoint paths to the boundary cycle. Equivalently, the graph becomes three-connected after adding a new vertex and connecting it to every vertex of C. These graphs cannot be treated directly by our method, but an approach which partitions the graph into three-connected components and puts them together at the end might work.

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