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Toric Ideals Generated by Quadratic Binomials

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A combinatorial criterion for the toric ideal arising from a finite graph to be generated by quadratic binomials is studied. Such a criterion guarantees that every Koszul algebra generated by squarefree quadratic monomials is normal. We present an example of a normal non-Koszul squarefree semigroup ring whose toric ideal is generated by quadratic binomials as well as an example of a non-normal Koszul squarefree semigroup ring whose toric ideal possesses no quadratic Gröbner basis. In addition, all the affine semigroup rings which are generated by squarefree quadratic monomials and which have 2-linear resolutions will be classified. Moreover, it is shown that the toric ideal of a normal affine semigroup ring generated by quadratic monomials is generated by quadratic binomials if its underlying polytope is simple.

INTRODUCTION

Let K be a field and $K[\mathbf{t}] = K[t_1, t_2, \ldots, t_d]$ the polynomial ring in d variables over K with each $\deg t_i = 1$. If $\mathscr{A} = \{f_1, f_2, \ldots, f_n\}$ is a finite set of monomials belonging to $K[\mathbf{t}]$ such that all the f_i's have the same degree, then we write $K[\mathscr{A}]$ for the subalgebra of $K[\mathbf{t}]$ which is generated by $f_1, f_2, \ldots, f_n$ over K. Let $K[\mathbf{x}] = K[x_1, x_2, \ldots, x_n]$ denote the polynomial ring in n variables over K and π: $K[\mathbf{x}] \to K[\mathscr{A}]$ the surjective homomorphism of semigroup rings defined by $\pi(x_i) = f_i$ for all $1 \leq i \leq n$. We write $I_{\mathscr{A}}$ for the kernel of π and call $I_{\mathscr{A}}$ the *toric ideal* associated with the affine semigroup ring $K[\mathscr{A}]$. It follows from, e.g., [11, Corollary 4.3] that the toric ideal $I_{\mathscr{A}}$ is generated by binomials.

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Recently, the following three properties on $\mathcal{A}$ were investigated by several commutative algebraists:

(i) $I_{\mathcal{A}}$ is generated by quadratic binomials;

(ii) $K[\mathcal{A}]$ is Koszul;

(iii) $I_{\mathcal{A}}$ possesses a quadratic Gröbner basis.

We refer the reader to, e.g., Backelin and Fröberg [1] for the foundation on Koszul algebras and to [2], [6], and [7] for detailed information about Gröbner bases. The hierarchy (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is known (e.g., [4]). The converse hierarchy is true for normal toric surfaces [5] and for affine semigroup rings arising from bipartite graphs [9]. On the other hand, it is shown in [10] that there exist a non-Koszul monomial curve whose toric ideal is generated by quadratic binomials and a Koszul monomial curve whose toric ideal has no quadratic Gröbner basis.

Regarding $K[\mathbf{x}]$ as a graded ring with each $\deg x_i = 1$ and writing $K[\mathbf{x}](-a)$, where a is an integer, for the graded module $K[\mathbf{x}]$ over itself with $\deg x_i = a$, we are interested in a graded minimal free resolution

$$0 \longrightarrow \bigoplus_{j=1}^{\beta_h} K[\mathbf{x}](-a_{h_j}) \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j=1}^{\beta_1} K[\mathbf{x}](-a_{1_j}) \xrightarrow{\varphi_1} K[\mathbf{x}] \xrightarrow{\pi} K[\mathcal{A}] \longrightarrow 0$$

of $K[\mathcal{A}]$ over $K[\mathbf{x}]$. See, e.g., [3] for detailed information about graded minimal free resolutions. Such a resolution is called *m-linear* if $a_{i_j} = m + i - 1$ for all i and j. We say that $K[\mathcal{A}]$ has m-linear resolution if a graded minimal free resolution of $K[\mathcal{A}]$ over $K[\mathbf{x}]$ is m-linear. Thus, in particular, if $K[\mathcal{A}]$ has m-linear resolution, then the toric ideal $I_{\mathcal{A}}$ is generated by binomials of degree m. It is known [1] that $K[\mathcal{A}]$ has 2-linear resolution if and only if $K[\mathcal{A}]$ is both a Koszul algebra and a Golod algebra.

The present manuscript follows our previous paper [8] and we are mainly interested in a finite set $\mathcal{A}$ consisting of squarefree quadratic monomials. First, in Section 1, we give a combinatorial criterion for $I_{\mathcal{A}}$ to be generated by quadratic binomials; see Theorem 1.2. Such a criterion guarantees that every Koszul algebra generated by squarefree quadratic monomials is normal; see Corollary 1.3. Second, in Section 2, we present an example of a normal non-Koszul algebra generated by squarefree quadratic monomials whose toric ideal is generated by quadratic binomials (cf. Example 2.1) and an example of a non-normal Koszul algebra generated by squarefree cubic monomials whose toric ideal possesses no quadratic Gröbner basis (cf. Example 2.2). On the other hand, a proof of Theorem 1.2 will be given in Section 3. Moreover, in Section 4, we classify all the affine semigroup rings $K[\mathcal{A}]$ such that $\mathcal{A}$ is a finite set of squarefree quadratic monomials and that $K[\mathcal{A}]$ has 2-linear resolutions. See Theorem 4.6. In Section 5, we prove that the toric ideals of a finite set $\mathcal{A}$ consisting of (not necessarily squarefree) quadratic monomials is generated by quadratic binomials if $K[\mathscr{A}]$ is normal and if the convex polytope associated with $\mathscr{A}$ is simple.

It would be, of course, of great interest to find all the Koszul algebras generated by squarefree quadratic monomials as well as to find all the Golod algebras generated by squarefree quadratic monomials. These problems belong to the research plan in our project about the combinatorial and algebraic study on binomial ideals arising from finite graphs.

1. BINOMIAL IDEALS ARISING FROM FINITE GRAPHS

The goal of this section is to give a combinatorial criterion for the toric ideal $I_{\mathscr{A}}$ associated with an affine semigroup ring $K[\mathscr{A}]$, where $\mathscr{A}$ is a finite set of squarefree quadratic monomials, to be generated by quadratic binomials. First, we recall fundamental material from [8] for the discussion of affine semigroup rings generated by squarefree quadratic monomials.

(1.1) Let G be a finite connected graph having no loop and no multiple edge on the vertex set $V(G) = \{1, 2, \ldots, d\}$ and $E(G) = \{e_1, e_2, \ldots, e_n\}$ be the set of edges of G. Let $K[\mathbf{t}] = K[t_1, t_2, \ldots, t_d]$ denote the polynomial ring in d variables over a field K. If $e = \{i, j\}$ is an edge of G combining $i \in V(G)$ with $j \in V(G)$, then we write $\mathbf{t}^e$ for the squarefree quadratic monomial $t_i t_j$ belonging to $K[\mathbf{t}]$. Let $K[G]$ denote the subalgebra of $K[\mathbf{t}]$ which is generated by $\mathbf{t}^{e_1}, \mathbf{t}^{e_2}, \ldots, \mathbf{t}^{e_n}$ over K. The affine semigroup ring $K[G]$ is called the *edge ring* of G. Let $K[\mathbf{x}] = K[x_1, x_2, \ldots, x_n]$ denote the polynomial ring in n variables over K and π: $K[\mathbf{x}] \to K[G]$ the surjective homomorphism of semigroup rings defined by by $\pi(x_i) = \mathbf{t}^{e_i}$ for all $1 \le i \le n$. We write I_G for the kernel of π and call I_G the *toric ideal* of G.

(1.2) A *walk of length q* of G connecting $v_1 \in V(G)$ and $v_{q+1} \in V(G)$ is a finite sequence of the form

$$\Gamma = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_q, v_{q+1}\}) \tag{1}$$

with each $\{v_k, v_{k+1}\} \in E(G)$. Such a walk Γ can be regarded as a subgraph of G in the obvious way, i.e., its vertex set $V(\Gamma)$ consists of all the vertices $v \in V(G)$ with $v = v_k$ for some $1 \le k \le q+1$ and its edge set $E(\Gamma)$ consists of all the edges $e \in E(G)$ with $e = \{v_k, v_{k+1}\}$ for some $1 \le k \le q$. An *even* (resp. *odd*) *walk* is a walk of even (resp. odd) length. A walk Γ of the form (1) is called *closed* if $v_{q+1} = v_1$.

A *cycle* is a closed walk

$$C = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_q, v_1\}) \tag{2}$$

with $v_i \neq v_j$ for all $1 \leq i < j \leq q$. A *chord* of a cycle (2) is an edge $e \in E(G)$ of the form $e = \{v_i, v_j\}$ for some $1 \leq i < j \leq q$ with $e \notin E(C)$. When a cycle (2) is even, an *even-chord* (resp. *odd-chord*) of (2) is an chord $e = \{v_i, v_j\}$ with $1 \leq i < j \leq q$ such that $j - i$ is odd (resp. even).

If $e = \{v_i, v_j\}$ and $e' = \{v_{i'}, v_{j'}\}$ are chords of a cycle (2) with $1 \leq i < j \leq q$ and with $1 \leq i' < j' \leq q$, then we say that e and e' *cross* in C if either $i < i' < j < j'$ or $i' < i < j' < j$ and if either $\{v_i, v_{i'}\}, \{v_j, v_{j'}\}$ are edges of C or $\{v_i, v_{j'}\}, \{v_j, v_{i'}\}$ are edges of C.

A *minimal* cycle of G is a cycle having no chords. If C_1 and C_2 are cycles of G having no common vertex, then a *bridge* between C_1 and C_2 is an edge $\{i, j\}$ of G with $i \in V(C_1)$ and $j \in V(C_2)$.

(1.3) Given an even closed walk

$$\Gamma = (e_{i_1}, e_{i_2}, \ldots, e_{i_{2q}})$$

of G with each $e_k \in E(G)$, we write f_Γ for the binomial

$$f_\Gamma = \prod_{k=1}^{q} x_{i_{2k-1}} - \prod_{k=1}^{q} x_{i_{2k}}$$

belonging to I_G. We often employ the abbreviated notation

$$f_\Gamma = f_\Gamma^{(+)} - f_\Gamma^{(-)},$$

where

$$f_\Gamma^{(+)} = \prod_{k=1}^{q} x_{i_{2k-1}}, \quad f_\Gamma^{(-)} = \prod_{k=1}^{q} x_{i_{2k}}.$$

Even though the following Lemma 1.1 must be an easy exercise, we give its proof for the sake of completeness.

LEMMA 1.1. *The toric ideal I_G is generated by all the binomials f_Γ, where Γ is an even closed walk of G.*

Proof. It is known, e.g., [11, Corollary 4.3] that, in general, every toric ideal is generated by binomials. Choose a binomial $f = \prod_{k=1}^{q} x_{i_k} - \prod_{k=1}^{q} x_{j_k}$ belonging to I_G with $i_k \neq j_{k'}$ for all $1 \leq k \leq q$ and for all $1 \leq k' \leq q$. Let, say, $\pi(x_{i_1}) = t_1 t_2$. Since $\pi(\prod_{k=1}^{q} x_{i_k}) = \pi(\prod_{k=1}^{q} x_{j_k})$, we have $\pi(x_{j_m}) = t_2 t_r$ for some $1 \leq m \leq q$ with $r \neq 1$. Say $m = 1$ and $r = 3$, i.e., $\pi(x_{j_1}) = t_2 t_3$. Then $\pi(x_{i_\ell}) = t_3 t_s$ for some $2 \leq \ell \leq q$ with $s \neq 2$. Repeated application of such procedure enables us to find an even closed walk, say, $\Gamma' = (e_{i_1}, e_{j_1}, e_{i_2}, e_{j_2}, \ldots, e_{i_p}, e_{j_p})$ with $f_{\Gamma'} = \prod_{k=1}^{p} x_{i_k} - \prod_{k=1}^{p} x_{j_k} \in I_G$. Since $\pi(\prod_{k=1}^{q} x_{i_k}) = \pi(\prod_{k=1}^{q} x_{j_k})$ and since $\pi(\prod_{k=1}^{p} x_{i_k}) = \pi(\prod_{k=1}^{p} x_{j_k})$, we have $\pi(\prod_{k=p+1}^{q} x_{i_k}) = \pi(\prod_{k=p+1}^{q} x_{j_k})$. Hence $\prod_{k=p+1}^{q} x_{i_k} - \prod_{k=p+1}^{q} x_{j_k}$ belongs to

I_G. Working with induction on q (≥ 2), we may assume that $\prod_{k=p+1}^{q} x_{i_k} - \prod_{k=p+1}^{q} x_{j_k}$ belongs to the ideal which is generated by all the binomials f_Γ, where Γ is an even closed walk of G. Now, we have

$$\begin{aligned} f &= \prod_{k=p+1}^{q} x_{i_k} \left(\prod_{k=1}^{p} x_{i_k} - \prod_{k=1}^{p} x_{j_k} \right) + \prod_{k=1}^{p} x_{j_k} \left(\prod_{k=p+1}^{q} x_{i_k} - \prod_{k=p+1}^{q} x_{j_k} \right) \\ &= f_{\Gamma'} \prod_{k=p+1}^{q} x_{i_k} + \prod_{k=1}^{p} x_{j_k} \left(\prod_{k=p+1}^{q} x_{i_k} - \prod_{k=p+1}^{q} x_{j_k} \right). \end{aligned}$$

Hence, the binomial f belongs to the ideal which is generated by all the binomials f_Γ, where Γ is an even closed walk of G. Thus, the toric ideal I_G is generated by all the binomials f_Γ, where Γ is an even closed walk of G as required. Q. E. D.

(1.4) We are now in the position to state a combinatorial criterion for the toric ideal I_G of G to be generated by quadratic binomials.

THEOREM 1.2. *Let G be a finite connected graph having no loop and no multiple edge. Then, the toric ideal I_G of G is generated by quadratic binomials if and only if the following conditions are satisfied*:

(i) *If C is an even cycle of G of length* ≥ 6, *then either C has an even-chord or C has three odd-chords e, e', e'' such that e and e' cross in C*;

(ii) *If C_1 and C_2 are minimal odd cycles having exactly one common vertex, then there exists an edge $\{i, j\} \notin E(C_1) \cup E(C_2)$ with $i \in V(C_1)$ and $j \in V(C_2)$*;

(iii) *If C_1 and C_2 are minimal odd cycles having no common vertex, then there exist at least two bridges between C_1 and C_2.*

The proof will be postponed to Section 3. We discuss complementary results and some examples related with Theorem 1.2. By virtue of [8], it follows immediately that every Koszul algebra generated by squarefree quadratic monomials is normal.

COROLLARY 1.3. *Let G be a finite connected graph having no loop and no multiple edge and suppose that the affine semigroup ring $K[G]$ is Koszul. Then $K[G]$ is normal.*

Proof. It is known, e.g., [8, Corollary 2.3] that $K[G]$ is normal if and only if G satisfies the following condition: If C_1 and C_2 are minimal odd cycles of G having no common vertex, then there exists a bridge between C_1 and C_2. Now, if $K[G]$ is Koszul, then the toric ideal I_G is generated by quadratic binomials. In particular, Theorem 1.2 guarantees that G satisfies the above condition for normality as desired. Q. E. D.

EXAMPLE 1.4. We give an example of a non-normal edge ring $K[G]$ whose toric ideal I_G is generated by quadratic binomials and cubic binomials. Let G be the graph on the vertex set $\{1, 2, 3, 4, 5, 6, 7\}$ with the edges $e_1 = \{1, 2\}$, $e_2 = \{2, 3\}$, $e_3 = \{3, 4\}$, $e_4 = \{1, 4\}$, $e_5 = \{4, 5\}$, $e_6 = \{4, 7\}$, $e_7 = \{6, 7\}$, $e_8 = \{5, 6\}$, $e_9 = \{1, 3\}$, and $e_{10} = \{5, 7\}$. Then I_G is not generated by quadratic binomials by Theorem 1.2 and $K[G]$ is non-normal by [8, Corollary 2.3]. The toric ideal I_G is generated by two quadratic binomials $x_1x_3 - x_2x_4$, $x_5x_7 - x_6x_8$ and one cubic binomial $x_3x_4x_{10} - x_5x_6x_9$.

EXAMPLE 1.5. We present an example of a non-normal semigroup ring generated by quadratic monomials whose toric ideal has a quadratic Gröbner basis. The affine semigroup ring $K[{t_1}^2, t_1t_2, {t_2}^2, t_2t_3, {t_3}^2]$ is non-normal; however, its toric ideal has a quadratic Gröbner basis $\{x_2^2 - x_1x_3, x_4^2 - x_3x_5\}$.

On the other hand, the result below guarantees that if an affine semigroup ring $K[\mathscr{A}]$ is generated by squarefree monomials of the same degree and if its toric ideal $I_{\mathscr{A}}$ possesses a quadratic Gröbner basis, then $K[\mathscr{A}]$ is normal.

PROPOSITION 1.6. *Let $K[\mathbf{t}] = K[t_1, t_2, \ldots, t_d]$ denote the polynomial ring in d variables over a field K with each* $\deg t_i = 1$ *and $K[\mathscr{A}]$ the subalgebra of $K[\mathbf{t}]$ generated by squarefree monomials $f_1, f_2, \ldots, f_n$ such that all the f_i's have the same degree. Suppose that the toric ideal $I_{\mathscr{A}}$ associated with $K[\mathscr{A}]$ has a quadratic Gröbner basis. Then $K[\mathscr{A}]$ is normal.*

Proof. If a binomial $f = {x_i}^2 - x_jx_k$ belongs to I_G, then ${f_i}^2 = f_jf_k$. Thus $f_i = f_j = f_k$ since f_i, f_j and f_k are squarefree. Hence, all quadratic binomials belonging to $I_{\mathscr{A}}$ are squarefree. Thus, in particular, if $I_{\mathscr{A}}$ has a quadratic Gröbner basis, then some initial ideal of $I_{\mathscr{A}}$ is squarefree. Hence, by [11, Corollary 8.9] the affine semigroup ring $K[\mathscr{A}]$ is normal. Q. E. D.

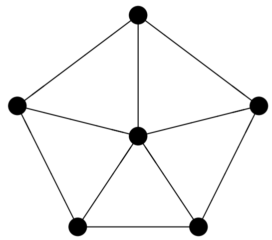
2. SOME EXAMPLES

We now present an example of a normal non-Koszul squarefree semigroup ring whose toric ideal is generated by quadratic binomials as well as an example of a non-normal Koszul squarefree semigroup ring whose toric ideal possesses no quadratic Gröbner basis.

EXAMPLE 2.1. Let G be the graph below with 6 vertices and 10 edges. Then $K[G]$ is normal by [8, Corollary 2.3] and its toric ideal I_G is generated by the quadratic binomials

$$x_4x_6 - x_5x_9, \qquad x_3x_{10} - x_4x_8, \qquad x_2x_9 - x_3x_7,$$
$$x_1x_{10} - x_5x_7, \qquad x_1x_8 - x_2x_6.$$

However, by an explicit computation with MACAULAY it turns out that $K[G]$ is non-Koszul. Thus, in particular, I_G has no quadratic Gröbner basis.



EXAMPLE 2.2. Let $K[\mathscr{A}]$ denote the affine semigroup which is generated by squarefree monomials

$$\begin{array}{cccc} t_1t_2t_3, & t_1t_3t_4, & t_1t_4t_5, & t_1t_2t_5, \\ t_2t_3t_6, & t_4t_5t_6, & t_3t_4t_7, & t_2t_5t_7. \end{array}$$

Then, its toric ideal $I_{\mathscr{A}}$ is generated by the quadratic binomials

$$x_2x_8 - x_4x_7, \qquad x_1x_6 - x_3x_5, \qquad x_1x_3 - x_2x_4.$$

Let f denote the squarefree monomial $t_2t_3t_4t_5t_6t_7$. Then f is integral over $K[\mathscr{A}]$ since

$$f^2 = (t_2t_3t_6)(t_4t_5t_6)(t_3t_4t_7)(t_2t_5t_7)$$

and f belongs to the quotient field of $K[\mathscr{A}]$ since

$$f = \frac{(t_2t_3t_6)(t_3t_4t_7)(t_1t_2t_5)}{t_1t_2t_3}.$$

However, $f \notin K[\mathscr{A}]$. Thus, the affine semigroup ring $K[\mathscr{A}]$ is non-normal. Hence, by Proposition 1.6 the toric ideal $I_{\mathscr{A}}$ has no quadratic Gröbner basis. For example,

$$\{x_3^2x_5x_8 - x_4^2x_6x_7, x_2x_8 - x_4x_7, x_2x_4x_6 - x_3^2x_5, x_1x_6 - x_3x_5, x_1x_3 - x_2x_4\}$$

is a Gröbner basis of the toric ideal $I_{\mathscr{A}}$.

We now prove that $K[\mathscr{A}]$ is Koszul. Let $K[\mathscr{A}']$ denote the subalgebra of $K[\mathscr{A}]$ generated by

$$t_1t_2t_3, \quad t_1t_3t_4, \quad t_1t_4t_5, \quad t_1t_2t_5, \quad t_2t_3t_6, \quad t_4t_5t_6, \quad t_3t_4t_7.$$

Then, the toric ideal $I_{\mathscr{A}'}$ has a quadratic Gröbner basis

$$\{x_2x_4 - x_1x_3,\ x_1x_6 - x_3x_5\}.$$

Hence $K[\mathscr{A}']$ is Koszul. Thus $K[\mathscr{A}'][x_8]$ is also Koszul. Since

$$K[\mathscr{A}] = K[\mathscr{A}'][x_8]/(x_2x_8 - x_4x_7)$$

and since the quadratic binomial $x_2x_8 - x_4x_7$ is a non-zerodivisor on $K[\mathscr{A}'][x_8]$, it follows from [1, Theorem 4] that the affine semigroup ring $K[\mathscr{A}]$ is Koszul as required.

Remark. (a) A non-Koszul monomial curve whose toric ideal is generated by quadratic binomials constructed in [10] is non-normal. Our non-Koszul squarefree semigroup ring whose toric ideal is generated by quadratic binomials discussed in Example 2.1 is normal.

(b) A Koszul monomial curve whose toric ideal has no quadratic Gröbner basis constructed in [10] is defined by 11 quadratic binomials. Our Koszul squarefree semigroup ring whose toric ideal possesses no quadratic Gröbner basis discussed in Example 2.2 is defined by three quadratic binomials. It is not difficult to show that every affine semigroup ring whose toric ideal is generated by at most two quadratic binomials is Koszul. It might be of interest to find an edge ring $K[G]$ such that (i) $K[G]$ is normal, (ii) $K[G]$ is Koszul, and (iii) I_G has no quadratic Gröbner basis.

3. PROOF OF THEOREM 1.2

The purpose of the present section is to give a proof of Theorem 1.2. For a while we keep the same notation as in Section 1. Every graph G to be studied is a finite connected graph having no loop and no multiple edge.

We say that an even closed walk $\Gamma = (e_{i_1}, e_{i_2}, \ldots, e_{i_{2q}})$ of G is *primitive* if there exists no even closed walk of G of the form $(e_{j_1}, e_{j_2}, \ldots, e_{j_{2p}})$ with $1 \leq p < q$ such that each j_{2k-1} belongs to $\{i_1, i_3, \ldots, i_{2q-1}\}$, each j_{2k} belongs to $\{i_2, i_4, \ldots, i_{2q}\}$, and $j_{2k-1} \neq j_{2\ell}$ for all $1 \leq k \leq p$ and for all $1 \leq \ell \leq p$.

LEMMA 3.1. *The toric ideal I_G of G is generated by the binomials f_Γ, where Γ is a primitive even closed walk of G.*

Proof. If an even closed walk Γ of G of length $2q$ is not primitive, then we can find an even closed walk Γ' of G of length $< 2q$ such that $f_{\Gamma'}^{(+)}$ divides $f_\Gamma^{(+)}$ and $f_{\Gamma'}^{(-)}$ divides $f_\Gamma^{(-)}$. It follows that the binomial $g = f_\Gamma^{(+)}/f_{\Gamma'}^{(+)} - f_\Gamma^{(-)}/f_{\Gamma'}^{(-)}$ belongs to I_G. Since $f_\Gamma = g f_{\Gamma'}^{(+)} + f_{\Gamma'} f_\Gamma^{(-)}/f_{\Gamma'}^{(-)}$, Lemma 1.1 guarantees that the toric ideal I_G is generated by the binomials associated with primitive even closed walks of G as required. Q. E. D.

It follows from the proof of Lemma 1.1 that the set of all binomials f_Γ, where Γ is a primitive even closed walk of G, coincides with the Graver basis [11, p. 35] of I_G.

LEMMA 3.2. *A primitive even closed walk* Γ *of* G *is one of the following*:

(i) Γ *is an even cycle of* G;

(ii) $\Gamma = (C_1, C_2)$, *where* C_1 *and* C_2 *are odd cycles of* G *having exactly one common vertex*;

(iii) $\Gamma = (C_1, \Gamma_1, C_2, \Gamma_2)$, *where* C_1 *and* C_2 *are odd cycles of* G *having no common vertex and where* Γ_1 *and* Γ_2 *are walks of* G *both of which combine a vertex* v_1 *of* C_1 *and a vertex* v_2 *of* C_2.

Proof. If Γ is a cycle of G, then Γ must be an even cycle. Let us assume that Γ is not a cycle. We then have $\Gamma = (\Gamma_1, \Gamma_2)$, where Γ_1 and Γ_2 are closed walks of G having a common vertex $v \in V(\Gamma)$. Since Γ is primitive, both Γ_1 and Γ_2 must be odd. If Γ_1 and Γ_2 have a common vertex w ($\neq v$), then it follows that Γ is not primitive. Hence, Γ_1 and Γ_2 have exactly one common vertex v. If both Γ_1 and Γ_2 are odd cycles, then Γ is of the form required in (ii). If Γ_1 is not a cycle, then $\Gamma_1 = (\Gamma_3, \Gamma_4, \Gamma_5)$, where Γ_3 is a walk of G connecting v and a vertex of Γ_1, say v', where Γ_4 is a closed walk of G and where Γ_5 is a walk of G connecting v' and v. Since Γ is primitive, we have $v \neq v'$. Moreover, Γ_4 must be odd with $v \notin V(\Gamma_4)$. If Γ_4 is a cycle, then Γ is of the form required in (iii). While, if Γ_4 is not a cycle, then by repeating the above technique Γ turns out to be of the desired form in (iii). Q. E. D.

If W is a subset of the vertex set $V(G)$ of G, then the *induced subgraph* of G on W is the subgraph of G whose vertex set is W and whose edge set is $\{\{v_1, v_2\} \in E(G);\ v_1, v_2 \in W, v_1 \neq v_2\}$.

An even closed walk Γ of G is called *fundamental* if every even closed walk Γ' of the induced subgraph on $V(\Gamma)$ satisfies either $f_\Gamma = f_{\Gamma'}$ or $f_\Gamma = -f_{\Gamma'}$.

LEMMA 3.3. *Let* Γ *be a fundamental even closed walk of* G *and suppose that the toric ideal* I_G *is generated by* $f_{\Gamma_1}, f_{\Gamma_2}, \ldots, f_{\Gamma_s}$, *where each* Γ_i *is an even closed walk of* G. *Then*, *either* $f_\Gamma = f_{\Gamma_i}$ *or* $f_\Gamma = -f_{\Gamma_i}$ *for some* $1 \leq i \leq s$.

Proof. Since $f_\Gamma \in I_G$ we can choose f_{Γ_i} such that $f_{\Gamma_i}^{(+)}$ divides either $f_\Gamma^{(+)}$ or $f_\Gamma^{(-)}$. It then follows that each vertex of Γ_i must belong to $V(\Gamma)$. Hence, Γ_i is an even closed walk of the induced subgraph on $V(\Gamma)$. Thus f_Γ coincides with f_{Γ_i} as required. Q. E. D.

If e is an edge of G, then we write x_e for the variable of $K[\mathbf{x}]$ with $\pi(x_e) = \mathbf{t}^e \in K[G]$.

Proof of Theorem 1.2. First, to show the "only if" part of Theorem 1.2 suppose that the toric ideal I_G of G is generated by quadratic binomials.

(i) Let C be an even cycle of length ≥ 6 of G. Since $f_C \in I_G$ and since I_G is generated by quadratic binomials, we can find two quadratic binomials f_{C_1} and f_{C_2}, where both C_1 and C_2 are even cycles of length 4, such that $f_{C_1}^{(+)}$ divides $f_C^{(+)}$ and $f_{C_2}^{(+)}$ divides $f_C^{(-)}$. It then follows that C has either an even-chord or two odd-chords which cross in C. If C has exactly two chords e and e' such that e and e' are odd-chords which cross in C and if $C' = (e, e_i, e', e_j)$ with $e_i, e_j \in E(C)$ is a cycle of length 4, then we have either $f_{C'} = f_{C''}$ or $f_{C'} = -f_{C''}$ for all even cycles C'' of length 4 of the induced subgraph on $V(C)$. Hence, it is impossible to find two even cycles C_1 and C_2 required above.

(ii) Let C_1 and C_2 be minimal odd cycles of G having exactly one common vertex and suppose that there exists no edge $\{i, j\} \notin E(C_1) \cup E(C_2)$ with $i \in V(C_1)$ and $j \in V(C_2)$. Since the even closed walk $\Gamma = (C_1, C_2)$ is fundamental, by Lemma 3.3 the toric ideal I_G cannot be generated by quadratic binomials.

(iii) Let C_1 and C_2 be minimal odd cycles of G having no common vertex and suppose that there exists no bridge between C_1 and C_2. Since G is connected, there exists a walk $\Gamma_1 = (\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{t-1}, v_t\})$ of length $t \geq 2$ with $v_0 \in V(C_1)$ and $v_t \in V(C_2)$. We may assume that t is the minimum length of a walk connecting a vertex of C_1 and a vertex of C_2. Let Γ denote the even closed walk $(C_1, \Gamma_1, C_2, \Gamma_1^\star)$, where $\Gamma_1^\star$ is a walk $(\{v_t, v_{t-1}\}, \{v_{t-1}, v_{t-2}\}, \ldots, \{v_1, v_0\})$. Let $G_{V(\Gamma)}$ denote the induced subgraph of G on $V(\Gamma)$. If $G_{V(\Gamma)} = \Gamma$, then by Lemma 3.3 a contradiction arises since the degree of the binomial f_Γ is at least $t + 3 \geq 5$. Thus $G_{V(\Gamma)} \neq \Gamma$ and we can find an edge $e \in E(G_{V(\Gamma)}) \setminus E(\Gamma)$. Since C_1 and C_2 are minimal odd cycles of G having no common vertex, since there exists no bridge between C_1 and C_2, and since t is the minimum length of a walk connecting a vertex of C_1 and a vertex of C_2, it follows that either $e = \{i, v_1\}$ with $i \in V(C_1)$ or $e = \{v_{t-1}, j\}$ with $j \in V(C_2)$. Let us assume that $e = \{i, v_1\}$ with $i \in V(C_1)$. We then find an odd cycle C_3 ($\neq C_1$) with $E(C_3) \subset E(C_1) \cup \{\{i, v_1\}, \{v_0, v_1\}\}$ and choose a minimal odd cycle C_4 with $V(C_4) \subset V(C_3)$. Note that v_1 belongs to $V(C_4)$ since C_1 is a minimal cycle. Let Γ' be the even closed walk $(C_4, \Gamma_2, C_2, \Gamma_2^\star)$, where $\Gamma_2 = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{t-1}, v_t\})$ and $\Gamma_2^\star = (\{v_t, v_{t-1}\}, \{v_{t-1}, v_{t-2}\}, \ldots, \{v_2, v_1\})$. Let $G_{V(\Gamma')}$ denote the induced subgraph of G on $V(\Gamma')$. If $G_{V(\Gamma')} = \Gamma'$, then by Lemma 3.3 a contradiction arises again since the degree of the binomial $f_{\Gamma'}$ is at least $t + 2 \geq 4$. Thus $G_{V(\Gamma')} \neq \Gamma'$ and we can find an edge $e' = \{v_{t-1}, j\} \in E(G_{V(\Gamma')}) \setminus E(\Gamma')$ with $j \in V(C_2)$. We then find an odd cycle C_5 ($\neq C_2$) with $E(C_5) \subset E(C_2) \cup \{\{v_{t-1}, j\}, \{v_{t-1}, v_t\}\}$ and choose a minimal odd cycle C_6 with $V(C_6) \subset V(C_5)$. Note that v_{t-1} belongs to $V(C_6)$ since C_6 is a minimal cycle. Let Γ'' denote the even closed walk $(C_4, \Gamma_3, C_6, \Gamma_3^\star)$, where $\Gamma_3 = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{t-2}, v_{t-1}\})$ and $\Gamma_3^\star = (\{v_{t-1}, v_{t-2}\}, \{v_{t-2}, v_{t-3}\}, \ldots, \{v_2, v_1\})$. It follows that Γ_3 is fundamental and by Lemma 3.3 I_G cannot be generated by quadratic binomials since the degree of f_{Γ_3} is at least $t+1 \geq 3$. Such a contradiction guarantees that there exists a bridge between C_1 and C_2. Now, if there exists exactly one bridge $b \in E(G)$ between C_1 and C_2, then the even closed walk $\Gamma = (C_1, b, C_2, b)$ is fundamental of length ≥ 8. By Lemma 3.3 again, the toric ideal I_G cannot be generated by quadratic binomials. Hence, there exist at least two bridges between C_1 and C_2 as desired.

Second, in order to see why the "if" part of Theorem 1.2 is true, by virtue of Lemma 3.1, given a primitive even closed walk Γ of G of length $2q \geq 6$, we must prove that the binomial f_Γ belongs to the ideal $(I_G)_{<q}$ $(\subset K[\mathbf{x}])$ which is generated by the binomials of degree $< q$ belonging to I_G.

(**First Step**) Let Γ be a primitive even closed walk of G of length $2q \geq 6$ which is of the form stated in (i) of Lemma 3.2; i.e., Γ is an even cycle $C = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{2q}, v_1\})$.

(a) Suppose that C has an even-chord $e = \{v_1, v_{2t}\}$ with $2 \leq t < q$. Let C_1 be the even cycle $(e, \{v_{2t}, v_{2t+1}\}, \{v_{2t+1}, v_{2t+2}\}, \ldots, \{v_{2q-1}, v_{2q}\}, \{v_{2q}, v_1\})$ and C_2 the even cycle $(e, \{v_{2t}, v_{2t-1}\}, \{v_{2t-1}, v_{2t-2}\}, \ldots, \{v_2, v_1\})$. Then $f_C = gf_{C_1} - hf_{C_2} \in (I_G)_{<q}$, where $g = f_{C_2}^{(+)}/x_e$ and $h = f_{C_1}^{(+)}/x_e$, as desired.

(b) Suppose that C has no even-chord and that C has three odd-chords e, e', and e'' such that e and e' cross in C. Let $e = \{v_1, v_t\}$ and $e' = \{v_2, v_{t+1}\}$ with $3 \leq t \leq 2q-1$. Let $\Gamma = (\{v_t, v_{t-1}\}, \{v_{t-1}, v_{t-2}\}, \ldots, \{v_3, v_2\})$ and $\Gamma' = (\{v_{t+1}, v_{t+2}\}, \{v_{t+2}, v_{t+3}\}, \ldots, \{v_{2q-1}, v_{2q}\}, \{v_{2q}, v_1\})$. If C_1 is the even cycle (e, Γ, e', Γ') and if C_2 is the even cycle $(e, \{v_t, v_{t+1}\}, e', \{v_2, v_1\})$, then $f_C = f_{C_1} - hf_{C_2}$ with $h = f_{C_1}^{(+)}/x_e x_{e'}$. Note that the binomial f_{C_1} is of degree q and f_{C_2} is a quadratic binomial. Let $e'' = \{v_i, v_j\}$, $S = \{v_1, v_{t+1}, v_{t+2}, \ldots, v_{2q}\}$, and $T = \{i_2, i_3, \ldots, i_t\}$. If $v_i \in S$ and $v_j \in T$, then e'' is an even-chord of C_1. Hence, $f_C \in (I_G)_{<q}$ as desired.

Let us assume that both v_i and v_j belong to T with $2 \leq i < j \leq t$. We choose a minimal odd cycle C_3 with $V(C_3) \subset S \cup \{v_t\}$ and a minimal odd cycle C_4 with $V(C_4) \subset S \cup \{v_2\}$. For a while, suppose that the cycle C has no chord $\{v_{i'}, v_{j'}\}$ with $2 \leq i' < j' \leq t$ such that either $i' = 2$ or $j' = t$. Note that $\{v_2, v_t\}$ cannot be a chord of C since C has no even-chord. Let C_5 denote a minimal odd cycle with $V(C_5) \subset \{v_i, v_{i+1}, \ldots, v_j\}$. Since C_3 and C_5 are odd cycles of G having no common vertex, we can find a bridge $b = \{v_k, v_\ell\}$ between C_3 and C_5. The bridge b must be an odd-chord of C with $v_k \in S$ and $v_\ell \in T$. On the other hand, suppose that the cycle C has a chord $\{v_2, v_{j'}\}$ with $2 < j' < t$ and that C has no chord $\{v_2, v_{j''}\}$ with $2 <$

$j'' < j'$. We choose a minimal odd cycle C_6 with $V(C_6) \subset \{v_2, v_3, \ldots, v_{j'}\}$. If C_4 and C_6 have exactly one common vertex ($= v_2$), then there exists a bridge $\{v_k, v_\ell\}$ between C_4 and C_6 with $v_k \in S$ and $v_\ell \in T$. If C_4 and C_6 have no common vertex, then there exist at least two bridges between C_4 and C_6. It then follows that one of the bridges between C_4 and C_6 is of the form $\{v_k, v_\ell\}$ with $v_k \in S$ and $v_\ell \in T$.

(**Second Step**) Let Γ be a primitive even closed walk of G of length ≥ 6 which is of the form stated in (ii) of Lemma 3.2, i.e., $\Gamma = (C_1, C_2)$, where $C_1 = (\{w, v_1\}, \{v_1, v_2\}, \ldots, \{v_{2s-1}, v_{2s}\}, \{v_{2s}, w\})$ is an odd cycle and $C_2 = (\{w, v'_1\}, \{v'_1, v'_2\}, \ldots, \{v'_{2t-1}, v'_{2t}\}, \{v'_{2t}, w\})$ is an odd cycle having exactly one common vertex w.

(a) Suppose that there exists an edge $e = \{v_i, v'_j\}$ of G with $1 \leq i \leq 2s$ and with $1 \leq j \leq 2t$. Let us assume that both i and j are even. Let Γ_1 denote the even closed walk

$$(\{w, v_1\}, \{v_1, v_2\}, \ldots, \{v_{i-1}, v_i\}, e, \{v'_j, v'_{j+1}\}, \ldots, \{v'_{2t-1}, v'_{2t}\}, \{v'_{2t}, w\})$$

and Γ_2 the even closed walk

$$(\{w, v'_1\}, \{v'_1, v'_2\}, \ldots, \{v'_{j-1}, v'_j\}, e, \{v_i, v_{i+1}\}, \ldots, \{v_{2s-1}, v_{2s}\}, \{v_{2s}, w\}).$$

Then $f_\Gamma = g f_{\Gamma_1} - h f_{\Gamma_2}$, where $g = f_{\Gamma_2}^{(+)}/x_e$ and $h = f_{\Gamma_1}^{(+)}/x_e$.

(b) Suppose that C_1 is not minimal and that the edge $\{v_i, v'_j\}$ does not belong to $E(G)$ for all $1 \leq i \leq 2s$ and for all $1 \leq j \leq 2t$. Let e be a chord of C_1. If $e = \{v_i, w\}$, where i is even, and if

$$\Gamma_1 = (\{w, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{i-1}, v_i\}, e, C_2)$$

and

$$\Gamma_2 = (e, \{v_i, v_{i+1}\}, \ldots, \{v_{2s-1}, v_{2s}\}, \{v_{2s}, w\}),$$

then $f_\Gamma = g f_{\Gamma_1} - h f_{\Gamma_2}$, where $g = f_{\Gamma_2}^{(+)}/x_e$ and $h = f_{\Gamma_1}^{(+)}/x_e$.

If $e = \{v_i, v_j\}$ with $1 \leq i < j \leq 2s$, then we write C_3 for the odd cycle of G with $e \in E(C_3) \subset E(C_1) \cup \{e\}$ and C_4 for the even cycle of G with $e \in E(C_4) \subset E(C_1) \cup \{e\}$. If $w \notin V(C_3)$ then, since C_2 and C_3 have no common vertex, there exist at least two bridges between C_2 and C_3. Thus, in particular, we can find a chord $e_1 = \{v_i, w\}$ of C_1 since $\{v_i, v'_j\} \notin E(G)$ for all $1 \leq i \leq 2s$ and for all $1 \leq j \leq 2t$. Suppose that $w \in V(C_3)$. Let Γ_1 denote the even closed walk (C_2, C_3). Assuming that x_e divides both $f_{\Gamma_1}^{(+)}$ and $f_{C_4}^{(+)}$, let $g = f_{\Gamma_1}^{(+)}/x_e$ and $h = f_{C_4}^{(+)}/x_e$. Then, either $f_\Gamma = g f_{C_4} - h f_{\Gamma_1}$ or $f_\Gamma = -g f_{C_4} + h f_{\Gamma_1}$.

(**Third Step**) Let Γ be a primitive even closed walk of G of length ≥ 6 which is of the form stated in (iii) of Lemma 3.2, i.e., $\Gamma = (C_1, \Gamma_1, C_2, \Gamma_2)$, where $C_1 = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{2s}, v_{2s+1}\}, \{v_{2s+1}, v_1\})$ is an odd cycle and $C_2 = (\{v'_1, v'_2\}, \{v'_2, v'_3\}, \ldots, \{v'_{2t}, v'_{2t+1}\}, \{v'_{2t+1}, v'_1\})$ is an odd cycle having no common vertex, and where Γ_1 and Γ_2 are walks of G both of which combine v_1 and v'_1. Since there exist at least two bridges between C_1 and C_2, we can find a bridge $e = \{v_i, v'_j\}$ between C_1 and C_2 with, say, $j \neq 1$. Since Γ is an even closed walk, the sum of the length of Γ_1 and the length of Γ_2 must be even. When both the length of Γ_1 and the length of Γ_2 are odd, we assume that both i and j are odd. When both the length of Γ_1 and the length of Γ_2 are even, we assume that i is odd and j is even. Let Γ_3 denote the even closed walk

$$(e, \{v'_j, v'_{j-1}\}, \ldots, \{v'_2, v'_1\}, \Gamma_1, \{v_1, v_2\}, \ldots, \{v_{i-1}, v_i\})$$

and Γ_4 the even closed walk

$$(e, \{v'_j, v'_{j+1}\}, \ldots, \{v'_{2t+1}, v'_1\}, \Gamma_2, \{v_1, v_{2s+1}\}, \ldots, \{v_{i+1}, v_i\}).$$

Then $f_\Gamma = g f_{\Gamma_1} - h f_{\Gamma_2}$, where $g = f^{(+)}_{\Gamma_2}/x_e$ and $h = f^{(+)}_{\Gamma_1}/x_e$. Q. E. D.

4. EDGE RINGS WITH 2-LINEAR RESOLUTIONS

We discuss the problem of finding the finite connected graphs G for which the edge ring $K[G]$ has 2-linear resolution. Let, as before, G be a finite connected graph having no loop and no multiple edge on the vertex set $V(G) = \{1, 2, \ldots, d\}$ and $E(G) = \{e_1, e_2, \ldots, e_n\}$ be the set of edges of G. If $e = \{i, j\}$ is an edge of G joining $i \in V(G)$ with $j \in V(G)$, then we define $\rho(e) \in \mathbf{R}^d$ by $\rho(e) = \mathbf{e}_i + \mathbf{e}_j$. Here $\mathbf{e}_i$ is the ith unit coordinate vector in $\mathbf{R}^d$. We write $\mathscr{P}_G \subset \mathbf{R}^d$ for the convex hull of the finite set $\{\rho(e); e \in E(G)\} \subset \mathbf{R}^d$ and call $\mathscr{P}_G$ the *edge polytope* of G. The edge polytope $\mathscr{P}_G$ of G is called *normal* if the edge ring $K[G]$ is normal. Let $\delta(\mathscr{P}_G)$ denote the normalized volume (e.g., [11, p. 36]) of $\mathscr{P}_G$.

LEMMA 4.1. *If the edge ring $K[G]$ of G is Cohen–Macaulay, then $\delta(\mathscr{P}_G) \geq n - (d - \varepsilon(G)) + 1$, where $\varepsilon(G) = 1$ if G is bipartite and $\varepsilon(G) = 0$ if G is non-bipartite.*

Proof. Since the Krull-dimension of $K[G]$ is equal to $d - \varepsilon(G)$ and since $K[G]$ is Cohen–Macaulay, it follows that the Hilbert series of $K[G]$ is

$$F(K[G], \lambda) = \frac{h_0 + h_1\lambda + \cdots + h_{d-\varepsilon(G)-1}\lambda^{d-\varepsilon(G)-1}}{(1-\lambda)^{d-\varepsilon(G)}},$$

where $h_0 = 1$, $h_1 = n - (d - \varepsilon(G))$, and each $h_i \geq 0$. Since the normalized volume $\delta(\mathscr{P}_G)$ of $\mathscr{P}_G$ coincides with $h_0 + h_1 + \cdots + h_{d-\varepsilon(G)-1}$, we have $\delta(\mathscr{P}_G) \geq n - (d - \varepsilon(G)) + 1$ as required. Q. E. D.

We say that the edge polytope $\mathscr{P}_G$ of G is *of minimal volume* if $\mathscr{P}_G$ is normal and its normalized volume $\delta(\mathscr{P}_G)$ is equal to $n - (d - \varepsilon(G)) + 1$. For example, if G is either a tree (i.e., a connected graph having no cycle) or an odd cycle, then $\mathscr{P}_G$ is of minimal volume with $\delta(\mathscr{P}_G) = 1$. While, if G is an even cycle of length $2q \geq 6$, then $\delta(\mathscr{P}_G) = q$. Hence, its edge polytope is of minimal volume if and only if $q = 2$. Moreover, the edge polytope of the complete graph with d (≥ 2) vertices is of minimal volume if and only if either $d = 2$ or $d = 3$. Let $G_{(p,q)}$ denote the complete bipartite (p, q)-graph with $2 \leq p \leq q$; i.e., $G_{(p,q)}$ is the graph on the vertex set $\{1, 2, \ldots, p, 1', 2', \ldots, q'\}$ with the edge set $\{\{i, j'\}; 1 \leq i \leq p, 1 \leq j \leq q\}$. Then, the edge polytope of $G_{(p,q)}$ is of minimal volume if and only if $p = 2$.

LEMMA 4.2. *The edge ring $K[G]$ of G has 2-linear resolution if and only if the edge polytope $\mathscr{P}_G$ is of minimal volume.*

Proof. If the edge ring $K[G]$ is normal, then $K[G]$ is Cohen–Macaulay. If $K[G]$ is Cohen–Macaulay, then it follows from, e.g., [3, Exercise 4.1.17] that $K[G]$ has 2-linear resolution if and only if the normalized volume $\delta(\mathscr{P}_G)$ of the edge polytope $\mathscr{P}_G$ is equal to $n - (d - \varepsilon(G)) + 1$. Now, by virtue of Theorem 1.2 the edge ring $K[G]$ must be normal if I_G is generated by quadratic binomials, in particular, if $K[G]$ has 2-linear resolution. Q. E. D.

LEMMA 4.3. *Let G' be a connected subgraph of G. If the edge polytope $\mathscr{P}_G$ of G is of minimal volume and if G' is a connected subgraph of G whose edge polytope $\mathscr{P}_{G'}$ is normal, then $\mathscr{P}_{G'}$ is of minimal volume.*

Proof. First, we choose a sequence $G_0, G_1, \ldots, G_m$ of subgraphs of G such that (i) $G_0 = G'$, (ii) $G_m = G$, and (iii) each G_i is obtained by adding one edge to G_{i-1}. Let d_i denote the number of vertices of G_i and n_i the number of edges of G_i. It then follows that $n_i - (d_i - \varepsilon(G_i)) + 1$ is equal to or one more than $n_{i-1} - (d_{i-1} - \varepsilon(G_{i-1})) + 1$ and that $\delta(\mathscr{P}_{G_i}) \geq \delta(\mathscr{P}_{G_{i-1}})$. Moreover, $\delta(\mathscr{P}_{G_i}) > \delta(\mathscr{P}_{G_{i-1}})$ if $n_i - (d_i - \varepsilon(G_i)) + 1 = (n_{i-1} - (d_{i-1} - \varepsilon(G_{i-1})) + 1) + 1$. Hence, if $\mathscr{P}_{G'}$ is not of minimal volume, then the edge polytope $\mathscr{P}_G$ of G cannot be of minimal volume. Q. E. D.

LEMMA 4.4. *If the edge polytope $\mathscr{P}_G$ of G is of minimal volume, then none of the following can be a subgraph of G*: (a) *the even cycle of length* 6, (b) *the complete graph with* 4 *vertices*, (c) $C \cup C'$, *where C and C' are odd cycles of G of length* 3 *having exactly one common vertex*, *or* (d) $C \cup C'$, *where C and C' are even cycles of G of length* 4 *having at most one common vertex*.

Proof. The edge polytope of each (a), (b), and (c) is normal but cannot be of minimal volume. Let us assume that G has a subgraph $C \cup C'$, where C and C' are even cycles of G of length 4 having at most one common vertex. We then find a subgraph G' of G such that the cycles of G' are only C and C'. The edge polytope of G' is normal; however, it cannot be of minimal volume. Hence, by Lemma 4.3, none of the graphs (a), (b), (c), and (d) can be a subgraph of G as desired. Q. E. D.

LEMMA 4.5. *Let C and C' be even cycles of G of length* 4 *and suppose that the edge polytope $\mathcal{P}_G$ of G is of minimal volume. Then, either* (i) *C and C' have exactly two common vertices with no common edge, or* (ii) *C and C' have exactly three common vertices and exactly two common edges.*

Proof. It follows from Lemma 4.4 that C and C' have at least two and at most three common vertices. If C and C' have exactly two common vertices and one common edge, then the cycle of length 6 must be a subgraph of G. If C and C' have exactly three common vertices and exactly one common edge, then G must contain a subgraph which consists of the two odd cycles of length 3 having exactly one common vertex. Hence, by Lemma 4.4 again we have either (i) or (ii) as required. Q. E. D.

We come to the main result on the edge rings with 2-linear resolutions.

THEOREM 4.6. *Let G be a finite connected graph having no loop and no multiple edge with d vertices and with n edges. Then, the edge ring $K[G]$ of G has* 2*-linear resolution if and only if $K[G]$ is isomorphic to the polynomial ring in $n-2\delta$ variables over the edge ring $K[G_{(2,\delta)}]$ of the complete bipartite* $(2, \delta)$*-graph $G_{(2,\delta)}$.* (*It then follows that δ coincides with the normalized volume of the edge polytope $\mathcal{P}_G$ of G.*)

Proof. Thanks to Lemma 4.2, the edge ring $K[G_{(2,\delta)}]$ of the complete bipartite $(2, \delta)$-graph $G_{(2,\delta)}$ has 2-linear resolution. Hence, if $K[G]$ is isomorphic to the polynomial ring in $n-2\delta$ variables over $K[G_{(2,\delta)}]$, then $K[G]$ has 2-linear resolution. Suppose that the edge ring $K[G]$ of G has 2-linear resolution. Let δ denote the largest integer for which the complete bipartite $(2, \delta)$-graph $G_{(2,\delta)}$ is contained in G and choose a subgraph G' of G with $G' = G_{(2,\delta)}$. Then, Lemma 4.5 guarantees that any even cycle of G of length 4 are contained in G'. Let us assume that the edges of G' are $e_1, e_2, \ldots, e_{2\delta}$. Since the toric ideal I_G is generated by all binomials associated with even cycles of G of length 4 and since any even cycles of G of length 4 are contained in G', it follows that $I_G = I_{G'}K[x_{2\delta+1}, x_{2\delta+2}, \ldots, x_n]$ and that $K[G] = K[G'][x_{2\delta+1}, x_{2\delta+2}, \ldots, x_n]$ as desired. Q. E. D.

5. SIMPLE EDGE POLYTOPES

In the final section, we discuss the edge polytope and the edge ring of a finite graph allowing loops (and having no multiple edges) studied in [8]. If G has a loop, then $K[G]$ is not necessarily normal even if I_G is generated by quadratic binomials. See Example 1.5.

A convex polytope $\mathscr{P}$ of dimension d is called *simple* if each vertex of $\mathscr{P}$ belongs to exactly d edges of $\mathscr{P}$.

THEOREM 5.1. *Let G be a finite connected graph allowing loops and having no multiple edge. If the edge ring $K[G]$ is normal and if the edge polytope $\mathscr{P}_G$ is simple, then the toric ideal I_G is generated by quadratic binomials.*

However, even though $\mathscr{P}_G$ is simple, I_G is not necessarily generated by quadratic binomials. For example, if G is a finite graph with $V(G) = \{1, 2, 3\}$ and with $E(G) = \{\{1, 1\}, \{1, 2\}, \{2, 3\}, \{3, 3\}\}$, then $\mathscr{P}_G$ is simple, $K[G]$ is not normal, and I_G is generated by one cubic binomial.

LEMMA 5.2. *Let G' be a connected induced subgraph of G. If $\mathscr{P}_G$ is simple, then $\mathscr{P}_{G'}$ is also simple.*

Proof. In general, all faces of a simple polytope are also simple. Since G' is a connected induced subgraph of G, the edge polytope $\mathscr{P}_{G'}$ is a face of $\mathscr{P}_G$. Thus, $\mathscr{P}_{G'}$ is a simple polytope as required. Q. E. D.

LEMMA 5.3. *Let G be a finite graph having no loop and no multiple edge. Let $e = \{i, j\}$ and $f = \{k, \ell\}$ be edges of G with $e \neq f$ and let $[\rho(e), \rho(f)]$ denote the convex hull of $\{\rho(e), \rho(f)\}$. Then, the segment $[\rho(e), \rho(e')]$ is a face of $\mathscr{P}_G$ if and only if the induced subgraph of G on $\{i, j\} \cup \{k, \ell\}$ contains no cycle of length 4. In particular, if e and f possess exactly one common vertex, then $[\rho(e), \rho(e')]$ is a face of $\mathscr{P}_G$.*

Proof. Let G' denote the induced subgraph of G on $\{i, j\} \cup \{k, \ell\}$ and $\mathscr{F} = \mathscr{P}_{G'}$. Since $\mathscr{F}$ is a face of $\mathscr{P}_G$, the segment $[\rho(e), \rho(e')]$ is a face of $\mathscr{P}_G$ if and only if $[\rho(e), \rho(e')]$ is a face of $\mathscr{F}$. If e and f have exactly one common vertex, then $\mathscr{F}$ is a simplex and $[\rho(e), \rho(e')]$ is a face of $\mathscr{F}$. If e and f have no common vertex, say $e = \{1, 2\}$ and $f = \{3, 4\}$, then $\mathscr{F}$ may be regarded as a subpolytope of the convex hull of $\{(1, 1, 0), (1, 0, 1), (1, 0, 0), (0, 1, 1), (0, 1, 0), (0, 0, 1)\}$ in $\mathbf{R}^3$. Then, $[(1, 1, 0), (0, 0, 1)]$ is a face of $\mathscr{F}$ if and only if $\mathscr{F}$ is a simplex. Moreover, $\mathscr{F}$ is a simplex if and only if G' contain no cycle of length 4. Hence, the segment $[\rho(e), \rho(e')]$ is a face of $\mathscr{P}_G$ if and only if G' contains no cycle of length 4 as desired. Q. E. D.

COROLLARY 5.4. *If G has no loop with $I_G \neq (0)$, then the edge polytope $\mathscr{P}_G$ is a simple polytope if and only if G is a complete bipartite graph.*

Proof. First, let us assume that G is a complete bipartite (p, q)-graph. Then $\dim \mathscr{P}_G = p + q - 2$. Moreover, Lemma 5.3 guarantees that $[\rho(e), \rho(e')]$ with $e, e' \in E(G)$ is a 1-face of $\mathscr{P}_G$ if and only if e and e' possess exactly one common vertex of G. Thus, $\mathscr{P}_G$ is a simple polytope.

Second, suppose that $\mathscr{P}_G$ is simple. By Lemma 5.3 again, if G possesses no cycle of length 4, then $\mathscr{P}_G$ is simple if and only if $\mathscr{P}_G$ is a simplex. Moreover, $\mathscr{P}_G$ is a simplex if and only if either G is a tree or G has exactly one odd cycle and it is a unique cycle of G. Thus, $\mathscr{P}_G$ is a simplex if and only if $I_G = (0)$. Hence, G has a cycle C of length 4 since $I_G \neq (0)$. It then follows from Lemma 5.2 that C has no chord. We now choose a complete bipartite subgraph Γ of G with $C \subset \Gamma$. Let $V(\Gamma) = V_1 \cup V_2$ be the partition of $V(\Gamma)$. If $\Gamma \neq G$, then we can find an edge $\{i, j\} \in E(G) \setminus E(\Gamma)$ with $i \in V(\Gamma)$, say $i \in V_1$. Then, $j \notin V(\Gamma)$ since every cycle of G of length 4 has no chord. Let $i' \in V_1$ with $i' \neq i$ and C' a cycle in Γ of length 4 with $i, i' \in V(C')$, say $V(C') = \{i, i', k, k'\}$ with $k, k' \in V_2$ and $k \neq k'$. Let G' denote the induced subgraph of G on $\{i, i', k, k', j\}$. It then follows from Lemma 5.2 again that $E(G') = E(C') \cup \{\{i, j\}, \{i', j\}\}$. Thus, the induced subgraph Γ' of G on $V(\Gamma) \cup \{j\}$ is a complete bipartite graph with $\{i, j\} \in E(\Gamma')$. Repeated application of such technique enables us to see that G itself is a complete bipartite graph. Q. E. D.

LEMMA 5.5. *Suppose that $K[G]$ is normal and $\mathscr{P}_G$ is simple. If $\mathscr{P}_G$ is not a simplex and if G has a loop at $i \in V(G)$, then all edges $\{i, j\}$ with $j \in V(G) \setminus \{i\}$ belong to G.*

Proof. Let W denote the subset of $V(G)$ consisting of all vertices j with $\{i, j\} \in E(G)$ and G' the induced subgraph of G on W. Let e be a loop at $i \in V(G)$. In particular, $i \in W$ and $e \in G'$. Since $K[G]$ is normal, all vertices j belong to W if G has a loop at j. If $f \in E(G)$, then $[\rho(e), \rho(f)]$ is a face of $\mathscr{P}_G$ if and only if either (i) f is a loop, or (ii) $f = \{i, k\}$ with $k \neq i$ such that G has no loop at k, or (iii) $f = \{k, \ell\}$ with $k \neq i$ and $\ell \neq i$ such that either $\{i, k\} \notin E(G)$ or $\{i, \ell\} \notin E(G)$. If $W \neq V(G)$, then the number of edges e' of G with $e' \notin E(G')$ is equal to the number of vertices i' of G with $i' \notin W$. Since $\mathscr{P}_G$ is not a simplex, we then find an edge $\{j, j'\}$ of G with $j, j' \in W \setminus \{i\}$ and $j \neq j'$. We choose an edge $f = \{j'', m\}$ of G with $j'' \in W$ and $m \notin W$ and write G'' for the induced subgraph of G on $W \cup \{m\}$. Then, the edge polytope $\mathscr{P}_{G''}$ of G'' cannot be simple. Hence, W coincides with $V(G)$ as required. Q. E. D.

LEMMA 5.6. *Let G be a finite connected graph allowing loops and having no multiple edge. If the edge ring $K[G]$ is normal and if the edge polytope $\mathscr{P}_G$ is simple, then*

(a) *Each of the even cycles of G of length ≥ 6 has an even-chord.*

(b) *If e is a loop at i and C is a minimal odd cycle of G of length ≥ 3 with $i \in E(C)$, then the length of C must be 3.*

(c) *If C_1 and C_2 are minimal odd cycles of G of length ≥ 3 having exactly one common vertex, then there exists an edge $\{i, j\} \notin E(C_1) \cup E(C_2)$ with $i \in V(C_1)$ and $j \in V(C_2)$.*

(d) *If e is a loop at i and f is a loop at j with $i \neq j$, then $\{i, j\}$ is an edge of G.*

(e) *Let* C_1 *be either a loop or a minimal odd cycle of length* ≥ 3 *and* C_2 *a minimal odd cycle of length* ≥ 3. *If* C_1 *and* C_2 *have no common vertex, then there exist at least two bridges between* C_1 *and* C_2.

Proof. (a) Let C be an even cycle of G and let G' be the induced subgraph of G on $V(C)$. Then, by Lemma 5.2 the edge polytope $\mathscr{P}_{G'}$ is simple. If G' has a loop, then Lemma 5.5 guarantees the existence of an even-chord of C. While, if G' has no loop, then Corollary 5.4 guarantees that G' is a complete bipartite graph. In particular, C has an even-chord.

(b) By Lemma 5.5 all edges of the form $\{i, j\}$ with $j \in V(C)$ must belong to G. Thus, C must be of length 3 since C has no chord.

(c) Let $i \in V(C_1) \cap V(C_2)$ and write G' for the induced subgraph of G on $V(C_1) \cup V(C_2)$. Since G' is non-bipartite and since $\mathscr{P}_{G'}$ is simple, it follows from Corollary 5.4 that G' has a loop. If G' has a loop at j $(\neq i)$, then the existence of a required edge is guaranteed by Lemma 5.5. While, if G' has a loop e at i and if $E(G') = E(C_1) \cup E(C_2) \cup \{e\}$, then both C_1 and C_2 are odd cycles of length 3 by (b) above. Then, the edge polytope of the induced subgraph of G on $V(C_1) \cup \{k\}$, where $k \in V(C_2)$ and $k \neq i$, cannot be simple, a contradiction.

(d) Since $K[G]$ is normal, the edge $\{i, j\}$ must belong to G.

(e) Let G' denote the induced subgraph of G on $V(C_1) \cap V(C_2)$. Then, G' is connected since $K[G]$ is normal. We know that G' has a loop since G' is non-bipartite and since $\mathscr{P}_{G'}$ is simple. Hence, the existence of required edges is guaranteed by Lemma 5.5. Q. E. D.

Proof of Theorem 5.1. By virtue of the technique appearing in the proof of the "if" part of Theorem 1.2, the toric ideal I_G is generated by quadratic binomials if the conditions (a), (b), (c), (d), and (e) in Lemma 5.6 are satisfied. Hence, if the edge ring $K[G]$ is normal and if the edge polytope $\mathscr{P}_G$ is simple, then the toric ideal I_G is generated by quadratic binomials. This completes the proof of Theorem 5.1. Q. E. D.

We close the present paper with a remark on the toric variety $\mathscr{X}_G :=$ Proj $(K[G]) \hookrightarrow \mathbf{P}^{n-1}$ associated with I_G. The toric variety $\mathscr{X}_G \hookrightarrow \mathbf{P}^{n-1}$ is projectively normal if and only if the edge ring $K[G]$ is normal. Moreover, if $\mathscr{X}_G \hookrightarrow \mathbf{P}^{n-1}$ is nonsingular, then the edge polytope $\mathscr{P}_G$ is simple. Hence, by virtue of Theorem 5.1, if the toric variety $\mathscr{X}_G \hookrightarrow \mathbf{P}^{n-1}$ associated with I_G is nonsingular and projectively normal, then I_G is generated by quadratic binomials. See, e.g., [11, p. 138].

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