Point Sets with Many Non-Crossing Perfect Matchings

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Abstract

The maximum number of non-crossing straight-line perfect matchings that a set of n points in the plane can have is known to be $O(10.0438^n)$ and $\Omega^*(3^n)$. The lower bound, due to García, Noy, and Tejel (2000), is attained by the *double chain*, which has $\Theta(3^n/n^{\Theta(1)})$ such matchings. We reprove this bound in a simplified way that uses the novel notion of *down-free matchings*. We then apply this approach to several other constructions. As a result, we improve the lower bound. First we show that the *double zigzag chain* with n points has $\Theta^*(\lambda^n)$ non-crossing perfect matchings with $\lambda \approx 3.0532$. Next we analyze further generalizations of double zigzag chains – *double r-chains*. The best choice of parameters leads to a construction that has $\Theta^*(\nu^n)$ matchings with $\nu \approx 3.0930$. The derivation of this bound requires an analysis of a coupled dynamic-programming recursion between two infinite vectors.

Keywords: Geometric graphs, perfect matchings, asymptotic enumeration.

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1. Introduction

Background. A *non-crossing straight-line matching* of a finite planar point set is a graph whose vertices are the given points, whose edges are realized by pairwise non-crossing straight segments, and where every vertex has degree at most 1. In what follows, such matchings will be simply called *matchings*. A matching is *perfect* if every point is matched – that is, has degree 1. Throughout the paper, all point sets are assumed to be in general position in the sense that no three points lie on a line.

In this paper we deal with bounds on the number of perfect matchings that a set of size n can have. This question arises in a broader context. Non-crossing straight-line matchings, either perfect or not necessarily perfect, are just two kinds of geometric plane graphs, others being triangulations, spanning trees, connected graphs, etc. A web page of Adam Sheffer [2] maintains the best up-to-date bounds on the maximum number of geometric plane graphs of several kinds.

First we recall that for the *minimum* number of perfect matchings that n points in general position can have, the exact solution is known. García, Noy, and Tejel [6] proved the number of perfect matching is minimized on point sets in convex position. It is well-known that the number of perfect matchings is then $C_{n/2}$, where $C_k = \frac{1}{k+1}\binom{2k}{k} = \Theta(4^k/k^{3/2})$ is the k-th Catalan number. The minimum number $C_{n/2}$ of perfect matchings is in fact attained *only* for point sets in convex position, with the exception of one configuration of six points [2].

Regarding the *maximum* number of perfect matchings that a point set of size n can have, only asymptotic bounds are known. The best upper bound to date, $O(10.0438^n)$, was proved by Sharir and Welzl [10]. The best previous lower bound was given by García, Noy, and Tejel in the above-mentioned paper [6]. They showed that for the so-called *double chain* with n points (denoted by DC_n, see Figure 1 below), the following holds:

Theorem 1 ([6, Theorem 4.1])**.** *The number of perfect matchings of the double chain with n points is $\Theta\left(3^n/n^{O(1)}\right)$.*

Actually, it follows from their proof that this number is $\Omega(3^n/n^4)$ and $O(3^n/n^3)$. In Section 2.3 we shall sketch this proof, and also determine the polynomial factor more precisely.

The double chain was used in [6] not only for improving the lower bounds on the maximum number of perfect matchings, but also for some other kinds of geometric graphs: triangulations, spanning trees and polygonizations. It was believed by some researchers in the field that it might give the true upper bound at least for some of these kinds [1, p. 78]. However, in 2006, Aichholzer, Hackl, Huemer, Hurtado, Krasser, and Vogtenhuber [1] introduced a new construction, the *double zigzag chain* with n points, denoted by DZZC_n, see Figure 3 below. They proved that DZZC_n improves the lower bound for the number of triangulations: it is $\Theta^*(8^n)$ for DC_n and $\Theta^*(8.48^n)$ for DZZC_n. (The notations O^* and Ω^* correspond to the usual O- and Ω-notations, but with polynomial factors $n^{\pm O(1)}$ omitted. The notation Θ^* is the conjunction of O^* and Ω^*, possibly with different hidden polynomial factors.) To our knowledge, the number of geometric graphs of other kinds mentioned above for DZZC_n was not found.

In this paper we determine asymptotically the number of perfect matchings for DZZC_n and its further generalizations, improving the existing lower bound.

Our results. In Section 4, we will first show that DZZC_n has asymptotically more perfect matchings than DC_n:

Theorem 2. *The number of perfect matchings of the double zigzag chain with n points is $\Theta^*(\lambda^n)$, where $\lambda = \sqrt{(\sqrt{93}+9)/2} \approx 3.0532$.*

In Sections 5 and 6, we will present a generalization of DC_n, which comes in two variations: *double r-chains without corners* and *double r-chains with corners*, see Figures 7 and 8 below. Our best results for these constructions are as follows:

Theorem 3. *The number of perfect matchings of the double* 11*-chain without corners with n points is $\Theta^*(\nu^n)$, where $\nu = \sqrt[11]{240054} \approx 3.0840$.*

[2] https://adamsheffer.wordpress.com/numbers-of-plane-graphs/

Theorem 4. *The number of perfect matchings of the double* 8*-chain with corners with* n *points is* $\Omega((\nu-\varepsilon)^n)$, *and* $O(\nu^n)$, *where* $\nu = \sqrt[8]{\left(8389+3\sqrt{7771737}\,\right)/2} \approx 3.0930$ *and* $\varepsilon > 0$ *is arbitrarily small.*

A double r-chain without corners has $n = 2rk$ vertices, and a double r-chain with corners has $n = 2rk+2$ vertices, for some natural k. Hence, these structures are defined only for particular values of n. However, the largest number of perfect matchings that an n-point set with n even can have is clearly monotone increasing in n. Hence, in particular, regarding Theorem 4, one derives easily that for every even n there is an n-point set with $\Omega((\nu-\varepsilon)^n)$ perfect matchings, with the constant $\nu \approx 3.0930$. This is currently the best asymptotic lower bound for the maximum number of perfect matchings that a point set can have.

We shall present proofs for all three theorems because they use different techniques. First, in Section 3 we introduce the notion of *down-free matchings* and show in Theorem 6 how one can generally reduce the problem of asymptotic enumeration of perfect matchings of a "double structure" to that of down-free matchings of the corresponding "single structure". In the proof of Theorem 2 (Section 4), we find a recursion for the number of down-free matchings of the zigzag chain, and translate it into a functional equation satisfied by the generating function. We solve this equation explicitly, which allows us to find the *exponential growth constant* (that is, the base of the exponential term in the asymptotic formula) by looking at the smallest singularity of the function. In the proof of Theorem 3 (Section 5) we use matchings which possibly have *runners* – edges with only one endpoint assigned. We define a sequence of infinite vectors whose entries are the numbers of down-free matchings of the r-chain of a certain size, sorted by the number of runners. These vectors can be computed recursively. We reformulate this recursion in term of lattice paths and obtain the desired growth constant ν with the help of a result of Banderier and Flajolet [3]. The proof of Theorem 4 (Section 6) starts similarly, but due to technical obstacles, we need *two* sequences of infinite vectors, defined by a coupled recursion. We find that the desired growth constant is determined by the dominant eigenvalue of certain 2×2 matrix.

Notation. We use the following notation and convention. A *construction* X is a family $\{X_n\}_{n\in I}$ for some infinite $I \subseteq \mathbb{N}$, where, for fixed n, X_n is a class of point sets of size n with certain common properties, for example, a certain order type (or, in some cases: one of several order types) and certain restrictions concerning position in the plane with respect to coordinate axes. The double chain (DC) mentioned above is one such construction. Occasionally we will abuse notation and denote by X_n not only such a class, but also any of its representatives. If we know that all members of X_n have, for example, the same number of matchings, we can speak unambiguously about "the number of matchings of X_n", and so on.

In what follows, $\mathsf{pm}(X_n)$ denotes the number of perfect matchings of X_n; $\mathsf{am}(X_n)$, the number of all (non-crossing straight-line, but not necessarily perfect) matchings of X_n; $\mathsf{dfm}(X_n)$, the number of down-free matchings of X_n. For some constructions it can happen that not all representatives of X_n have the same number of (for example) perfect matchings and, thus, $\mathsf{pm}(X_n)$ is not well-defined, but the common asymptotic bound still can be given, which enables us to write expressions like $\mathsf{pm}(X_n) = \Theta^*(\mu^n)$ in such cases as well.

For two distinct points p and q, the straight line through p and q will be denoted by $\ell(p,q)$.

A set of points is *in downward position* (respectively, *in upward position*) if the points lie on the graph of a convex (respectively, concave) function. In particular, three points with different x-coordinates are in downward position (respectively, in upward position) if they form a counter-clockwise (respectively, clockwise) oriented triangle when sorted by x-coordinate.

A point of X not matched by a matching will be called a *free point.*

2. Double chains and double zigzag chains

In this section we recall the definitions of a double chain and a double zigzag chain, and recall how the bound $\mathsf{pm}(\mathrm{DC}_n) = \Theta^*(3^n)$ from Theorem 1 was obtained in [6].

2.1. One set high above another and general "double constructions"

Double constructions are constructed by putting a point set "high above" another point set:

Definition. Let P and Q be two point sets in the plane. We say that P is *high above* Q if the points in each of the two sets have distinct x-coordinates, P lies completely above any line through two points of Q, and Q lies completely below any line through two points of P.

It is easy to see that, for any two point sets P and Q, it is possible to put a translate of P high above a translate of Q, provided that the points in each set have distinct x-coordinates.

Let X_n be a construction. A *double* X_n (denoted by $\mathrm{D}X_{2n}$) is the family of sets obtained by taking a representative point set P of X_n, another representative Q of X_n reflected across a horizontal line, and placing P high above Q. Examples of such double constructions follow below. In Theorem 6, we will see that the perfect matchings in $\mathrm{D}X_{2n}$ are related to *down-free* matchings of X_n, which will be introduced in Section 3.1.

An edge between a point of P and a point of Q will be called a *PQ-edge*.

2.2. Double chains

A (single) *downward chain* (respectively, *upward chain*) of size n is a set of n points in downward (respectively, upward) position. A downward chain of size n will be denoted by SC_n.

Let n be an even number. A *double chain* of size n consists of a downward chain of size $n/2$, $P = \{p_1, p_2, \ldots, p_{n/2}\}$, placed high above an upward chain of size $n/2$, $Q = \{q_1, q_2, \ldots, q_{n/2}\}$. See Figure 1 for an example. A double chain of size n will be denoted by DC_n.

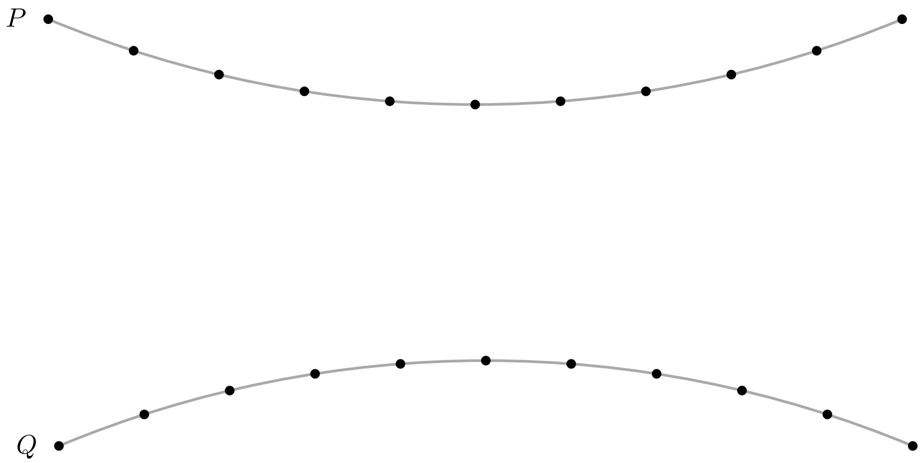

Figure 1: A double chain of size 22.

2.3. Perfect matchings in the double chain

Theorem 1 was proved in [6] as follows. Denote by $\mathsf{pm}_j(\mathrm{DC}_n)$ the number of perfect matchings of DC_n that have exactly j PQ-edges between the upper and the lower chain. If $n/2 - j$ is odd, then no perfect matching exists, so we assume that $n/2 - j$ is even. One can construct a perfect matching with j PQ-edges in the following way. First choose any j points of P and j points of Q and connect them by j non-intersecting PQ-edges. It is easy to see that there is a unique way to connect the chosen points (see also Proposition 5 below). Then, choose any perfect matching of the free points in each chain. Alternatively, one can first choose $n/2 - j$ points of P and $n/2 - j$ points of Q, then take any matching of P and any matching of Q that uses the chosen points; after that, the free points can be matched by PQ-edges in a unique way. Since Q has the same order type as P, it follows that

$$\mathsf{pm}_j(\mathrm{DC}_n) = \big(\mathsf{am}_j(\mathrm{SC}_{n/2})\big)^2 = \left(\binom{n/2}{j} \cdot C_{(n/2-j)/2}\right)^2, \tag{1}$$

where $\mathsf{am}_j(P)$ denotes the number of matchings of P (or equivalently, of any set of $n/2$ points in convex position) with exactly j free points. Finally, the total number of perfect matchings of DC_n is

$$\mathsf{pm}(\mathrm{DC}_n) = \sum_{\substack{0 \le j \le n/2 \\ j \equiv n/2 \pmod 2}} \left(\binom{n/2}{j} \cdot C_{(n/2-j)/2}\right)^2. \tag{2}$$

An analysis shows that the dominant term in this sum is the term corresponding to $j \approx n/6$, that is $\left(\binom{n/2}{n/6} \cdot C_{n/6}\right)^2$ ($n/6$ should be rounded in one way or the other). Using the estimates $C_k = \Theta(4^k/k^{3/2})$ and $\binom{ak}{bk} = \Theta\Big(\big(\frac{a^a}{b^b(a-b)^{a-b}}\big)^k/k^{1/2}\Big)$ for any constants $a > b > 0$, which follow from Stirling's formula, one obtains $\mathsf{pm}(\mathrm{DC}_{n,n/6}) = \Theta(3^n/n^4)$, and therefore, $\mathsf{pm}(\mathrm{DC}_n) = \Omega(3^n/n^4)$ and $O(3^n/n^3)$. With the help of Stirling's formula, and replacing the sum (2) by an integral, one can obtain the more precise estimate $\mathsf{pm}(\mathrm{DC}_n) = 3^n/n^{7/2}(182/\pi^{3/2} + o(1))$. (We omit the details.)

2.4. Double zigzag chains

In this section we recall the definitions of a (single) zigzag chain SZZC and a double zigzag chain DZZC. The concept is elementary and obvious from Figures 2 and 3, but the precise definitions suffer somewhat from a multitude of variations due to parity conditions. These variations will, however, be needed in the recursions in Section 4.

Let $P = \{p_1, p_2, \dots, p_n\}$ be a downward chain (SC_n) sorted by x-coordinate. For each even i, $1 < i < n$, we move the point p_i vertically up, very slightly above the segment $p_{i-1}p_{i+1}$, so that all consecutive triples $p_{i-1}p_ip_{i+1}$ with even i ($1 < i < n$) are now in upward position, whereas all other triples $p_ip_jp_k$ of points remain in downward position. After this modification, the points $p_1, p_2, \dots, p_n$ are still sorted by x-coordinate. A set obtained in this way will be called an *even (single) downward zigzag chain* of size n and denoted by eSZZC_n. If instead of even i-s we perform this transformation for each odd i, $1 < i < n$, we obtain an *odd (single) downward zigzag chain* (oSZZC_n). If n is even, then eSZZC_n and oSZZC_n are reflections of each other with respect to a vertical line; but if n is odd, then eSZZC_n and oSZZC_n have different order types, and – as one can verify on some small examples – different numbers of (perfect or not necessarily perfect) matchings. See Figure 2 for an example. A zigzag chain of size n, denoted by SZZC_n, is either an eSZZC_n or an oSZZC_n. For both types of SZZC_n, we shall derive the same asymptotic bound on the number of perfect matchings.

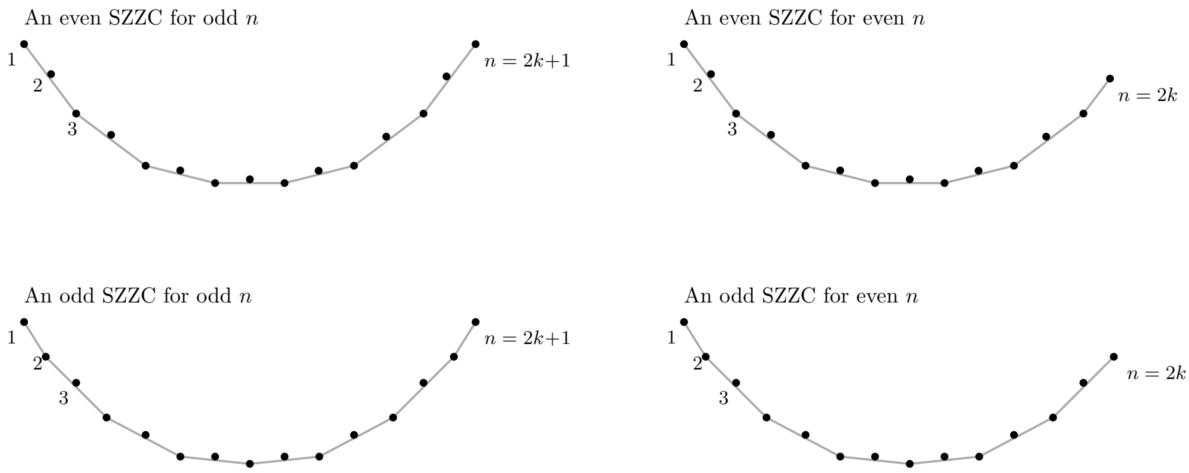

Figure 2: A (single) zigzag chain – several cases.

An *upward zigzag chain* (of either kind) is a downward zigzag chain reflected across a horizontal line. The construction of a double zigzag chain from zigzag chains is analogous to the construction of the double chain from two single chains: A *double zigzag chain* of (even) size n (DZZC_n) consists of a downward zigzag chain $P = \{p_1, p_2, \dots, p_{n/2}\}$ high above an upward zigzag chain $Q = \{q_1, q_2, \dots, q_{n/2}\}$. We can combine even and odd zigzag chains in various ways, but as mentioned above, this will make no difference for the asymptotic number of perfect matchings. See Figure 3 for an example of double zigzag chain obtained from two even zigzag chains of odd size.

3. Down-free matchings and perfect matchings

3.1. Down-free matchings

Suppose that we want to adapt the argument that was used for estimating $\mathsf{pm}(\mathrm{DC}_n)$ for the case of $\mathsf{pm}(\mathrm{DZZC}_n)$ (of any kind). That is: for fixed j (such that $n/2 - j$ is even) we want to choose j PQ-edges, and to complete this matching to a perfect matching by choosing edges that connect free points

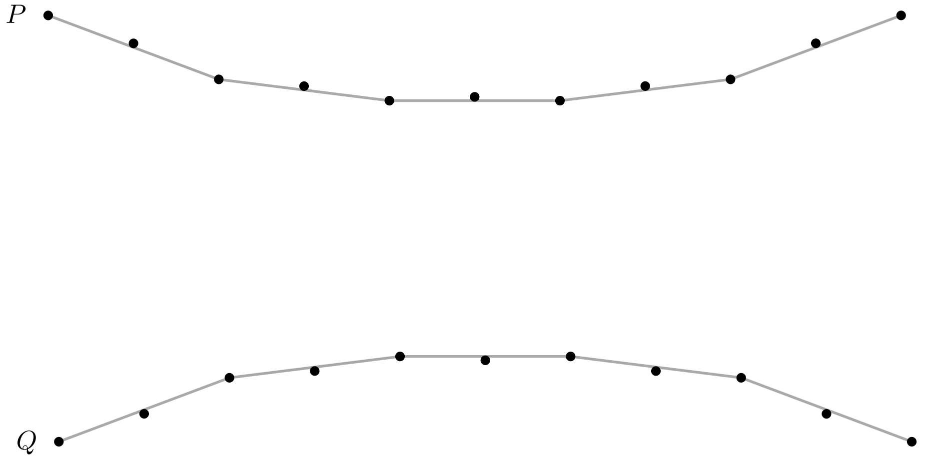

Figure 3: A double zigzag chain of size $n = 22$.

of the same chain in all possible ways. One can hope for improvement since the number of perfect matchings in SZZC_n (of any kind) is $\Theta^*(\nu^n)$ with $\nu = \sqrt{2+2\sqrt{2}} \approx 2.1974$, in contrast to $\Theta^*(2^n)$ for SC_n. (This bound for SZZC_n was proven in [1] for a slightly different construction, the so called *double circle*. The order type of SZZC is different from that of a double circle only in one triple of points; it is easy to show that they have the same asymptotic number of perfect matchings.) However, in comparison with the case of DC_n, here we have less freedom and no uniformity in constructing the matchings inside P and Q, once PQ-edges are chosen. Indeed, the j chosen PQ-edges may block visibility between certain pairs of free points from P or from Q. Moreover, for different choices of j PQ-edges, we have in general different numbers of ways to complete them to a perfect matching of DZZC_n. This follows from the fact that sets of points that remain free after choosing j PQ-edges have in general various order types, and, so, it seems hopeless to enumerate them in this way. On the other hand, if we first choose $(n-2j)/4$ edges between two points of P and $(n-2j)/4$ edges between two points of Q, then – as we prove below in Proposition 5 – there is *at most* one way to complete such a matching to a perfect matching of DZZC_n. More precisely, if the free points of P "see" all free points of Q, there is exactly one way of complete a matching to a perfect one, otherwise it is impossible. Next we define a property of matchings which – for two sets being one high above another – ensures the desired visibility of free points.

Definition. Let P be a set of points with distinct x-coordinates. A *down-free matching* is a matching of P in which no edge passes below an unmatched point. In other words: for each free point $p \in P$, the vertical ray going down from p does not cross any edge of the matching. Similarly, one defines an *up-free matching.*

Proposition 5. *Let P and Q be two point sets in general position such that P is high above Q.*

1. *Every perfect matching of $P \cup Q$ with j PQ-edges gives rise, after removing the PQ-edges, to a down-free matching M_P of P and an up-free matching M_Q of Q with j free points each.*
2. *Conversely, let M_P be a down-free matching of P and M_Q an up-free matching of Q. If M_P and M_Q have the same number of free points, then there is a unique way to complete $M_P \cup M_Q$ to a perfect matching of $P \cup Q$ by adding PQ-edges.*

Proof. We assume that the points $P = \{p_1, p_2, p_3, \dots\}$ and $Q = \{q_1, q_2, q_3, \dots\}$ are sorted by x-coordinate.

1. We only have to show that M_P is down-free and M_Q is up-free. For contradiction, assume without loss of generality that M_P is not down-free. Then there is a free point p_β in M_P so that the vertical downward ray from p_β crosses an edge $p_\alpha p_\gamma$, with $\alpha < \beta < \gamma$. See Figure 4(a) for an illustration. Since P is high above Q, the set Q must lie below $\ell(p_\alpha, p_\beta)$, $\ell(p_\alpha, p_\gamma)$, and $\ell(p_\beta, p_\gamma)$. There is no way to connect p_β to a point $q \in Q$ without crossing the edge $p_\alpha p_\gamma$.
2. Assume that M_P is down-free and M_Q is up-free.
   First we observe that for any four points the points $p_\alpha, p_\beta, q_\delta, q_\gamma$ with $p_\alpha, p_\beta \in P$, $q_\delta, q_\gamma \in Q$, $\alpha < \beta$, and $\gamma < \delta$ lie on the boundary of their convex hull in this clockwise order, see Figure 4(b)

for an illustration. Since P lies high above Q, the points of Q lie below $\ell(p_\alpha, p_\beta)$ and thus the points p_α and p_β lie on the convex hull consecutively and in this clockwise order. Similarly q_δ and q_γ lie on the convex hull consecutively and in this clockwise order. This implies the claim.

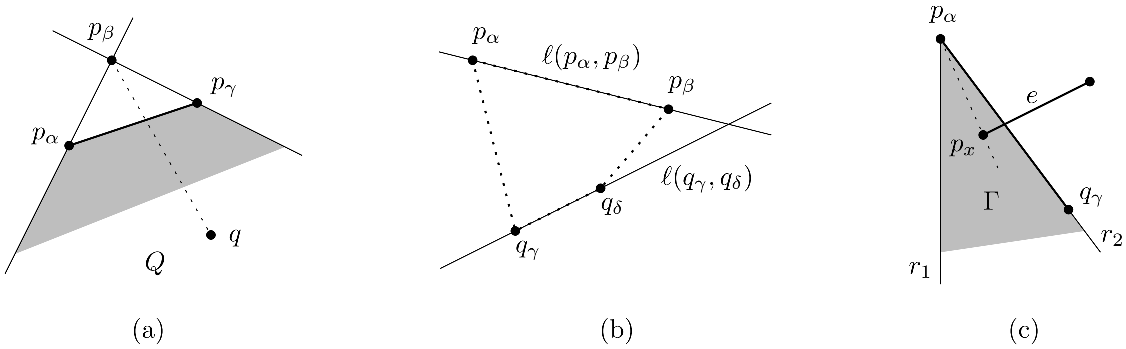

Figure 4: Illustrations to the proof of Proposition 5.

Let $p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_j}$ be the free points of P and let $q_{\gamma_1}, q_{\gamma_2}, \ldots, q_{\gamma_j}$ be the free points of Q, sorted from left to right. We complete $M_P \cup M_Q$ to a perfect matching of $P \cup Q$ by connecting p_{α_i} with q_{γ_i} for $i = 1, 2, \ldots, j$. By the just-proven claim about the cyclic order of $p_\alpha, p_\beta, q_\delta, q_\gamma$, these new PQ-edges do not cross each other. Moreover, they do not cross the edges of M_P and of M_Q. Indeed, assume that an edge $p_\alpha q_\gamma = p_{\alpha_i} q_{\gamma_i}$ crosses an edge $e \in M_P$. Consider the angular sector Γ bounded by the downward vertical ray r_1 with the origin p_α and the ray r_2 from p_α through q_γ, see Figure 4(c). The edge e crosses the ray r_2 by assumption and does not cross the ray r_1, because the matching M_P is down-free. Therefore, one of the endpoints of e, say p_x, lies in the interior of Γ. However, this is impossible because in such a case q_γ lies above the line $\ell(p_\alpha, p_x)$, which contradicts P being high above Q.
Finally, we need to show that this is the only way to complete $M_P \cup M_Q$ to a perfect matching of $P \cup Q$. Indeed, for any other possibility to match the free points we would have a pair of edges $p_\alpha q_\delta$ and $p_\beta q_\gamma$ with $\alpha < \beta$, $\gamma < \delta$. However, it follows from the claim about the cyclic order of $p_\alpha, p_\beta, q_\delta, q_\gamma$ that such edges necessarily cross. □

3.2. Down-free matchings of X and perfect matchings of double X

In Section 2.1, we have shown how to construct a double structure $\mathrm{D}X$ from any point set structure X. In the following theorem we show how asymptotic bounds on dfm for a structure X imply those on pm for the corresponding double structure $\mathrm{D}X$.

Theorem 6. *Let X be a construction so that $\mathsf{dfm}(X_n) = \Theta^*(\lambda^n)$. Then for the double structure $\mathrm{D}X$ we have $\mathsf{pm}(\mathrm{D}X_n) = \Theta^*(\lambda^n)$ for even n.*

More precisely: If $\mathsf{dfm}(X_n) = \Theta(\lambda^n/n^\alpha)$, then $\mathsf{pm}(\mathrm{D}X_n) = \Omega(\lambda^n/n^{2\alpha+1})$ and $O(\lambda^n/n^{2\alpha})$.

Proof. Denote by $\mathsf{dfm}_j(X_{n/2})$ the number of down-free matchings of $X_{n/2}$ with exactly j free points, for $0 \le j \le n/2$, and let $p_j = \mathsf{dfm}_j(X_{n/2})/\mathsf{dfm}(X_{n/2})$. Now

$$
\begin{aligned}
\mathsf{pm}(\mathrm{D}X_n) &= \sum_{j=0}^{n/2} \mathsf{pm}_j(\mathrm{D}X_n) && \text{by the definition of } \mathsf{pm}_j \\
&= \sum_{j=0}^{n/2} \mathsf{dfm}_j(X_{n/2})^2 && \text{by Proposition 5, see explanation below} \\
&= \mathsf{dfm}(X_{n/2})^2 \cdot \sum_{j=0}^{n/2} p_j^2 && \text{by the definition of } p_j \\
&= \Theta\Big(\frac{\lambda^{n/2}}{(n/2)^\alpha}\Big)^2 \cdot \sum_{j=0}^{n/2} p_j^2 && \text{by assumption} \\
&= \Theta(\lambda^n/n^{2\alpha}) \cdot \sum_{j=0}^{n/2} p_j^2.
\end{aligned}
$$

The second equation follows from Proposition 5, which relates perfect matchings for a set $P \cup Q$ of type $\mathrm{D}X_n$ to pairs of down-free matchings of P with up-free matchings of Q, which conform to the structure $X_{n/2}$.

Since $\sum_{j=0}^{n/2} p_j = 1$, we immediately get an upper bound for the last factor: $\sum_{j=0}^{n/2} p_j^2 \leq 1$. For a lower bound, we apply Jensen's equality with the convex function $x \mapsto x^2$, which gives $\sum_{j=0}^{n/2} p_j^2 \geq \frac{1}{n/2+1}$. These bounds imply the claims. □

As the first application of Theorem 6, we show how one can reprove Theorem 1 without need to determine the dominant term in (2). We use the following well-known fact.

Proposition 7 ([9] A001006)**.** *The number of all matchings in a set of n points in convex position is the n-th Motzkin number M_n. Asymptotically, $M_n = \Theta(3^n/n^{3/2})$.*

Moreover, every matching of a downward chain is obviously down-free. Therefore, Theorem 6, with $\lambda = 3$ and $\alpha = 3/2$ gives immediately $\mathsf{pm}(\mathrm{DC}_n) = \Omega(3^n/n^4)$ and $O(3^n/n^3)$.

In the next sections we use Theorem 6 for estimating pm for other constructions.

4. Zigzag chains

By Theorem 6, asymptotic bounds on $\mathsf{dfm}(\mathrm{SZZC}_n)$ imply those on $\mathsf{pm}(\mathrm{DZZC}_n)$. Thus, we analyze the number of down-free matchings of SZZC_n. We defined above two kinds of double chains: even and odd. We introduce three generating functions depending on the kind of chain and on the parity of n:

1. $A(x) = \sum_{k \geq 0} a_k x^k$, where $a_k = \mathsf{dfm}(\mathrm{eSZZC}_{2k+1})$;
2. $B(x) = \sum_{k \geq 0} b_k x^k$, where $b_k = \mathsf{dfm}(\mathrm{oSZZC}_{2k+1})$;
3. $C(x) = \sum_{k \geq 0} c_k x^k$, where $c_k = \mathsf{dfm}(\mathrm{eSZZC}_{2k}) = \mathsf{dfm}(\mathrm{oSZZC}_{2k})$.

We find recursive relationships between the coefficients of these functions.

Recursion for a_k. For every $k \geq 0$ we have the following cases (Figure 5).

1. p_1 is not matched. This contributes c_k matchings.
2. p_1 is matched to p_{2i+1} with $2 \leq i \leq k$. This contributes $\sum_{2 \leq i \leq k} b_{i-1} c_{k-i}$ matchings.
3. p_1 is matched to p_{2i} with $1 \leq i \leq k$, p_{2i-1} and p_{2i+1} are not matched to each other. This contributes $\sum_{1 \leq i \leq k} c_{i-1} a_{k-i}$ matchings.
4. p_1 is matched to p_{2i} with $2 \leq i \leq k$, p_{2i-1} and p_{2i+1} are matched to each other. This contributes $\sum_{2 \leq i \leq k} b_{i-2} c_{k-i}$ matchings.
5. p_1 is matched to p_3. Then p_2 must be matched to some point p_{2i+1} with $2 \leq i \leq k$. This contributes $\sum_{2 \leq i \leq k} b_{i-2} c_{k-i}$ matchings.
6. p_1 is matched to p_3, p_2 is matched to p_{2i} with $2 \leq i \leq k$, and p_{2i-1} and p_{2i+1} are not matched to each other. This contributes $\sum_{2 \leq i \leq k} c_{i-2} a_{k-i}$ matchings.
7. p_1 is matched to p_3, p_2 is matched to p_{2i} with $3 \leq i \leq k$, and p_{2i-1} and p_{2i+1} are matched to each other. This contributes $\sum_{3 \leq i \leq k} b_{i-3} c_{k-i}$ matchings.

Thus we obtain

$$a_k = c_k + \sum_{2 \leq i \leq k} b_{i-1} c_{k-i} + \sum_{1 \leq i \leq k} c_{i-1} a_{k-i} + 2 \sum_{2 \leq i \leq k} b_{i-2} c_{k-i} + \sum_{2 \leq i \leq k} c_{i-2} a_{k-i} + \sum_{3 \leq i \leq k} b_{i-3} c_{k-i}. \quad (3)$$

Recursion for b_k. For every $k \geq 0$ we have the following cases, see Figure 6, left side.

1. p_1 is not matched. This contributes c_k matchings.
2. p_1 is matched to p_{2i} with $1 \leq i \leq k$. This contributes $\sum_{1 \leq i \leq k} c_{i-1} b_{k-i}$ matchings.
3. p_1 is matched to p_{2i+1} with $1 \leq i \leq k$, p_{2i} and p_{2i+2} are not matched to each other. This contributes $\sum_{1 \leq i \leq k} a_{i-1} c_{k-i}$ matchings.
4. p_1 is matched to p_{2i+1} with $1 \leq i \leq k-1$, p_{2i} and p_{2i+2} are matched to each other. This contributes $\sum_{1 \leq i \leq k-1} c_{i-1} b_{k-i-1}$ matchings.

This yields

$$b_k = c_k + \sum_{1 \leq i \leq k} c_{i-1} b_{k-i} + \sum_{1 \leq i \leq k} a_{i-1} c_{k-i} + \sum_{1 \leq i \leq k-1} c_{i-1} b_{k-i-1}. \quad (4)$$

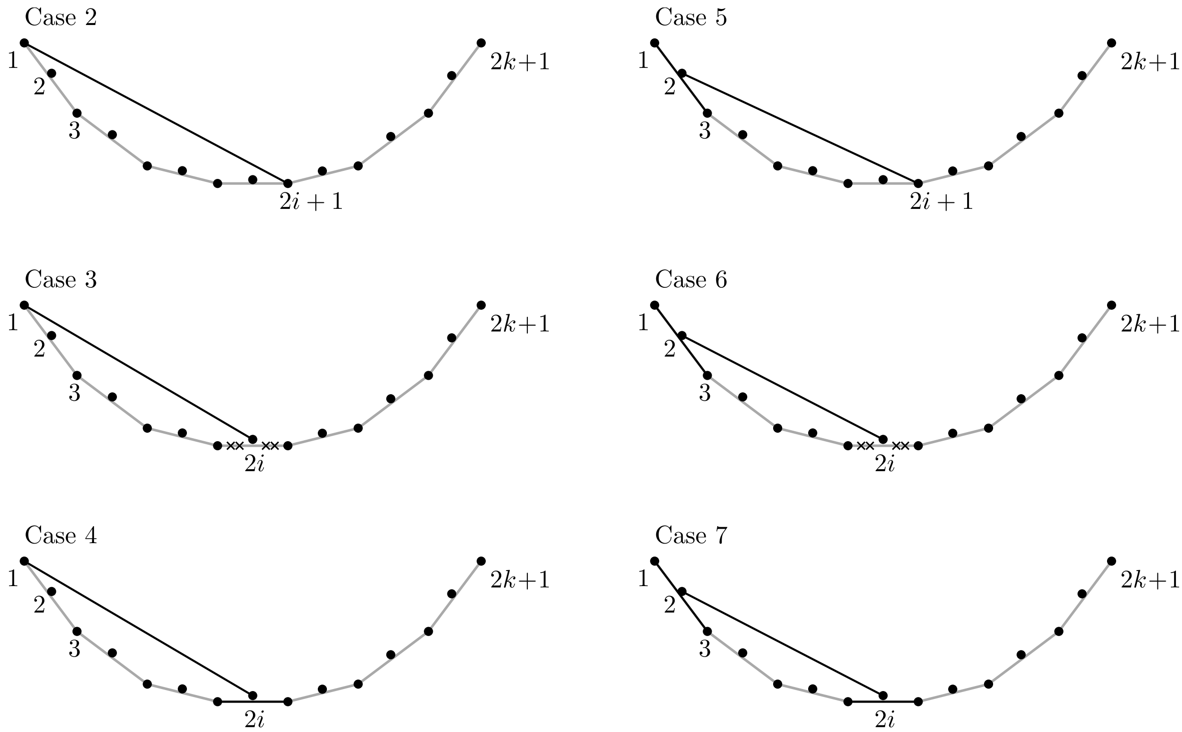

Figure 5: The cases in the recursion for a_k.

Recursion for c_k. Clearly, $c_0 = 1$. For $k \geq 1$ we have the following cases, see Figure 6, right side.

1. p_1 is not matched. This contributes a_{k-1} matchings.
2. p_1 is matched to p_{2i} with $1 \leq i \leq k$. This contributes $\sum_{1\leq i\leq k} c_{i-1}c_{k-i}$ matchings.
3. p_1 is matched to p_{2i+1} with $1 \leq i \leq k-1$, p_{2i} and p_{2i+2} are not matched to each other. This contributes $\sum_{1\leq i\leq k-1} a_{i-1}a_{k-i-1}$ matchings.
4. p_1 is matched to p_{2i+1} with $1 \leq i \leq k-1$, p_{2i} and p_{2i+2} are matched to each other. This contributes $\sum_{1\leq i\leq k-1} c_{i-1}c_{k-i-1}$ matchings.

This gives

$$c_k = a_{k-1} + \sum_{1\leq i\leq k} c_{i-1}c_{k-i} + \sum_{1\leq i\leq k-1} a_{i-1}a_{k-i-1} + \sum_{1\leq i\leq k-1} c_{i-1}c_{k-i-1}. \tag{5}$$

After simplifying equations (3–5), we obtain:

$$a_k = c_k - c_{k-1} + \sum_{i=0}^{k-1} b_i c_{k-1-i} + \sum_{i=0}^{k-1} c_i a_{k-1-i} + 2\sum_{i=0}^{k-2} b_i c_{k-2-i} + \sum_{i=0}^{k-2} c_i a_{k-2-i} + \sum_{i=0}^{k-3} b_i c_{k-3-i}$$

$$b_k = c_k + \sum_{i=0}^{k-1} c_i b_{k-1-i} + \sum_{i=0}^{k-1} a_i c_{k-1-i} + \sum_{i=0}^{k-2} c_i b_{k-2-i}$$

$$c_k = a_{k-1} + \sum_{i=0}^{k-1} c_i c_{k-1-i} + \sum_{i=0}^{k-2} a_i a_{k-2-i} + \sum_{i=0}^{k-2} c_i c_{k-2-i}$$

We translate these equations into generating functions $A(x) = \sum_{k=0}^{\infty} a_k x^k$, etc., and obtain the following system, where we write A, B, C for $A(x), B(x), C(x)$:

$$A = C((1-x) + x(1+x)A + x(1+x)^2 B)$$

$$B = C(1 + xA + x(1+x)B)$$

$$C = 1 + xA + x^2A^2 + x(1+x)C^2$$

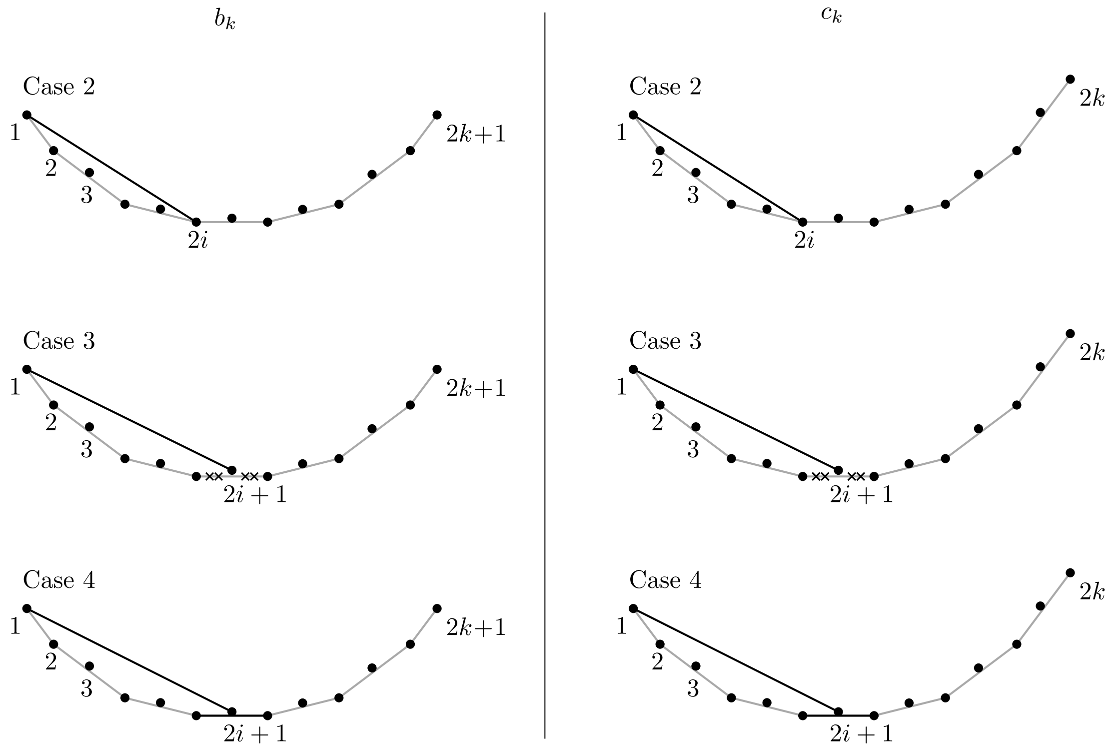

Figure 6: The cases in the recursions for b_k and c_k.

We eliminate A and B from this system and find that C satisfies the equation

$$\begin{aligned}1-(1+3x+5x^2)C+x(5+8x+8x^2+9x^3)C^2-8x^2(1+x)(1+x+x^3)C^3+\\ +4x^3(1+x+x^3)(1+x)^2C^4=0, \quad (6)\end{aligned}$$

and that A and B are related to C as follows:

$$A=\frac{C(1-x+2x^2C+2x^3C)}{1-2xC-2x^2C} \qquad (7)$$

$$B=\frac{C(1-2x^2C)}{1-2xC-2x^2C}$$

Equation (6) has four solutions. Only one of them can be written as a formal power series:

$$C=\frac{2(1+x+x^3)-\sqrt{2(1+x+x^3)\left(1-2x-8x^2-3x^3+(1+x)\sqrt{(1-x-3x^2)(1-9x-3x^2)}\right)}}{4x(1+x)(1+x+x^3)}$$

The other three solutions have different combinations of signs before the two square roots. For those combinations, the numerator has a non-zero constant term, and this cannot balance the absence of a constant term in the denominator. For the series $C(x)$ given above, the singularity closest to 0 occurs in $\mu=\frac{\sqrt{93}}{6}-\frac{3}{2}$, one of the roots of $1-9x-3x^2$. It is a square root singularity, and there is no other singularity with the same absolute value; thus, by the exponential growth formula [5, Thm. IV.7] and a transfer theorem [5, Thm. VI.1], the asymptotics of the sequence is $c_k=\Theta((1/\mu)^k\,k^{-3/2})$ with $1/\mu=(\sqrt{93}+9)/2\approx 9.3218$.

Since c_k counts matchings of $2k$ points, it follows that the number of down-free matchings of SZZC_n of this kind is $\Theta(\lambda^n/n^{3/2})$, where $\lambda=\sqrt{1/\mu}=\sqrt{(\sqrt{93}+9)/2}\approx 3.0532$. It is easy to see that the same bound holds for all kinds of zigzag chains: for the proof, note that a zigzag chain of kind C with $2k$ points includes a zigzag chain of kind A with $2k-1$ points and is included in a zigzag chain of kind A with $2k+1$ points; similarly for kind B.

Finally, it follows from Theorem 6 that the number of perfect matchings of DZZC_n (of either kind) is $\Omega(\lambda^n/n^4)$ and $O(\lambda^n/n^3)$. This proves Theorem 2.

5. r-chains without corners

5.1. Definition of r-chains with and without corners

In the following two sections we deal with further generalizations of the double chain. An upward single chain will be called an *arc*. As usual, the size of an arc is the number of its points. Recall that three points with distinct x-coordinates are in *upward position* if they form a clockwise oriented triangle when sorted by x-coordinate.

We define an *r-chain (with corners) with k arcs*, to be denoted by $\mathrm{CH}(r,k)$, see Figure 7(a) for an example. It consists of k arcs of size $r+1$, the rightmost point of the ith arc ($1 \le i \le k-1$) coinciding with the leftmost point of the $(i+1)$st arc, such that any three points are in upward position if and only if they belong to the same arc. An r-chain $\mathrm{CH}(r,k)$ has $rk+1$ points. As a special case, a simple (downward) chain is a 1-chain, and an even zigzag chain of odd size is a 2-chain.

One can construct an r-chain $\mathrm{CH}(r,k)$ with k arcs as follows:

- Take $k+1$ points $V_0, V_1, V_2, \ldots, V_k$, sorted by x-coordinate, in downward position. These points will be called the *corners*.
- For each $i = 1, 2, \ldots, k$, add $r-1$ points on the segment $V_{i-1}V_i$.
- Replace each segment $V_{i-1}V_i$ by a very flat upward circular arc through V_{i-1} and V_i. Move the $r-1$ points from the segment vertically upwards so that they lie on this circular arc. The radius of the circular arc must be sufficiently big so that the orientation of triples of points that do not lie on the same segment is not changed.

We shall often use a compact schematic drawing of r-chains as in Figure 7(b). In such drawings we have to draw some matching edges as curved lines rather than as straight-line segments, to avoid crossings.

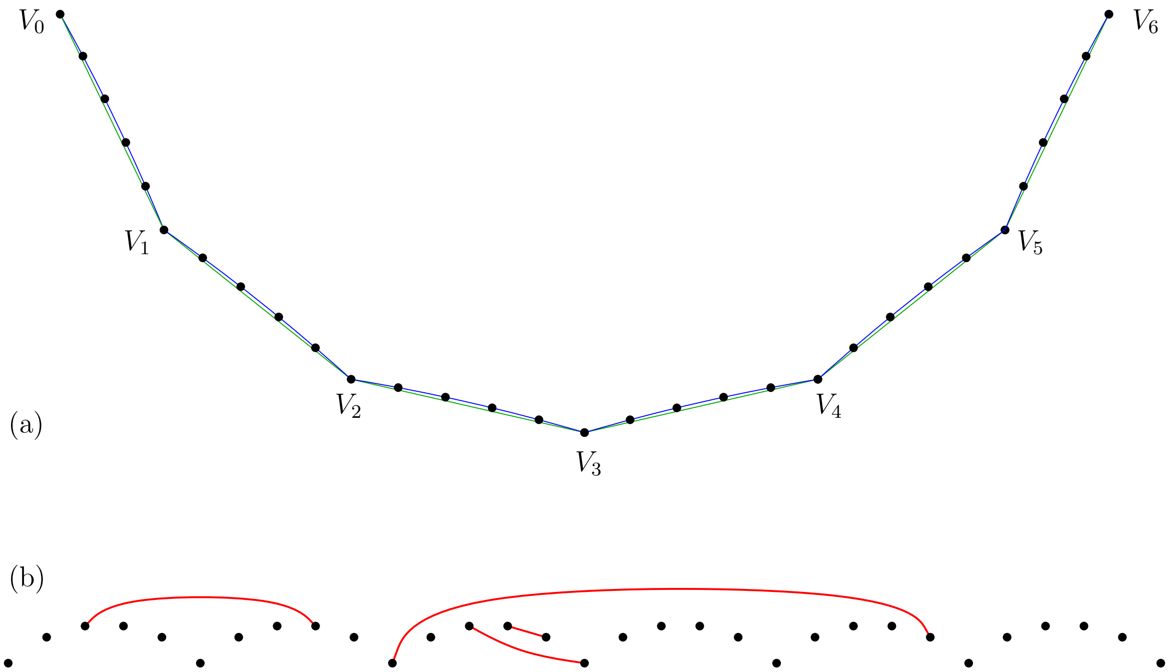

Figure 7: A 5-chain (with corners) with six arcs: (a) a precise drawing; (b) a schematic drawing.

The class of (double) r-chains was earlier used for finding lower bounds on the maximal number of *triangulations* (tr) of point sets in the plane. García, Noy, and Tejel [6] showed that $\mathsf{tr}(\mathrm{DC}_n) = \Theta^*(8^n)$. Aichholzer, Hackl, Huemer, Hurtado, Krasser, and Vogtenhuber [1] improved this bound by showing that $\mathsf{tr}(\mathrm{DZZC}_n) = \Theta^*(8.48^n)$. This result was further improved by Dumitrescu, Schulz, Sheffer, and Tóth [4], who showed that a double 4-chain of size n, denoted in their work by $D(n, 3^{n/8})$, has $\Omega(8.65^n)$ triangulations.

Now we define a variation of this structure whose analysis is easier. An *r-chain without corners with k arcs*, denoted by $\mathrm{CH}^*(r,k)$, is a set obtained from $\mathrm{CH}(r+1,k)$ by deleting the corners. It consists of rk points. See Figure 8 for an example. In this section, we will analyze r-chains without corners, and we will find precise asymptotic estimates for the number of down-free matchings. In the next section, we will turn to r-chains with corners. They give even stronger lower bounds, but the analysis will be more laborious and not so precise.

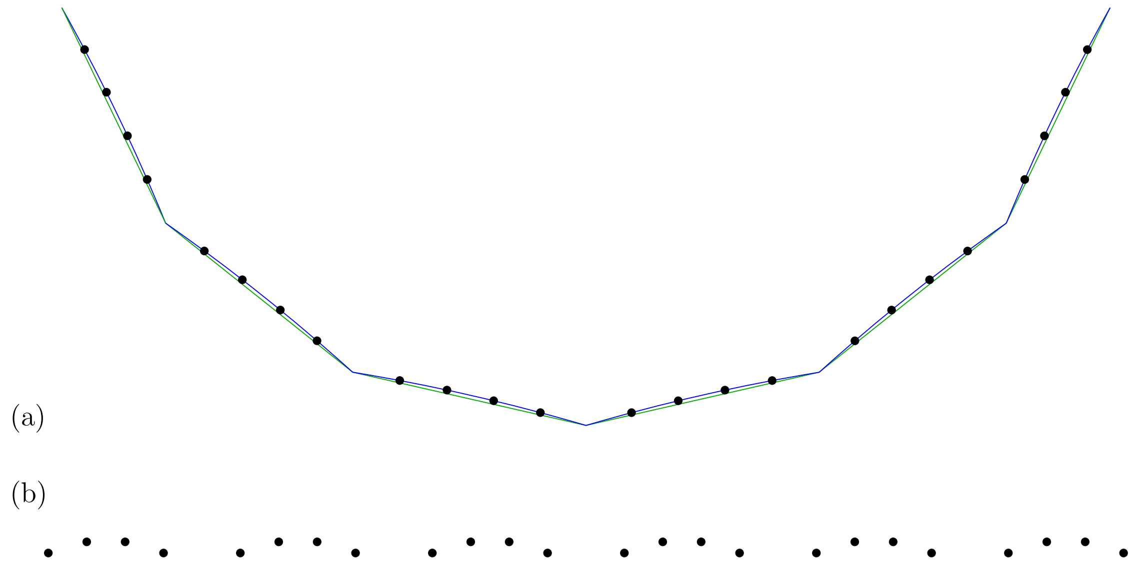

Figure 8: A 4-chain without corners with six arcs: (a) precise drawing; (b) schematic drawing.

5.2. *Matchings with runners*

Consider a matching of $X = \mathrm{CH}^*(r,k)$. We want to build down-free matchings incrementally from left to right by adding one arc at a time. If we cut such a matching between two arcs, then we possibly have some edges cut into two "half-edges", which we call *runners*. (In botany, runners are shoots that connect individual plants.) A runner can be formally defined as a *marked vertex*. Such a vertex must not be matched by "proper" edges and must be visible from above. These requirements ensure that, in the course of the incremental construction, two runners can be joined into one edge. Runners are visualized as half-edges that have one endpoint in X and the other end dangling, see Figure 9(a). Note that it is not assigned in advance whether a runner will be matched to the left or to the right.

A matching which possibly has runners will be called a *ρ-matching*. Extending our previous definition of free points, we call a point *free* in a ρ-matching if it is neither matched by a "proper" edge nor marked as an endpoint of a runner. A ρ-matching is *down-free* if all free vertices are visible from below.

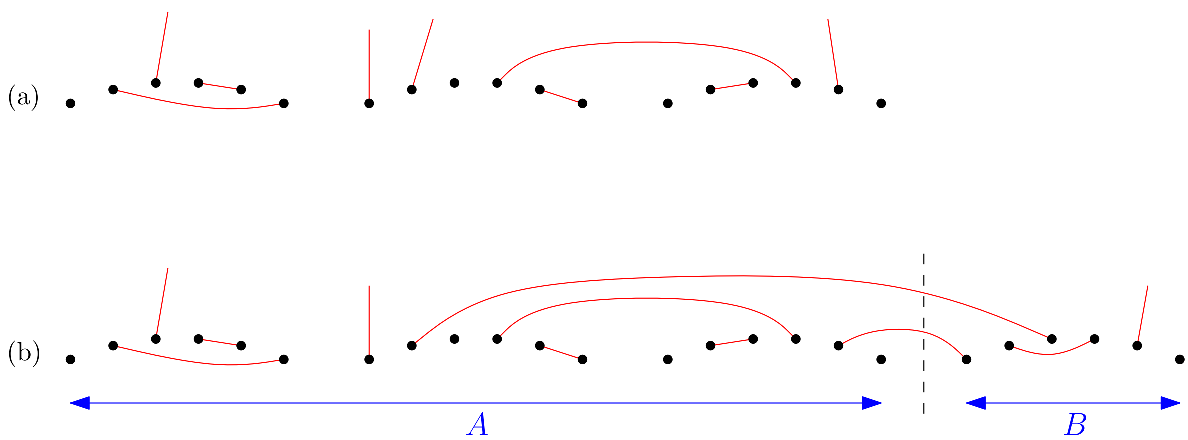

Figure 9: (a) A down-free ρ-matching M_A with four runners of $A = \mathrm{CH}^*(6,3)$. (b) Combining M_A with a down-free ρ-matching with three runners of $B = \mathrm{CH}^*(6,1)$.

In the course of the recursive construction of down-free ρ-matchings, runners from different arcs can be matched as follows. Let A and B be two r-chains without corners, and let M_A and M_B be down-free ρ-matchings of these sets. We place B to the right of A. If M_A has j runners and M_B has β runners, then for each ℓ in the range $0 \le \ell \le \min\{j, \beta\}$ we can match, in a unique way, the rightmost ℓ runners of M_A to the leftmost ℓ runners of M_B. The obtained ρ-matching M is also down-free; the runners which were not matched in this procedure remain runners in M; the number of such runners is $j + \beta - 2\ell$. Conversely, each down-free ρ-matching of $A \cup B$ can be obtained by this procedure from two uniquely determined down-free ρ-matchings of A and B. Figure 9(b) shows an example with $j = 4$, $\beta = 3$, $\ell = 2$.

We summarize these observations for the special case that we will use in the recursive construction of r-chains: adding one new arc to the right of a given r-chain, see Figure 10.

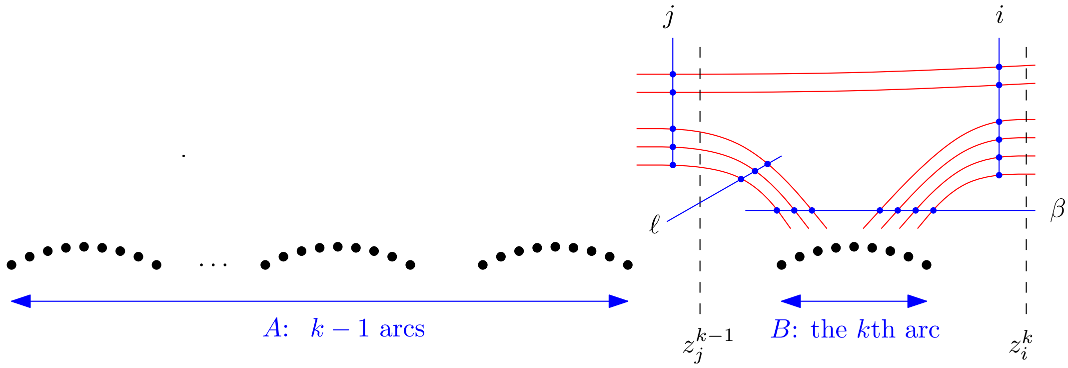

Figure 10: Runners in a recursive construction of a ρ-matching of an r-chain without corners.

Observation 8. *Let $X = \mathrm{CH}^*(r, k)$ be an r-chain without corners with $k \ge 1$ arcs. Let B be the rightmost arc of X, and let $A = X \setminus B$. Let M_A be a down-free ρ-matching of A with j runners, and let M_B be a down-free ρ-matching of B with β runners. For each $0 \le \ell \le \min\{j, \beta\}$ there exists a unique down-free ρ-matching $M_{X,\ell}$ of X obtained by matching the rightmost ℓ runners of M_A with the leftmost ℓ runners of M_B. The number of runners in $M_{X,\ell}$ is $i = j + \beta - 2\ell$.*

Conversely, each down-free ρ-matching M of X can be obtained in this way from uniquely determined matchings M_A and M_B (of A and B) as above. If M has i runners, M_A has j runners, and M_B has β runners, then the number of edges obtained by matching of pairs of runners is $\ell = (j + \beta - i)/2$.

For $k = 1$, this observation holds trivially: A is empty, and the only possibility is $j = \ell = 0$, $\beta = i$. From the above relations between the parameters i, j, β, ℓ, one can work out the constraints on the possible values of β for given i and j: The equation $i = j + \beta - 2\ell$ together with $0 \le \ell \le \min\{j, \beta\}$ implies that β must satisfy $|i - j| \le \beta \le i + j$ and $\beta \equiv i - j \pmod 2$.

5.3. Recursion for matchings with runners in r-chains without corners

Denote the number of down-free ρ-matchings of $\mathrm{CH}^*(r, k)$ with i runners by $z_i^k(r)$ or simply by z_i^k, since we will regard r as fixed (see the right part of Figure 10). The down-free matchings of $X = \mathrm{CH}^*(r, k)$ are just the down-free ρ-matchings without runners. Since the size of X is rk, the growth constant for the number of its down-free matchings is $\lim_{k\to\infty} \sqrt[rk]{z_0^k(r)}$.

For $k = 0$ we have $z_0^0 = 1$ and $z_i^0 = 0$ for $i > 0$. The numbers z_i^1 for a chain consisting of a single arc (or equivalently, a single upward chain of size r) will serve as a basis of the recursion. They are determined in the following proposition.

Proposition 9. 1. *The number of down-free matchings (without runners) of a single arc of size r is*

$$z_0^1 = z_0^1(r) = \binom{r}{\lfloor r/2 \rfloor}.$$

2. *The number of down-free ρ-matchings of a single arc of size r that have i runners is*

$$z_i^1 = z_i^1(r) = \binom{r}{i}\binom{r-i}{\lfloor (r-i)/2 \rfloor} = \binom{r}{i, \lfloor (r-i)/2 \rfloor, \lceil (r-i)/2 \rceil}.$$

Proof. 1. For the first equation, let $f(x) = \sum_{r=0}^{\infty} z_0^1(r)x^r$ be the generating function for the number of such matchings in terms of the size r of an arc. We will show that $f(x)$ satisfies the equation

$$f(x) = \frac{1}{1-x}\left(x^2 \cdot c(x^2) \cdot f(x) + 1\right), \tag{8}$$

where $c(x) = (1-\sqrt{1-4x})/2x$ is the generating function of the Catalan numbers. Therefore, we have

$$f(x) = \frac{1}{1-x-x^2c(x^2)},$$

and this is known to be the generating function for $\binom{r}{\lfloor r/2 \rfloor}$ [9, A001405].

To see why (8) holds, consider the leftmost matched point p (if there is any). Suppose that p is matched with q, see Figure 11 for illustration. Then all points to the left of p are free, which contributes $1/(1-x)$ to the generating function. The points between p and q are not visible from below and, therefore, they are matched by a *perfect* matching; this contributes $c(x^2)$. Finally, the points to the right of q are matched by a down-free matching, whose generating function is again $f(x)$. The factor x^2 accounts for the two points p and q, and the additive term $+1$ accounts for the case that p does not exist.

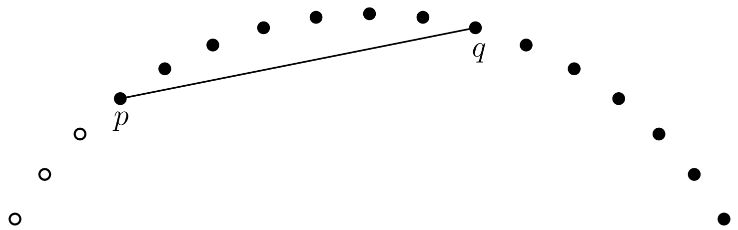

Figure 11: The leftmost edge pq in the proof of Proposition 9.1.

We give another proof – a bijective one. For a given matching, we mark the left and right endpoints of each edge by `L` and `R`, respectively. We leave the free points unmarked for the moment. Then the non-crossing matching can be reconstructed from the labels: We traverse the points from left to right, and we match each `R` that we meet with the closest previous unmatched `L`. Moreover, since the matching is down-free, free vertices can only appear when there are no previous unmatched `L`-vertices. Now we label the free points: If there are γ free points, we label the first $\lfloor \gamma/2 \rfloor$ free points by `R` and the last $\lceil \gamma/2 \rceil$ free points by `L`, see Figure 12 for illustration. The free points marked `R` can be recovered in a left-to-right sweep as those `R`-vertices for which we find no previous matching `L`-vertex in the above procedure. The free points marked `L` can be recovered similarly in a right-to-left sweep, and finally, the matching among the non-free points can be found as described above. Thus we have established a bijection with sequences of length r over the alphabet $\{\mathtt{L}, \mathtt{R}\}$ that contain $\lfloor r/2 \rfloor$ many `R`'s.

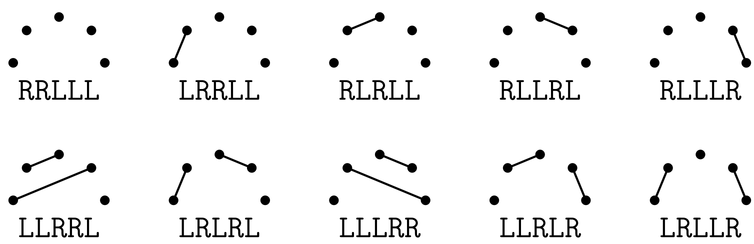

Figure 12: The coding of down-free matchings in the second proof of Proposition 9.1.

2. Let us turn to the second equation. Once we choose i endpoints of runners, the whole matching is down-free if and only if its restriction on the remaining $r - i$ points is down-free. Therefore, the result follows directly from the first part. □

Now we find a recursion for z_i^k, $k \geq 1$.

Proposition 10. *For fixed r, we have the recursion*

$$z_i^k = \sum_{j \geq 0} a_{ij} z_j^{k-1}, \tag{9}$$

with coefficients

$$a_{ij} = \sum_{\substack{0 \leq \beta \leq r, \\ |i-j| \leq \beta \leq i+j, \\ \beta \equiv i-j \pmod 2}} z_\beta^1 = z_{|i-j|}^1 + z_{|i-j|+2}^1 + \cdots + z_{\min\{r^*, i+j\}}^1, \tag{10}$$

where r^* is r or $r-1$ and has the same parity as $i-j$.

Proof. For $k = 1$, (9) can be verified directly. Assume now $X = \mathrm{CH}^*(r,k)$ with $k > 1$, let B be the rightmost arc of X, and let $A = X \setminus B$. For each $j \geq 0$ and each possible β we will find the number of ρ-matchings of X with i runners whose restriction to A has j runners and restriction to B has β runners. By Observation 8, ρ-matchings of A and B and the values of i, j and β determine uniquely an ρ-matching of X. Therefore ρ-matchings of A and B with (respectively) j and β runners contribute $z_j^{k-1} \cdot z_\beta^1$ ρ-matchings of X with i runners.

For given i and j, the bounds $|i-j| \leq \beta \leq i+j$ and the restriction $\beta \equiv i-j \pmod 2$ given in (10) are explained in the remark after Observation 8. ☐

5.4. Analysis of the recursion

For each $k \geq 0$, denote $v_k = (z_0^k, z_1^k, z_2^k, z_3^k, \dots)^\top$. In particular we have $v_0 = (1,0,0,0,\dots)^\top$. Consider the infinite coefficient matrix $A = (a_{ij})_{i,j \in \mathbb{N}_0}$ with a_{ij} given by (10). By Proposition 10, we have $Av_{k-1} = v_k$ for each $k \geq 1$. One easily verifies that the matrix A has the following properties:

- A is symmetric.
- A is a band matrix of bandwidth r: for $|i-j| > r$ we have $a_{ij} = 0$.
- The entries of the first row and column are $a_{i0} = a_{0i} = z_i^1 = \binom{r}{i}\binom{r-i}{\lfloor (r-i)/2 \rfloor}$.
- For $i+j \geq r^*$ we have $a_{i+1,j+1} = a_{ij}$. That is, the diagonals – sets of entries with fixed $q := j-i$, $|q| \leq r$ – *stabilize* starting from the entry $a_{(r^*-q)/2,(r^*+q)/2}$. For these entries we have:

  $$a_{ij} = a_{i,i+q} = \sum_{\substack{|q| \leq \beta \leq r \\ \beta \equiv q \pmod 2}} z_\beta^1. \tag{11}$$

  In particular, starting from the rth row (resp. column), the rows (resp. columns) are shifts of each other, and therefore, have the same sum of elements.
- The elements in the upper-left corner ($i+j < r^*$) are positive and smaller than the elements in the same diagonal after stabilization – since in this case we have a partial sum of (11).

For example, for $r = 5$, the matrix is

$$A = \begin{pmatrix}
10 & 30 & 30 & 20 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
30 & 40 & 50 & 35 & 21 & 5 & 1 & 0 & 0 & 0 & 0 & \cdots \\
30 & 50 & 45 & 51 & 35 & 21 & 5 & 1 & 0 & 0 & 0 & \cdots \\
20 & 35 & 51 & 45 & 51 & 35 & 21 & 5 & 1 & 0 & 0 & \cdots \\
5 & 21 & 35 & 51 & 45 & 51 & 35 & 21 & 5 & 1 & 0 & \cdots \\
1 & 5 & 21 & 35 & 51 & 45 & 51 & 35 & 21 & 5 & 1 & \cdots \\
0 & 1 & 5 & 21 & 35 & 51 & 45 & 51 & 35 & 21 & 5 & \cdots \\
0 & 0 & 1 & 5 & 21 & 35 & 51 & 45 & 51 & 35 & 21 & \cdots \\
0 & 0 & 0 & 1 & 5 & 21 & 35 & 51 & 45 & 51 & 35 & \cdots \\
0 & 0 & 0 & 0 & 1 & 5 & 21 & 35 & 51 & 45 & 51 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & 5 & 21 & 35 & 51 & 45 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \tag{12}$$

The column sum λ_r after stabilization of the columns, that is, starting from the $(r+1)$st column, is as follows:

$$\lambda_r = \sum_{i=0}^{r}(i+1)z_i^1 = \sum_{i=0}^{r}(i+1)\binom{r}{i}\binom{r-i}{\lfloor (r-i)/2 \rfloor} = \sum_{i=0}^{r}(i+1)\binom{r}{i, \lfloor (r-i)/2 \rfloor, \lceil (r-i)/2 \rceil} \qquad (13)$$

In the spirit of the Perron-Frobenius theorem for non-negative stochastic matrices, one can expect that λ_r is the growth constant for $(z_0^k)_{k\geq 0}$. This is indeed the case. We will prove it by using a result by Banderier and Flajolet [3] about enumeration of certain kinds of colored lattice paths.

Proposition 11. *For fixed r, we have $z_0^k = \Theta^*\big((\lambda_r)^k\big)$.*

Note that the superscript k in the left-hand side denotes an index, whereas in the right-hand side it is a power.

Proof. We begin with some notion for lattice paths. Families of lattice paths are usually defined by indicating a starting point – normally $(0,0)$ – and a set of possible moves of the form $(1,\beta)$. For many familiar families it is additionally required that the paths never go below the x-axis and/or end at the x-axis. The paths that start at $(0,0)$ and satisfy both these restrictions are called *excursions*. For example, Motzkin paths [9, A001006] are excursions that use the moves $(1,1)$, $(1,0)$, $(1,-1)$.

In a more general setting, a set of possible moves may depend on the point reached by a path. Moreover, each move $(1,\beta_i)$ starting in a certain point can have a non-negative integer *multiplicity* m_i. This is sometimes expressed by saying that these are m_i copies of the same move that are distinguished by different “colors”.

In summary, to each lattice point (a,b) we assign a *rule* – a set of moves that can be used for the next step once a path has reached this point, together with multiplicities. It is assumed that for each lattice point the number of moves with non-zero multiplicity is finite. The condition of not crossing the x-axis can be expressed in terms of such rules: for each point (a,b) there must be only moves $(1,\beta)$ with $\beta \geq -b$.

Consider now the case that all points that lie on the same horizontal line have the same rule. Namely, for $y=j$ and $i\geq 0$ we denote by d_{ij} the multiplicity of the move $(1,i-j)$ at (any) point (a,j). We collect these data in the infinite matrix $D=(d_{ij})_{i,j\in\mathbb{N}_0}$. Let $u=(1,0,0,\dots)^\top$. Then the number of paths that start at $(0,0)$, do not cross the x-axis, and end at a point (a,b) is equal to the bth component of $D^a u$ – this follows directly from matrix multiplication. In particular, the upper-left entry of D^k is the number of excursions of length k, which we will denote by $\mathrm{Ex}(D,k)$. The quantity in which we are interested, the number z_0^k of down-free matchings, is then given by $z_0^k = \mathrm{Ex}(A,k)$, where A is the coefficient matrix given above (11).

Suppose now that we have an even more restricted case: all points have the same rules; yet still we want to consider only paths that remain weakly above the x-axis, so we exclude the moves that violate this requirement. For such families, a result of [3, Theorem 3] can be applied. It states that the number of excursions of length k with moves $\{(1,b_1),(1,b_2),\dots,(1,b_m)\}$ and associated multiplicities $w_1,\dots,w_m$, is of the form $\Theta(C^k/k^{3/2})$, where the base C of the exponential growth is determined as follows: For the Laurent polynomial $P(u)=\sum_{j=1}^m w_j u^{b_j}$, let τ be the unique positive number such that $P'(\tau)=0$; then $C=P(\tau)$. The situation is particularly easy for families with a symmetric set of moves, that is, if $(1,b)$ is a move then $(1,-b)$ is also a move with the same multiplicity, or equivalently, $P(u)=P(u^{-1})$. In this case, $\tau=1$, and consequently, $C=P(\tau)=\sum_{j=1}^m w_j$.

The situation for our matrix A is very similar to this case, except that the first $r-1$ horizontal lines of the lattice follow different rules, in accordance with the fact that the first $r-1$ rows of A are different from the others. However, this does not affect the asymptotic growth constant. Indeed, let us look at the matrix A' in which the first r rows and columns of A have been removed. It coincides with A for $i+j\geq r$, but the rule $a_{i+1,j+1}=a_{ij}$ holds for all entries – also in the upper-left corner. Since $A\leq A'$ elementwise, we clearly have $\mathrm{Ex}(A,k)\leq \mathrm{Ex}(A',k)=\Theta(\lambda_r^k/k^{3/2})$. To see that we have a lower bound of the same asymptotic form, consider only those excursions that start with the move $(1,+r)$, end with the move $(1,-r)$, and never go below level r. The intermediate part of the excursion is governed by the matrix A from which the first r rows and columns have been removed, which coincides with the matrix A'. Thus $\mathrm{Ex}(A,k)\geq \mathrm{Ex}(A',k-2)=\Theta(\lambda_r^k/k^{3/2})$. □

5.5. Asymptotic growth constants

Since $A = \mathrm{CH}^*(r,k)$ has $n = rk$ points, it follows from Proposition 11 that the growth constant for the number of down-free matchings of the r-chain without corners of size n is $\sqrt[r]{\lambda_r}$. In order to estimate λ_r, we note that the expression (13), when the factor $(i+1)$ is ignored, counts partitions of r elements into three subsets (the latter two being of almost equal size). The total number of such partitions is 3^r. Hence, $\lambda_r \le (r+1)3^r$, and $\sqrt[r]{\lambda_r}$ converges to 3. Computations suggest that the maximum of $\sqrt[r]{\lambda_r}$ is obtained for $r = 11$: $\sqrt[11]{\lambda_{11}} = \sqrt[11]{240054} \approx 3.0840$; after that it apparently decreases monotonically to 3, see the left part of Table 1 for the first few values. To prove that $r = 11$ gives indeed the maximum, one estimates that $\sqrt[r]{\lambda_r} \le 3\sqrt[r]{r+1} < 3.0838$ for $r \ge 191$, and the finitely many values up to $r = 190$ can be checked individually. This completes the proof of Theorem 3.

In order to find a more precise estimate for λ_r, we notice that the middle expression in (13) expresses λ_r as the binomial convolution of the sequence of natural numbers and the sequence $\binom{m}{\lfloor m/2 \rfloor}$. It follows that the exponential generating function for $(\lambda_r)_{r\ge 0}$ is

$$(1+x)\,e^x\,(I_0(2x) + I_1(2x)),$$

where $I_0(x)$ and $I_1(x)$ are the modified Bessel functions of the first kind. From this we can conclude that the sequence $(\lambda_r)_{r\ge 0}$ is the sum of the sequence A005773 and a shifted copy of A132894 in [9]. The ordinary generating function for this sequence is then

$$\frac{1}{2x}\left(\frac{1-2x-x^2}{(1+x)^{1/2}(1-3x)^{3/2}} - 1\right),$$

and it follows from the exponential growth formula that $\lambda_r = \Theta(3^r r^{1/2})$. By Theorem 6 this number is also the growth constant of the number of perfect matchings for the corresponding double structure.

6. *r*-chains with corners

6.1. Definitions and notation

In this section, we will treat r-chains *with corners*, but we will simply refer to them as r-chains. The analysis of these r-chains is more complicated due to the fact that the corners belong to two arcs. As before, we will incrementally build the r-chain and estimate the number of matchings of the r-chain with k arcs, which possibly have runners. We extend the notions of runners, free points, and ρ-matchings to r-chains with corners in the obvious way.

Recall that the corners of the chain are denoted by $V_0, V_1, \ldots, V_k, \ldots$: V_k is the rightmost point of the kth arc. We cut a down-free ρ-matching M of $\mathrm{CH}(r,k)$ to the *right* of V_{k-1} and obtain two down-free ρ-matchings: the first, M_A, of A – the set consisting of the first $k-1$ arcs of $\mathrm{CH}(r,k)$; and the second, M_B, of B – the rightmost arc of $\mathrm{CH}(r,k)$ without the point V_{k-1}, see Figure 13 for an example. Note that in the case of r-chains with corners a runner incident to V_{k-1}, upon adding B on the right, can be also connected to a point of B: in such a case we say that it is *matched internally.*

We distinguish whether M has a runner incident to V_k or not. Let C_i^k be the number of down-free ρ-matchings of $\mathrm{CH}(r,k)$, where V_k has a runner and there are i runners in addition to the runner at V_k. Let F_i^k be the number of down-free ρ-matchings of $\mathrm{CH}(r,k)$, where V_k has no runner and there are i runners. (C stands for "corner", F for "free".) For $k = 0$, there is a single vertex, and we have $C_0^0 = F_0^0 = 1$ and $C_i^0 = F_i^0 = 0$ for all $i > 0$. The number that we are interested in, the number of matchings in $\mathrm{CH}(r,k)$, is F_0^k.

6.2. Recursions

Next we find interdependent recursive expressions for C_i^k and for F_i^k.

Recursion for C_i^k. For C_i^k, the new corner V_k has a runner and is not available for receiving edges from the left. Thus for the formulae below, it can be treated as if it were not present in the kth arc. We have the following three cases:

1. (Figure 13.) The previous corner V_{k-1} has a runner which is not matched internally in the kth arc. It is thus matched to the right. Suppose there are α runners originating in the k-th arc, in addition to the runner originating in V_{k-1}. These runners must also be matched to the right. The contribution to C_i^k is
$$\sum_{0\le\alpha\le\min\{r-1,i-1\}} Z_\alpha C_{i-1-\alpha}^{k-1}, \tag{14}$$

	without corners		with corners	
r	λ_r	$\sqrt[r]{\lambda_r}$	M_r	T_r
1	3	3	$\left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$	3
2	9	3	$\left(\begin{smallmatrix} 3 & 3 \\ 7 & 6 \end{smallmatrix}\right)$	3.0532
3	28	3.0366	$\left(\begin{smallmatrix} 10 & 9 \\ 21 & 19 \end{smallmatrix}\right)$	3.0711
4	87	3.0541	$\left(\begin{smallmatrix} 31 & 28 \\ 66 & 59 \end{smallmatrix}\right)$	3.0819
5	271	3.0662	$\left(\begin{smallmatrix} 97 & 87 \\ 204 & 184 \end{smallmatrix}\right)$	3.0877
6	843	3.0735	$\left(\begin{smallmatrix} 301 & 271 \\ 632 & 572 \end{smallmatrix}\right)$	3.0909
7	2619	3.0783	$\left(\begin{smallmatrix} 933 & 843 \\ 1952 & 1776 \end{smallmatrix}\right)$	3.0925
8	8123	3.0812	$\left(\begin{smallmatrix} 2885 & 2619 \\ 6022 & 5504 \end{smallmatrix}\right)$	**3.0930**
9	25153	3.0829	$\left(\begin{smallmatrix} 8907 & 8123 \\ 18550 & 17040 \end{smallmatrix}\right)$	3.0929
10	77763	3.0837	$\left(\begin{smallmatrix} 27457 & 25153 \\ 57071 & 52610 \end{smallmatrix}\right)$	3.0923
11	240054	**3.0840**	$\left(\begin{smallmatrix} 84528 & 77763 \\ 175381 & 162291 \end{smallmatrix}\right)$	3.0915
12	740017	3.0839	$\left(\begin{smallmatrix} 259909 & 240054 \\ 538386 & 499963 \end{smallmatrix}\right)$	3.0904
13	2278329	3.0835	$\left(\begin{smallmatrix} 798295 & 740017 \\ 1651140 & 1538312 \end{smallmatrix}\right)$	3.0893
14	7006093	3.0829	$\left(\begin{smallmatrix} 2449435 & 2278329 \\ 5059251 & 4727764 \end{smallmatrix}\right)$	3.0880
15	21520872	3.0822	$\left(\begin{smallmatrix} 7508686 & 7006093 \\ 15489221 & 14514779 \end{smallmatrix}\right)$	3.0867
16	66039651	3.0813	$\left(\begin{smallmatrix} 22997907 & 21520872 \\ 47384904 & 44518779 \end{smallmatrix}\right)$	3.0854
17	202462113	3.0804	$\left(\begin{smallmatrix} 70382811 & 66039651 \\ 144857454 & 136422462 \end{smallmatrix}\right)$	3.0841
18	620164491	3.0794	$\left(\begin{smallmatrix} 215240265 & 202462113 \\ 442540653 & 417702378 \end{smallmatrix}\right)$	3.0828
19	1898109900	3.0785	$\left(\begin{smallmatrix} 657780918 & 620164491 \\ 1351126551 & 1277945409 \end{smallmatrix}\right)$	3.0815
20	5805127269	3.0774	$\left(\begin{smallmatrix} 2008907469 & 1898109900 \\ 4122747150 & 3907017369 \end{smallmatrix}\right)$	3.0803

Table 1: Summary of results for r-chains without and with corners, for $1 \leq r \leq 20$. For r-chains without corners, λ_r is the row sum of the matrix A (Section 5.4), and $\sqrt[r]{\lambda_r}$ is the growth constant for pm. For r-chains with corners, the *condensed coefficient matrix* M_r is derived from the recursion (Section 6.5), and T_r, the r-th root of its dominant eigenvalue, is the growth constant for pm. In both cases, the values for $r = 1$ and $r = 2$ reproduce the known bounds. Indeed, a 1- and a 2-chain without corners, as well as a 1-chain with corners, is just a downward chain, and thus the growth constant of 3 agrees with Theorem 1. A 2-chain with corners is a zigzag chain, and thus $T_2 \approx 3.0532$ agrees with Theorem 2.

where

$$Z_\alpha = \binom{r-1}{\alpha}\binom{r-1-\alpha}{\lfloor (r-1-\alpha)/2 \rfloor}.$$

The expression for Z_α is similar to z^1_α from Proposition 9.2, but here we have only $r-1$ points: all the points of the kth arc, excluding the corners.

2. (Figure 14.) V_{k-1} has no runner. This possibility contributes

$$\sum_{j \geq 0} \sum_{\substack{|i-j| \leq \alpha \leq i+j \\ \alpha \equiv i-j \pmod 2 \\ 0 \leq \alpha \leq r-1}} Z_\alpha F^{k-1}_j. \tag{15}$$

This formula (as well as some of the formulae in the following cases) has the same pattern as (9), with appropriate changes.

3. (Figure 15.) V_{k-1} has a runner matched internally in the k-th arc. The contribution to C^k_i is

$$\sum_{j \geq 0} \sum_{\substack{|i-j| \leq \alpha \leq i+j \\ \alpha \equiv i-j \pmod 2 \\ 0 \leq \alpha \leq r-1}} I_\alpha C^{k-1}_j, \tag{16}$$

where

$$I_\alpha = \binom{r-1}{\alpha}\left[\binom{r-\alpha}{\lfloor (r-\alpha)/2 \rfloor} - \binom{r-1-\alpha}{\lfloor (r-1-\alpha)/2 \rfloor}\right] = \binom{r-1}{\alpha}\binom{r-1-\alpha}{\lfloor (r-2-\alpha)/2 \rfloor}.$$

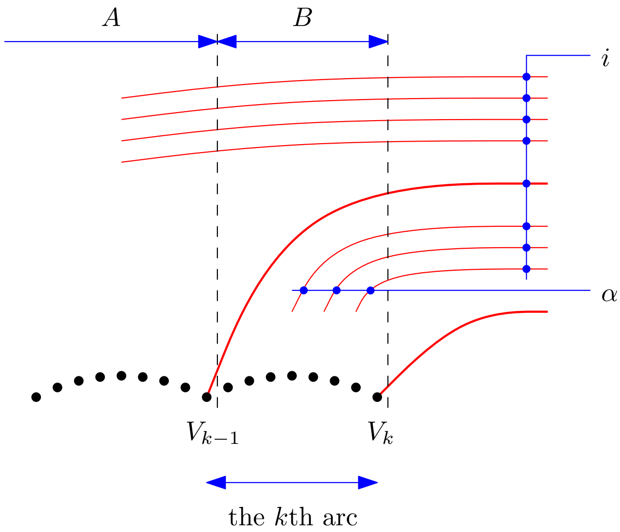

Figure 13: Case 1 in the recursion for C_i^k: V_{k-1} has a runner not matched internally in the kth arc.

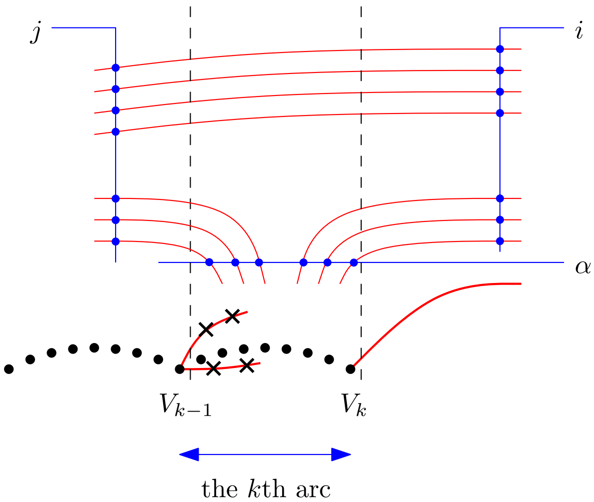

Figure 14: Case 2 in the recursion for C_i^k: V_{k-1} has no runner.

In the expression for I_α, the first factor counts the choices of α runners from the $r-1$ points. In the second factor, we subtract from all down-free ρ-matchings on the remaining $r-\alpha$ points (including V_{k-1}) those in which V_{k-1} is unmatched, which is the same as down-free ρ matchings on $r-1-\alpha$ points.

C_i^k is the sum of the three expressions (14–16).

Recursion for F_i^k. For F_i^k, we have again three cases:

1. (Figure 16.) V_{k-1} has a runner not matched internally in the kth arc. In this case, all the additional α runners originating in the interior of the k-th arc must be matched to the right. V_k is either free or matched internally to the left. The contribution to F_i^k is

$$\sum_{0 \le \alpha \le \min\{r-1, i-1\}} W_\alpha C_{i-1-\alpha}^{k-1}, \tag{17}$$

where

$$W_\alpha = \binom{r-1}{\alpha}\binom{r-\alpha}{\lfloor (r-\alpha)/2 \rfloor}$$

is again similar to z_α^1 from Proposition 9.2, but here we have $r-1$ in the first factor because no runner originates from V_k.

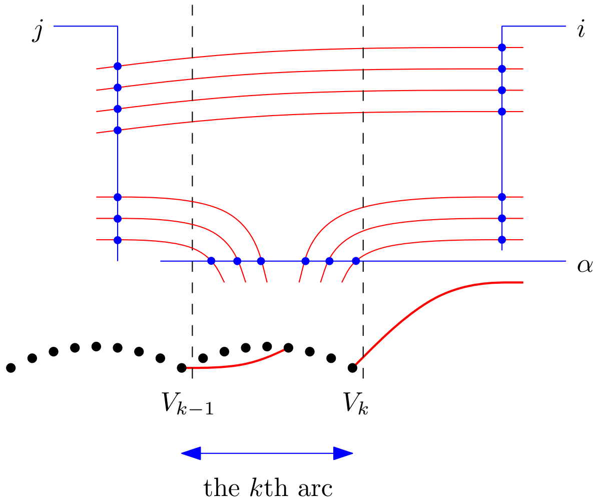

Figure 15: Case 3 in the recursion for C_i^k: V_{k-1} has a runner matched internally in the kth arc.

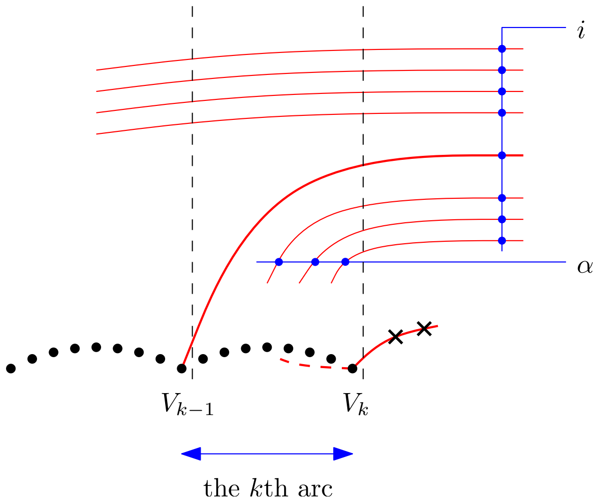

Figure 16: Case 1 in the recursion for F_i^k: V_{k-1} has a runner not matched internally in the kth arc.

2. (Figure 17.) V_k has a runner connected to a point of $A \setminus \{V_{k-1}\}$. In this case, all α runners originating in the k-th arc must be matched to the left. The contribution to F_i^k is

$$\sum_{0 \le \alpha \le r-1} (I_\alpha C_{i+1+\alpha}^{k-1} + Z_\alpha F_{i+1+\alpha}^{k-1}). \tag{18}$$

The two terms – with C^{k-1} and with F^{k-1} – correspond to the subcases where V_{k-1} is internally matched or, respectively, not matched to a point of the kth arc.

3. (Figure 18.) V_{k-1} has no runner, and V_k has no runner matched to a point of $A \setminus \{V_{k-1}\}$. The contribution to F_i^k is

$$\sum_{j \ge 0} \sum_{\substack{|i-j| \le \alpha \le i+j \\ \alpha \equiv j-i \pmod 2 \\ 0 \le \alpha \le r-1}} (U_\alpha C_j^{k-1} + W_\alpha F_j^{k-1}), \tag{19}$$

where

$$U_\alpha = \binom{r-1}{\alpha}\left[\binom{r+1-\alpha}{\lfloor (r+1-\alpha)/2 \rfloor} - \binom{r-\alpha}{\lfloor (r-\alpha)/2 \rfloor}\right] = \binom{r-1}{\alpha}\binom{r-\alpha}{\lfloor (r-1-\alpha)/2 \rfloor}$$

The two terms correspond to the same possibilities as in the previous case. The factor U_α is similar to I_α in the third case for C_i^k, but here we count the down free ρ-matchings of the whole kth arc with both its corners, hence we have $r+1$ instead of r in the second factor.

F_i^k is the sum of the three expressions (17–19).

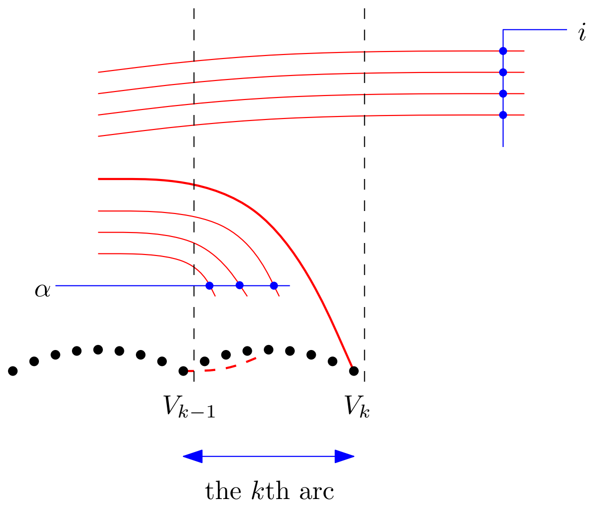

Figure 17: Case 2 in the recursion for F_i^k: V_k is connected to a point to the left of V_{k-1}.

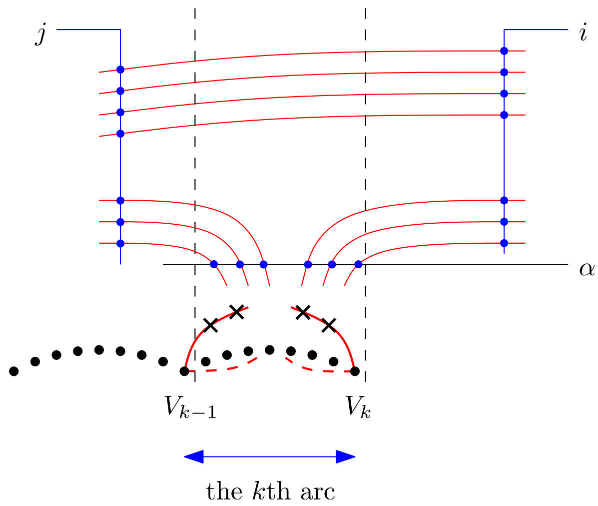

Figure 18: Case 3 in the recursion for F_i^k: V_{k-1} has no runner and V_k is not connected to a point to the left of V_{k-1}.

6.3. Analysis of the recursion

The expressions above imply a coupled mutual recurrence between two sequences of vectors $C^k = (C_0^k, C_1^k, C_2^k, \ldots)^\top$ and $F^k = (F_0^k, F_1^k, F_2^k, \ldots)^\top$. The initial values are $C^0 = F^0 = (1, 0, 0, \ldots)^\top$. C^k and F^k are expressed in terms of C^{k-1} and F^{k-1} as follows. For $i \geq r$, we have:

$$\begin{aligned} C_i^k &= \sum_{\beta=-r}^{r} a_\beta^{CC} C_{i+\beta}^{k-1} + \sum_{\beta=-r}^{r} a_\beta^{CF} F_{i+\beta}^{k-1} \\ F_i^k &= \sum_{\beta=-r}^{r} a_\beta^{FC} C_{i+\beta}^{k-1} + \sum_{\beta=-r}^{r} a_\beta^{FF} F_{i+\beta}^{k-1}, \end{aligned} \tag{20}$$

where the numbers a^{CC}, a^{CF}, a^{FC}, a^{FF} are to be read out from the expressions in Section 6.2. For the small indices $i < r$, we have irregularities, like for r-chains without corners: The coefficients in (20) must be replaced by smaller coefficients which depend also on i. In matrix notation, the recursion is written as

$$\begin{aligned} C^k &= A^{CC} C^{k-1} + A^{CF} F^{k-1} \\ F^k &= A^{FC} C^{k-1} + A^{FF} F^{k-1}, \end{aligned} \tag{21}$$

where there are four band matrices A^{CC}, A^{CF}, A^{FC}, A^{FF} of bandwidth r, similar to the matrix A from (12).

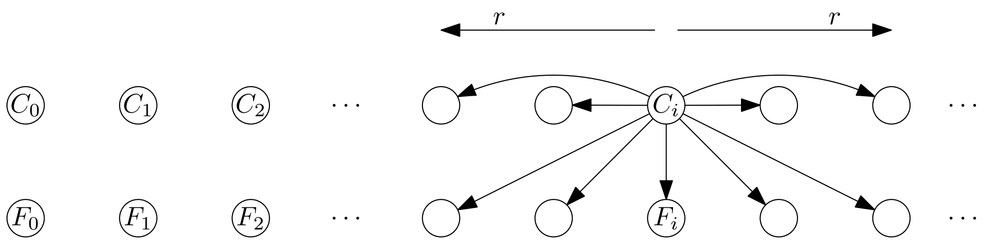

Figure 19: The recursion (20) gives the number of paths on this network. The neighborhood of a typical vertex C_i is shown in a schematic way.

This system can be interpreted as a set of lattice paths on a two-layer lattice, see Figure 19. We have a row of nodes $C_0, C_1, C_2, \dots$ and another row of nodes $F_0, F_1, F_2, \dots$ immediately below it. The possible jumps and their multiplicity depend only on the row, with irregularities close to the left edge. In this representation, the lattice paths considered in the proof of Proposition 11 in Section 5.3 correspond to walks on a ray $0, 1, 2, \dots$. The x-coordinate of the two-dimensional lattice in Section 5.3 is now represented as time.)

We are not able to provide as precise estimates for the growth constant as for chains without corners, where we had a single recursion. One would expect a similar behaviour. However, we can still pin down the base of the exponential growth as an eigenvalue of an associated 2×2 matrix.

First, we can get rid of the irregularities by cutting off the first r rows and columns of the coefficient matrices. As for the case of a single matrix, this does not affect the asymptotic growth. We can now assume that the diagonals are constant, and the recursion (20) holds for all i, with the convention that C_j^{k-1} and F_j^{k-1} in the right-hand side are taken as 0 for $j < 0$.

For better readability, we will now replace the vectors C^k and F^k by more generic names x^k and y^k:

$$\begin{aligned} x_i^k &= \sum_{\beta=-r}^{r} a_\beta^{XX} x_{i+\beta}^{k-1} + \sum_{\beta=-r}^{r} a_\beta^{XY} y_{i+\beta}^{k-1} \\ y_i^k &= \sum_{\beta=-r}^{r} a_\beta^{YX} x_{i+\beta}^{k-1} + \sum_{\beta=-r}^{r} a_\beta^{YY} y_{i+\beta}^{k-1}, \end{aligned} \tag{22}$$

for all i, with the understanding that quantities x_j^{k-1} and y_j^{k-1} with negative subscripts j are regarded as zero on the right-hand side.

We start with the vectors

$$x^0 = y^0 = (1, 0, 0, 0, \dots), \tag{23}$$

but any other nonnegative start vectors different from the zero vector will lead to the same asymptotic growth.

Our analysis below relies on the fact that the coefficients of the recursion don't exhibit a tendency to favor larger or smaller indices, or in other words, that the Markov chain associated to the system does not systematically drift to the left or to the right. (In the one-vector recursion analyzed in the proof of Proposition 11, this no-drift condition was not an issue because the set of moves was symmetric.) To formulate this condition precisely, we have to set up some notation and establish some terms.

Let us denote the coefficient sums in the terms of the recursion (22) as follows:

$$A^{XX} = \sum_{\beta=-r}^{r} a_\beta^{XX},\ A^{XY} = \sum_{\beta=-r}^{r} a_\beta^{XY},\ A^{YX} = \sum_{\beta=-r}^{r} a_\beta^{YX},\ A^{YY} = \sum_{\beta=-r}^{r} a_\beta^{YY}.$$

These numbers are the column sums of the coefficient matrices after stabilization. These sums form the *condensed coefficient matrix*

$$\begin{pmatrix} A^{XX} & A^{XY} \\ A^{YX} & A^{YY} \end{pmatrix}. \tag{24}$$

Let M denote its dominant eigenvalue. Let (ρ_X, ρ_Y) be the corresponding left eigenvector and $(\pi_X, \pi_Y)^\top$ be the corresponding right eigenvector, with the normalization $\rho_X + \rho_Y = \pi_X + \pi_Y = 1$. Since the matrix is positive, these two vectors are positive.

We define the *total group-to-group jump sizes* of the system:

$$D^{XX} = \sum_{\beta=-r}^{r} a_\beta^{XX}\beta,\ D^{XY} = \sum_{\beta=-r}^{r} a_\beta^{XY}\beta,\ D^{YX} = \sum_{\beta=-r}^{r} a_\beta^{YX}\beta,\ D^{YY} = \sum_{\beta=-r}^{r} a_\beta^{YY}\beta.$$

The *weighted total jump size* D of the system is then defined as follows:

$$\begin{aligned} D &= \rho_X\pi_X D^{XX} + \rho_X\pi_Y D^{XY} + \rho_Y\pi_X D^{YX} + \rho_Y\pi_Y D^{YY} \\ &= \begin{pmatrix}\rho_X & \rho_Y\end{pmatrix} \begin{pmatrix} D^{XX} & D^{XY} \\ D^{YX} & D^{YY}\end{pmatrix} \begin{pmatrix}\pi_X \\ \pi_Y\end{pmatrix} \end{aligned} \tag{25}$$

Now we can state the main result of the analysis.

Theorem 12. *Suppose the system* (22) *has non-negative coefficients, and the weighted total jump size D is zero. Assume that the coefficients $a_\beta^{XX}, a_\beta^{XY}, a_\beta^{YX}, a_\beta^{YY}$ are positive for $\beta = -1, 0, 1$. Let M be the dominant eigenvalue of the condensed coefficient matrix* (24). *Then*

$$x_0^k = O(M^k),\ y_0^k = O(M^k),$$

and

$$x_0^k = \Omega((M-\varepsilon)^k),\ y_0^k = \Omega((M-\varepsilon)^k)$$

for every $\varepsilon > 0$.

Since the proof is quite substantial, we devote a separate section to it.

6.4. Proof of the theorem about mutually coupled recursions

We will transform the problem to a recursion in which the left eigenvector is $(\rho_X, \rho_Y) = (1, 1)$, and thus the column sums of the coefficient matrix (after stabilization) are constant. We achieve this by rescaling the vectors x and y to $\tilde{x}_i^k = \rho_X x_i^k$ and $\tilde{y}_i^k = \rho_Y y_i^k$. Clearly, the asymptotic growth of x and y is unaffected by this multiplication with a constant. For these new vectors, the coefficients of the recursion change to $\tilde{a}_\beta^{XY} = \rho_X/\rho_Y \cdot a_\beta^{XY}$ and $\tilde{a}_\beta^{YX} = \rho_Y/\rho_X \cdot a_\beta^{YX}$, while $\tilde{a}_\beta^{XX} = a_\beta^{XX}$, $\tilde{a}_\beta^{YY} = a_\beta^{YY}$ are unchanged. Consequently, the first column sum of the condensed coefficient matrix (24) becomes $A^{XX} + \rho_Y/\rho_X \cdot A^{YX} = (\rho_X \cdot A^{XX} + \rho_Y \cdot A^{YX})/\rho_X = (M\rho_X)/\rho_X = M$, and similarly for the second column. Theorem 12 follows therefore from the following theorem, which is a special case of Theorem 12 with the additional assumption that the matrix (24) has constant column sums.

Theorem 13. *Suppose the system* (22) *has non-negative coefficients and constant column sums*

$$M = A^{XX} + A^{YX} = A^{XY} + A^{YY}. \tag{26}$$

Suppose that

$$\pi_X\left(D^{XX} + D^{YX}\right) + \pi_Y\left(D^{XY} + D^{YY}\right) = 0, \tag{27}$$

where (π_X, π_Y) is a right eigenvector of the matrix (24) *with eigenvalue M. Suppose further that the coefficients $a_\beta^{XX}, a_\beta^{XY}, a_\beta^{YX}, a_\beta^{YY}$ are positive for $\beta = -1, 0, 1$. Then*

$$x_0^k = O(M^k),\ y_0^k = O(M^k),$$

and

$$x_0^k = \Omega((M-\varepsilon)^k),\ y_0^k = \Omega((M-\varepsilon)^k)$$

for every $\varepsilon > 0$.

Theorem 13 is formulated in terms of the original recursion (22), but it must be applied to $\tilde{x}$ and $\tilde{y}$ instead of x and y in order to prove Theorem 12. Our rescaling modifies group-to-group jump sizes in the same way as the coefficients: $\tilde{D}^{XY} = \rho_X/\rho_Y \cdot D^{XY}$, etc.; the eigenvectors of the modified condensed coefficient matrix are $(\tilde{\pi}_X, \tilde{\pi}_Y) = (\rho_X\pi_X, \rho_Y\pi_Y)$ and $(\tilde{\rho}_X, \tilde{\rho}_Y) = (1, 1)$ (without normalization), and with these substitutions, the condition that D from (25) is zero translates into (27), after erasing the tildes. This concludes the proof of Theorem 12. □

Proof of Theorem 13. The upper bound is easy: by summing all equations of (22), one sees that $\sum_{i\geq 0} x_i^k + \sum_{i\geq 0} y_i^k$ can grow at most by the factor M in each iteration, since the column sums of the coefficient matrix are bounded by M. It follows that $x_0^k, y_0^k \leq \sum_i x_i^k + \sum_i y_i^k \leq M^k(\sum_i x_i^0 + \sum_i y_i^0) = 2M^k$.

Let us now turn to the lower bound: To have a compact notation for the linear operator expressing in the recursion (22), we denote it by ϕ:

$$(x^k, y^k) = \phi(x^{k-1}, y^{k-1})$$

As an intermediate lemma, we will show that any "sub-eigenvector" with eigenvalue λ is enough for a lower bound on the growth.

Lemma 14. *Suppose there is a pair of non-negative non-zero vectors $\bar{x}$ and $\bar{y}$ with finitely many non-zero elements such that the inequality*

$$\phi(\bar{x}, \bar{y}) \geq \lambda \cdot (\bar{x}, \bar{y}) \tag{28}$$

holds componentwise for some $\lambda > 0$. Then there is a constant $K > 0$ such that $x_0^n, y_0^n \geq K\lambda^n$ for all $n \in \mathbb{N}$.

Proof. Since ϕ is a monotone operator, the inequality (28) remains fulfilled if we repeatedly apply ϕ to each side:

$$\phi^{k+1}(\bar{x}, \bar{y}) = \phi(\phi^k(\bar{x}, \bar{y})) \geq \lambda \cdot \phi^k(\bar{x}, \bar{y})$$

Applying ϕ to $(\bar{x}, \bar{y})$ sufficiently many times, we eventually obtain a vector $(\tilde{x}, \tilde{y}) = \phi^k(\bar{x}, \bar{y})$ whose components $\tilde{x}_0$ and $\tilde{y}_0$ are positive, since the coefficients $a_1^{XX}, a_1^{XY}, a_1^{YX}, a_1^{YY}$ are positive by assumption. Moreover, by scaling we can obtain a vector in which these components are bigger than 1 and (28) still holds. Thus, renaming the new vector to $(\bar{x}, \bar{y})$, we can assume that $\bar{x}_0 \geq 1$ and $\bar{y}_0 \geq 1$.

Now, we find n_1 and K such that the following inequality holds componentwise for $n = n_1$:

$$(x^n, y^n) \geq K\lambda^n \cdot (\bar{x}, \bar{y}) \tag{29}$$

To see that this is possible, we use the assumption that $a_\beta^{XX}, a_\beta^{XY}, a_\beta^{YX}, a_\beta^{YY}$ are positive for $\beta = 0$ and $\beta = -1$. Thus, by making n_1 big enough, we can ensure that (x^{n_1}, y^{n_1}) has positive components wherever $(\bar{x}, \bar{y})$ has positive components. We can then fulfill (29) by choosing K small enough.

The inequality (29) carries over to all larger n by induction, using monotonicity of the operator ϕ and the assumption (28). Since $\bar{x}_0 \geq 1$ and $\bar{y}_0 \geq 1$, the desired inequalities follow from (29) for all $n \geq n_1$. Finally, for the finitely many values $n < n_1$, we can fulfill the inequalities $x_0^n, y_0^n \geq K\lambda^n$ by decreasing K if necessary. □

Let us explain the idea for getting "sub-eigenvectors" $\bar{x}$ and $\bar{y}$ for Lemma 14. If we wish to fulfill (28) for $\lambda = M$, vectors $\bar{x}$ and $\bar{y}$ with constant entries will do the job. However, they have infinitely many non-zero entries. Thus, we aim for a smaller $\lambda = M - \varepsilon$, and we make an *ansatz* where the entries are determined by a concave quadratic function. This has to be adjusted later because the vectors have to be non-negative, and because the recursion (22) has some irregularities for the small values $i < r$. Moreover, the two coupled sequences $\bar{x}$ and $\bar{y}$ depend on each other in a non-symmetric way, and therefore we cannot use the same quadratic function for both sequences. They have to be scaled differently, and shifted horizontally relative to each other.

We define the shift constant

$$\delta = \frac{\pi_X D^{XX} + \pi_Y D^{XY}}{-\pi_Y A^{XY}} = \frac{-(\pi_X D^{YX} + \pi_Y D^{YY})}{\pi_Y (A^{YY} - M)}.$$

In this definition, equality of the numerators follows from the assumption (27), which expresses that the weighted total jump size is zero. The denominators are equal because the column sums are M (26).

We take two real parameters that are to be determined later, the *peak value* p and the *shift value* s, and define the quadratic functions h_X and h_Y and two auxiliary vectors $\hat{x}$ and $\hat{y}$ as follows:

$$h_X(i) = \pi_X(p - i^2) \tag{30}$$

$$h_Y(i) = \pi_Y(p - (i + \delta)^2) \tag{31}$$

$$\hat{x}_i = h_X(i - s)$$

$$\hat{y}_i = h_Y(i - s)$$

for all $i \in \mathbb{Z}$. The two quadratic functions have their peaks at $i = 0$ and $i = -\delta$, with respective values $p\pi_X$ and $p\pi_Y$. These function are shifted to the right by s before they are used as entries of $\hat{x}$ and $\hat{y}$. The setup (30–31) and the shift constant δ have been chosen to make the following statement true, which expresses the deviation of the vectors $\hat{x}$ and $\hat{y}$ from being an eigenvector with eigenvalue M:

Lemma 15. *Each of the two expressions*

$$Q_X = M \cdot \hat{x}_i - \left(\sum_{\beta=-r}^{r} a_\beta^{XX} \hat{x}_{i+\beta} + \sum_{\beta=-r}^{r} a_\beta^{XY} \hat{y}_{i+\beta} \right), \tag{32}$$

$$Q_Y = M \cdot \hat{y}_i - \left(\sum_{\beta=-r}^{r} a_\beta^{YX} \hat{x}_{i+\beta} + \sum_{\beta=-r}^{r} a_\beta^{YY} \hat{y}_{i+\beta} \right) \tag{33}$$

has a constant positive value independent of i, p, and s.

Proof. First, we replace the quadratic function in each of the summation terms by a Taylor series around the weighted average point. The linear terms will then cancel, and the quadratic terms have a constant value. We carry this out by way of example for the sum of the a^{XX} terms. The parameters i and s always occur together in the combination $i - s$, and thus we express our terms in terms of the parameter $t := i - s$.

$$\sum_{\beta=-r}^{r} a_\beta^{XX} \hat{x}_{i+\beta} = \sum_{\beta=-r}^{r} a_\beta^{XX} h_X(i + \beta - s) = \sum_{\beta=-r}^{r} a_\beta^{XX} h_X(t + \beta)$$

We denote the *average jump size* from group X to group X by

$$\bar{D}^{XX} = \frac{\sum_{\beta=-r}^{r} a_\beta^{XX} \beta}{\sum_{\beta=-r}^{r} a_\beta^{XX}} = \frac{D^{XX}}{A^{XX}}.$$

Then we rewrite h_X as a Taylor series in the point $t + \bar{D}^{XX}$.

$$h_X(t + x) = h_X(t + \bar{D}^{XX}) + h'_X(t + \bar{D}^{XX})(x - \bar{D}^{XX}) - \pi_X(x - \bar{D}^{XX})^2$$

We get

$$\begin{aligned}
&\sum_{\beta=-r}^{r} a_\beta^{XX} h_X(t + \beta) \\
&= h_X(t + \bar{D}^{XX}) \sum_\beta a_\beta^{XX} + h'_X(t + \bar{D}^{XX}) \sum_\beta a_\beta^{XX} (\beta - \bar{D}^{XX}) - \pi_X \sum_\beta a_\beta^{XX} (\beta - \bar{D}^{XX})^2 \\
&= h_X(t + \bar{D}^{XX}) A^{XX} + h'_X(t + \bar{D}^{XX}) \cdot 0 - C^{XX},
\end{aligned}$$

with a constant $C^{XX} > 0$. We transform the other sum in the expression (32) analogously, using the average jump size $\bar{D}^{XY} = D^{XY}/A^{XY}$, and then we can rewrite (32) as follows:

$$\begin{aligned}
Q_X &= M \cdot h_X(t) - h_X(t + \bar{D}^{XX}) A^{XX} + C^{XX} - h_Y(t + \bar{D}^{XY}) A^{XY} + C^{XY} \\
&= M \cdot \pi_X(p - t^2) - \pi_X\big(p - (t + \bar{D}^{XX})^2\big) A^{XX} - \pi_Y\big(p - (t + \bar{D}^{XY} + \delta)^2\big) A^{XY} + (C^{XX} + C^{XY}) \\
&= (p - t^2)(\pi_X M - \pi_X A^{XX} - \pi_Y A^{XY}) \\
&\qquad + 2t(\pi_X \bar{D}^{XX} A^{XX} + \pi_Y \bar{D}^{XY} A^{XY} + \pi_Y \delta A^{XY}) + (C^{XX} + C^{XY})
\end{aligned}$$

The coefficient of $(p - t^2)$ is zero because (π_X, π_Y) is an eigenvector, and the coefficient of t is zero by the definition of δ. Thus, the expression Q_X has a constant positive value $C^{XX} + C^{XY}$, as claimed.

For the expression (33), the calculation is slightly different:

$$\begin{aligned}
Q_Y &= M \cdot h_Y(t) - h_X(t + \bar{D}^{YX})A^{YX} + C^{YX} - h_Y(t + \bar{D}^{YY})A^{YY} + C^{YY} \\
&= M \cdot \pi_Y\big(p - (t+\delta)^2\big) - \pi_X\big(p - (t + \bar{D}^{YX})^2)\big)A^{YX} - \pi_Y\big(p - (t + \bar{D}^{YY} + \delta)^2\big)A^{YY} + (C^{YX} + C^{YY}) \\
&= (p - t^2)(\pi_Y M - \pi_X A^{YX} - \pi_Y A^{YY}) \\
&\quad + 2t(-\pi_Y \delta M + \pi_X \bar{D}^{YX} A^{YX} + \pi_Y \bar{D}^{YY} A^{YY} + \pi_Y \delta A^{YY}) + (C^{YX} + C^{YY})
\end{aligned}$$

The coefficients of $(p - t^2)$ and t vanish for the same reasons as above. This concludes the proof of the lemma. □

The quadratic functions h_X and h_Y are unbounded from below, and hence the vectors $\hat{x}$ and $\hat{y}$ have negative values. To get our desired vectors $\bar{x}$ and $\bar{y}$, we will clip these values to 0. We determine the parameters p and s in such a way that the resulting vectors $\bar{x}$ and $\bar{y}$ start with a big jump from 0 to a positive value, big enough to accommodate the "perturbation" resulting from modifying the negative values to 0. Let $\varepsilon > 0$ be given, and let $K := \max\{Q_X, Q_Y\} > 0$ be the maximum of Q_X and Q_Y. We look at the sorted set of values

$$\{\, i^2 \mid i \in \mathbb{Z} \,\} \cup \{\, (i - \delta)^2 \mid i \in \mathbb{Z} \,\}$$

and find p as a positive value in this set such that the gap to the largest value which is smaller than p is at least $K/(\varepsilon \min\{\pi_X, \pi_Y\})$. Since the functions i^2 and $(i - \delta)^2$ are quadratic, there must be larger and larger gaps as the numbers get bigger, and therefore such a value p exists. For the functions $h_X(i) = \pi_X(p - i^2)$ and $h_Y(i) = \pi_Y(p - (i + \delta)^2)$ in (30–31), this implies that the smallest positive value in their range is at least K/ε. Now we shift the functions horizontally such that positive values occur only at positive arguments, by choosing $s \geq \sqrt{p} + |\delta|$. Finally, we clip the negative values and define

$$\bar{x}_i = \max\{\hat{x}_i, 0\} = \max\{h_X(i - s), 0\}, \quad \bar{y}_i = \max\{\hat{y}_i, 0\} = \max\{h_Y(i - s), 0\},$$

for all $i \in \mathbb{Z}$. This will set $\bar{x}_i = \bar{y}_i = 0$ for $i < 0$, in accordance with the interpretation that is given in (22) when these values appear on the right-hand side.

We will show that

$$\phi(\bar{x}, \bar{y}) \geq (M - \varepsilon) \cdot (\bar{x}, \bar{y}), \tag{34}$$

thus establishing condition (28) and proving the lower bound of the theorem with the help of Lemma 14.

In concrete terms, our desired relation (34) looks as follows:

$$(M - \varepsilon) \cdot \bar{x}_i \leq \sum_{\beta=-r}^{r} a_\beta^{XX} \bar{x}_{i+\beta} + \sum_{\beta=-r}^{r} a_\beta^{XY} \bar{y}_{i+\beta} \tag{35}$$

$$(M - \varepsilon) \cdot \bar{y}_i \leq \sum_{\beta=-r}^{r} a_\beta^{YX} \bar{x}_{i+\beta} + \sum_{\beta=-r}^{r} a_\beta^{YY} \bar{y}_{i+\beta} \tag{36}$$

We concentrate on the first inequality (35). When $\bar{x}_i$ is 0, the inequality is trivially fulfilled. Thus, we can restrict ourselves to the case when $\bar{x}_i > 0$, and hence $\bar{x}_i = \hat{x}_i$. If we set $\varepsilon = 0$ and replace $(\bar{x}, \bar{y})$ by $(\hat{x}, \hat{y})$ everywhere, the difference between the left side and the right side of (35) is the quantity Q_X in Lemma 15, and hence it is bounded by K. Going back from $(\hat{x}, \hat{y})$ to $(\bar{x}, \bar{y})$ cannot make the right-hand side smaller. Hence we are done if we prove that the "slack term" term $\varepsilon \cdot \bar{x}_i$ is at least K. This is true by construction, since the non-zero values of $\bar{x}_i$ are at least K/ε. The other inequality (36) follows similarly.

This concludes the proof of the lower bound and, thus, of Theorem 13. □

The theorem can be extended to more than two coupled recursive sequences. Then we need a separate parameter δ for each function in (30–31). These parameters must be determined from a system of equations, and the no-drift condition ensures that this system has a solution.

The technical condition of Theorems 12 and 13 that certain coefficients are positive has the purpose to exclude periodicity and can be replaced by weaker conditions.

6.5. *Asymptotic growth constants*

We apply Theorem 12 to the recursion describing the r-chain with corners. It is straightforward to compute the 2×2 condensed coefficient matrix (24) with a computer by accumulating all terms derived in Section 6.2, and to compute its dominant eigenvalue. Since $n = rk+1$, the growth constant T_r in terms of n is r-th root of this eigenvector. We observe the same phenomenon as for chains without corners, see the right-most column of Table 1: The values increase to some maximum, and then the taper off and converge to 3 as r increases further. The first two entries in the table reproduce the results for the double-chain (the condensed coefficient matrix is $\left(\begin{smallmatrix}1 & 1\\ 2 & 2\end{smallmatrix}\right)$, which gives $T_1 = 3$) and the double zigzag chain (the condensed coefficient matrix is $\left(\begin{smallmatrix}3 & 3\\ 7 & 6\end{smallmatrix}\right)$, which gives $T_2 \approx 3.0532$). We observe that the maximum is achieved for the 8-chain with corners ($r = 8$).

To establish this bound rigorously as a lower bound, we have to check the conditions of Theorem 12. It is easy to check that the coefficients $a_\beta^{CC}, a_\beta^{CF}, a_\beta^{FC}, a_\beta^{FF}$ are indeed positive for $\beta = -1, 0, 1$. The condensed coefficient matrix (24) is

$$\begin{pmatrix}\bar{A}^{CC} & \bar{A}^{CF}\\ \bar{A}^{FC} & \bar{A}^{FF}\end{pmatrix} = \begin{pmatrix}2885 & 2619\\ 6022 & 5504\end{pmatrix}. \tag{37}$$

Its dominant eigenvalue is $M = (8389 + \sqrt{69945633}\,)/2 \approx 8376.175$, with corresponding left (unnormalized) eigenvector $(\rho_X, \rho_Y) = (6022, M - 2885)$ and right eigenvector $(\pi_X, \pi_Y)^\top = (2619, M - 2885)^\top$. The matrix of total group-to-group jump sizes is

$$\begin{pmatrix}D^{CC} & D^{CF}\\ D^{FC} & D^{FF}\end{pmatrix} = \begin{pmatrix}-2619 & 0\\ -2619 & 2619\end{pmatrix}.$$

Weighting these numbers with the eigenvectors and summing them up (25) yields that the weighted total jump size D is zero. The conditions of Theorem 12 are thus fulfilled.

An intuitive explanation of the equality $D = 0$ might be as follows. The recursion between the two vectors C^k and F^k is not symmetric, as witnessed, for example, by the non-symmetric condensed matrix (37). This asymmetry comes from the arbitrary decision to cut the construction to the *right* of each corner point. However, on the whole, this irregularity should not cause a systematic "drift" in the recursion, which would favor a tendency towards larger or smaller numbers i of unfinished runners crossing the cut. Thus, it is not surprising that $D = 0$. We expect that $D = 0$ should hold for all r, but we have only checked it numerically for small values of r, and we have established it rigorously only for the concrete case $r = 8$.

By Theorem 12, the sequence F_0^k grows at most like M^k and at least like $(M-\varepsilon)^k$, for any $\varepsilon > 0$. Since $n = 8k+1$, the growth constant in terms of n is $T_8 = \sqrt[8]{M} \approx 3.093005695$.

Corollary 16. *The 8-chains with corners have $O(T_8^n)$ and $\Omega((T_8-\varepsilon)^n)$ down-free matchings, for every $\varepsilon > 0$.* □

This implies Theorem 4 with the help of Theorem 6.

Numerical data suggest the more precise estimate $F_0^k = M^k/k^{3/2}(u_0 + u_1/k + O(1/k^2))$ with $u_0 \approx 0.1321$ and $u_1 \approx -0.102$. This has been computed by Moritz Firsching (personal communication) using the so-called "asymp$_k$" trick of Don Zagier [11], see also [7, Section 5.1]. This method estimates the coefficients by interpolation from successive elements of the sequence, assuming that the sequence has the asymptotic form $F_0^k = C^k/k^\alpha(u_0 + u_1/k + u_2/k^2 + \cdots)$. In our case, we used the elements $F_0^{785}, F_0^{786}, F_0^{787}, \ldots, F_0^{1000}$. The number of decimal digits of $C = M$ that were correctly predicted in this way was larger than 300, and $\alpha = 3/2$ was also determined to a precision of more than 300 digits. By comparison, for the sequence a_k of down-free matching numbers of the zigzag-chain (Section 4), for which the explicit generating function (7) and hence the form $a_k = C^k/k^{3/2}(u_0 + O(1/k))$ of the asymptotic growth is known, the same method gave estimates for the coefficients that were accurate also to more than 300 digits, both regarding the growth constant $C = 1/\mu = (\sqrt{93}+9)/2$ and the power $\alpha = 3/2$ of the polynomial factor. The constant factor was identified as $u_0 = [(\sqrt{57017277} + 7551)/1984\pi]^{1/2} \approx 1.5566$, but we did not check whether this agrees with the result from the generating function.

The asymptotic growth of the form $F_0^k = u_0 M^k/k^{3/2}(1+o(1))$ is not unexpected; it is in accordance with the behaviour of r-chains without runners, which has been derived in the proof of Proposition 11 (Section 5.4) by the lattice path method [3, Theorem 3].

7. Concluding remarks

7.1. *Table of results for* pm, dfm *and* am

In Table 2 we summarize asymptotic bounds on different structures for three kinds of matchings considered in this paper – pm, dfm and am. Some of them do not follow from results proven or mentioned in this paper, and we explain them below. First we want to point out some observations that can be seen in the table.

Obviously, $\mathsf{pm}(X_n) \le \mathsf{dfm}(X_n) \le \mathsf{am}(X_n)$, but is $\mathsf{dfm}(X_n)$ more likely to behave similarly to $\mathsf{pm}(X_n)$ or to $\mathsf{am}(X_n)$? Table 2 shows that different possibilities exist. For a downward chain SC_n, every matching is down-free and thus $\mathsf{dfm}(\mathrm{SC}_n) = \mathsf{am}(X_n)$, but for an upward chain dfm is equal, up to a polynomial factor, to the lower bound. For SZZC_n, the three growth constants are all different, but the intermediate basis for dfm is closer to the upper bound. However for r-chains without corners, as r grows, the growth constant for pm and dfm tends to the same value, 3, from below and from above respectively; whereas that for am tends to 4.

X_n	$\mathsf{pm}(X_n)$	$\mathsf{dfm}(X_n)$	$\mathsf{am}(X_n)$
SC_n (downward)	$C_{n/2} = \Theta^*(2^n)$	$M_n = \Theta^*(3^n)$	$M_n = \Theta^*(3^n)$
SC_n upside down	$C_{n/2} = \Theta^*(2^n)$	$\binom{n}{\lfloor n/2 \rfloor} = \Theta^*(2^n)$	$M_n = \Theta^*(3^n)$
SZZC_n	$\Theta^*(2.1974^n)$	$\Theta^*(3.0532^n)$	$\Theta^*(3.1022^n)$
$\mathrm{CH}^*(11, n/11)$	$\Theta^*(2.5517^n)$	$\Theta^*(3.0840^n)$	$\Theta^*(3.4614^n)$
$\mathrm{CH}^*(r, n/r)$, $r \to \infty$	$\Theta^*(\alpha^n)$, $\alpha \nearrow 3$	$\Theta^*(\beta^n)$, $\beta \searrow 3$	$\Theta^*(\gamma^n)$, $\gamma \nearrow 4$
$\mathrm{CH}(8, (n-1)/8)$		$\Theta^*(3.0930^n)$	
$\mathrm{CH}(r, (n-1)/r)$, $r \to \infty$		$\Theta^*(\delta^n)$, $\delta \searrow 3$?	
DC_n	$\Theta^*(3^n)$	?	$\Theta^*(4^n)$

Table 2: pm, dfm, am for several structures.

Now we describe the entries of the table. The first two lines are classical results, except for the formula $\mathsf{dfm} = \binom{n}{\lfloor n/2 \rfloor}$ for an upward chain, which has been proved in Proposition 9.

The estimate $\mathsf{pm}(\mathrm{SZZC}_n) = \Theta^*(2.1974^n)$ from [1] was mentioned in Section 3. Actually, it was the fact that pm increases from SC to SZZC which initially prompted us to try whether the old record of the double structure DC could be beaten by the corresponding double structure DZZC. The formula $\mathsf{dfm}(\mathrm{SZZC}_n) = \Theta^*(3.0532^n)$ is the main result of Section 4. The estimate $\mathsf{am}(\mathrm{SZZC}_n) = \Theta^*(3.1022^n)$ can be derived in a similar way, by adding an appropriate term to the recursion (3) for a_k: the only difference is that when P_1 is matched to P_3, the point P_2 can be free. The singularity closest to 0 of the resulting generating functions occurs now in $(\sqrt{105} - 9)/12$, one of the roots of $1 - 9x - 6x^2$. Thus, in this case the base is $\sqrt{12/(\sqrt{105} - 9)} \approx 3.1022$.

For r-chains without corners, $\mathrm{CH}^*(r, k)$, the growth constant for dfm has been determined in Section 5.3, and, as was discussed in Section 5.4, it converges to 3 from above as $r \to \infty$. The other entries in the line for CH^* can be obtained by modifying the analysis of Section 5.3; we only need to replace appropriately in the formula for z_i^1 in Proposition 9 the factor $\binom{r-i}{\lfloor (r-i)/2 \rfloor}$, representing the number of down-free matching on an arc of $r - i$ points. For pm, we have to replace it by the Catalan number $C_{(r-i)/2}$ when $r - i$ is even and by 0 when $r - i$ is odd; for am, we replace it by the Motzkin number M_{r-i}. The row sums of the recursion matrix can be obtained by plugging these modified expressions for z_i^1 into (13). For pm, the resulting sequence of row sums is the sequence A189912 from [9], and for am, it is the sequence A077587. (We omit the proofs.) From the asymptotic behavior of these sequences it follows that their r-th roots, which are the growth constants, converge to 3 and 4 from below.

The growth constant for dfm for r-chains with corners, $\mathrm{CH}(r, k)$, was treated in Section 6. Empirically, they seem to be better than r-chains without corners. The monotone convergence to 3 from above is not proved. It seems plausible that the difference between r-chains with corners and r-chains without corners should become negligible as $r \to \infty$, and therefore the growth constant should converge to the same constant 3. That the convergence should be monotonically decreasing is only based on the empirical observation from Table 1. We have not extended the analysis to pm and am, although this would be feasible with some effort. We expect that the results would be the same as for r-chains without corners.

The formula $\mathsf{pm}(\mathrm{DC}_n) = \Theta^*(3^n)$ is the classical result of García, Noy, and Tejel [6], in accordance with $\mathsf{dfm}(\mathrm{SC}_n) = \Theta^*(3^n)$ from the first line. The estimate $\mathsf{am}(\mathrm{DC}_n) = \Theta^*(4^n)$ is due to Sharir and Welzl [10], and it is currently the best lower bound on the maximum number of am. The growth of $\mathsf{dfm}(\mathrm{DC}_n)$ remains unknown, but it is $\Omega^*(3^n)$ and $O^*(4^n)$.

We see no reason to think that our best construction $\mathrm{CH}(8, k)$ is optimal in the sense that it has the maximal possible dfm and/or that the corresponding double construction has the maximal possible pm. Sets with asymptotically higher bounds may very well be more complicated – both in terms of their description and their analysis. An obvious continuation from single chains to r-chains would be to insert a third level of downward arcs between the vertices of r-chains, possibly continuing towards a fractal-like pattern. We have not attempted to analyze these structures.

7.2. Summary and Outlook

We have found new constructions of point sets with a larger number of perfect matchings than previously known. More importantly, we show that, like for triangulations, the true bound for perfect matchings is not given by the double chain. For the analysis of these sets, the notion of down-free matchings was crucial. It allowed us to concentrate on one half of a double-construction.

We have shown that methods from analytic combinatorics are useful for counting problems for geometric plane graphs. However, the results from analytic combinatorics that we are aware of cannot be readily applied for r-chains with corners. In this case, the analysis leads to coupled recursions involving two sets of variables. For these recursions, we had to develop our own methods. These somewhat pedestrian methods give the growth constant only up to an arbitrarily small error ε. We hope that the methods of analytic combinatorics will be further developed to encompass such cases as well.

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