An Exclusion Region for Minimum Dilation Triangulations

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Abstract

Given a planar graph G, the *graph theoretic dilation* of G is defined as the maximum ratio of the shortest-path distance and the Euclidean distance between any two vertices of G. Given a planar point set S, a triangulation of S that achieves minimum graph theoretic dilation is called a *minimum dilation triangulation* of S. In this paper, we show that a simple exclusion region for an edge e of the minimum dilation triangulation is given by the disk of radius $\alpha|e|$ centered at the midpoint of e, where α is any constant $< 3\cos(\pi/6)/(4\pi) \approx 0.2067$.

1 Introduction

In this paper, we are going to consider minimum dilation triangulations. The problem is as follows: Given a set S of points in the Euclidean plane, find a triangulation T of S such that the maximum dilation between any pair of these points in T is minimal, where the *dilation* between a pair of points (u, v) in S is defined as the ratio between the shortest path distance of u and v in T and the Euclidean distance $|uv|$ (see Section 2 for formal definitions of these terms). In Figure 1, we can see an example of a planar point set and two triangulations, one of which achieves a very low maximum dilation, while the other triangulation has a very high maximum dilation.

The maximum dilation between any pair of points in S with respect to a triangulation T of S is called the *graph theoretic dilation* of T, and the minimum graph theoretic dilation that any triangulation of S can achieve is called the graph theoretic dilation of S.

When considering optimal triangulations, it is instructive to look at local properties of the edges of these triangulations, since local properties improve our understanding of the structure of optimal triangulations and sometimes lead to efficient algorithms to compute them. One important class of local properties that has been studied for minimum weight triangulations and greedy triangulations is constituted by *exclusion regions*. Exclusion regions give us a necessary condition for the inclusion of an edge into an optimal triangulation: If u and v are two points in a given planar point set S, then the edge $e := \overline{uv}$ can only be contained in an optimal triangulation of S if no other points of S lie in certain parts of the exclusion region of S. For example, Das and Joseph [3] proved that e can only be included in the minimum weight triangulation of a point set S, if at least one of the two equilateral triangles with base e and base angle $\frac{\pi}{8}$ is empty (see Figure 2). This result was improved by Drysdale *et al.* [5], who proved that the base angle can be increased to $\pi/4.6$ and that also the disk of diameter $|e|/\sqrt{2}$ centered at the midpoint of e is an exclusion region for the minimum weight triangulation. A similar result with slightly different parameters also holds for the greedy triangulation [6]. In this paper, we are going to show that an analogous result applies to the minimum dilation triangulation. More specifically, we show that an edge e can only be included in the minimum dilation triangulation of S, if at least one of the two half circles with radius $\alpha|e|$ whose center is the center of e is empty (see Figure 2). Here α denotes any constant such that $0 < \alpha < 3\cos(\pi/6)/(4\pi) \approx 0.2067$.

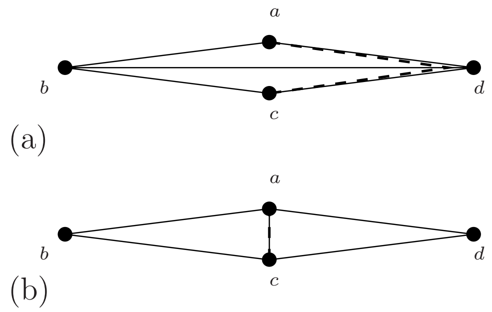

Figure 1: Two triangulations of point set $\{a, b, c, d\}$. In triangulation (a), the dilation between points a and c is very high, whereas triangulation (b) achieves a very low dilation. The bold dashed lines represent a shortest path between a and c in the respective triangulation.

Previous Work. Up to now, very little research has been done on minimum dilation triangulations, but there has been some work on estimating the dilation of certain types of triangulations that had already been studied in other contexts. Chew [2] shows that the rectilinear Delaunay triangulation has dilation at most $\sqrt{10}$. A similar result for the Euclidean Delaunay triangulation is given by Dobkin *et al.* [4]. They show that the dilation of the Euclidean De-

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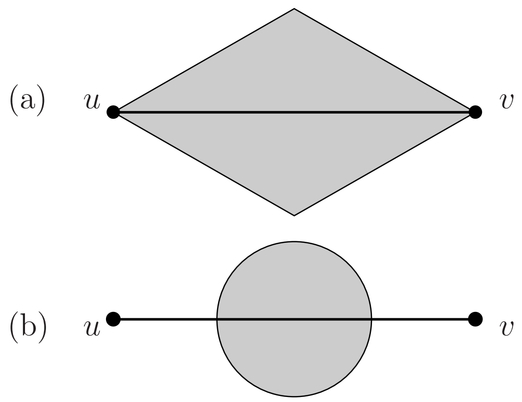

Figure 2: (a) shows the standard exclusion region for the minimum weight triangulation. (b) shows our exclusion region for the minimum dilation triangulation.

launay triangulation can be bounded from above by $\left(1+\sqrt{5}\right)\pi/2 \approx 5.08$. This bound was further improved to $2\pi/(3\cos(\pi/6)) \approx 2.42$ by Keil and Gutwin [8], and we are going to use this bound as an essential ingredient in our proof.

Das and Joseph [3] generalize these results by identifying two properties of planar graphs such that if A is an algorithm that computes a planar graph from a given set of points and if all the graphs constructed by A meet these properties, then the dilation of all the graphs constructed by A is bounded by a constant.

A more comprehensive survey of results on the graph theoretic dilation of planar and general graphs can be found in Eppstein's survey [7].

Surprisingly, very few results are known about the triangulations which actually achieve the optimum graph theoretic dilation. In his master's thesis, Mulzer [9] investigates the structure of minimum dilation triangulations for the regular n-gon, but beyond that not much is known.

2 Preliminaries

Let S be a finite set of points in the Euclidean plane, and let T be a triangulation of S. For any two points $u, v \in S$, the ratio between the shortest path distance $\pi_T(u,v)$ and the Euclidean distance $|uv|$ is called the *(relative) dilation* between u and v with respect to T, which we shall denote by $\delta_T(u,v)$. Formally, the dilation is defined as follows:

$$\delta_T(u,v) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } u = v, \\ \frac{\pi_T(u,v)}{|uv|}, & \text{if } u \neq v. \end{cases}$$

The convention to define $\delta_T(v,v) = 1$ for any $v \in S$ is very natural, since from the definition it is immediate that $\delta_T(u,v) \geq 1$ for every $u, v \in S$, as clearly we have $\pi_T(u,v) \geq |uv|$ for all $u, v \in S$.

Intuitively, the dilation is a measure for the quality of the connection between u and v in T. If the dilation is large, this means that we have to travel a long way along the edges in T in order to reach v from u even though the direct route would be much shorter.

In order to get a measure for the quality of the connection between any two vertices of T, it is natural to take the maximum over all the dilations between pairs of vertices in T. This quantity is called the *graph theoretic dilation* of T. We will denote it by $\delta(T)$. The formal definition is this:

$$\delta(T) \stackrel{\text{def}}{=} \max_{u,v \in S} \delta_T(u,v).$$

If T has the property that its graph theoretic dilation is minimal among all triangulations of S, we call T a *minimum dilation triangulation* of S.

3 An Exclusion Region

Let $0 < \alpha < 3\cos(\pi/6)/(4\pi)$ be a constant, S a planar point set, $u, v \in S$ two points in the plane, and let D be the disk of radius $\alpha|uv|$ centered at the midpoint of line segment $e = \overline{uv}$. We are going to show that D is an exclusion region for e.

The basic idea is very simple: Even though we do not know much about the actual minimum dilation triangulation of a planar point set S, we know that the graph theoretic dilation of the Delaunay triangulation of S is bounded by the constant $\gamma = 2\pi/(3\cos(\pi/6))$ [8]. Furthermore, it is obvious that if we have an edge e and two points that are quite close to the center of e and that lie on opposite sides of e, then the dilation between these two points is very large, because the line segment e constitutes an obstacle that any path between these two points needs to circumvent (see Figure 1(a) for an example). Thus, all we need to check is that the dilation between any pair of points in the disk that lie on opposing sides of e is larger than γ, and then we know that if such a pair of points exists, then e cannot be contained in the minimum dilation triangulation of S, since the Delaunay triangulation would give us a better graph theoretic dilation than any triangulation containing e.

Thus, we assume that there exist two points $a, b \in S$ in D on opposite sides of e (see Figure 3). We need to show that $\delta_T(a,b) > \gamma$ for any triangulation T of S that contains line segment e. For this we need to know the shortest path distance between a and b in T, $\pi_T(a,b)$. Since the only thing we know about T is that T contains e, the best thing we can do is to lowerbound $\pi_T(a,b)$ by $\min(|au|+|ub|, |av|+|vb|)$.

The first thing we observe is that we can assume that the two points lie on the boundary of D, since the dilation between the intersection points of the line through a and b with the boundary of D is smaller than the dilation between a and b.

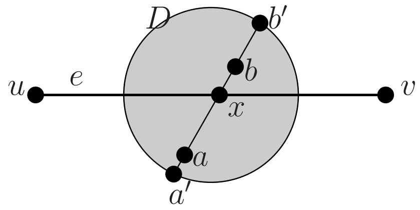

Figure 3: The situation described in Observation 1. The dilation between a' and b' is smaller than the dilation between a and b. x is the intersection point of $\overline{ab}$ and e.

Observation 1 *Let a be a point in D to the right of line segment $e = uv$, and let x be a point on e and in D. For $d > 0$, let $b(d)$ be the point to the left of line segment e on the half line ax such that $|xb(d)| = d$. Then the dilation $\delta(d)$ between a and $b(d)$ decreases as d increases.*

Proof. Due to the triangle inequality, the shortest path between a and b cannot include e, and hence $\delta(d)$ is given by

$$\delta(d) = \frac{\min\left(|ua| + |ub(d)|, |va| + |vb(d)|\right)}{|ax| + d}.$$

First, we are going to check that $\ell(d) := (|ua| + |ub(d)|)/(|ax| + d)$ is monotonically decreasing. By the law of cosines, the numerator can be written as $\mathrm{num}(d) = |ua| + \sqrt{|ux|^2 + d^2 - 2|ux|\cos\delta}$, where δ denotes the angle between $\overline{ux}$ and $\overline{xb(d)}$. An easy calculation shows $\mathrm{num}'(d) \leq 1$. The derivative of the denominator is 1. Therefore, by the mean value theorem, it follows that $\ell(d)$ is monotonically decreasing (note that the numerator is never smaller than the denominator), and the observation follows, since by a similar argument we can check that also $d \mapsto (|va| + |vb(d)|)/(|ax| + d)$ decreases monotonically, and hence $\delta(d)$ decreases. □

Now we are left with the task of bounding the dilation between two points on the boundary of D. First of all, it is clear that dilation $(2\alpha)^{-1}$ can be achieved when a and b are infinitesimally close to the two intersection points of D and e, respectively. We are going to show that this is already an optimal configuration. For our calculations we need a propitious parameterization. We proceed as follows: Let z be the center of D. By symmetry, we may assume that $\overline{ab}$ lies to the right of z. We describe the line segment $\overline{ab}$ by looking at the angle $\beta = \angle bza$ and the angle $x = \angle bzv - \beta/2$. The angle x describes the rotation of $\overline{ab}$ with respect to the position in which $\overline{ab}$ is perpendicular to e (see Figure 4). By our assumptions, we have $\beta \in (0, \pi]$ and $x \in (-\beta/2, \beta/2)$. Our parameterization is chosen in such a way that the following equations can be written in a symmetric manner, which simplifies some of the calculations.

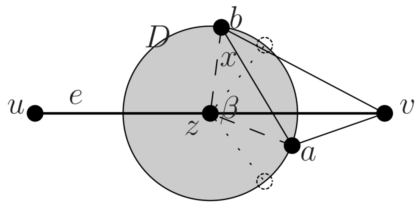

Figure 4: Our parameterization. The angle $\angle azb$ is called β. The offset x denotes the rotation of $\overline{ab}$ with respect to the vertical position (dashed lines).

The angle $\angle bza$ is at most π, and hence the shortest path between a and b passes v. Thus, the dilation between a and b is given by

$$\delta(x, \beta) := \frac{f(x) + f(-x)}{2\alpha\sin(\beta/2)},$$

where

$$f(x, \beta) = \sqrt{0.25 + \alpha^2 - \alpha\cos(\beta/2 + x)}.$$

Here, $f(x)$ and $f(-x)$ denote the length of line segment $|vb|$ and $|va|$, respectively.

First, we fix $\beta \in (0, \pi]$ and optimize $x \mapsto \delta(x, \beta)$. An elementary yet tedious calculation yields the following observation:

Observation 2 *Let $\beta \in (0, \pi]$ be fixed. If we have $\cos(\beta/2) \leq 2\alpha$, the function $x \mapsto \delta(x, \beta)$ is minimal for $\cos(x) = (2\alpha)^{-1}\cos(\beta/2)$. Otherwise, $x \mapsto \delta(x, \beta)$ is minimal for $x = 0$.*

Now there are two cases to consider. If $\cos(\beta/2) \geq 2\alpha$, we need to look at $\delta(0, \beta) = f(0)/(\alpha\sin(\beta/2))$. Again, it turns out that this function is minimal if $\cos(\beta/2) = 2\alpha$, for this value of β we get that the dilation between a and b is exactly $(2\alpha)^{-1}$. What happens if $\cos(\beta/2) < 2\alpha$? In this case, we need to consider the value of $\delta(x, \beta)$, where x has the property that $\cos x = (2\alpha)^{-1}\cos(\beta/2)$. By using this property and by some trigonometric manipulations, we find that $\delta(x, \beta) = (2\alpha)^{-1}$. It follows that the dilation between a and b exhibits quite a remarkable behavior. If a and b are diametrically opposed, the minimum configuration with minimum dilation occurs when a and b are infinitesimally close to the two intersection points between D and e. As the chord $\overline{ab}$ gets shorter, the angle between e and $\overline{ab}$ in the optimal configuration becomes larger, until e and $\overline{ab}$ are perpendicular. As soon as this configuration is reached, the dilation between a and b increases as $\overline{ab}$ gets shorter.

Consequently, the minimum dilation between any points a and b in the two halves of D is $(2\alpha)^{-1}$, and by our choice of α and the upper bound on the graph theoretic dilation of the Delaunay triangulation [8], we can conclude with the following theorem:

Theorem 1 *Let $0 < \alpha < 3\cos(\pi/6)/(4\pi)$ be a constant, and let a and b be two points in the plane. Then the circle of radius $\alpha|ab|$ centered at the midpoint of $\overline{ab}$ is an exclusion region for the minimum dilation triangulation.*

Note that this exclusion region can be enlarged a little bit on the upper and lower boundary. For example, the dilation between the north- and south-pole of D is strictly less than γ. However, this would give us some curve of order 4 that is more difficult to handle than a simple circle.

4 Conclusion

We have made some progress in the field of minimum dilation triangulations and have shown that the concept of exclusion regions also makes sense for the minimum dilation triangulation. Usually, exclusion regions are applied as an initial filter of algorithms that compute minimum weight triangulations or greedy triangulations [1, 6]. It is easy to see that if the point set S is drawn independently and uniformly from a convex set C, then only an expected number of $O(n)$ edges pass the exclusion region test, so a large amount of edges can be discarded. In the algorithms for the other optimal triangulations, the remaining edges are processed using the greedy property (for the greedy triangulation) or some other local properties that give sufficient conditions for the inclusion of an edge (for the minimum weight triangulation). For the minimum dilation triangulation, however, it is not yet clear what to do with the remaining edges, since no other useful local properties are known that could be used in further processing steps. Finding such local properties remains an open problem.

It may also be interesting to look for configurations of points that show how tight our exclusion region is and whether it can be enlarged. At least it is clear that the exclusion region cannot come arbitrarily close to the endpoints of the edge, since otherwise the dilation between two points in the exclusion region can be arbitrarily close to 1.

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