# FINITELY MANY SMOOTH $d$-POLYTOPES WITH $n$ LATTICE POINTS

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ABSTRACT. We prove that for fixed $n$ there are only finitely many embeddings of $\mathbb{Q}$-factorial toric varieties $X$ into $\mathbb{P}^n$ that are induced by a complete linear system. The proof is based on a combinatorial result that implies that for fixed nonnegative integers $d$ and $n$, there are only finitely many smooth $d$-polytopes with $n$ lattice points. We also enumerate all smooth 3-polytopes with $\leq 12$ lattice points.

## 1. Introduction

The present paper has two target audiences: combinatorialists and algebraic geometers. We give combinatorial proofs of results motivated by the algebraic geometry of toric varieties. We provide two introductions with statements of the main results in the language of divisors on toric varieties on the one hand, and in the language of lattice polytopes on the other. In Section 2, we collect the relevant entries from the dictionary translating between the two worlds. In Section 3 we prove our theorems, and in Section 4 we report on first classification results.

### 1.1. Introduction (for algebraic geometers).

The purpose of this paper is to show the following finiteness theorem about embeddings of toric varieties[1] into projective space of a fixed dimension $n$.

**Theorem 1.** *Let $n$ be a nonnegative integer. Then there exist only finitely many embeddings of $\mathbb{Q}$-factorial toric varieties into $\mathbb{P}^n$ that are induced by a complete linear series.*

Neither of the two conditions ($\mathbb{Q}$-factorial and embedded via a complete linear series) can be omitted in the statement:

- For complete linear series embeddings of non-$\mathbb{Q}$-factorial varieties of dimension at least three, the embedding dimension does not even bound the degree, see Example 17.
- Every Hirzebruch surface $X(\mathbb{F}_a)$ (which is smooth, hence $\mathbb{Q}$-factorial) admits an embedding into $\mathbb{P}^5$; see Example 12.

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[1] All toric varieties appearing in this paper are normal by construction.

Furthermore, there exist infinitely many polarized toric varieties $(X, L)$ where $X$ is $\mathbb{Q}$-factorial and $L$ is ample with $h^0(X, L) = 3$, see Example 7.

When we assume that $X$ is smooth we obtain an even stronger result. For an ample line bundle $L = \mathcal{O}(D)$ on a toric variety $X$, we let $d(L) = \sum D \cdot C$, where the sum runs over all torus invariant curves $C$ in $X$.

**Theorem 2.** *For fixed $n$, there are only finitely many smooth polarized toric varieties $(X, L)$ such that $d(L) \leq n$.*

Example 18 shows that this theorem is false for very ample line bundles on $\mathbb{Q}$-factorial toric surfaces.

In Section 4 we use Oda's classification of smooth 3-dimensional toric varieties that are minimal with respect to equivariant blow-ups to classify all embeddings of smooth 3-dimensional toric varieties into $\mathbb{P}^{\leq 11}$ using a complete linear series. In the appendix we present the complete list of the corresponding 3-polytopes with $\leq 12$ lattice points up to equivalence.

Our motivation for this classification is a hierarchy of long standing open questions on toric embeddings, for example Oda's question [Oda08] whether an ample line bundle on a smooth projective toric variety is normally generated (see Section 4.5).

1.2. **Introduction (for polyhedral geometers).** The purpose of this paper is to show that there is only a finite number of classes (modulo integral equivalence) of smooth lattice polytopes once we fix some properties of them. For example, let us call a lattice polytope *smooth* if it is simple and all its normal cones (equivalently, all its tangent cones) are unimodular. Then Theorem 2 is equivalent to:

**Theorem 3.** *Let $n$ be a nonnegative integer. Then, modulo integral equivalence, there are only finitely many smooth lattice polytopes with $n$ lattice points on their edges.*

We prove several versions of this theorem; the most general one (Theorem 20) says that instead of requiring our polytopes to be smooth, as in the above in Theorem 3, it suffices to fix a finite list of possible tangent cones for the vertices (modulo integral equivalence).

Our proofs are based on a statement that transfers finiteness from dimension two to dimension $n$ (Lemma 9), together with a detailed analysis of the case of dimension two. In dimension two, simply using Pick's theorem already implies that there is a finite number of polygons with a fixed number of lattice points (see the proof of Theorem 10), but by using the classification of 2-dimensional unimodular fans we get that it is in fact enough to fix the number of lattice points on edges, as long as the *multiplicity* of the tangent cones is also bounded (Theorem 19, see Section 2.3.1 for the definition of multiplicity). In the smooth case, we also give bounds on how many polygons there are and how big their area can be in terms of the number of lattice points on edges (Theorems 25 and 26).

In Section 4 we use Oda's classification of 3-dimensional unimodular fans with $\leq 8$ rays that are minimal with respect to stellar subdivisions to classify all 3-dimensional polytopes with unimodular normal fan and $\leq 12$ lattice points, up to equivalence. They are listed in the appendix. In subsequent work, Anders Lundman has extended this classification to 16 lattice points [Lun13].

Also from the combinatorial viewpoint, our motivation for this classification is a hierarchy of long standing open questions about smooth polytopes (see Section 4.5).

1.3. **Related Results.** Let us briefly give an overview of related finiteness and classification results.

The first finiteness theorem goes back to Hensley [Hen83], with the current best bound due to Pikhurko [Pik01, (9)].

**Theorem 4.** *For a positive integer $d$, there is a bound $V(d)$ so that the volume of every lattice $d$-polytope with $k \geq 1$ interior lattice points is bounded by $k \cdot V(d)$.*

The second result, due to Lagarias and Ziegler [LZ91, Theorem 2], implies that bounding the volume automatically bounds the number of lattice points.

**Theorem 5.** *A family of lattice $d$-polytopes with bounded volume contains only a finite number of integral equivalence classes.*

Putting these two results together we get:

**Corollary 6.** *Any family of lattice polytopes with bounded number of lattice points contains only finitely many integral equivalence classes of polytopes with interior lattice points.*

**Example 7.** Without the assumption on interior lattice points the result is not true. For example, it is well-known (and was first observed by John Reeve [Ree57]) that there are simplices such as

$$P_k = \operatorname{conv}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & k \end{pmatrix}$$

with only 4 lattice points but unbounded volume. In particular, this shows that the number of lattice points of a lattice polytope does not give a bound on its volume.

On the classification side, most of the known results concern toric Fano varieties. Equivalently, on the polyhedral side the classifications deal with polytopes for which the primitive ray generators of the normal fan are the vertices of a convex polytope. In dimension two, $\mathbb{Q}$-Gorenstein toric Fano surfaces are known for Gorenstein index $\leq 17$ [KKN10]. In dimension three, the finite list of canonical toric Fano varieties was obtained by A. Kasprzyk [Kas06]. We refer the interested reader to the Graded Ring Database `grdb.lboro.ac.uk` for these and other classification results. Gorenstein toric Fano varieties, corresponding to so-called reflexive polytopes [Bat94], are completely classified in dimension $\leq 4$ [KS98, KS00]. Toric Fano manifolds are classified up to dimension 8 [Bat99, Sat00, KN09, Øbr07]; recently, B. Lorenz computed dimension 9. The complete list of the corresponding smooth reflexive polytopes can be found in the database at `polymake.org`.

Higher-dimensional classification results of toric varieties are only known in two cases: in the Gorenstein Fano case under strong symmetry assumptions [VK85, Ewa96, Nil06a] or if the Picard number of a toric manifold is at most 3, i.e., the $d$-dimensional fan has at most $d+3$ rays, in which case the variety is automatically projective [KS91, Bat91].

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## 2. Polarized toric varieties and lattice polytopes.

In this section we introduce notation and recall some basic facts about toric varieties. For more details we refer to [CLS11, §2.3] or [Ful93, Section 3.4].

2.1. **Lattice Polytopes.** Let $N \cong \mathbb{Z}^d$ be a lattice with dual lattice $M = \operatorname{Hom}(N, \mathbb{Z})$ and associated vector spaces $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$. A *lattice polytope* $P \subseteq M_\mathbb{R}$ is the convex hull of a finite number $u_1, u_2, \ldots, u_r$ of points in $M$. Any lattice polytope is the intersection of finitely many affine half spaces with primitive normal vectors $v_1, \ldots, v_s$ in $N$:

$$P \;=\; \operatorname{conv}(m_1, \ldots, m_r) \;=\; \{u \in M_\mathbb{R} \mid \langle v_j, u\rangle \geq -\alpha_j \,,\; 1 \leq j \leq s\}$$

for integral $\alpha_j$'s. A *face* of $P$ is the intersection of $P$ with an affine hyperplane $H$ such that $P$ is completely contained in one of the affine half spaces defined by $H$. Faces of a lattice polytope are lattice polytopes themselves.

For a vertex $u$ of $P$, let $T_uP := \operatorname{cone}(u' - u \mid u' \in P)$ be the (inner) *tangent cone* to $P$ at $u$. It is dual to the (inner) *normal cone* $\sigma(P, u) := \{v \in N_\mathbb{R} : \langle u' - u, v\rangle \geq 0 \;\forall u' \in P\}$ of $P$ at $u$. The normal cones of the different vertices of $P$ together with their faces form a polyhedral decomposition of $N_\mathbb{R}$ called the *normal fan* of $P$.

For a subset $S$ of $M_\mathbb{R}$, let $\operatorname{aff}(S)$ denote the affine span of $S$. We say that two lattice polytopes $P \subset M_\mathbb{R}$ and $P' \subset M'_\mathbb{R}$ for lattices $M$ and $M'$ are *integrally equivalent* if there is a lattice preserving affine map $\operatorname{aff} P \to \operatorname{aff} P'$ that maps $M \cap \operatorname{aff} P$ bijectively to $M' \cap \operatorname{aff} P'$ and $P$ to $P'$. Up to this integral equivalence, we can (and will) always assume that our polytope $P$ is full dimensional, i.e. $\operatorname{aff} P = M_\mathbb{R}$.

Let $e_1, \ldots, e_d$ be any basis of the lattice $M$. The *normalized volume* $V(P)$ is the volume that assigns 1 to the simplex $\operatorname{conv}(0, e_1, \ldots, e_d)$. In dimension 2 Pick's formula [Pic99] relates the normalized volume $V$ with the number $i$ of interior lattice points and the number $b$ of boundary lattice points via

$$V + 2 \;=\; 2i + b\,. \tag{1}$$

2.2. **Line bundles and polytopes.** Let $k$ be an arbitrary field and let $\Sigma$ be a complete rational fan of dimension $d$ in $N_{\mathbb{R}}$. Let $X = X(\Sigma)$ be the associated *toric variety*, a normal equivariant compactification of the algebraic torus $T \cong (k^*)^d$. The dual lattice $M$ is naturally isomorphic to the character lattice of $T$. Assume that $X$ is *projective* (equivalently, that $\Sigma$ is the *normal fan* of a polytope), and let $\mathcal{L}$ be an ample line bundle on $X$. The polarized toric variety $(X, \mathcal{L})$ corresponds to a lattice polytope $P \subseteq M_{\mathbb{R}}$ of dimension $d$ with its normal fan equal to $\Sigma$. Moreover, we have an isomorphism

$$H^0(X, \mathcal{L}) \cong \bigoplus_{u \in P \cap M} k\chi^u,$$

where $\chi^u\colon T \to k^*, (t_1, \ldots, t_d) \mapsto t_1^{u_1} \cdots t_d^{u_d}$ is the character corresponding to $u \in M$.

A linear series $W \subseteq H^0(X, \mathcal{L})$ induces a rational map $X \dashrightarrow \mathbb{P}(W)$, which is equivariant if and only if $W$ is torus invariant, that is, $W \cong \oplus_{u \in S} k\chi^u$ for some $S \subseteq P \cap M$. Letting $S = \{u_1, \ldots, u_{|S|}\}$, the induced map is given by $x \mapsto [\chi^{u_1}(x) \colon \cdots \colon \chi^{u_{|S|}}(x)]$. The degree of this map turns out to be the *normalized volume* of $\operatorname{conv}(S)$ – the volume measured in volumes of unimodular simplices. The map is induced by a *complete linear series* $W$ if and only if $W = H^0(X, \mathcal{L})$, that is, $S = P \cap M$. See [CLS11, §6].

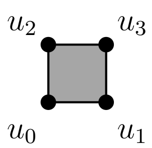



FIGURE 1. The Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ via $\mathcal{O}(1,1)$

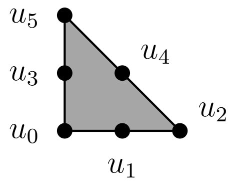



FIGURE 2. The Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ via $\mathcal{O}(2)$

If $P$ and $P'$ are integrally equivalent, and if $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ are the corresponding polarized toric varieties, then there exists a torus equivariant isomorphism $\phi\colon X \to X'$ such that $\phi^*\mathcal{L}' \cong \mathcal{L}$.

2.3. **Singularities and cones.** Let $\mathcal{L}$ be an ample line bundle on the toric variety $X(\Sigma)$ with corresponding lattice polytope $P \subseteq M_{\mathbb{R}}$. Then $X(\Sigma)$ is covered by torus invariant affine pieces $U_u$ which correspond to the vertices $u$ of $P$.

For each tangent cone, the semigroup $T_uP \cap M$ is finitely generated. Its unique minimal set of generators $\operatorname{Hilb}(T_uP)$ is called the *Hilbert basis* of the cone [CLS11, Proposition 1.2.22]. The coordinate ring of the affine variety $U_u$ is the semigroup ring $k[U_u] = k[T_uP \cap M]\,.$

The line bundle $\mathcal{L}$ is called *very ample* if its global sections induce an embedding into projective space. The combinatorial condition for $\mathcal{L}$ to be very ample is that for every vertex $u$ of $P$, the shifted polytope $P-u$ contains the Hilbert basis, i.e., $\mathrm{Hilb}(T_uP) \subseteq P-u$, see [Ful93, Section 3.4]. We call $P$ *very ample* if this happens.

**Example 8** (Example 7 continued)**.** The line bundle corresponding to Reeve's simplex $P_k$ is not very ample. The line bundle corresponding to $2P_k$ is normally generated, so in particular it is very ample (see [BGT97, Theorem 1.3.3] or [ON02]). It induces an embedding into $\mathbb{P}^{k+8}$.

2.3.1. $\mathbb{Q}$-*Gorenstein cones.* Let $\sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^d$ be a pointed rational $d$-cone with primitive generators $v_1, \ldots, v_r$. We call $\sigma$ $\mathbb{Q}$-*Gorenstein* if the $v_i$ lie in an affine hyperplane in $N_{\mathbb{R}}$. That is, if there is a linear functional on $\sigma$ which takes the value 1 on all $v_i$. This functional is called *height* and denoted $\mathrm{ht}_\sigma \in M_{\mathbb{R}}$ . The *index* of $\sigma$ is the smallest $k \in \mathbb{Z}_{>0}$ so that $k \cdot \mathrm{ht}_\sigma \in M$. We call $\sigma$ *Gorenstein* if this index is equal to 1.

These notions agree with the notions ($\mathbb{Q}$-)Gorenstein and index for the toric singularity associated with $\sigma$. We define the *multiplicity* $\mathrm{mult}(\sigma)$ as the normalized volume of the *nib* of $\sigma$

$$\mathrm{nib}(\sigma) := \mathrm{conv}(0, v_1, \ldots, v_r) = \{x \in \sigma \mid \langle \mathrm{ht}_\sigma, x\rangle \leq 1\}$$

which equals the product of the index with the normalized volume of $\mathrm{conv}(v_1, \ldots, v_r)$. Observe that every simplicial cone is $\mathbb{Q}$-Gorenstein, and its multiplicity equals $\det(v_1, \ldots, v_r)$.

Let $P$ be a lattice polytope with $\mathbb{Q}$-Gorenstein normal fan. We define the *multiplicity* of $P$ to be

$$\mathrm{mult}(P) = \max_u \mathrm{mult}(\sigma(P, u)),$$

the maximal multiplicity of a normal cone to $P$.

Note that for a projective toric variety $X$, the multiplicity does not depend on the polarization, so we can define the multiplicity $\mathrm{mult}(X) = \mathrm{mult}(P)$, where $P$ is a lattice polytope corresponding to an ample line bundle on $X$.

2.3.2. *Simplicial cones.* The toric singularity $U_u$ is $\mathbb{Q}$-factorial if the tangent cone $T_uP$ of $P$ at $u$ is simplicial, that is, it is generated by a linearly independent set $\{v_1, \ldots, v_d\}$ of primitive vectors. In this case, the singularity $U_u$ is a quotient $k^d/G$ of affine space by a finite abelian group, and the multiplicity is the cardinality of that group. The *box* of $T_uP$ is the half open parallelepiped

$$\square(T_uP) := \left\{ \sum_{i=1}^{d} \lambda_i v_i \mid \lambda_i \in [0, 1) \text{ for } i = 1, \ldots, d \right\},$$

and a *box point* is one of the $\mathrm{mult}(T_uP)$ many lattice points in $\square(T_uP)$. Every Hilbert basis element that is not one of the generators of $T_uP$ is a box point, and has smaller height than $d$. In particular, we have $\mathrm{Hilb}(T_uP) \setminus \{v_1, \ldots, v_d\} \subset \square(T_uP)$.

A cone is called *unimodular* if its primitive minimal generators form a lattice basis. Unimodularity is equivalent to having multiplicity 1. We call a lattice polytope $P$ *smooth* if every cone in its normal fan is unimodular.

A lattice polytope is smooth if and only if the associated projective toric variety $X$ is smooth (see for example [Ful93, Section 2.1]). Moreover, every ample line bundle on a smooth toric variety is very ample.

## 3. Finiteness Theorems

When we bound the number of lattice points, we arrive fairly quickly at the desired finiteness result for smooth polytopes (see Section 3.1). The case of simple and very ample polytopes is treated in Section 3.2. Finally, in Section 3.3, we show that for polytopes with restricted normal cones it suffices to bound the number of lattice points on the edges.

### 3.1. Few polytopes with $n$ lattice points.

Our finiteness theorems are based on the analysis of what happens in dimension two and then applying the following Lemma.

**Lemma 9.** *Let $n > d \geq 2$ be positive integers, let $\mathcal{F}$ be a finite family of $d$-dimensional lattice cones and let $\mathcal{P}$ be a finite family of lattice polygons. There are, up to integral equivalence, finitely many lattice $d$-polytopes with less than $n$ vertices such that every 2-dimensional face is integrally equivalent to a polygon from $\mathcal{P}$ and every normal cone is integrally equivalent to a cone from $\mathcal{F}$.*

*Proof.* We first observe that there is a finite number of combinatorial types of fans $\Sigma$ with $\leq n$ maximal cones. Here, the *combinatorial type* is given by the set of faces, partially ordered by inclusion. Once the combinatorial type of $\Sigma$ is fixed, there are only finitely many choices to assign an element $[\sigma_u]$ of $\mathcal{F}$ to a maximal cone of $\Sigma$ and to embed the face poset of $\sigma_u$ into the face poset of $\Sigma$ (if possible at all). So, we only need to prove finiteness of the number of polytopes $P$ with a fixed combinatorial type so that at every vertex $u$ the edges containing $u$ are assigned facets of $[\sigma_u]$ and such that every two-dimensional face is integrally equivalent to a polygon in $\mathcal{P}$. There are only finitely many ways to embed the combinatorial type of a polygon from $\mathcal{P}$ into the combinatorial type of a 2-face of $P$. We claim that these choices actually determine $P$ up to equivalence.

To this end, fix a vertex $u$ of $P$ and an element $\sigma_u$ of the equivalence class in $\mathcal{F}$ that we assigned to $u$. This determines all 2-dimensional faces of $P$ incident to $u$. In particular, if $u'$ is another vertex of $P$ adjacent to $u$, $u'$ together with all edges which are incident to $u'$ and contained in a common 2-face with $u$ are determined. The directions of these edges, together with the edge $uu'$ span $\mathbb{R}^d$ as a vector space. They thus pin down the normal cone $\sigma_{u'}$ in its class.

In summary, fixing a vertex and its normal cone also fixes all adjacent vertices and their normal cones. As the vertex-edge graph of $P$ is connected, this determines $P$. $\square$

**Theorem 10.** *Let $n > d \geq 2$ be positive integers, and let $\mathcal{F}$ be a finite family of $d$-dimensional lattice cones. There are, up to integral equivalence, finitely many lattice $d$-polytopes with at most $n$ lattice points such that every normal cone is equivalent to a cone from $\mathcal{F}$.*

*Proof.* In dimension two the statement follows from Theorem 5. Indeed, Pick's formula (1) implies that the volume $V$ and the number of lattice points $n$ of a lattice polygon bound each other:

$$n \leq V + 2 \leq 2n - 3.$$

Then Lemma 9 implies the theorem. $\square$

By taking $\mathcal{F}$ to consist of a single element, the unimodular cone, Theorem 10 implies the following weak version of Theorem 3:

**Corollary 11.** *Let $n$ be a nonnegative integer. Then, there are only finitely many smooth lattice polytopes with $n$ lattice points.*

**Example 12.** Corollary 11 does not imply that there are only finitely many projective torus equivariant embeddings into a fixed projective space. If we don't require the linear series to be complete, Figure 3 shows how to embed an arbitrary Hirzebruch surface torically into $\mathbb{P}^5$.

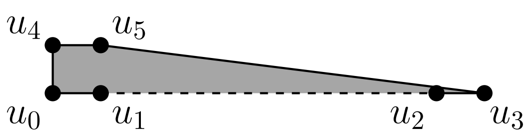



FIGURE 3. Hirzebruch surface $X(\mathbb{F}_a) \hookrightarrow \mathbb{P}^5$

**Corollary 13.** *For nonnegative integers $m$ and $n$, there are only finitely many lattice polytopes with $\mathbb{Q}$-Gorenstein normal cones of multiplicity bounded by $m$ and with $n$ lattice points.*

*Proof.* Applying Theorem 5 to the convex hull of 0 and the primitive generators of a $\mathbb{Q}$-Gorenstein cone, we see that the family of $\mathbb{Q}$-Gorenstein cones with multiplicity $\leq m$ contains only finitely many equivalence classes. Now apply Theorem 10. $\square$

We consider two morphisms to $\mathbb{P}^n$ the same if they differ by an automorphism of $\mathbb{P}^n$. Using the dictionary between toric morphisms and lattice polytopes, Corollary 13 implies the following corollary.

**Corollary 14.** *Let $n$ and $m$ be nonnegative integers. There are finitely many morphisms from some $\mathbb{Q}$-Gorenstein toric variety $X$ with* $\operatorname{mult}(X) \leq m$ *to $\mathbb{P}^n$ that are induced by a complete linear series.*

Example 7 shows that the assumption that the multiplicities are bounded in Corollaries 13 and 14 is needed.

3.2. **Simple, very ample polytopes with $n$ lattice points.** In this section we will show that when $P$ is simple and very ample, then the multiplicity of $P$ is bounded and so the corresponding assumption in Corollary 13 comes for free.

In order to deduce Theorem 1 from Corollary 14, we need another lemma.

**Lemma 15.** *For nonnegative integers $n$ and $d$ there are only finitely many $\mathbb{Q}$-Gorenstein cones $\sigma \subset \mathbb{R}^d$ so that*

$$\# \operatorname{Hilb}(\sigma) \; + \; \#(\operatorname{nib}(\sigma) \cap \mathbb{Z}^d) \; \leq \; n\,. \tag{$*$}$$

Observe that bounding $\# \operatorname{Hilb}(\sigma)$ or $\#(\operatorname{nib}(\sigma) \cap \mathbb{Z}^d)$ alone is not enough. Examples 17 and 7 show infinitely many cones with bounded $\#(\operatorname{nib}(\sigma) \cap \mathbb{Z}^d)$, and the following cones have only three Hilbert basis elements, and multiplicity $2a$:

$$C_a := \operatorname{cone}\{(1,a),(-1,a)\} \subseteq \mathbb{R}^2, \quad \operatorname{Hilb}(C_a) = \{(1,a),(-1,a),(0,1)\}.$$

*Proof of Lemma 15.* We will show by induction on $d$ that $(*)$ implies that $\operatorname{mult}(\sigma)$ is bounded. Then Theorem 5 implies that there are only finitely many choices for $\sigma$.

For $d = 1$ there is only one cone. For $d = 2$, Pick's formula (1) tells us that $\operatorname{mult}(\sigma) \leq 2\#(\operatorname{nib}(\sigma) \cap \mathbb{Z}^2) - 5$. So let us assume that the lemma is true for $d-1$. Because of Corollary 6, we can assume that $\operatorname{nib}(\sigma)$ has no interior lattice points. This implies that all interior Hilbert basis elements of $\sigma$ have height $\geq 1$. By induction, there is a minimal height $\epsilon(d-1,n) > 0$, depending only on $d-1$ and $n$, of a Hilbert basis element in the boundary of $\sigma$. Let $\epsilon = \min\{\epsilon(d-1,n), 1\}$.

Triangulate $\sigma = \cup_{i=1}^r \sigma_i$ into simplicial cones using only rays of $\sigma$. Every Hilbert basis element of $\sigma$ is a box point of one of the $\sigma_i$. As $\sigma$ has at most $n$ rays, every box point belongs to less than $\binom{n}{d}$ of the $\sigma_i$.

Now, every box point of every $\sigma_i$ has a representation $\sum_{v \in \operatorname{Hilb}(\sigma)} a_v v$ with $a_v \in \mathbb{Z}_{\geq 0}$. On the other hand, any box point has height $< d$, so that in the above representation we must have $\epsilon \cdot \sum_{v \in \operatorname{Hilb}(\sigma)} a_v < d$ which leaves at most $\binom{n+\lfloor d/\epsilon \rfloor}{n}$ possibilities for the coefficients $a_v$. In other words,

$$\operatorname{mult}(\sigma) = \sum_{i=1}^{r} \# \square(\sigma_i) < \binom{n}{d} \cdot \# \left( \bigcup_{i=1}^{r} \square(\sigma_i) \right) < \binom{n}{d} \binom{n + \lfloor d/\epsilon \rfloor}{n}.$$

□

The following statement is equivalent to Theorem 1.

**Theorem 16.** *Let $n$ be a nonnegative integer. Then there exist only finitely many simple and very ample polytopes with $n$ lattice points.*

*Proof.* Since $P$ is simple, every tangent cone to $P$ is $\mathbb{Q}$-Gorenstein. Moreover, since $P$ is very ample, a translate of the Hilbert basis for each tangent cone is a subset of the lattice points of $P$. Since $P$ has $n$ lattice points, it follows from Lemma 15 that there are only finitely many equivalence classes of tangent cones. So there are only finitely many equivalence classes of normal cones. Now the claim follows from Theorem 10. □

The following example shows that we need to assume that $P$ is simple (resp., that $X$ is $\mathbb{Q}$-factorial) in Theorem 16 (resp., Theorem 1).

**Example 17.** In [MFO07, p.2290] Winfried Bruns gives an example of a very ample divisor on a toric 3-fold whose complete linear series does not yield a projectively normal embedding. This example generalizes to a family of very ample polytopes

$$Q_k := \operatorname{conv}\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & k & k+1 \end{pmatrix}$$

with 8 lattice points but unbounded volume. Observe that these polytopes have a Gorenstein normal fan with $\operatorname{mult}(Q_k) = k+1$. However, the tangent cone $T_{(0,1,0)}(Q_k) = \operatorname{cone}((0,-1,0),(0,0,1),(1,-1,0),(1,0,k))$ is not $\mathbb{Q}$-Gorenstein for $k \geq 2$.

3.3. **Polytopes with $n$ lattice points on their edges.** The proof of Theorem 10 and, hence, those of Corollaries 11 and 13/14, were based on Pick's formula (1), which allowed us to bound the number of equivalence classes of polygons with a given number of lattice points. We now show that bounding the number of lattice points *along the edges* of the polygons is enough, if we also put a bound on the multiplicity of the cones. The following example shows that bounding the multiplicity is necessary.

**Example 18.** The polygons $P_{pq} = \operatorname{conv}((-1,-1),(p,0),(0,q))$ for $p$ and $q$ relatively prime positive integers form an infinite family of polygons having only 3 lattice points on their edges.

We call a lattice polygon $P$ a *$(m,n)$-polygon* if $P$ has at most $n$ lattice points on the boundary and $\operatorname{mult}(P) \leq m$. Then our most general finiteness result for polygons is the following theorem.

**Theorem 19.** *Let $m$ and $n$ be positive integers. There are only finitely many integral equivalence classes of $(m,n)$-polygons.*

Before proving Theorem 19 in Section 3.3.1, the following strong versions of Theorem 10 and Corollaries 11 and 13 are derived.

**Theorem 20.** *Let $n > d \geq 2$ be positive integers, and let $\mathcal{F}$ be a finite family of $d$-dimensional lattice cones. There are, up to integral equivalence, finitely many lattice $d$-polytopes with at most $n$ lattice points on edges and such that every normal cone is equivalent to a cone from $\mathcal{F}$.*

*Proof.* Let $P$ be a $d$-polytope such that every normal cone is in $\mathcal{F}$. This implies that the normal cones to every two-dimensional face of $P$ are contained in a finite family of two-dimensional cones $\mathcal{F}'$. In particular, each such two-dimensional face has bounded multiplicity. Since the number of lattice points on the edges of a face of a polytope $P$ are bounded by the number of lattice points on the edges of $P$, it then follows from Theorem 19 that the set $\mathcal{P}$ of polygons that can occur as two-dimensional faces of a polytope $P$ satisfying the assumptions of the theorem is finite. Now the claim follows from Lemma 9. $\square$

We can now apply this to prove our main Theorem.

*Proof of Theorem 2/ 3.* Fix $n$. Then any lattice polytope with less than $n$ lattice points on its edges has less than $n$ vertices, so $d = \dim(P) \leq n-1$. So it is enough to show that there are finitely many smooth lattice polytopes of dimension $d$ with less than $n$ lattice points on their edges. When $P$ is smooth, then every normal cone to $P$ is unimodular, so this follows from applying Theorem 20 to $n \geq n' > d$ and $\mathcal{F}$ consisting of the unimodular cone of dimension $d$. □

Theorem 2 implies the following. We are not aware of a more direct proof of this statement.

**Corollary 21.** *For smooth lattice polytopes, bounding the number of lattice points on the edges bounds the number of total lattice points. In other words, for a line bundle on a smooth polarized toric variety $(X, L)$ bounding $d(L)$ bounds $h^0(X, L)$.*

As lattice polygons are always $\mathbb{Q}$-Gorenstein and very ample, Example 18 shows that the statement of Theorem 2 does not hold when we only assume that $P$ is simple and very ample. In fact, in the proof of Theorem 16, we used the total number of lattice points to bound the multiplicity. If we assume in addition that the multiplicity is bounded, we obtain the following.

**Corollary 22.** *For nonnegative integers $m$ and $n$, there are only finitely many lattice polytopes with $\mathbb{Q}$-Gorenstein normal cones of multiplicity bounded by $m$ and with $n$ lattice points on edges.*

*Proof.* The first part is like the proof of Corollary 13, but then apply Theorem 20. □

3.3.1. *Finitely many polygons.* In this section we prove Theorem 19, arguing on the normal fan of a polygon. A 2-dimensional polyhedral fan $\Sigma$ is a *$[k, n]$-fan* if it is complete, has $k' \leq k$ rays (one-dimensional faces) with primitive generators $v_1, \ldots, v_{k'} \in N$ such that there are non-negative integers $\lambda_1, \ldots, \lambda_{k'}$ with $\sum \lambda_i v_i = 0$, $\sum \lambda_i \leq n$ and $\operatorname{span}(v_j \mid \lambda_j \neq 0) = N_{\mathbb{R}}$. The last condition means that there exists a polygon whose normal fan is refined by $\Sigma$ with at most $n$ lattice points on its boundary.

**Lemma 23.** *Let $P$ be an $(m, n)$-polygon with normal fan $\Sigma$. Then there is a unimodular $[nm, n]$-fan $\overline{\Sigma}$ refining $\Sigma$.*

*Proof.* The minimal unimodular subdivision $\overline{\Sigma}$ of $\Sigma$ (as discussed in detail in [CLS11, §10.2]) introduces less than $m$ new rays for each cone of multiplicity $\leq m$. So $\overline{\Sigma}$ has at most $nm$ rays.

Let $\lambda_v$ be the lattice length of the edge of $P$ dual to a ray $v$. In particular, $\lambda_v = 0$ for the extra generators introduced in the refinement from $\Sigma$ to $\overline{\Sigma}$. This choice of coefficients certifies $\overline{\Sigma}$ as an $[nm, n]$-fan by Minkowski's Theorem (cf. [Nil06b, Lemma 4.9], [Grü03, p. 332]). □

For what follows, we need a classification result for complete two-dimensional unimodular fans (cf. [Ewa96, Theorem V.6.6] or [Ful93, Section 2.5]).

(1) Any complete two-dimensional unimodular fan $\Sigma$ is integrally equivalent to either the fan of $\mathbb{P}^2$, which is generated by the three vectors $(1,0)$, $(0,1)$, and $(-1,-1)$, or to a refinement of the fan $\mathbb{F}_a$ of a *Hirzebruch surface* for $a \geq 0$: $\mathbb{F}_a$ is the complete fan with rays generated by $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (-1,0)$, and $v_4 = (a,-1)$.

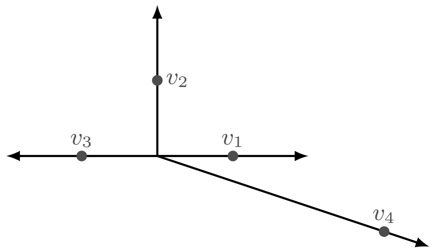



FIGURE 4. The fan $\mathbb{F}_a$ of the $a$-th Hirzebruch surface.

(2) The refinement from $\mathbb{F}_a$ to $\Sigma$ can be done introducing one ray at a time and in such a way that all intermediate fans are also unimodular. That is, in each refinement step a certain cone $\mathrm{cone}(v_1, v_2)$ with $|\det(v_1, v_2)| = 1$ is subdivided into $\mathrm{cone}(v_1, v_3)$ and $\mathrm{cone}(v_3, v_2)$ with $v_3 = v_1 + v_2$. In polyhedral terms this is an example of a *stellar subdivision*. In algebraic geometry terms, this corresponds to a blow-up at the fixed point corresponding to the cone $\mathrm{cone}(v_1, v_2)$.

The following result bounds the parameter $a$ of the starting Hirzebruch surface in terms of the parameters $m$ and $n$ of the fan:

**Lemma 24.** *Every unimodular $[k,n]$-fan $\Sigma$ is equivalent to the fan of $\mathbb{P}^2$ or to a stellar subdivision of a Hirzebruch fan $\mathbb{F}_a$ with $0 \leq a \leq n-2$.*

*Proof.* Assume that $\Sigma$ is not the fan of $\mathbb{P}^2$ and let $a \geq 0$ be the minimal integer such that $\Sigma$ is a stellar subdivision of (a fan equivalent to) $\mathbb{F}_a$. In the case $a = 0$ there is nothing to prove. So let us assume $a > 0$. Then the cones $\mathrm{cone}((0,1),(-1,0))$ and $\mathrm{cone}((-1,0),(a,-1))$ of $\mathbb{F}_a$ must be unsubdivided in $\Sigma$. Otherwise $\Sigma$ would contain the ray generated by either $(-1,1)$ or $(a-1,-1)$ and, hence, it would be a refinement of a fan equivalent to $\mathbb{F}_{a-1}$ as well. Hence, schematically, $\Sigma$ looks as in Figure 5. Now consider the rays with non-zero coefficient $\lambda$ in the expression certifying that $\Sigma$ is a $[k,n]$-fan. Since they positively span $N_{\mathbb{R}}$, at least one of them must have negative first coordinate and at least one of them must have negative second coordinate. The discussion above implies that

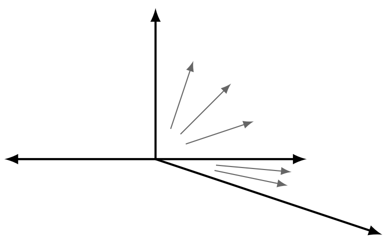
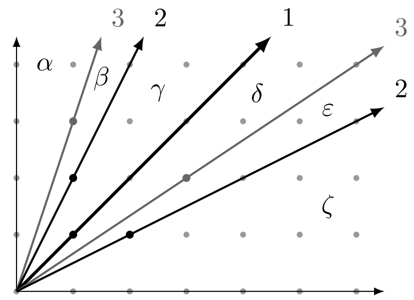


FIGURE 5. A schematic picture of the fan in the proof of Lemma 24



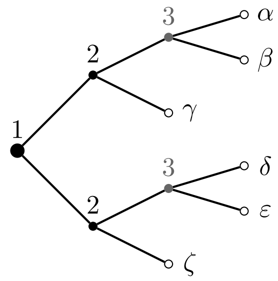




FIGURE 6. The binary tree corresponding to a unimodular refinement of a unimodular cone.

the only generator with negative first coordinate is $(-1,0)$, and that every generator with negative second coordinate has first coordinate $\geq a$. Hence, in order to have $\sum \lambda_v v = 0$ we must have $\lambda_{(-1,0)} \geq a$. Hence, the sum of all coefficients, $n$, is at least $a+2$. $\square$

*Proof of Theorem 19.* As the $t$-th dilation of a unimodular triangle has $3t$ lattice points on the boundary, there are at most $\lfloor n/3 \rfloor$ of them within our class. So, for the rest of the proof we bound the number of polygons which are not dilations of unimodular triangles. By Lemma 23, for any such polygon $P$ there exists a smooth $[nm,n]$-fan $\overline{\Sigma}$ refining the dual fan of $P$.

By Lemma 24, $\overline{\Sigma}$ is obtained from a Hirzebruch surface $\mathbb{F}_a$ with $a \leq n-2$ by a sequence of at most $nm-4$ unimodular stellar subdivisions. Since the number of possible unimodular stellar subdivisions in a unimodular 2-dimensional fan with $i$ rays is finite (it actually equals $i$) there is only a finite number of possibilities for the fan $\overline{\Sigma}$, hence also for the polygon $P$. $\square$

3.3.2. *How many polygons, and how big?* We now look at the refinement process described above in more detail in order to give estimates of how many polygons arise in Theorem 19 and what their maximum area is. We do so only in the smooth case and obtain these results:

**Theorem 25.** *Let $k \leq n$ be positive integers. Then, the number of smooth $k$-gons with $n$ boundary points is bounded above by $4^k \binom{n}{k} \frac{n}{k}$.*

**Theorem 26.** *Let $k \leq n$ be positive integers. Then, every smooth $k$-gon with $n$ boundary points has area bounded above by $\phi^{2k} n^2$, where $\phi = 1.618\ldots$ is the golden ratio.*

The starting point for the bound of Theorem 25 is that the different unimodular refinements of a unimodular cone can be recorded via binary trees. Remember that a binary tree is a rooted tree in which every node other than the leaves has exactly two children, labeled as "left" and "right". The number of different binary trees with $k$ leaves is the Catalan number $C_{k-1} = \frac{1}{k}\binom{2k-2}{k-1} \leq \frac{1}{k}4^k$ [Sta99, Ex. 6.19(d)]. If $\Sigma$ is a unimodular refinement of a unimodular 2-dimensional cone, we associate to $\Sigma$ the binary tree $T_\Sigma$ that has one internal node for each ray introduced in the refinement process and one leaf for each unimodular cone of $\Sigma$. See Figure 6 for an illustration. That is:

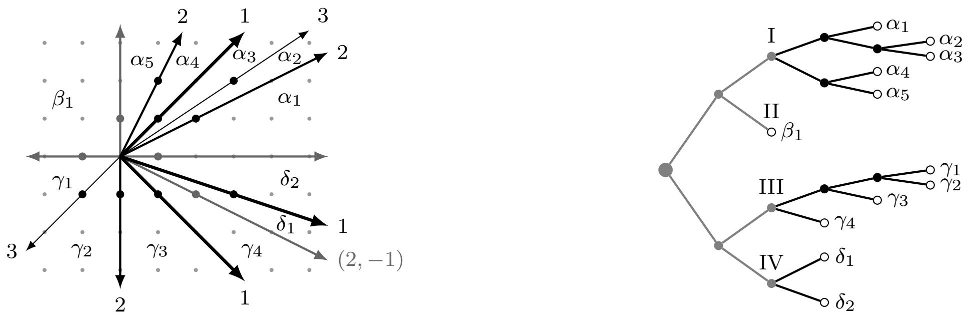



FIGURE 7. A subdivision of $\mathbb{F}_2$ and the corresponding binary tree

**Lemma 27.** *There is a bijection between stellar subdivisions of a unimodular cone with $k-1$ interior rays and binary trees with $k$ leaves.* □

With this we can prove Theorem 25:

*Proof of Theorem 25.* Apart from the case of the fan of a unimodular triangle, we need to count how many refinements there are of $\mathbb{F}_a$ with $k$ rays in total, and then how many ways to choose the coefficients $\lambda_v$ in such a way that $\sum \lambda_v v = 0$ and $\sum \lambda_v = n$. To bound this number, we combine the four binary trees that refine the four cones of $\mathbb{F}_a$ into a single tree with $k$ leaves, as shown in Figure 7. The number of ways of doing this is clearly smaller than $C_{k-1} \leq 4^k/k$. Observe that we are over-counting for several reasons: first, the trees we get are only those that have no leaf at depth 1. Second, in the case $a > 0$ we actually only need two binary trees, not four (put differently, the trees labeled $\beta$ and $\gamma$ in Figure 7 are empty). Third, in the case $a = 0$ there may be several copies of $\mathbb{F}_0$ in the fan $\Sigma$, which means there are different binary trees giving the same $\Sigma$.

We need to count the number of choices for the $\lambda$'s. We can bound this by the number of ways of partitioning $n$ into $k$ positive summands $\lambda_1 + \lambda_2 + \cdots + \lambda_n$, which equals $\binom{n-1}{k-1} \leq \binom{n}{k}$. Again, this is an overcount, because we do not care about the condition $\sum \lambda_v v = 0$. So, we get a bound of $\binom{n}{k}4^k/k$ for the number of polygons that come from a given $\mathbb{F}_a$. Since by Lemma 24 $a < n$, multiplying that bound for $n$ gives a global bound. □

In order to work towards the proof of Theorem 26, we first show an example that illustrates two points. On the one hand it shows that the upper bound given is not that bad; more precisely, it shows that the maximum area of a smooth $k$-gon with $n$ boundary points lies in $2^{\Theta(k)}n^{\Theta(1)}$. On the other hand, it shows where the golden ratio in the statement comes from.

On the other end of the range, Imre Bárány and Norihide Tokushige [BT04, Remark 2] constructed smooth lattice $n$-gons with area less than $n^3/54$.

**Example 28** (A smooth $k$-gon with area $\Omega(\phi^{2k/3})$)**.** We start with the normal fan of a unimodular triangle, whose rays we label as follows:

$$a_0 = c_1 = (1,0), \qquad b_0 = a_1 = (0,1), \qquad c_0 = b_1 = (-1,-1).$$

Starting with this fan, we refine the three cones in an iterative and symmetric manner. More precisely, choose an integer $\ell \geq 2$ and introduce:

$$a_i = a_{i-1} + a_{i-2}, \quad b_i = b_{i-1} + b_{i-2}, \quad c_i = c_{i-1} + c_{i-2}, \qquad \forall i = 2, \dots, \ell.$$

Since, by symmetry, the sum of these $k = 3\ell$ vectors is zero, this is the normal fan of a smooth polygon with all edges of lattice length 1. The (normalized) area of this polygon is at least the determinant of any pair of rays in the fan, since the convex hull of the corresponding edges contains a triangle with that area. Let us compute, for example, $\det(a_\ell, b_\ell)$. By construction we have

$$a_\ell = (F_{\ell-1}, F_\ell), \quad b_\ell = (-F_\ell, -F_{\ell-2}),$$

where $F_i$ denotes the $i$-th Fibonacci number. That is, $F_0 = 0$, $F_1 = F_2 = 1$, $F_3 = 2$, $F_{i+1} = F_{i-1} + F_i$. Hence, the determinant we are interested in equals

$$F_\ell^2 - F_{\ell-1}F_{\ell-2} \simeq c\phi^{2\ell} = c\phi^{2k/3}$$

for a certain constant $c$. Since the perimeter of the polygon is $O(F_\ell)$, this lower bound gives the correct area, up to the value of $c$.

*Proof of Theorem 26.* If the normal fan of $P$ is the fan of $\mathbb{P}^2$, $P$ is a unimodular triangle dilated by a factor of $n/3$, so its area is $n^2/9$. Thus, assume that the normal fan $\Sigma$ of $P$ is a refinement of the Hirzebruch fan $\mathbb{F}_a$ and, as in Lemma 24, assume that $a \geq 0$ is minimal with that property. If $a \neq 0$, then the three rays $v_2 = (0,1)$, $v_3 = (-1,0)$ and $v_4 = (a,-1)$ are consecutive in the normal fan of $P$. Let $e_2$, $e_3$ and $e_4$ be the corresponding edges. Then $P$ is inscribed in the triangle with base $e_3$ and third vertex in the intersection of the lines containing $e_2$ and $e_4$. That triangle (which may not be a lattice triangle) has normalized area $l^2/(2a) \leq n^2$, where $l \leq n$ is the length of $e_3$.

So, for the rest of the proof we assume $a = 0$; that is, $\Sigma$ refines the fan $\mathbb{F}_0$ of $\mathbb{P}^1 \times \mathbb{P}^1$. Let $k_1$, $k_2$, $k_3$ and $k_4$ denote the number of unimodular cones in $\Sigma$ that refine the four cones of $\mathbb{F}_0$.

The crucial observation is that, as in Example 28, in each quadrant, the $i$-th vector introduced by the refinement process is bounded from above by the $i$-th Fibonacci number $F_i$ in each coordinate. Here, as in Example 28, we reserve the indices $i = 0$ and $i = 1$ for the two boundary primitive vectors in the quadrant, so that the first vector refining the quadrant has $i = 2$. In particular, every coordinate of every ray is bounded above by $F_{k-3}$, since $k = k_1 + k_2 + k_3 + k_4$. On the other hand, the polygon $P$ is contained in the zonotope obtained as the Minkowski sum of its edges, and the (normalized) area of that zonotope is the sum of the absolute values of the determinants of all pairs of rays in the fan, where each ray is counted with a multiplicity equal to the length of the corresponding edge [Zie95, Ex. 7.19]. The stated bound then follows from these facts:

- The absolute value of each such determinant is bounded above by $2F_{k-3}^2$, which is smaller than $\phi^{2k}$.

- The number of subdeterminants (counting rays with multiplicity) is bounded above by $\binom{n}{2} < n^2$. $\square$

**Remark 29.** We believe $n^2\phi^{2k/3}$ to be also an upper bound for the area, which means that the construction of Example 28 is optimal, modulo a constant factor. The reason for this is that in order to get the vectors in $\Sigma$ to sum up to zero (when counted with multiplicity) we need to either have extremely high multiplicities in some of them (making $n$ exponentially big) or have at least two of the four cones of $\mathbb{F}_0$ be refined in basically the same way (making the Fibonacci numbers involved bounded by $F_{k/2}$ rather than $F_k$). But if only two (opposite) cones of $\mathbb{F}_0$ have this property then the Fibonacci-long vectors obtained will be almost opposite, making the area small. Three of the cones need to have vectors with big entries with respect to the basis of the starting $\mathbb{F}_0$, which should give the bound of $\phi^{2k/3}$.

## 4. Classification in Dimension 3

This section summarizes the strategy to classify smooth 3-polytopes with at most 12 lattice points. We don't follow the proof of Corollary 11 directly but use a modified strategy. For full details, including source code, see [Lor09, HLP10]. In subsequent work, Anders Lundman has extended this classification to 16 lattice points [Lun13].

### 4.1. Generating Normal Fans.

Katsuya Miyake and Tadao Oda classified smooth 3-dimensional fans which are minimal with respect to equivariant blow-ups [Oda88, Theorem 1.34]. This classification goes up to at most eight rays or equivalently, 12 full-dimensional cones. Starting from this list, all possible sequences of blow-ups had to be enumerated until no fan of a polytope with $\leq 12$ lattice points could occur further down the search tree. In order to prune the search tree, we used bounds based on the two-dimensional classification.

### 4.2. Generating Polytopes.

The next step is to find the polytopes corresponding to ample divisors, given the normal fan $\Sigma$. Let $\Sigma(1)$ denote the set of rays in $\Sigma$, and for $b \in \mathbb{R}^{|\Sigma(1)|}$, we let

$$P_b = \{u \mid \langle u, v_\rho\rangle \leq b_\rho\},$$

where $v_\rho$ is the primitive generator of the ray $\rho \in \Sigma(1)$. Note that for $b \in \mathbb{Z}^{|\Sigma(1)|}$, $P_b$ is the lattice polytope corresponding to the torus invariant prime divisor $D_b := \sum_{\rho\in\Sigma(1)} b_\rho D_\rho$. For a curve $C$ on $X$, the function $\mathrm{Div}(X)_\mathbb{R} \to \mathbb{R}\colon D \mapsto D\cdot C$ is linear. On a toric variety $D$ is ample if and only if $D\cdot C > 0$ for all torus invariant curves $C$, so these inequalities cut out the preimage of the ample cone in $\mathrm{Div}(X)_\mathbb{R}$. This preimage consists of the vectors $b$ such that the normal fan to $P_b$ is $\Sigma$. Note that when $D_b$ is ample, then $D_b\cdot C$ is the lattice length of the edge of $P_b$ corresponding to the $(n-1)$-dimensional cone in $\Sigma$ corresponding to $C$, see [Lat96, 1.4].

Bounding the sum of the edge lengths $d(\mathcal{O}(D))$ as a lower bound for the total number of lattice points, the search space of possible $b$-vectors which yield at most $n$ lattice points becomes itself the set of lattice points in a polytope.

The last step is to remove all polytopes that are integrally equivalent to another one in the list.

All these computations can be done with the `polymake` lattice polytope package by Benjamin Lorenz, Andreas Paffenholz and Michael Joswig [GJ, GJ00, JMP09] using interfaces to `4ti2` by the 4ti2 team [4ti2], `Latte` by Jesús De Loera et al. [LHTY04, LHTY] and `normaliz2` by Winfried Bruns et al. [BK01, BIS].

### 4.3. **Classification Results.**

**Theorem 30.** *There are 41 equivalence classes of smooth lattice polygons with at most 12 lattice points.*

| *Vertices* | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|
| *Polygons* | 3 | 30 | 3 | 4 | 0 | 1 | 0 | 0 | 0 | 0 |

**Theorem 31.** *There are 33 equivalence classes of smooth 3-dimensional lattice polytopes with at most 12 lattice points.*

| *Vertices* | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|
| *Polytopes* | 2 | 25 | 6 | 0 | 0 |

Note that a short parity argument shows that every *simple* (and hence every smooth) 3-polytope has an even number of vertices. Lists of all smooth polygons and smooth 3-polytopes with at most 12 lattice points can be found in the appendix.

4.4. **Comments.** We now have a list of smooth lattice polytopes in dimensions two and three with at most 12 lattice points. The bound 12 may seem rather low – the smallest smooth 3-polytope with one interior lattice point has 21 lattice points total [Kas10]. The classification carried out here serves as a proof of concept – it can be done. There are several points in the algorithm where it could be improved (compare [Lun13]).

In the current implementation, the generation of the normal fans is the bottleneck. By implementing a different way to directly generate all smooth normal fans one could skip the big recursion of calculating all blowing-ups, as well as overcome the limits of at most 12 vertices imposed by the Miyake/Oda classification. The second point to work on is the calculation of lattice points of the polytope containing all right-hand sides $b$. The dimension of this polytope is equal to the Picard number of the toric variety: the number of rays of the fan minus the ambient dimension. Of course, better theoretical bounds for all steps of the algorithm will directly improve the performance.

4.5. **Conjectures on smooth toric varieties.** There is an entire hierarchy of successively stronger conjectures concerning embeddings of smooth projective toric varieties which are open even in dimension 3, (compare [MFO07, p. 2313]). The weakest conjecture is Oda's question whether every smooth lattice polytope is *integrally closed*, i.e., every lattice point in $mP$ can be written as a sum of $m$ lattice points in $P$. The principal obstacle to theoretical progress on Oda's question on normality and the related conjectures is a serious lack of well understood examples. Recently, Gubeladze [Gub09] has shown that any lattice polytope with sufficiently long edges (depending on the dimension) gives rise to a projectively normal embedding. In view of this result, if there exists a counterexample, it is more likely to be a small polytope. Yet, all polytopes in our classification up to 12 lattice points satisfy even the strongest of these conjectures (see Corollary 34). In particular, the homogeneous coordinate ring is a Koszul algebra.

The following proposition shows that Oda's question implies Theorem 1 for smooth toric varieties.

**Proposition 32.** *There are only finitely many integrally closed lattice polytopes $P$ with $n$ lattice points.*

*Proof.* If $P$ is normal, then the semigroup in $M \times \mathbb{Z}$ generated by $(u, 1)$, where $u$ is a lattice point in $P$, is normal. This implies that the associated semigroup algebra is integrally closed and thus a Cohen-Macaulay standard graded algebra [Hoc72] with $\leq n$ generators. Thus, the coefficients of its Hilbert function (the Ehrhart polynomial of $P$) are bounded (compare, e.g. [Hib92, Lemma 18.1]). This bounds the degree (the normalized volume of $P$). By Theorem 5, there are only finitely many such $P$. □

Furthermore, using our classification, we were able to confirm the strongest conjecture for smooth polytopes with at most 12 lattice points.

**Theorem 33.** *If $P$ is a 3-dimensional smooth polytope with at most 12 lattice points, then $P$ has a regular unimodular triangulation with minimal non-faces of size two.*

**Corollary 34.** *Let $X$ be a smooth toric threefold embedded in $\mathbb{P}^{\leq 11}$ using a complete linear series. Then the defining ideal of $X$ has an initial ideal generated by square-free quadratic monomials.*

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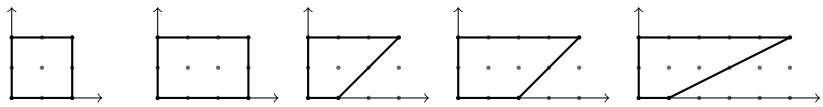
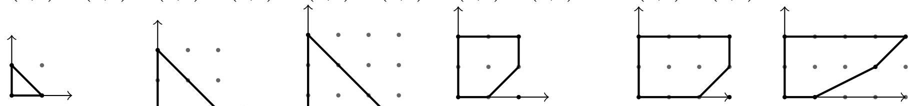
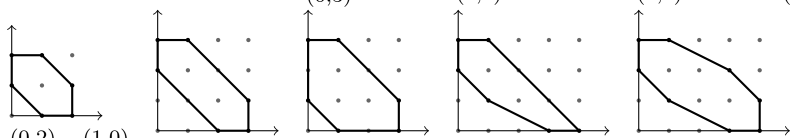
List of Smooth Polygons with $\leq 12$ Lattice Points



(2,0) (0,0) (0,2) (2,2) | (0,2) (0,0) (3,0) (3,2) | (0,0) (1,0) (5,2) (0,2) | (0,0) (1,0) (5,2) (0,2) | (0,0) (1,0) (5,2) (0,2)



(0,1) (1,0) (0,0) | (0,0) (2,0) (0,2) | (0,0) (3,0) (0,3) | (1,0) (0,0) (0,2) (2,1) (2,2) | (0,2) (0,0) (3,1) (3,2) (2,0) | (1,1) (3,0) (0,2) (0,4) (4,0)



(0,2) (1,0) (0,1) (2,0) (2,1) (1,2) | (0,3) (2,0) (0,2) (3,0) (3,1) (1,3) | (0,3) (1,0) (0,1) (3,0) (3,1) (1,3) | (0,3) (0,2) (3,0) (1,1) (4,0) (1,3) | (4,1) (0,3) (0,2) (3,0) (1,1) (4,0) (1,3) (3,2)

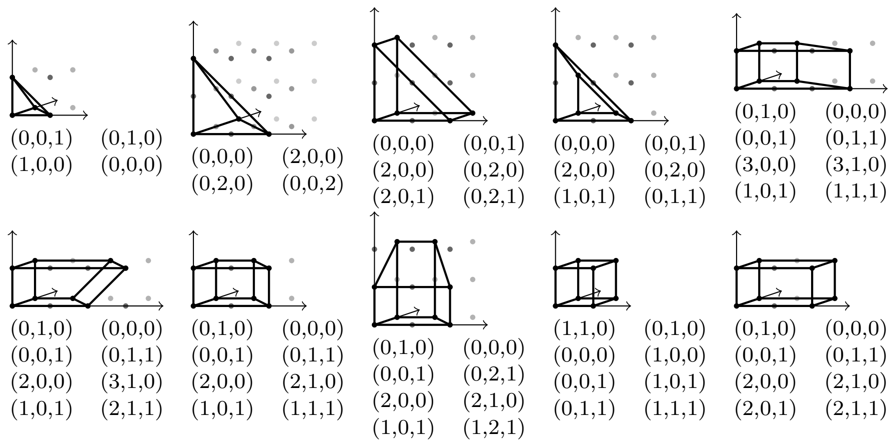
There are also 25 Hirzebruch quadrangles, omitted from the list:

$$Q_{a,b} := \operatorname{conv}\{(0,0), (0,a), (1,0), (1,b)\},$$

for all $a, b \geq 1$ with $a + b \leq 10$:

$$(a,b) \in \left\{ \begin{array}{c} (1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),\ (1,7),\ (1,8),\ (1,9) \\ (2,2),\ (2,3),\ (2,4),\ (2,5),\ (2,6),\ (2,7),\ (2,8) \\ (3,3),\ (3,4),\ (3,5),\ (3,6),\ (3,7),\ (4,4),\ (4,5),\ (4,6),\ (5,5) \end{array} \right\}.$$

List of Smooth 3-Polytopes with $\leq$ 12 Lattice Points




There are also the following 23 prisms, omitted from the list:

$$Q_{a,b,c} := \operatorname{conv}\{(0,0,0),(0,0,a),(1,0,0),(1,0,b),(0,1,0),(0,1,c)\},$$

for all $a, b, c \geq 1$ with $a + b + c \leq 9$:

$$(a,b,c) \in \left\{ \begin{array}{c} (1,1,1),\ (1,1,2),\ (1,1,3),\ (1,1,4),\ (1,1,5),\ (1,1,6),\ (1,1,7) \\ (1,2,2),\ (1,2,3),\ (1,2,4),\ (1,2,5),\ (1,2,6),\ (1,3,3),\ (1,3,4),\ (1,3,5),\ (1,4,4) \\ (2,2,2),\ (2,2,3),\ (2,2,4),\ (2,2,5),\ (2,3,3),\ (2,3,4),\ (3,3,3) \end{array} \right\}.$$

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