Frequently Asked Questions

to

A Concise Introduction to Mathematical Logic

Question 1. In **5.5** (page 200, 3rd edition) it is claimed that the existence of a model complete extension is necessary for the existence of a model completion of a theory but not sufficient. I don't see how the latter follows from Exercise 1.

Answer. Clearly, DO_{00} is a model complete extension of the theory T_{irr} of all irreflexive relations. But DO_{00} cannot be the model completion of T_{irr} because this theory is not model compatible with DO_{00}: The 2-element model of T_{irr} with a symmetric relation cannot be ordered, hence is not embeddable into a DO_{00}-model.

Question 2. A complete theory with a finite model is easily seen to be model complete but does not allow quantifier elimination in general (e.g., the theory of a finite order with more than one element). Is there a complete and model complete theory without finite models that does not allow quantifier elimination?

Answer. A fairly easy example is the theory T in the language $\mathcal{L}_\sim$ with the axioms

A1: $\sim$ is symmetric, i.e. $\forall xy(x \sim y \rightarrow y \sim x)$,

A2: there is precisely one reflexive isolated point, i.e. $\exists! x(x \sim x \wedge (\forall y{\sim}x)y{=}x)$,

A3: there are precisely two reflexive points x, y with $x \sim y$,

A4: all irreflexive points are isolated, i.e. $\forall x(x \not\sim x \rightarrow \forall y\, x \not\sim y)$,

A5: (schema) there are infinitely many points.

Note that each $\mathcal{A} \vDash T$ contains precisely three reflexive, and infinitely many irreflexive points. This easily entails that T is $\aleph_0$-categorical and hence complete by Vaught's criterion. Moreover, if $\mathcal{A}, \mathcal{B} \vDash T$, $\mathcal{A} \subseteq \mathcal{B}$, $\vec{a} \in A^n$, and $b \in B \setminus A$ (so that b is irreflexive) then there is an automorphism $f : \mathcal{B} \to \mathcal{B}$ with $f\vec{a} = \vec{a}$ and $fb \in A$. Hence $\mathcal{A} \preccurlyeq \mathcal{B}$ by Theorem **5.1.4**. Thus, T is model complete. We show that T does not permit quantifier elimination by proving that T is not substructure complete. Let $\mathcal{C}$ denote the one-element reflexive structure, so that $D\mathcal{C} = \{c{=}c, c \sim c\}$ with a new constant symbol c. This c may be interpreted in any fixed (for simplicity countable) $\mathcal{A} \vDash D\mathcal{C} \cup T$ in two different ways: c may denote either the reflexive isolated point of $\mathcal{A}$, or else one of the two non-isolated reflexive points of $\mathcal{A}$. Thus,

$DC \cup T$ has two not elementarily equivalent models: In the one holds $(\forall x{\sim}a)\, x = a$, in the other one $(\exists x{\sim}a)\, x \neq a$. Consequently, $DC \cup T$ is incomplete what we wanted to show. Actually, the quantifier is not eliminable, for instance, from $\exists x\, x \sim y$.

This example shows that Theorem **5.6.4** is very useful for obtaining both, positive and negative results on quantifier elimination. In the same way as above one shows, for instance, that of the four types of densely ordered sets the one without edge elements is the *only* one that allows quantifier elimination.

Question 3. Can't we always demand $\mathit{free}\,\varphi' = \mathit{free}\,\varphi$ on page 202 line 2?

Answer. No. Consider the theory T in the language with a single unary predicate P and the axioms $\forall x Px \vee \forall x \neg Px$ plus the schema 'there are infinitely many points'. T is substructure complete, hence quantifiers are eliminable. Note that $\exists x\, Px \equiv_T P\boldsymbol{v}_0$. In fact, $\{P\boldsymbol{v}_0\}$ is a Boolean base for the set of sentences modulo T.

Question 4. Is the theory N in **6.3** identical with the theory of discretely ordered commutative semirings, called also PA^- in the literature?

Answer. Yes, it is, provided the axiom $\mathsf{S}x = x + 1$ has been included in the axioms.

Question 5. Should't the theory Q in Remark 1 in **6.3** be replaced by N as a base theory for $I\Delta_0$?

Answer. You may replace Q by N because all axioms of N are provable in Q plus induction for Δ_0-formulas. Even induction on open formulas would probably suffice. You may omit axiom Q3 which is provable with Δ_0-induction, see **3.3**.

Question 6. In the proof of Theorem **7.1.2**, a treatment of the formula $x{\cdot}y = z$ is left to the reader. I can't figure out the proof of the claim $\vdash_{\mathsf{PA}} x{\cdot}y = z \rightarrow \partial\, x{\cdot}y = z$.

Answer. It suffices to prove $\vdash_{\mathsf{PA}} \forall yz(x{\cdot}y = z \rightarrow \partial\, x{\cdot}y = z)$ by induction on x. The initial step $x = 0$ follows from $\vdash_{\mathsf{PA}} 0 = z \rightarrow \partial\, 0 = z$ which is equivalent to $\vdash_{\mathsf{PA}} \partial\, 0 = 0$ and hence is obvious. As regards the induction step, we write the induction claim as $\mathsf{S}x{\cdot}y = z \vdash_{\mathsf{PA}} \partial\, \mathsf{S}x{\cdot}y = z$. Clearly, $\mathsf{S}x{\cdot}y = z \equiv_{\mathsf{PA}} \exists u(x{\cdot}y = u \wedge u + y = z)$. Hence, in view of $d00$ it suffices to prove for $\varphi := x{\cdot}y = u \wedge u + y = z$,

$$(*) \quad \exists u\varphi \vdash_{\mathsf{PA}} \partial\, \exists u\varphi.$$

Now, $\varphi \vdash_{\mathsf{PA}} \partial\, x{\cdot}y = u \wedge \partial\, u + y = z \vdash_{\mathsf{PA}} \partial\varphi$ by the induction hypothesis, the already treated case of the formula $u + y = z$, and $d\wedge$. Moreover, $\partial\varphi \vdash_{\mathsf{PA}} \partial\exists u\varphi$ by $d2$ since $\varphi \vdash_{\mathsf{PA}} \exists u\varphi$. Thus, $\varphi \vdash_{\mathsf{PA}} \partial\, \exists u\varphi$. This yields $(*)$ by anterior particularization.

Question 7. The definition of preference orders on page 296 of your book differs essentially from that used in economics. For instance, I found the definition that a preference order is a total preorder (i.e., a reflexive and transitive relation $\leqslant$) which in addition satisfies $(*)$ $P \leqslant Q$ or $Q \leqslant P$, for any two points P, Q or the domain. What is the relationship to the corresponding notion in your book?

Answer. For simplicity, let us consider finite preorders $(g, \leqslant)$ only. As is easily checked, the relation of mutual comparability, given by $P \leqslant Q \ \& \ Q \leqslant P$ $(P, Q \in g)$ is an equivalence relation on g whose equivalence classes are called *clusters*. In the presence of $(*)$ these clusters are totally ordered in a natural way as we shall see. Call a preference order as mentioned in the question *reflexive* during our discussion, while a preference order as defined in the book is called a *strict* preference order.

Let $(g, \leqslant)$ be any given reflexive preference order. If we define a relation $<$ on g by $P < Q \Leftrightarrow Q \nleqslant P$, then $(g, <)$ will turn out to be a strict preference order. Indeed, $(g, <)$ is a poset, i.e. it is irreflexive and transitive. Irreflexivity is obvious. To verify transitivity let $P < Q < R$, that is, $R \nleqslant Q \nleqslant P$. Assume that $P \not< R$, i.e. $R \leqslant P$. Since $P \leqslant Q$ by $(*)$, we get $R \leqslant Q$, a contradiction to $R \nleqslant Q$. Thus, indeed $P < R$. Instead of constructing a preference function π as defined on page 296 directly, it suffices to verify $(\mathbf{p})$, or equivalently, $(\mathbf{q})$: $P \not< Q \not< R \Rightarrow P \not< R$, the transitivity of the ordinary incomparability. This verification is trivial, for $(\mathbf{q})$ is clearly equivalent to $R \leqslant Q \leqslant P \Rightarrow R \leqslant P$ according to the definition of $<$.

If starting from a strict preference order $(g, <)$ one can also easily construct a corresponding reflexive preference order, just by defining $P \leqslant Q \Leftrightarrow Q \not< P$. The proof that $(g, \leqslant)$ is indeed a reflexive preference order is very easy if starting from a characterization of a strict preference as a poset with the property $(\mathbf{q})$.

It should be noticed that if we look at a cluster C of g in terms of $<$ then C has the property $P \not< Q \ \& \ Q \not< P$ for all $P, Q \in C$. For the latter means just $Q \leqslant P \leqslant Q$ which is obvious by $(*)$. Thus, in $(g, <)$, a cluster can be regarded as an equivalence class of the *mutual incomparability relation*, given by $P \not< Q \not< P$. It should also be observed that the clusters $C, D, \ldots$ of g are strictly ordered by defining $C < D \Leftrightarrow P < Q$, with $P \in C$ and $Q \in D$ (this definition is independent on the representatives P, Q as is readily seen). Indeed, $C \neq D \Rightarrow Q \nleqslant P \vee P \nleqslant Q$, hence $C \neq D \Rightarrow C < D \vee D < C$ by definition. The rest is obvious.

The above remarks reveal a very close relationship between strict and reflexive preference orders. Which one of these are preferred in practice, for instance in economics, depends on taste and habit. We mention that strict preference orders may have another name in certain descriptions, for instance, *strict weak orderings*. But it seems that the name *strict preference order* is a good choice.