PL MORSE THEORY IN LOW DIMENSIONS

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Abstract. We discuss a PL analogue of Morse theory for PL manifolds. There are several notions of regular and critical points. A point is homologically regular if the homology does not change when passing through its level, it is strongly regular if the function can serve as one coordinate in a chart. Several criteria for strong regularity are presented. In particular we show that in low dimensions $d \leq 4$ a homologically regular point on a PL d-manifold is always strongly regular. Examples show that this fails to hold in higher dimensions $d \geq 5$. One of our constructions involves an 8-vertex embedding of the dunce hat into a polytopal 4-sphere with 8 vertices such that a regular neighborhood is Mazur's contractible 4-manifold.

1. Introduction

What is nowadays called *Morse Theory* after its pioneer Marston Morse (1892–1977) has two roots: One from the calculus of variations [31], the other one from the differential topology of manifolds [32]. In both cases, the idea is to consider stationary points for the first variation of smooth functions or functionals. Then the second variation around such a stationary point describes the behavior in a neighborhood. In finite-dimensional calculus this can be completely described by the Hessian of the function provided that the Hessian is non-degenerate. In the global theory of (finite-dimensional) differential manifolds, smooth Morse functions can be used for a decomposition of the manifolds into certain parts. Here the basic observation is that generically a smooth real function has isolated critical points (that is, points with a vanishing gradient), and at each critical point the Hessian matrix is non-degenerate. The index of the Hessian is then taken as the *index* of the critical point. This leads to the Morse lemma and the Morse relations, as well as a handle decomposition of the manifold [31, 30, 35, 36]. Particular cases are height functions of submanifolds of Euclidean spaces. Almost all height functions are non-degenerate, and for compact manifolds the average of the number of critical points equals the total absolute curvature of the submanifold. Consequently, the infimum of the total absolute curvature coincides with the *Morse number* of a manifold, which is defined as the minimum possible number of critical points of a Morse function [21].

Already in the early days of Morse theory, this approach was extended to non-smooth functions on suitable spaces [33, 34, 21, 22]. One branch of that development led to several possibilities of a Morse theory for PL manifolds or for polyhedra in general.

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First of all, it has to be defined what a critical point is supposed to be since there is no natural substitute for the gradient and the Hessian of a function. Instead the typical behavior of such a function at a critical or non-critical point has to be adapted to the PL situation. Secondly, it cannot be expected that non-degenerate points are generic in the same sense as in the smooth case, at least not extrinsically for submanifolds of Euclidean space: For example, a monkey saddle of a height function on a smooth surface in 3-space can be split by a small perturbation of the direction of the height vector into two non-degenerate saddle points. By contrast, a monkey saddle on a PL surface in space is locally stable under such perturbations [1]. Abstractly, one can split the monkey saddle into an edge with two endpoints that are ordinary saddle points, see [11, Fig. 3]. Finally, in higher dimensions we have certain topological phenomena that have no analogue in classical Morse theory like contractible but not collapsible polyhedra, homology points that are not homotopy points, non-PL triangulations and non-triangulable topological manifolds.

From an application viewpoint, piecewise linear functions on domains of high dimensions arise in many fields, for example from simulation experiments or from measured data. One powerful way to explore such a function that is defined, say, on a three-dimensional domain, is by the interactive visualization of level sets. In this setting, it is interesting to know the topological changes between level sets, and critical points are precisely those points where such changes occur.

After an introductory section about polyhedra and PL manifolds (Section 2), we review the definitions of regular and critical points in a homological sense in Section 3. In Section 4, we contrast this with what we call *strongly regular* points (Definition 4.1). In accordance with classical Morse Theory, we distinguish the points that are not strongly regular into non-degenerate critical points and degenerate critical points, and we define PL Morse functions as functions that have no degenerate critical points. Section 5 briefly discusses the construction of a PL isotopy between level sets across strongly regular points. Section 6 extends the treatment to surfaces with boundary.

Another branch of the development was established by Forman's *Discrete Morse theory* [12]. Here in a purely combinatorial way functions are considered that associate certain values to faces of various dimensions in a complex. These Morse functions are not a priori continuous functions in the ordinary sense. However, as we show in Section 7, they can be turned into PL Morse functions in the sense defined above.

While in low dimensions up to 4, the weaker notion of H-regularity is sufficient to guarantee strong regularity (Section 8), this is no longer true in higher dimensions. Sections 9 and 10 give various examples of phenomena that arise in high dimensions. Finally, in Section 11, we discuss the algorithmic questions that arise around the concept of strong regularity. In particular, we show some undecidability results in high dimensions.

The results of Sections 4, 5, 7 and 11 are based on the Ph.D. thesis of R. Grunert [14]. Some preliminary approaches to these questions were earlier sketched in [37].

2. Polyhedra and PL manifolds

Definition 2.1. *A topological manifold M is called a* PL manifold *if it is equipped with a covering $(M_i)_{i\in I}$ of* charts M_i *such that all coordinate transformations between two overlapping charts are piecewise linear homeomorphisms of open parts of Euclidean space.*

From the practical point of view, a compact PL n-manifold M can be interpreted as a finite polytopal complex K *built up by convex d-polytopes such that $|K|$ is homeomorphic with M and such that the star of each (relatively open) cell is piecewise linearly homeomorphic with an open ball in d-space. Since every polytope can be triangulated, any compact PL d-manifold can be triangulated such that the link of every k-simplex is a combinatorial $(d-k-1)$-sphere. Such a simplicial complex is often called a* combinatorial d-manifold [24].

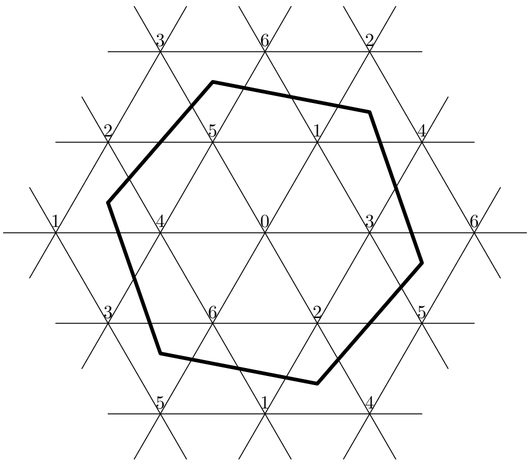

Figure 1. The unique 7-vertex triangulation of the torus

In greater generality, one can consider finite polytopal complexes. In the sequel we will consider a Morse theory for polytopal complexes in general as well as for combinatorial manifolds. If the polytopal complex is embedded into Euclidean space such that every cell is realized by a convex polytope of the same dimension, then we have the *height functions* defined as restrictions of linear functions.

A particular case is the abstract 7-vertex triangulation of the torus (see Figure 1) and its realization in 3-space [25]. Observe that a generic PL function with $f(1) < f(2) < f(4) < f(0) < \cdots$ has a monkey saddle at the vertex 0 since in the link of 0 the sublevel consists of the three isolated vertices 1, 2, 4. Therefore, passing through the level of 0 from below will attach two 1-handles simultaneously to a disc around the triangle 124. Compare Fig. 11 in [19, p.99].

For a general outline and the terminology of PL topology we refer to [38], where – in particular – Chapter 3 introduces the notion of a *regular neighborhood* of a subpolyhedron of a polyhedron.

Occasionally, results in PL topology depend on the Hauptvermutung or the Schoenflies Conjecture.

THE HAUPTVERMUTUNG: This conjecture stated that two PL manifolds that are homeomorphic to one another are also PL homeomorphic to one another.

This conjecture is true for dimensions $d \leq 3$ but systematically false in higher dimensions. However, it holds for d-spheres with $d \neq 4$ and for other special manifolds, compare [39].

THE PL SCHOENFLIES CONJECTURE: This states the following: A combinatorial $(d-1)$-sphere embedded into a combinatorial d-sphere decomposes the latter into two combinatorial d-balls.

The PL Schoenflies Conjecture is true for $d \leq 3$ and unknown in higher dimensions. If however the closure of each component of $S^d \setminus S^{d-1}$ is a manifold with boundary, then the conclusion of the Schoenflies Conjecture is true for all $d \neq 4$ [38, Ch.3].

3. REGULAR AND CRITICAL POINTS OF PL FUNCTIONS

The simplest way to carry over the ideas of Morse theory to PL is to consider functions that are linear on each polyhedral cell (or simplex in the simplicial case) and *generic*, meaning that no two vertices have the same image under the function. Such a theory was sketched in [6, 19] for obtaining lower bounds for the number of vertices of combinatorial manifolds of certain type.

We now define genericity for finite abstract polytopal complexes (for a definition see [42, Ch.5]). Examples are simplicial complexes and cubical complexes. Moreover, any subcomplex of the boundary complex of a convex d-polytope is a polytopal complex embedded in $\mathbb{E}^d$.

Definition 3.1. *Let P be a finite (abstract) polytopal complex. A function $f\colon P \to \mathbb{R}$ is called* generic PL *if it is linear on each polytopal cell separately and if $f(v) \neq f(w)$ for any two distinct vertices v, w of P. As a consequence, f is not constant on any edge or higher-dimensional cell.*

Similarly, if $P \subset \mathbb{E}^n$ is a compact polyhedron with the structure of a polytopal complex, then any linear function on $\mathbb{E}^n$ induces a height function *on P. This height function f is called* generic *if the same condition is satisfied. It is clear that for almost all directions in space (with respect to the Lebesgue measure) the associated height function is generic.*

We denote by f_a and f^a the sublevel set and the superlevel set:

$$f_a := \{x \mid f(x) \leq a\}, \quad f^a := \{x \mid f(x) \geq a\}$$

Lemma 3.2. *If $f\colon P \to \mathbb{R}$ is generic PL and if $f^{-1}[a, b]$ contains no vertex of P, then f_a is a strong deformation retract of the sublevel f_b.*

Proof. If P is a convex polytope then the assertion is obviously true. Therefore it holds for any single cell of P and – in combination – for the entire complex P. □

It is easy to construct an isotopy that smoothly interpolates between the level sets $f^{-1}(a)$ and $f^{-1}(b)$, resulting in mappings between different level sets $f^{-1}(t)$, $f^{-1}(t')$, for $a \le t, t' \le b$, that are piecewise linear. With more technical effort one can construct such an isotopy that is piecewise linear even when considered as a function of all variables, including the interpolation parameter $t \in [a, b]$ [14, Section 4.2.3, Lemma 4.13 and Theorem 4.20]. We will make some more remarks about this topic in Section 5.

Lemma 3.2 tells us that all points p other than vertices satisfy the regularity condition in Morse theory: The topology of the sublevel does not change when passing through p. It remains to talk about the vertices since passing through a vertex can definitely change the topology of the sublevel, as simple examples show. The topology can be measured preferably by topological invariants. Therefore the following definition is suitable:

Definition 3.3. *Let $f\colon P \to \mathbb{R}$ be generic PL and let v be a vertex with the level $f(v) = a$. Then v is called* homologically critical for f *or* H-critical *for short if $H_*(f_a, f_a \setminus \{v\}; \mathbb{F}) \neq 0$ where H_* denotes an appropriate homology theory with coefficients in a field $\mathbb{F}$. The total rank of $H_*(f_a, f_a \setminus \{v\})$ is called the* total multiplicity *of v with respect to f. If*

$$H_k(f_a, f_a \setminus \{v\}) \neq 0$$

then we say that v is H-critical of index k, *and the rank of $H_k(f_a, f_a \setminus \{v\})$ is referred to as the corresponding* multiplicity *of v restricted to the index k.*

Remark: The idea behind this notion is that the homological type of the sublevel set changes when passing through an H-critical point. Since no two vertices have the same level under f, the homology of $f_a \setminus \{v\}$ is the same as that for the open sublevel $(f_a)^\circ = \{x \mid f(x) < a\}$.

By excision and the long exact sequence for the reduced homology $\widetilde{H}$ in a simplicial complex P we can detect criticality in the link $lk(v)$ and the star $st(v)$ of a vertex v:

$$\widetilde{H}_k(f_a, f_a \setminus \{v\}) \cong \widetilde{H}_k(f_a \cap st(v), f_a \cap lk(v)) \cong \widetilde{H}_{k-1}(f_a \cap lk(v)) \cong \widetilde{H}_{k-1}(lk^-(v))$$

for $k \ge 1$ where $lk^-(v)$ denotes

$$lk^-(v) := \{x \in lk(v) \mid f(x) \le f(v)\} = lk(v) \cap f_a.$$

The homology of $lk^-(v)$ is the same as that of the full span of those vertices in the link of v whose level lies below $f(v)$. Similarly we will use the notation

$$lk^+(v) := \{x \in lk(v) \mid f(x) \ge f(v)\} = lk(v) \cap f^a.$$

This definition is also applicable to classical smooth Morse functions on a smooth manifold. Then a critical point of index k is also critical with respect to Definition 3.3 with the same index, and the total multiplicity is always 1. Even for polyhedral surfaces the case of higher total multiplicity occurs, as the example of a polyhedral monkey saddle shows. It is easy to construct polyhedra such that there are critical vertices of several indices simultaneously: Take the 1-point union of a 1-sphere with a 2-sphere.

Remark: For polyhedra the homological definition used in [8] is equivalent to our definition above. It compares the homology of the $(a - \epsilon)$-level with that of the $(a + \epsilon)$-level if a is the critical level. However, for topological spaces in general both definitions do not agree, as pointed out in [13]. The problem with the incorrect *Critical Value Lemma* in [8] is that a nested sequence of closed intervals can converge to a common boundary point. Then no open ϵ-neighborhood around the critical level can fit into any of the closed intervals. Instead of the definition above one could compare the open sublevel $(f_a)^\circ = f_a \setminus f^{-1}(a)$ to the closed sublevel f_a. For polytopal complexes (with closed polytopal faces) this will lead to the same definition.

There remains the possible case of $H_*(f_a, f_a \setminus \{v\}) = 0$ for some vertex v. Since homology does not detect that it is critical we would like to call it *non-critical* or *regular*. However, we have to be careful since regularity in the sense of Lemma 3.2 is different. The question is: Can $f_{a+\epsilon}$ and $f_{a-\epsilon}$ be topologically distinct in this case?

Definition 3.4. *A vertex v with $f(v) = a$ is called* homologically regular for f *or* H-regular *for short if $H_*(f_a, f_a \setminus \{v\}; \mathbb{F}) = 0$ for an arbitrary field $\mathbb{F}$.*

In classical Morse theory any H-regular point is actually regular in a stronger sense (compare Section 4). We will see below that this is still true in dimensions $d \leq 4$ but it does not hold in general for PL manifolds and generic PL functions.

Theorem 3.5. (Morse relations, duality [36, 21, 19])
Let $f \colon M \to \mathbb{R}$ be a generic PL function on a compact PL d-manifold M, and let $v_1, \ldots, v_n$ be the vertices. By a_i we denote the level $a_i = f(v_i)$. Then the Morse inequality

$$\sum_i \mathrm{rk} H_k(f_{a_i}, f_{a_i} \setminus \{v_i\}; \mathbb{F}) \geq \mathrm{rk} H_k(M; \mathbb{F}) \tag{1}$$

holds for any k and any field $\mathbb{F}$. Moreover,

$$\sum_k (-1)^k \sum_i \mathrm{rk} H_k(f_{a_i}, f_{a_i} \setminus \{v_i\}; \mathbb{F}) = \sum_k (-1)^k \mathrm{rk} H_k(M, \mathbb{F}) = \chi(M). \tag{2}$$

The expression $\mathrm{rk} H_k(f_{a_i}, f_{a_i} \setminus \{v_i\}; \mathbb{F})$ is nothing but the multiplicity of v_i restricted to the index k, and $\sum_i \mathrm{rk} H_k(f_{a_i}, f_{a_i} \setminus \{v_i\}; \mathbb{F})$ is the number $\mu_k(f)$ of critical points of index k, weighted by their multiplicities. Therefore the Morse inequality can also be written in the form

$$\mu_k(f) \geq \mathrm{rk} H_k(M; \mathbb{F}).$$

Concerning the duality:
By Alexander duality in the link of a vertex v one has $\widetilde{H}_{d-k-1}(lk^+(v)) \cong \widetilde{H}_{k-1}(lk^-(v))$ for $1 \leq k \leq d-1$ and consequently

$$\widetilde{H}_{d-k}(f^a, f^a \setminus v) \cong \widetilde{H}_k(f_a, f_a \setminus v). \tag{3}$$

Clearly a local minimum of f $(k = 0)$ is a local maximum $(k = d)$ for $-f$ and conversely. This means that the number of critical points of f of index k coincides with the number of critical points of $-f$ of index $d - k$ (weighted with multiplicities).

Definition 3.6. (perfect functions, tight triangulations)
If a function f satisfies the Morse inequality (1) *in Theorem 3.5 with equality, for each k, then it is usually called a* perfect function *or a* tight function*. A* tight triangulation *of a manifold is a triangulation such that*

any generic PL function f with arbitrarily chosen levels of the vertices is a tight function [20].

EXAMPLES: A generic PL function f on a compact surface without boundary is perfect if and only f_a is connected for any a. On a simply connected compact 4-manifold without boundary it is perfect if and only if f_a is connected and simply connected for any a. A triangulation of a surface is tight if and only if it is 2-neighborly, one of a simply connected 4-manifold is tight if and only if it is 3-neighborly. For any combinatorial sphere K with n vertices the power complex 2^K is a tightly embedded cubical manifold in $\mathbb{E}^{n+1}$, see [20, 3.24].

4. PL Morse functions

By emphasizing the critical behavior of classical Morse functions (attaching a cell at each critical point) one can adapt the classical Morse theory to the PL case as follows:

Definition 4.1. *Let M be a PL d-manifold and $f\colon M \to \mathbb{R}$ a generic PL function.*

- *A point p is called* strongly regular *if there is a chart around p such that the function f can be used as one of the coordinates, i.e., if in those coordinates*

$$f(x_1, \dots, x_d) = f(p) + x_d. \tag{4}$$

  If in a concrete polyhedral decomposition of M distinct vertices have distinct values of f, then f is also generic PL, and moreover all points are strongly regular except possibly the vertices.
- *A vertex v is called* non-degenerate critical *if there is a PL chart around v such that in those coordinates $x_1, \dots, x_d$ the function f can be expressed as*

$$f(x_1, \dots, x_d) = f(v) - |x_1| - \cdots - |x_k| + |x_{k+1}| + \cdots + |x_d|. \tag{5}$$

  The number k is then uniquely determined and coincides with the index of v. The multiplicity is always 1 in this case: $H_k(f_a, f_a \setminus \{v\}; \mathbb{F}) \cong \mathbb{F}$ and $H_j(f_a, f_a \setminus \{v\}) = 0$ for any $j \neq k$. The change by passing through the critical level can be either $H_k(f_{a+\epsilon}) \cong H_k(f_{a-\epsilon}) \oplus \mathbb{F}$ or $H_{k-1}(f_{a-\epsilon}) \cong H_{k-1}(f_{a+\epsilon}) \oplus \mathbb{F}$. A function such that the second case never occurs is called a perfect function.
- *The function f is called a* PL Morse function *if all vertices are either non-degenerate critical or strongly regular. In the terminology of* [33] *these are called* topologically ordinary *and* topologically critical, *respectively. The function itself is called* topologically non-degenerate *in this case.*

The definitions of strongly regular and non-degenerate critical points have in common that they require a local homeomorphism that transforms f into a certain PL map g. It turns out that determining the topological type of the embedding of $lk^-(v)$ into $lk(v)$ suffices to verify such a requirement. The connection between a characterization in terms of local charts and equivalent characterizations in terms of $lk^-(v)$ is established by the following general fact: There is a PL homeomorphism between neighborhoods N_v and N_w mapping v to w and transforming a PL map f on N_v with $f(v) = 0$ to a PL map g with $g(w) = 0$ if and only if there is a PL homeomorphism between $lk(v)$ and $lk(w)$ such that the signs of f and g at corresponding points agree.

For strongly regular points, this observation leads to the following result:

Lemma 4.2. (strongly regular points)
Let f be a generic PL function on a combinatorial d-manifold. Then a vertex v with $f(v) = a$ is strongly regular for f if and only if $lk^-(v)$ is a PL $(d-1)$-ball.

In particular, we obtain for strongly regular vertices v an embedding of a $(d-2)$-sphere into a $(d-1)$-sphere that separates the latter into two $(d-1)$-balls, namely, the boundary sphere $f^{-1}(a) \cap lk(v)$ of $lk^-(v)$ separates $lk(v)$ into the balls $lk^-(v)$ and $lk^+(v)$. Such an embedding is called an unknotted $(d-1, d-2)$-sphere pair. Thus, we can rephrase the previous characterization in terms of unknotted sphere pairs:

Corollary 4.3. *For dimension $d > 1$, a vertex v is strongly regular if and only if the pair $(lk(v), f^{-1}(a) \cap lk(v))$ is an unknotted $(d-1, d-2)$-sphere pair.*

The question whether all embeddings of $(d-2)$-spheres into $(d-1)$-spheres are unknotted is the Schoenflies problem. Since f is generic, the embedding of $f^{-1}(a) \cap lk(v)$ in $lk(v)$ is locally flat. Therefore another characterization for strongly regular vertices is possible for the cases where the Schoenflies problem in the PL locally flat category is known to have an affirmative answer.

Corollary 4.4. *Let v be a vertex of a combinatorial d-manifold M with $d > 1$ and $d \neq 5$. Then v is strongly regular if and only if $f^{-1}(a) \cap lk(v)$ is a $(d-2)$-sphere.*

Similar considerations for non-degenerate critical points yield the following characterizations:

Lemma 4.5. (non-degenerate critical points)
Let f be a generic PL function on a combinatorial d-manifold. Then a vertex v is non-degenerate critical for f with index k if and only if $lk^-(v)$ is a regular neighborhood of an unknotted $(k-1)$-sphere embedded into the $(d-1)$-sphere $lk(v)$.

Corollary 4.6. *Let f be a generic PL function on a combinatorial d-manifold. Assume that the vertex v is H-critical of index k. Then v is non-degenerate critical for f with index k if and only if the embedding of $f^{-1}(a) \cap lk(v)$ into $lk(v)$ is PL-homeomorphic to the embedding of $S^{k-1} \times S^{d-k-1}$ into the sphere S^{d-1} given by the boundary of a regular neighborhood of an unknotted S^{k-1} in S^{d-1}.*

Note that without the assumption of H-criticality, the criterion still implies that v is non-degenerate critical with index k or index $d-k$.

Lemma 4.7. (Morse Lemma)
Let $f\colon M \to \mathbb{R}$ be a PL Morse function and assume that there are no critical points with f-values in the interval $[a, b]$. Then f_a and f_b are PL homeomorphic to each other, and $f^{-1}([a, b])$ is PL homeomorphic with the "collar" $f^{-1}(a) \times [a, b]$.

Corollary 4.8. (Morse relations, duality)
Let $f\colon M \to \mathbb{R}$ be a PL Morse function on a compact PL manifold M, and let $\mu_k(f)$ be the number of critical vertices of index k, then the Morse inequality

$$\mu_k(f) \geq \mathrm{rk} H_k(M; \mathbb{F}) \tag{6}$$

holds for any k and any field $\mathbb{F}$. Moreover we have the Euler-Poincaré equation

$$\sum_k (-1)^k \mu_k(f) = \chi(M)$$

and the duality

$$\mu_{d-k}(f) = \mu_k(-f).$$

For a perfect function,

$$\mu_k(f) = \mathrm{rk} H_k(M; \mathbb{F})$$

for all k. This notion depends on the choice of $\mathbb{F}$.

This follows from Theorem 3.5.

Corollary 4.9. (Reeb theorem, [17])
Let M be a compact PL d-manifold and $f\colon M \to \mathbb{R}$ be a PL Morse function with exactly two critical vertices. Then M is PL homeomorphic to the sphere S^d.

Proof. Since the minimum p and maximum q are always critical the assumption can be reformulated by saying that any point between minimum and maximum is strongly regular. Let us consider the restriction

$$f_| \colon M \setminus \{p, q\} \to \mathbb{R}$$

without critical points. For any level $f^{-1}(c)$ with $f(p) < c < f(q)$ the Morse lemma tells us that there is an $\epsilon > 0$ such that $f^{-1}(c - \epsilon, c + \epsilon)$ is PL homeomorphic with the cylinder $f^{-1}(c) \times (-\epsilon, \epsilon)$. Furthermore there is a $\delta > 0$ such that $f^{-1}[f(p), f(p) + \delta]$ and $f^{-1}[f(q) - \delta, f(q)]$ are PL homeomorphic with d-balls. Consequently $f^{-1}(f(p) + \delta)$ and $f^{-1}(f(p) - \delta)$ are PL homeomorphic with the $(d-1)$-sphere. This implies that $f^{-1}[f(p) + \delta, f(q) - \delta]$ is PL homeomorphic with the cylinder

$$f^{-1}(c) \times [p + \delta, q - \delta] \cong S^{d-1} \times [p + \delta, q - \delta].$$

Putting the three parts together we see that M is PL homeomorphic with the d-sphere S^d. □

Remark: (a) In the smooth theory the same kind of proof leads only to a homeomorphism to the standard S^d but not to a diffeomorphism. There are exotic 7-spheres admitting a Morse function with two critical points, thus providing a counterexample. By contrast it is well known that the d-sphere ($d \neq 4$) admits a unique PL structure [23, Thm. 7]. Therefore this problem could occur only for $d = 4$. But gluing together two standard 4-balls along their boundaries leads to the standard 4-sphere. Therefore the proof above gives a PL homeomorphy even for $d = 4$.

(b) For the case of compact PL manifolds admitting a PL Morse function with exactly three critical points see [10]. The only possibilities occur in dimensions $d = 2, 4, 8, 16$ with an intermediate critical point of index $k = 1, 2, 4, 8$, respectively.

Consequence: (1) If there is an exotic PL 4-sphere then any PL Morse function on it must have at least four critical points.

(2) If M is a homology sphere that is not a sphere, then any PL Morse function f on M has at least six critical points. Consequently, it cannot admit a perfect function.

Proof of (2). M has a non-trivial fundamental group with a trivial commutator factor group. Therefore f must have a critical point of index 1. This leads to a free fundamental group in the critical sublevel f_a. If a critical point of index 2 introduces a relation in that group, the quotient will be abelian. A non-abelian group requires a second generator, and this requires a second critical point of index 1. Since the fundamental group is not free, there must be a critical point of index 2 introducing a relation between the generators. By the Euler relation the number of critical points must be even, so there are two critical points of index 1, minimum and maximum and two others. □

Example: (3 critical points)

For the unique (and 3-neighborly and tight) 9-vertex triangulation of the complex projective plane [20, Sect. 4B] any generic PL function assigning distinct levels to the 9 vertices is a PL Morse function with three critical points: minimum, maximum and a saddle point of index 2 in between. Since 123 is a 2-face of the triangulation, for the special case $f(1) < f(2) < f(3) < f(4) < \cdots < f(9)$ the sublevel f_a will be a 4-ball for $f(1) < a < f(4)$ and the complement of a 4-ball for $f(4) < a < f(9)$. Since 1234 is not a 3-face of the triangulation, the critical sublevel $f_{f(4)}$ consists of the boundary of the tetrahedron spanned by 1234 extended by sections through all 4-simplices except 56789.

Example: (4 critical points)

There is a highly symmetric (and 3-neighborly and tight) 13-vertex triangulation of the simply connected 5-manifold $M^5 = SU(3)/SO(3)$ [24, Ex.5_13_3_2]. Any generic PL function assigning distinct values to the 13 vertices will have total multiplicity 4, for special choices it will be a PL Morse function with minimum, maximum one saddle point of index 2 and one of index 3. Since 135 is a 2-face of the triangulation, for a beginning sequence with $f(1) < f(3) < f(5) < f(7)$ any sublevel f_a will be a 5-ball for $f(1) < a < f(7)$, the first critical level is $b = f(7)$ since 1357 is not a 3-face. Again f_b will be the boundary of the tetrahedron 1357 extended by sections through 5-simplices. According to $H_2(M^5; \mathbb{Z}) \cong \mathbb{Z}_2$ this empty tetrahedron 1357 generates the second homology but twice the generator is homologous to zero. Clearly 7 will be a saddle point for f of index 2. However we extend this sequence, by the Morse inequality $H_3(M^5;\mathbb{Z}_2) \cong \mathbb{Z}_2$ implies that there must be a critical point of index 3 also.

5. Isotopy

We have mentioned after Lemma 3.2 that successive level sets can be connected by an isotopy if there is no vertex between them. Such an isotopy can be used for visualization, by putting some texture on the level sets in order to make it clear how a level set moves as the level changes.

From an application viewpoint, there are also quantitative aspects that play a role here. One might look for isotopies that deform the level sets as little as possible and that are PL while using few additional vertices. Some results in this direction are given in [14, Section 6.2].

But already establishing the mere existence of a PL isotopy, in particular for the case when the level set passes over a strongly regular vertex, is not a trivial matter. As suggested in [37], such a PL isotopy can be represented by a PL homeomorphism

$$\phi\colon f^{-1}(b) \times [a,b] \to f^{-1}[a,b]$$

such that $f(\phi(x,t)) = t$ holds for all arguments. We sketch an existence proof following [14, Section 4.2.3].

If $f^{-1}[a,b]$ contains no vertices, $f^{-1}(b) \times [a,b]$ and $f^{-1}[a,b]$ are combinatorially equivalent polytopal complexes. Triangulating these complexes by starring at each vertex in corresponding orders yields combinatorially equivalent simplicial complexes and hence a PL homeomorphism by simplexwise linear interpolation.

It suffices to consider intervals $[a,b]$ such that $f^{-1}[a,b]$ contains a single regular vertex v with f-value a or b. Since the case $f(v) = a$ can be treated analogously, we assume $f(v) = b$.

First, apply the isotopy construction for intervals without vertices outlined above for $M \setminus (st(v))^\circ$, that is, M with the open star of v removed. This isotopy restricts to a PL homeomorphism from $(lk(v) \cap f^{-1}(b)) \times \{a\}$ to $lk(v) \cap f^{-1}(a)$. Since v is regular, $(st(v) \cap f^{-1}(b)) \times \{a\}$ is a ball bounded by the sphere $(lk(v) \cap f^{-1}(b)) \times \{a\}$ and $st(v) \cap f^{-1}(a)$ is a ball bounded by the sphere $lk(v) \cap f^{-1}(a)$. The PL homeomorphism between the boundary spheres can be extended to a PL homeomorphism between the balls $(st(v) \cap f^{-1}(b)) \times \{a\}$ and $st(v) \cap f^{-1}(a)$. This PL homeomorphism matches on $(lk(v) \cap f^{-1}(b)) \times \{a\}$ with the isotopy on the deletion of v. Therefore we obtain a PL homeomorphism between $(((M \setminus (st(v))^\circ) \cap f^{-1}(b)) \times [a,b]) \cup (st(v) \cap f^{-1}(b)) \times \{a\}$ and $((M \setminus (st(v))^\circ) \cap f^{-1}[a,b]) \cup (st(v) \cap f^{-1}(a))$ Now $(st(v) \cap f^{-1}(b)) \times [a,b]$ can be considered as a cone on $((lk(v) \cap f^{-1}(b)) \times [a,b]) \cup (st(v) \cap f^{-1}(b)) \times \{a\}$ with apex (v,b), and $st(v) \cap f^{-1}[a,b]$ as a cone on $(lk(v) \cap f^{-1}[a,b]) \cup (st(v) \cap f^{-1}(a))$ with apex v. Thus a cone construction defined by mapping (v,b) to v and interpolating between apices and bases extends the given PL homeomorphism to a PL homeomorphism between $f^{-1}(b) \times [a,b]$ and $f^{-1}[a,b]$ as desired.

6. Manifolds with boundary

The classical Morse theory was extended to smooth manifolds with boundary $(M, \partial M)$ in [5]. Here a *Morse function* is defined as a smooth function having only non-degenerate critical points in $M \setminus \partial M$ and no critical points on ∂M, i.e., $\mathrm{grad} f \neq 0$ on ∂M. Furthermore the restriction $f|_{\partial M}$ is assumed to be a Morse function on ∂M.

Definition 6.1. *A critical point p of $f|_{\partial M}$ is called* (+)-critical for f *if* $\mathrm{grad} f|_p$ *is an interior vector on M (pointing into M). It is called* (−)-critical for f *if* $\mathrm{grad} f|_p$ *is an exterior vector on M (pointing away from M).*

Proposition 6.2. (Braess [5])
Let M be a compact smooth manifold with boundary, and let $\mu^+(f)$ and $\mu^-(f)$ denote the number of (+)- and (−)-critical points. Only the (+)-critical points are H-critical and change the sublevel by attaching a cell, the (−)-critical points are H-regular. Moreover $f_{a-\epsilon}$ is a deformation retract of $f_{a+\epsilon}$ if $f^{-1}[a-\epsilon, a+\epsilon]$ contains only a (−)-critical point on ∂M and no critical point in $M \setminus \partial M$. Then the Morse inequality reads as

$$\mu(f|_{M \setminus \partial M}) + \mu^+(f) \geq rkH_*(M).$$

Moreover by duality on the boundary one has

$$\mu^+(f) + \mu^-(f) = \mu(f|_{\partial M}) \geq \mathrm{rk} H_*(\partial M).$$

However, there is no duality on M since a point is (+)-critical for f if and only if it is (−)-critical for $-f$.

For a proof see [5, Satz 4.1 and Satz 7.1]. In Satz 4.1 the assumption should be that the interval contains no critical point in the interior and no (+)-critical point on the boundary.

In the case of a generic PL function we can directly apply Definition 3.3 with the following result for a vertex $v \in \partial M$ with $f(v) = a$ [18]:

$$\mathrm{rk} H_*(f_a, f_a \setminus \{v\}) + \mathrm{rk} H_*(f^a, f^a \setminus \{v\}) \geq \mathrm{rk} H_*((f|_{\partial M})_a, (f|_{\partial M})_a \setminus \{v\})$$

Example: Simple 2-dimensional examples show that the last inequality is not always an equality: It can happen that a boundary point is H-critical for f but H-regular for $f|_{\partial M}$. By integrating the number of critical points over all directions of height functions we see that the contribution of the boundary is half the integral over the boundary separately in the smooth case and greater or equal to half this integral in the PL case [18].

By combining the definitions for PL Morse functions in Section 4 with the ideas of Definition 6.1 above we can formulate a theory of PL Morse functions on manifolds with boundary as follows.

Definition 6.3. *Let M be a compact PL d-manifold with boundary and $f\colon M \to \mathbb{R}$ a generic PL function. Then f is called a* PL Morse function *if all interior vertices are either non-degenerate critical or strongly regular in the sense of Definition 4.1 and all vertices on ∂M are either (+)-critical or (−)-critical or strongly regular.*

A point $p \in \partial M$ is called strongly regular *if there is a chart around p such that M is described by $x_1 \leq 0$ and the function f can be used as the coordinates x_d in ∂M, i.e., if in those coordinates*

$$f(x_1, \ldots, x_d) = f(p) + x_d \tag{7}$$

for $x_1 \leq 0$. If in a concrete polyhedral decomposition of M distinct vertices have distinct f-values, then f is also generic PL, and moreover all points are strongly regular except possibly the vertices.

A vertex $v \in \partial M$ is called non-degenerate (+)-critical *(or* (−)-critical, *respectively) if there is a PL chart with coordinates $x_1, \ldots, x_d$ around v for which the set M is described by the constraint*

$$x_d \geq -|x_1| - \cdots - |x_k| + |x_{k+1}| + \cdots + |x_{d-1}|$$
$$(\textit{or } x_d \leq -|x_1| - \cdots - |x_k| + |x_{k+1}| + \cdots + |x_{d-1}| \textit{ respectively})$$

and the function f can be expressed as

$$f(x_1, \ldots, x_d) = f(v) + x_d. \tag{8}$$

See Figure 2 for an illustration. In this case the boundary is represented by the equation

$$x_d = -|x_1| - \cdots - |x_k| + |x_{k+1}| + \cdots + |x_{d-1}|,$$

and the restriction $f|_{\partial M}$ is written as

$$f(x_1, \ldots, x_{d-1}) = f(v) - |x_1| - \cdots - |x_k| + |x_{k+1}| + \cdots + |x_{d-1}|, \tag{9}$$

so v is non-degenerate critical for $f|_{\partial M}$.

Corollary 6.4. *In the situation of Definition 6.3 only (+)-critical points on the boundary are H-critical, necessarily with multiplicity 1 and index k. Any (−)-critical point on the boundary is H-regular.*

Proof. The number k in Definition 6.3 is uniquely determined and coincides with the index of v if $v \in \partial M$ is (+)-critical, and the multiplicity is always 1 in this case: $H_k(f_a, f_a, \setminus\{v\}; \mathbb{F}) \cong \mathbb{F}$ and $H_j(f_a, f_a, \setminus\{v\}) = 0$ for any $j \neq k$. The change by passing through the critical level can be either $H_k(f_{a+\epsilon}) \cong H_k(f_{a-\epsilon}) \oplus \mathbb{F}$ or $H_{k-1}(f_{a-\epsilon}) \cong H_{k-1}(f_{a+\epsilon}) \oplus \mathbb{F}$. A function such that the second case never occurs is called a *perfect function.* For a (−)-critical vertex $v \in \partial M$ the homotopy types of f_a and $f_a \setminus \{v\}$ coincide. □

Corollary 6.5. *Proposition 6.2 remains valid for PL Morse functions on PL manifolds with boundary.*

7. Discrete Morse functions induce PL Morse functions

The above characterizations of strongly regular, non-degenerate, (+)- and (−)-critical points also allow an easy proof for a construction of PL Morse functions from discrete Morse functions. For the connection between classical Morse theory and discrete Morse theory see [2]. In particular for any smooth d-manifold with $d \leq 7$ the set of smooth Morse vectors coincides with the set of discrete Morse vectors.

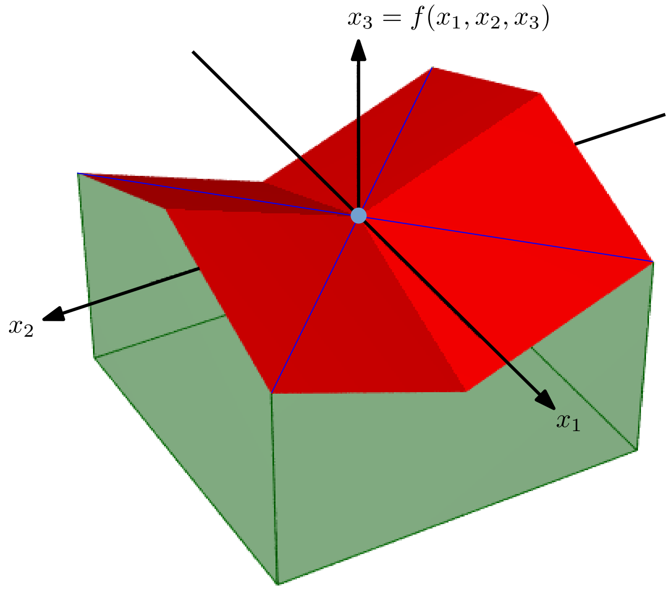

FIGURE 2. A non-degenerate critical point (blue) of index 1 on the boundary of a 3-manifold M. The boundary ∂M is the corrugated red saddle surface. If M consists of the volume under the "roof", as indicated by the green "walls", then this is a $(-)$-critical point. If M lies above the red surface, then it is a $(+)$-critical point. The blue cross is the level set at the critical value.

Definition 7.1. (Forman [12])

A discrete Morse function *maps cells of a complex to real numbers such that for each k-cell, there is at most one exceptional $(k-1)$-face whose value is not strictly smaller and at most one exceptional $(k+1)$-coface whose value is not strictly larger. A k-cell is called* critical *if it has no exceptional $(k-1)$-face and no exceptional $(k+1)$-coface.*

Fact: No cell has both an exceptional face and an exceptional coface, hence pairing each non-critical cell with its exceptional face or coface yields a partial matching of immediate face/coface pairs.

We call a discrete Morse function generic *if it has the following additional properties: The function is injective. Any non-immediate face of a cell has smaller value.*

Fact: Any discrete Morse function is equivalent to a generic one in the sense that it has the same critical cells and induces the same matching.

Lemma 7.2. *Any discrete Morse function on a combinatorial manifold M induces a generic PL Morse function linear on cells of a derived subdivision of M such that non-critical cells correspond to strongly regular vertices and critical cells of dimension k correspond to non-degenerate vertices of index k.*

Proof. Let K be the underlying complex of M and $g\colon K \to \mathbb{R}$ a discrete Morse function, without loss of generality generic. Define f on the domain of a derived subdivision of K by linearly interpolating the values at the vertices given by the assignment $f(v_S) = g(S)$ for each cell $S \in K$ and its corresponding vertex v_S in the derived. Observe that for a k-simplex S in K, the link of v_S in a derived subdivision is the join of two spheres, namely the derived of $bd(S)$, formed by vertices corresponding to proper faces of S, and a sphere formed by the vertices corresponding to proper cofaces of S. In particular, the embedding of the $(k-1)$-sphere formed by the derived of $bd(S)$ is unknotted in $lk(v_S)$. For a critical cell S, this implies already the claim that v_S is non-degenerate critical of index k, because the subcomplex of $lk(v)$ spanned by the vertices with f-value smaller than $g(S)$ agrees with the derived of $bd(S)$ in this case and hence $lk^-(v_S)$ is a regular neighborhood of an unknotted $(k-1)$-sphere.

For a non-critical cell S however, the subcomplex of $lk(v)$ spanned by the vertices with f-value smaller than $g(S)$ is either the derived of $bd(S)$ with the open star of a vertex v_T removed, where T is the exceptional face of S, or the join of the derived of $bd(S)$ with a single vertex v_{ST}, where ST is the exceptional coface of S. In any case, the subcomplex is a ball and its regular neighborhood $lk^-(v_S)$ is a ball as well, showing that v_S is strongly regular. □

The construction from Lemma 7.2 also works for generic discrete Morse functions g on a combinatorial manifold M with boundary. Then the boundary cells produce the following types of vertices for the induced PL Morse function: A critical boundary cell of dimension k corresponds to a $(+)$-critical point of index k. A non-critical cell that is paired with a cell in the boundary, i.e., the cell is also non-critical with respect to the restriction of g to the boundary of M, corresponds to a strongly regular point. A non-critical cell of dimension k that is paired with a cell not belonging to the boundary, i.e., the cell is critical with respect to the restriction of g to the boundary of M, corresponds to a $(-)$-critical point of index k.

8. The special case of low dimensions

Under the assumption that distinct vertices have distinct f-levels, only vertices can be critical. The critical vertices play the role of the critical points in classical Morse theory, either in the version of non-degenerate points or – more generally – for generic PL functions where higher multiplicities are admitted. However, the H-regular vertices that are not strongly regular do not fit this analogy: They do not contribute to the Morse inequalities and they have no analogue in the classical theory since they do not allow the cylindrical decomposition in a neighborhood with an isotopy between the upper and the lower sublevel. In some sense they are the most exotic objects to be considered here. Therefore the question is whether they can occur or not. In low dimensions $d \leq 4$ this is indeed not the case.

Proposition 8.1. *A* 1*-dimensional finite polyhedral complex is a graph. Any generic PL function has only minima (index* 0*) or critical vertices of index* 1*, possibly with higher multiplicity. Any vertex which is H-regular for* f *and for* $-f$ *simultaneously is also strongly regular for both of them.*

For a 1*-dimensional manifold we have only minima (index* 0*), maxima (index* 1*) and strongly regular vertices otherwise.*

Proof. Let v be a vertex and $a = f(v)$. The link of v is a finite set of points, some below the a-level, some above. If $lk^-(v)$ is empty we have a local minimum, the total multiplicity is 1. If $lk^-(v)$ consists of $r \geq 2$ points then v is critical of index 1 with the multiplicity $r - 1$. In the special case $r = 1$ the point is H-regular. For $-f$ we have to interchange $lk^-(v)$ and $lk^+(v)$. If in addition $lk^+(v)$ consists of only one point then v is a vertex of valence 2 between one upper and one lower vertex. Obviously v is strongly regular in this case. For a 1-manifold $lk(v)$ consists always of precisely two points, so the condition follows from $r = 1$ for one of the functions f or $-f$. □

Proposition 8.2. *Let* M *be a PL* 2*-manifold (a surface) with a generic PL function* $f : M \to \mathbb{R}$*. The critical points (vertices) are only of the following types:*

1. *Local minima (index* 0*, multiplicity* 1*),*
2. *local maxima (index* 2*, multiplicity* 1*),*
3. *saddle points (index* 1*, multiplicity arbitrary).*

Any H-regular vertex is also strongly regular, and any saddle point is non-degenerate critical in the sense of Definition 4.1 if its (total) multiplicity is 1 *in the sense of Definition 3.3.*

A splitting process of saddle points with higher multiplicity into ordinary saddle points is described in [11, p. 93].

Corollary 8.3. *Any generic PL function on a PL* 2*-manifold is a PL Morse function if the multiplicity of every saddle point is* 1*.*

Proof of Proposition 8.2. The link of a vertex v is a closed circuit of edges. If $lk^-(v)$ is empty we have a minimum, if $lk^-(v) = lk(v)$ we have a maximum ($lk^+(v)$ is empty), in all other cases $lk^-(v)$ and $lk^+(v)$ have the same number of components, say r components. Then v is critical of index 1 and multiplicity $r - 1$. An ordinary (non-degenerate) saddle point has $r = 2$, a monkey saddle $r = 3$.

The case of a H-regular vertex corresponds to the case $r = 1$. Since $st(v)$ is a topological disc, this implies that both $st^-(v)$ and $st^+(v)$ are discs, fitting together along the a-level which is an interval. Then we can apply Lemma 4.2.

The case of an ordinary saddle point corresponds to the case $r = 2$. These two components in $lk^-(v)$ and $lk^+(v)$ determine one coordinate line each such that the function f is linearly decreasing or increasing, respectively. The $f(v)$-level in between is the cross of the two diagonals in that coordinate system. □

Theorem 8.4. *Let M be a PL 3-manifold with a generic PL function $f\colon M \to \mathbb{R}$. The critical points (vertices) are only of the following types:*

1. Local minima (index 0, multiplicity 1),

2. local maxima (index 3, multiplicity 1),

3. mixed saddle points (index 1 or 2 or both, multiplicity arbitrary).

Any H-regular vertex is also strongly regular, and any saddle point is non-degenerate critical in the sense of Definition 4.1 if its (total) multiplicity is 1.

Proof. Let v be a H-regular vertex (not a local minimum) with

$$H_0(lk^-(v);\mathbb{F}) \cong \mathbb{F}, \quad H_1(lk^-(v)) = 0 \ \text{ and } \ H_2(lk^-(v)) = 0.$$

Therefore $lk^-(v) = f_a \cap lk(v)$ is a subset of $lk(v) \cong S^2$ which is a homology point. This implies that it is a homotopy point also, hence contractible. Consequently, $lk^-(v) \subset S^2$ is a disc since it is also a compact 2-manifold with boundary. Its complement is a disc also. Then we can apply Lemma 4.2.

Now let v be a saddle point with total multiplicity 1. This means that $lk^-(v)$ and $lk^+(v)$ are subsets of a 2-sphere with homology of a 0-sphere and a 1-sphere, respectively (in any order). So there are two discs in $lk^-(v)$ and a cylinder in $lk^+(v)$ or vice versa. Let us pick one point in each disc and a circle in the cylinder as “souls”. Then the cones from v determine one coordinate direction with decreasing f and two directions with increasing f (or vice versa). This defines the chart according to Definition 4.1. □

Theorem 8.5. *Let M be a PL 4-manifold with a generic PL function $f\colon M \to \mathbb{R}$. Then any H-regular vertex is also strongly regular.*

Proof. Let v be a H-regular vertex (not a local minimum) with

$$H_0(lk^-(v);\mathbb{F}) \cong \mathbb{F}, \quad H_1(lk^-(v)) = 0, \quad H_2(lk^-(v)) = 0 \ \text{ and } \ H_3(lk^-(v)) = 0$$

for any field $\mathbb{F}$. Therefore $lk^-(v)$ is a subset of $lk(v) \cong S^3$ which is a homology point for arbitrary $\mathbb{F}$, hence it is also a homology point for $\mathbb{Z}$, in other words: it is $\mathbb{Z}$-acyclic. The following argument is taken from [26]: $lk^-(v)$ is a compact 3-manifold which is $\mathbb{Z}$-acyclic, so the Euler characteristic is $\chi(lk^-(v)) = 1$. The Euler characteristic of the boundary is twice the Euler characteristic of the entire manifold, so $\chi = 2$ for the boundary which therefore contains a 2-sphere as one connected component, tamely (or locally flat) embedded into a polyhedral S^3. Then by the 3-dimensional Schoenflies theorem in PL [23] it bounds a 3-ball in S^3 on either side. This in turn shows that in our case there is no other component of the boundary since it would contradict the assumption that $lk^-(v)$ is acyclic. Then we can apply Lemma 4.2. □

It is remarkable that embeddings of the dunce hat into the 3-sphere cannot provide counterexamples since their regular neighborhoods must be 3-balls [3].

Remark: In higher dimensions $d \geq 5$ one obstruction is that a homology point contained in a vertex link is not necessarily a homotopy point, see Section 6 below. In particular there are acyclic 2-complexes in the 4-sphere that are not contractible [26], moreover there are particular embeddings of the contractible dunce hat into the 4-sphere with regular neighborhoods that are again contractible but not 4-balls [41]. These phenomena make it impossible to carry over the proofs above to dimensions higher than $d = 4$.

9. Counterexamples in higher dimensions

Example 1: (Critical point of total multiplicity 1 containing a knot)

We start with an ordinary knot built up by edges in a combinatorial 3-sphere. A concrete example is the 6-vertex trefoil knot in the 1-skeleton of the Brückner-Grünbaum sphere with 8 vertices, see [19, Fig.4]. After barycentric subdivision the knot coincides with the full subcomplex spanned by its vertices. This combinatorial 3-sphere can be the link of a vertex v in a 4-manifold. Define a generic PL function f with $f(v) = 0, f(x) < 0$ for all vertices x on the knot, and $f(y) > 0$ for all the other vertices y in the 3-sphere. This vertex v will be critical for f of index 2 and multiplicity 1, so homologically it behaves like a non-degenerate critical point of index 2 of a PL Morse function. However, the critical level will be a cone from v to a knotted torus in $lk(v)$. Therefore v is not a non-degenerate critical point in the sense of Definition 4.1.

Example 2: (H-regular point that is not strongly regular)

There are homology spheres that are not homotopy spheres. The most prominent example is the Poincaré sphere Σ^3 that can be defined as the quotient of the 3-sphere S^3 by the standard action of the binary icosahedral group (this action can be visualized in the symmetry group of the 120-cell). It admits a simplicial triangulation with only 16 vertices [4]. By removing an open 3-ball we obtain a space that is a homology point but not a homotopy point since its fundamental group does not vanish. By removing one open vertex star we find an example with 15 vertices $v_1, \ldots, v_{15}$. This simplicial complex C can be embedded into a high dimensional combinatorial sphere S^n_k with vertices $v_1, \ldots, v_k$, $k > 15$ such that C is the full complex spanned by those 15 vertices $v_1, \ldots, v_{15}$. Then we can build a combinatorial $(n+1)$-manifold M such that the star of one vertex v_0 is this combinatorial sphere S^n_k. The simplest example seems to be the suspension $S(S^n_k)$ of this combinatorial sphere S^n_k with altogether $k+2$ vertices. Next we define a simplexwise linear function f on M in such a way that

$$f(v_1) < f(v_2) < \cdots < f(v_{15}) < f(v_0) < f(v_{16}) < f(v_{17}) < \cdots < f(v_k)$$

and with arbitrary but distinct values for all the other vertices of M. Then the vertex v_0 is H-regular for f since in the link below the level and above the level the homology is trivial. However, it is not strongly regular since in the open vertex star the sublevel of v_0 is not contractible and is therefore not an open ball. In other words: Homology is unable to detect that v_0 is a non-regular point. It behaves exactly like any of the points in the interior of a top-dimensional simplex (which of course is strongly regular).

Example 3: (H-regular point that is not strongly regular)

There is a $\mathbb{Z}$-acyclic but not contractible 2-dimensional simplicial complex K with 23 vertices polyhedrally embedded into a polyhedral 4-sphere [26]. This can be extended to a triangulation of the 4-sphere with additional vertices outside K such that K coincides with the full subcomplex spanned by the 23 original vertices. As in Example 1 above one can define a generic

PL function f on some PL 5-manifold such that in the link of a vertex v_0 the sublevel is spanned by those 23 vertices. Consequently $lk^-(v_0)$ is acyclic, so v_0 is H-regular for f. It is not strongly regular since $lk^-(v_0)$ is not contractible, so it cannot be a 4-ball and $f_a \cap st(v_0)$ cannot be a 5-ball.

By further embedding of K into higher dimensional spheres it follows that a regular neighborhood of K is always homologically trivial but not contractible. Consequently, for any $d \geq 5$ there is an example of a generic PL function on a PL d-manifold with a H-regular critical point that is not strongly regular. This bound is optimal by the results of Section 5.

EXAMPLE 4: (Degenerate critical point of total multiplicity 1)
It is well known that the double suspension $S(S(\Sigma^3))$ of the Poincaré sphere Σ^3 in Example 2 is homeomorphic with the sphere S^5 (the so-called *Edwards sphere* [24]). However, since the link of certain edges is precisely Σ^3, the triangulation is not combinatorial and does not induce a PL structure. Nevertheless, we can define generic PL functions adapted to this 20-vertex triangulation of $S(S(\Sigma^3))$. If this 5-sphere occurs as the link of a vertex v in a 6-manifold, then we can find a generic PL function such that $f(v) = 0$, $f(x) < 0$ for all vertices of Σ^3 and $f(x) > 0$ for the others. Then v is a H-critical point that homologically behaves like a non-degenerate critical point of index 4 and multiplicity 1 but it is degenerate, so f will not be a PL Morse function.

10. A SPECIAL OBSTRUCTION: THE DUNCE HAT

Homology is a weaker concept than homotopy. So one might conjecture that a vertex v is strongly regular whenever both $lk^-(v)$ and $lk^+(v)$ are contractible, so that no homotopy group would detect anything critical (one might call this *homotopically regular*). The results of Section 5 show that this is true for generic PL functions on k-manifolds with $k \leq 4$. Here we are going to show that this systematically fails to hold in dimensions $k \geq 5$.

The dunce hat is known to be a 2-dimensional space that is contractible [41]. Any triangulation of it is not collapsible since there is no edge to start the collapse. There are embeddings into the k-sphere for any $k \geq 3$ [3]. If such a triangulated dunce hat occurs as the spanning full subcomplex of $lk^-(v)$ then neither homology nor homotopy will detect that v is a critical point. However, v will be strongly regular if and only if a regular neighborhood of the embedded dunce hat is a k-ball.

By the results of [28, 41], there are embeddings of the dunce hat into S^4 such that a tubular neighborhood is not a 4-ball, but Mazur's contractible 4-manifold with boundary. The boundary must be a homology 3-sphere. Here we present a simple model based on an 8-vertex triangulation. We start with the triangulation shown in Figure 3. It is equivalent to the triangulation used in [3]. Here is the list of triangles:

$$124, 234, 346, 136, 126, 256, 235, 135, 127, 147, 278, 457, 578, 238, 138, 158, 456.$$

It has the special property that any triangle contains either 1 or 8 or two vertices with consecutive labels $j, j+1$. This implies that it can be embedded into the boundary complex of the cyclic 5-polytope $C_5(8)$ with 8 vertices 1,

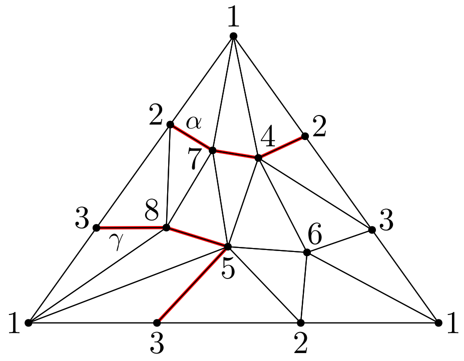

Figure 3. A triangulated dunce hat, and two cycles α and γ in the link of vertex 1.

2, 3, . . . , 7, 8 in that order. Using Gale's evenness condition [42], we find the missing triangles: 246, 247, 257, 357. The main question is: Is a tubular neighborhood of the 2-complex in the 4-dimensional boundary complex of the cyclic 5-polytope a 4-ball or not? It is certainly contractible since the dunce hat is. One special property of the embedding is easily seen: The two cycles α and γ in [41] are (2472) and (3583), and these two are linked in the link of the vertex 1. In fact, this is the cyclic 4-polytope $C_4(7)$ with 7 vertices, and that contains the 7-vertex torus (see Figure 1). The two cycles represent $(1, 1)$-knots on this torus, and any two of them are linked like Hopf fibers. Then [41, Conjecture 3] would imply that a tubular neighborhood of the embedded dunce hat is not a 4-ball. However, since we do not know whether this conjecture has been decided, we constructed a tubular neighborhood M, using the Sage[1] mathematics software system, and checked the fundamental group of its boundary ∂M. The fundamental group turned out to have a presentation with two generators u, v and the relations $uvu^{-4}v = 1 = (v^2u^{-1}v^{-1}u^{-1})^2v$. By introducing the extra relation $u^5 = 1$ we obtain $uv = (uv)^{-1} = v^{-1}u^{-1}$ and consequently

$$u^5 = v^7 = (uv)^2 = 1.$$

This group is known to be infinite [9, Sect. 5.3]. It coincides with the group of orientation preserving automorphisms of the regular $(7, 5)$-tessellation of the hyperbolic plane, in accordance with [28].

As an independent confirmation, Benjamin Burton (private communication) analyzed M with the Regina software for low-dimensional topology[2]. Regina could simplify ∂M to 9 tetrahedra, which it could recognize in its built-in census database as a Seifert fibred space, SFS [S2: (2,1) (5,1) (7,−5)]. In summary, the result was in both cases that the boundary ∂M of the tubular neighborhood is not a 3-sphere.

Corollary 10.1. *A regular neighborhood of the* 8*-vertex dunce hat above in the boundary complex of the cyclic polytope* $C_5(8)$ *is a contractible* 4*-manifold with boundary but not a* 4*-ball since its boundary is not a sphere.*

[1] `http://www.sagemath.org/`

[2] `https://regina-normal.github.io/`

Corollary 10.2. (explicit triangulation)
The second barycentric subdivision of the cyclic polytope $C_5(8)$ contains an explicit triangulation of a contractible 4-manifold with boundary which is not a 4-ball.

For the construction one just has to take the closed subcomplex of all simplices that meet the embedded dunce hat in $C_5(8)$ above. According to [2] this triangulation is not locally constructible.

Corollary 10.3. *There is a generic PL function on a 5-manifold with a vertex v that is H-regular but not strongly regular and – in addition – with the special property that both $lk^-(v)$ and $lk^+(v)$ are contractible. There are examples of this kind in every dimension $d \geq 6$ [16].*[3]

For the construction we start with a combinatorial 5-manifold containing a vertex v whose link is the boundary of the cyclic polytope $C_5(8)$; a concrete example is the cyclic polytope $C_6(9)$. Then we define a generic PL function f on the second barycentric subdivision such that the open regular neighborhood of the embedded dunce hat lies below $f(v)$ and its open complement lies above. Then the level of v itself in $lk(v)$ is a homology sphere but not a sphere, in contrast with the characterization of Lemma 4.2.

11. Computational aspects: Is regularity decidable?

The first problem is the *manifold recognition problem*: Given a pure simplicial complex of dimension d, can we algorithmically decide whether it is the triangulation of a combinatorial manifold? More precisely, can we algorithmically decide whether all vertex links are $(d-1)$-dimensional combinatorial spheres? This is trivial for $d = 1$ and fairly easy for $d = 2$. For $d = 3$ we can decide whether a vertex link is a connected 2-manifold, and then the Euler characteristic $\chi = 2$ is a sufficient criterion for being a 2-sphere. For $d = 4$ we can first decide whether a certain vertex link is a connected 3-manifold. Then we can apply the sphere recognition algorithm of A. Mijatović [29] and obtain:

Corollary 11.1. *It is algorithmically decidable whether a given simplicial complex of dimension d is a combinatorial d-manifold whenever $d \leq 4$.*

For a generic PL function on a PL manifold it is clearly decidable whether a vertex v is H-regular: One just has to compute the integral homology of $lk^-(v)$. There are software packages to do so. It is a much more delicate question to decide whether a vertex v is strongly regular. By the results of Section 5 H-regularity is a sufficient criterion in low dimensions. Therefore we can state part (1) as follows:

Corollary 11.2. *(1) For a PL manifold M of dimension $d \leq 4$ and a generic PL function f on M it is decidable whether a particular vertex v is strongly regular.*
(2) Moreover, for $d \leq 4$ it is decidable whether a generic PL function on M is a PL Morse function or not.

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[3] see `https://en.wikipedia.org/wiki/Mazur_manifold`

Proof of (2). By the results in Section 5 this is clear if $d \leq 3$. For $d = 4$ we have to look at possible saddle points v of index 1, 2 or 3 with total multiplicity 1. This can be decided by the homology. In the case of index 1 $lk^-(v)$ consists of two homology points, and $lk^+(v)$ consists of a homology 2-sphere, embedded into $lk(v) \cong S^3$. By the argument used in Theorem 8.5 each homology point is a 3-ball, and the homology 2-sphere is a regular neighborhood of an embedded 2-sphere. From this situation one can reconstruct a chart with 1 direction of decreasing f and 3 directions with increasing f. the case of index 3 is mirror symmetric to this situation (just interchange $-$ and $+$). It remains to discuss the case of index 2 where both $lk^-(v)$ and $lk^+(v)$ are homology 1-spheres that are linked in $lk(v) \cong S^3$. But that means that on the critical level $f_a \cap f^a \cap lk(v)$ we have an embedded (connected) surface with $\chi = 0$, so it is a torus. However, this torus can be knotted, see Example 1 in Section 6. So in addition we have to decide whether this torus is unknotted. This is known to be algorithmically decidable. If it is unknotted then it defines the chart according to Definition 4.1. If it is knotted then f is not a PL Morse function. □

Concerning 5-manifolds we run into several problems: The Schoenflies problem is unsolved for embeddings of the 3-sphere into the 4-sphere, the Hauptvermutung is unknown for the 4-sphere, and an algorithm for recognizing the 4-sphere (and hence: 5-manifolds) is not available. (See however [15] for practical approaches.)

For d-manifolds of higher dimension $d \geq 6$, we even obtain undecidability results. Novikov proved [40, 7, 27] that recognition of spheres in dimension 5 and above is an undecidable problem. In particular the manifold recognition problem is undecidable for d-manifolds with $d \geq 6$.

What are the consequences of Novikov's result for the recognition of strongly regular points? Let us consider the suspension $S(K')$ of an input K' for the sphere recognition problem and define f on $S(K')$ by choosing a negative f-value for a single vertex w of K', the f-value 0 for one vertex v added by taking the suspension, and distinct positive f-values for the remaining vertices. If K' is a sphere, then this construction yields a strongly regular vertex v, because $lk^-(v)$ is a regular neighborhood of the vertex w in $lk(v) = K'$, hence a ball. If K' is not a sphere however, not only the vertex v fails to be strongly regular, its link K' witnesses that $S(K')$ fails to be a (closed) manifold as well.

This shows that the above construction yields a reduction from the d-sphere recognition problem to the recognition problem of strongly regular vertices in arbitrary $(d+1)$-dimensional simplicial complexes. Novikov's result renders the latter problem undecidable for complexes of dimension at least 6.

Proposition 11.3. *For arbitrary simplicial d-complexes with $d \geq 6$, the problem of recognizing strongly regular vertices is undecidable.*

This reduction and its implied undecidability result are somewhat unsatisfactory however. The reduction produces manifold instances only from positive instances of the sphere recognition problem, whereas negative instances are reduced to non-manifold instances. Hence the reduction establishes undecidability only if verifying the manifold property is considered to be part of the problem. But, as noted above, recognizing d-manifolds for $d \geq 6$ is already known to be undecidable in itself.

Therefore we would prefer a reduction that produces manifold instances for the regular vertex recognition problem from all instances of the sphere recognition problem. For the proof of the following undecidability result, we present a reduction that achieves this, but at the cost of requiring higher dimension: Instead of producing $(k+1)$-dimensional instances from k-dimensional ones, it produces $2(k+1)$-dimensional instances.

Proposition 11.4. *Recognizing strongly regular vertices of combinatorial d-manifolds with dimension $d \geq 12$ is undecidable.*

Proof. We sketch a reduction from Novikov's sphere recognition problem. The input instances for this undecidable problem are 5-dimensional simplicial homology spheres, with positive instances being PL spheres and negative instances having a non-trivial fundamental group [27, Theorem 3.1].

Consider a simplicial complex K' as input for Novikov's sphere recognition problem. Remove a maximal simplex from K'. Embed the result as a subcomplex into the boundary sphere S' of a 6-neighborly simplicial d-polytope for $d \geq 12$ (more generally: a $(\dim(K')+1)$-neighborly simplicial d-polytope for $d \geq 2(\dim(K')+1)$). Subdivide S' to obtain an embedding as a full subcomplex. Denote the subdivided complex by S and the full subcomplex representing K' minus a simplex by K.

The suspension on S is a combinatorial d-manifold, in fact, a d-sphere, with S being the link of each of the two additional vertices. Define a function f by choosing distinct values at the vertices such that one vertex v of the additional vertices has f-value 0, the vertices from K have negative f-value, and the remaining vertices from S have positive f-value. Then $lk^-(v)$ is a regular neighborhood of K embedded into S.

If K' is a sphere, then K is a ball, and its regular neighborhood $lk^-(v)$ is a ball as well. Hence v is a strongly regular vertex. On the other hand, if K' has a non-trivial fundamental group, then, by the Seifert–van Kampen theorem, K has the same non-trivial fundamental group. Since K and $lk^-(v)$ are homotopy equivalent, the latter is not a ball, thus v is not strongly regular. □

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