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Normal Polytopes Arising from Finite Graphs

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Let G be a finite connected graph on the vertex set $\{1, \dots, d\}$ allowing loops and having no multiple edge. Let $K[t_1, \dots, t_d]$ denote the polynomial ring in d indeterminates over a field K and let $K[G]$ be the subalgebra of $K[t_1, \dots, t_d]$ generated by all quadratic monomials $t_i t_j$ such that $\{i, j\}$ is an edge of G and by all quadratic monomials t_i^2 such that G has a loop at i. We describe the normalization of $K[G]$ explicitly and we give a combinatorial criterion for $K[G]$ to be normal.

INTRODUCTION

Let K be a field and let $K[t_1, t_1^{-1}, \dots, t_N, t_N^{-1}, s]$ be the Laurent polynomial ring over K. Given an *integral* convex polytope $\mathscr{P} \subset \mathbb{R}^N$, i.e., a convex polytope any of whose vertices has integer coordinates, we may associate the subalgebra $K[\mathscr{P}]$ of $K[t_1, t_1^{-1}, \dots, t_N, t_N^{-1}, s]$ which is generated by all monomials $t_1^{\alpha_1} \cdots t_N^{\alpha_N} s$ with $(\alpha_1, \dots, \alpha_N) \in \mathscr{P} \cap \mathbb{Z}^N$. We say that an integral convex polytope $\mathscr{P}$ is *normal* if the subalgebra $K[\mathscr{P}]$ is a normal domain. We are interested in the problem of finding a combinatorial criterion for a convex polytope to be normal. One of the most effective conditions which guarantees the normality of $\mathscr{P}$ is the existence of a unimodular covering of $\mathscr{P}$. Thus, in particular, if the toric ideal of $K[\mathscr{P}]$ has a square-free initial ideal, then $\mathscr{P}$ is a normal polytope. See, e.g., [7].

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Every graph G is a finite connected graph on the vertex set $V(G) = \{1, \ldots, d\}$ allowing loops and having no multiple edge. Let $E(G)$ denote the set of edges and loops of G. If $e = \{i, j\}$ is an edge of G joining $i \in V(G)$ with $j \in V(G)$, then we define $\rho(e) \in \mathbb{R}^d$ by $\rho(e) = \mathbf{e}_i + \mathbf{e}_j$. Here $\mathbf{e}_i$ is the ith unit coordinate vector in $\mathbb{R}^d$. Moreover, if e is a loop at $i \in V(G)$, then $\rho(e) := \mathbf{e}_i + \mathbf{e}_i$. We write $\mathscr{P}_G \subset \mathbb{R}^d$ for the convex hull of the finite set $\{\rho(e); e \in E(G)\} \subset \mathbb{R}^d$ and we call $\mathscr{P}_G$ the *edge polytope* of G. Some fundamental combinatorial structures on edge polytopes are discussed. Among other things, the facets of edge polytopes are determined. See Theorem 1.7.

Let $K[t_1, \ldots, t_d]$ denote the polynomial ring in d indeterminates over a field K and let $K[G]$ be the subalgebra of $K[t_1, \ldots, t_d]$ generated by all quadratic monomials $t_i t_j$ such that $\{i, j\}$ is an edge of G and by all quadratic monimials t_i^2 such that G has a loop at i. The affine semigroup ring $K[G]$ is called the *edge ring* of G. Let $\tilde{G}$ denote the graph obtained by adding those edges $\{i, j\}$ to G (if $\{i, j\} \notin E(G)$) such that G has both a loop at i and a loop at j. It then follows that $\mathscr{P}_G = \mathscr{P}_{\tilde{G}}$ and that $K[\mathscr{P}_G]$ is isomorphic to $K[\tilde{G}]$.

The main purpose of the present article is to describe the normalization of the edge ring $K[G]$ of a graph G explicitly in terms of combinatorics on G and to give a combinatorial criterion for the edge polytope $\mathscr{P}_G$ to be normal. See Theorem 2.2 and Corollary 2.3. A cycle C in a graph is called *minimal* if C possesses no chord. An *odd* cycle in a graph is a cycle whose length is odd. Thus, in particular, a loop is minimal and is an odd cycle (of length 1). We say that a graph G satisfies the *odd cycle condition* [2], if for arbitrary two minimal odd cycles C and C' in G, either C and C' have a common vertex or there exists an edge of G joining a vertex of C with a vertex of C'.

COROLLARY 2.3. *Let G be a finite connected graph allowing loops and having no multiple edge. Then, the following conditions are equivalent*:

(i) *the edge ring $K[G]$ is normal*;

(ii) *the edge polytope $\mathscr{P}_G$ possesses a unimodular covering*;

(iii) *the graph G satisfies the odd cycle condition.*

In particular, the edge polytope $\mathscr{P}_G$ of G is a normal polytope if and only if the graph $\tilde{G}$ satisfies the odd cycle condition.

Corollary 2.3 is an affirmative answer to the conjecture in Simis, Vasconcelos, and Villarreal [5, p. 412]. See also Fulkerson, Hoffman, and McAndrew [2] and Stanley [6] for some information about the graphs satisfying the odd cycle condition. We are grateful to Richard P. Stanley for bringing the articles [2] and [6] to our attention.

1. EDGE POLYTOPES OF FINITE GRAPHS

Every graph G is a finite connected graph on the vertex set $V(G) = \{1, \dots, d\}$ allowing loops and having no multiple edge. Let $E(G)$ denote the set of edges and loops of G. If $i \in V(G)$, then we write $N(G; i)$ for the set of vertices which are joined with i by edges or loops of G. Thus, $i \in N(G; i)$ if and only if G has a loop at i. The *degree* of $i \in V(G)$ in G is $\deg_G i := |N(G; i)|$. Here, $|X|$ is the cardinality of a finite set X. An *odd* cycle in G is a cycle whose length is odd. In particular, a loop is an odd cycle (of length 1). A *spanning* subgraph of G is a subgraph of G whose vertex set coincides with that of G. A *tree* is a connected graph having no cycle and having at least one edge. A *simple* graph is a graph having no loop and no multiple edge.

If $e = \{i, j\}$ is an edge of a graph G joining $i \in V(G)$ with $j \in V(G)$, then we define $\rho(e) \in \mathbb{R}^d$ by $\rho(e) = \mathbf{e}_i + \mathbf{e}_j$. Here $\mathbf{e}_i$ is the ith unit coordinate vector in $\mathbb{R}^d$. Moreover, if e is a loop at i, then $\rho(e) := \mathbf{e}_i + \mathbf{e}_i$. We write $\mathscr{P}_G \subset \mathbb{R}^d$ for the convex hull of the finite set $\{\rho(e);\ e \in E(G)\} \subset \mathbb{R}^d$ and we call $\mathscr{P}_G$ the *edge polytope* of G. Let $\tilde{G}$ denote the graph obtained by adding those edges $\{i, j\}$ to G (if $\{i, j\} \notin E(G)$) such that G has both a loop at i and a loop at j. It then follows that $\mathscr{P}_G = \mathscr{P}_{\tilde{G}}$. Moreover,

PROPOSITION 1.1. $\mathscr{P}_G \cap \mathbb{Z}^d = \{\rho(e);\ e \in E(\tilde{G})\}$.

Proof. First, the edge polytope $\mathscr{P}_G \subset \mathbb{R}^d$ is contained in the hyperplane,

$$\mathscr{H} = \left\{ (x_1, \dots, x_d) \in \mathbb{R}^d;\ \sum_{i=1}^{d} x_i = 2 \right\},$$

in $\mathbb{R}^d$. Because each $\rho(e)$ with $e \in E(G)$ satisfies the linear inequalities,

$$\mathscr{F}_i\colon\ x_i \le \sum_{j \in N(G;\, i)} x_j, \qquad 1 \le i \le d,$$

the edge polytope $\mathscr{P}_G$ is contained in the closed half-space in $\mathbb{R}^d$ defined by $\mathscr{F}_i$ for every $1 \le i \le d$. Let $(a_1, \dots, a_d) \in \mathscr{P}_G \cap \mathbb{Z}^d$. Because each $a_i \ge 0$ and because $\sum_{i=1}^{d} a_i = 2$, we have either

$$(a_1, \dots, a_d) = \Bigl(0, \dots, 0, \overset{i}{\breve{2}}, 0, \dots, 0\Bigr),$$

for some $1 \le i \le d$, or

$$(a_1, \dots, a_d) = \Bigl(0, \dots, 0, \overset{i}{\breve{1}}, 0, \dots, 0, \overset{j}{\breve{1}}, 0, \dots, 0\Bigr),$$

for some $1 \leq i < j \leq d$. The first vector satisfies the inequality $\mathscr{F}_i$ if and only if G has a loop at i, while the second vector satisfies the inequalities $\mathscr{F}_i$ and $\mathscr{F}_j$ if and only if either $\{i, j\}$ is an edge of G, or G has both a loop at i and a loop at j. Hence, every $(a_1, \ldots, a_d) \in \mathscr{P}_G \cap \mathbb{Z}^d$ is of the form $\rho(e)$ for some $e \in E(\tilde{G})$ as desired. Q.E.D.

A graph G is called *reduced* if G possesses no subgraph having two vertices and consisting of two loops together with one edge. If G is reduced, then none of the $\rho(e)$s with $e \in E(G)$ belongs to the convex hull of $\{\rho(e'); e \neq e' \in E(G)\}$ in $\mathbb{R}^d$. Hence,

PROPOSITION 1.2. *If a graph G is reduced, then each $\rho(e)$ with $e \in E(G)$ is a vertex of the edge polytope $\mathscr{P}_G$.*

A simple graph G is called *bipartite* if $V(G)$ has a partition $V(G) = V_1 \cup V_2$ with $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, and $V_1 \cap V_2 = \emptyset$ such that each edge of G is of the form $\{i, j\}$ with $i \in V_1$ and $j \in V_2$. Such a partition $V(G) = V_1 \cup V_2$ is unique and, in what follows, is called *the* partition of $V(G)$ if G is connected. A graph G is bipartite if and only if G has no odd cycle.

PROPOSITION 1.3. *We have*

$$\dim \mathscr{P}_G = \begin{cases} d-2, & \text{if } G \text{ has no odd cycle;} \\ d-1, & \text{if } G \text{ has at least one odd cycle.} \end{cases}$$

Proof. Because the edge polytope $\mathscr{P}_G \subset \mathbb{R}^d$ of G is contained in the hyperplane $\mathscr{H} \subset \mathbb{R}^d$ defined in the proof of Proposition 2.1, it follows that $\dim \mathscr{P}_G \leq d-1$. Because G is connected, we can find a spanning subtree G_0 of G. Because $\mathscr{P}_{G_0}$ is a $(d-2)$-simplex, we have $\dim \mathscr{P}_G \geq \dim \mathscr{P}_{G_0} = d-2$.

First, if G has no odd cycle, then G is a bipartite graph. Let $V(G) = V_1 \cup V_2$ denote the partition of $V(G)$ and define the hyperplanes $\mathscr{H}_k$ $(k = 1, 2)$ in $\mathbb{R}^d$ by

$$\mathscr{H}_k = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d; \sum_{i \in V_k} x_i = 1 \right\}.$$

Because $\rho(e) \in \mathscr{H}_1 \cap \mathscr{H}_2$ for every $e \in E(G)$, we have $\mathscr{P}_G \subset \mathscr{H}_1 \cap \mathscr{H}_2$. Thus, $\dim \mathscr{P}_G = d-2$.

Second, suppose that G has at least one odd cycle. Then, we can find a connected subgraph G' of G such that

(i) G' is a spanning subgraph of G;

(ii) G' has d edges;

(iii) G' has exactly one odd cycle and it is a unique cycle in G'.

Let $e_1, e_2, \ldots, e_d$ denote the edges of G' and let B denote the $d \times d$ matrix whose row vectors are $\rho(e_1), \rho(e_2), \ldots, \rho(e_d)$, i.e., B is the *incidence matrix* of G'. If G' is an odd cycle, then $\det(B) = \pm 2$. Otherwise, we can find $i \in V(G')$ with $\deg_{G'} i = 1$. We may assume that $\deg_{G'} d = 1$ and $e_d = \{d-1, d\}$. Let G'' denote the subgraph of G' with $V(G'') = \{1, 2, \ldots, d-1\}$ and let $E(G'') = \{e_1, e_2, \ldots, e_{d-1}\}$. Then, G'' has exactly one odd cycle and it is a unique cycle in G''. If C is the incidence matrix of G'', then $|\det(B)| = |\det(C)|$. Thus, it follows by induction on d that $\det(B) = \pm 2$. Hence, the convex hull of $\mathscr{P}_{G'} \cup \{(0, \ldots, 0)\}$ is a d-simplex. Here, $(0, \ldots, 0)$ is the origin of $\mathbb{R}^d$. Thus, $\mathscr{P}_{G'}$ is a $(d-1)$-simplex in $\mathbb{R}^d$. Because $\mathscr{P}_{G'} \subset \mathscr{P}_G$, we have $\dim \mathscr{P}_G = d - 1$ as required. Q.E.D.

Even though Lemma 1.4 is a simple modification of [7, Lemma 9.5], we give its proof for the convenience of the reader.

LEMMA 1.4. *Let G' be a reduced subgraph of G and let $\mathscr{P}_{G'}$ be its edge polytope. Then, $\mathscr{P}_{G'}$ is a $(d-1)$-simplex if and only if G' satisfies the following conditions*:

(i) *G' is a spanning subgraph of G*;

(ii) *G' has d edges*;

(iii) *every cycle in G' is odd*;

(iv) *every connected component of G' has exactly one odd cycle.*

Proof. First, suppose that the edge polytope $\mathscr{P}_{G'}$ of a reduced subgraph G' of G is a $(d-1)$-simplex in $\mathbb{R}^d$. Then, because G' is reduced, Propositions 1.1 and 1.2 guarantee that the conditions (i) and (ii) are satisfied. If B is the incidence matrix of G', then $\det(B) \neq 0$. Let $H_1, H_2, \ldots, H_p$ denote the connected components of G' and let B_k denote the incidence matrix of H_k for each $1 \le k \le p$. Let us assume that

$$B = \begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_p \end{pmatrix},$$

after relabeling the vertices of G'. Because $\det(B) \neq 0$, every B_i must be a square matrix with $\det(B_k) \neq 0$. Hence, for every $1 \le k \le p$, the number of the vertices of H_k is equal to the number of edges of H_k. Thus, each H_k has exactly one cycle. Because $\det(B_k) = 0$ if the cycle in H_k is even, the cycle in H_k must be odd.

Second, if G' is a reduced subgraph of G satisfying (i)–(iv), then the technique appearing in the proof of Proposition 1.3 enables us to see that

$|\det(B)| = 2^p \neq 0$, where B is the incidence matrix of G' and where p is the number of the connected components of G'. Hence, $\dim \mathscr{P}_{G'} = d - 1$. Thus, $\mathscr{P}_{G'}$ is a $(d-1)$-simplex as desired. Q.E.D.

An easy modification, i.e., replacing $\det(B) \neq 0$ with $\operatorname{rank}(B) = d - 1$; in the proof of Lemma 1.4 enables us to find the reduced subgraphs G' of G for which $\mathscr{P}_{G'}$ is a $(d-2)$-simplex.

LEMMA 1.5. *Let G' be a reduced subgraph of G and let $\mathscr{P}_{G'}$ be its edge polytope. Then, $\mathscr{P}_{G'}$ is a $(d-2)$-simplex if and only if either the following four conditions*

(i) *G' has $d-1$ vertices*;
(ii) *G' has $d-1$ edges*;
(iii) *every cycle in G' is odd*;
(iv) *every connected component of G' has exactly one odd cycle*

are satisfied, or the following five conditions

(i′) *G' is a spanning subgraph of G*:
(ii′) *G' has $d-1$ edges*;
(iii′) *every cycle in G' is odd*;
(iv′) *every connected component of G' has at most one odd cycle*;
(v′) *exactly one connected component is a tree*

are satisfied.

When G is a bipartite graph, i.e., $\dim \mathscr{P}_G = d - 2$, it would be required to find the subgraphs G' of G for which $\mathscr{P}_{G'}$ is a $(d-3)$-simplex.

LEMMA 1.6. *Suppose that G has no odd cycle, i.e.*, $\dim \mathscr{P}_G = d - 2$. *If G' is a subgraph of G, then $\mathscr{P}_{G'}$ is a $(d-3)$-simplex if and only if either G' is a tree with $d-1$ vertices, or if G' is a spanning subgraph of G consisting of two connected components each of which is a tree.*

Proof. If G' has k connected components, then it follows that $\dim \mathscr{P}_{G'} \leq d - k - 1$. Hence, G' has at most two connected components if $\mathscr{P}_{G'}$ is a $(d-3)$-simplex. Q.E.D.

We now turn to the problem of finding the facets of the edge polytope $\mathscr{P}_G$ of a graph G. Let G be a finite connected graph on the vertex set $\{1, \ldots, d\}$ allowing loops and having no multiple edge. In general, given a subset $W (\neq \emptyset)$ of $\{1, \ldots, d\}$, we write G_W for the subgraph of G having the vertex set W and consisting of all edges $\{i, j\}$ of G with $i, j \in W$ and of all loops of G at i with $i \in W$. We say that $i \in \{1, \ldots, d\}$ is *regular* (resp., *ordinary*) in G if every connected component of $G_{\{1,\ldots,d\}\setminus\{i\}}$ has at least one odd cycle (resp., if $G_{\{1,\ldots,d\}\setminus\{i\}}$ is connected).

A subset $\emptyset \neq T \subset \{1, \ldots, d\}$ is called *independent* in G if $N(G; i) \cap T = \emptyset$ for every $i \in T$ (in other words, no edge $\{i, j\}$ with $i, j \in T$ belongs to G and no loop at i with $i \in T$ belongs to G). Let $N(G;T) = \bigcup_{i \in T} N(G;i)$. If $\emptyset \neq T \subset \{1, \dots, d\}$ is independent in G, then the bipartite graph *induced by* T in G is defined to be the bipartite graph having the vertex set $T \cup N(G;T)$ and consisting of all edges $\{i, j\}$ of G with $i \in T$ and $j \in N(G;T)$.

When G has at least one odd cycle, we say that a subset $\emptyset \neq T \subset \{1, \dots, d\}$ is *fundamental* in G if (i) T is independent in G and the bipartite graph induced by T in G is connected, and (ii) either $T \cup N(G;T) = \{1, \dots, d\}$ or every connected component of the subgraph $G_{\{1, \dots, d\} \setminus (T \cup N(G;T))}$ has at least one odd cycle.

When G is a bipartite graph, we say that a subset $\emptyset \neq T \subset \{1, \dots, d\}$ is *acceptable* in G if (i) T is independent in G and the bipartite graph induced by T in G is connected, and (ii) $G_{\{1, \dots, d\} \setminus (T \cup N(G;T))}$ is a connected graph with at least one edge. If $V(G) = V_1 \cup V_2$ is the partition of $V(G)$ and if T is acceptable in G, then it follows easily that either $T \subset V_1$ or $T \subset V_2$.

Let $\mathscr{H}_i$ denote the hyperplane

$$\mathscr{H}_i = \left\{(x_1, \dots, x_d) \in \mathbb{R}^d;\ x_i = 0\right\},$$

in $\mathbb{R}^d$ and $\mathscr{H}_i^{(+)}$ the closed half-space,

$$\mathscr{H}_i^{(+)} = \left\{(x_1, \dots, x_d) \in \mathbb{R}^d;\ x_i \geq 0\right\},$$

in $\mathbb{R}^d$ for each $1 \leq i \leq d$. Then, $\mathscr{P}_G \subset \mathscr{H}_i^{(+)}$ for every $1 \leq i \leq d$. If $\emptyset \neq T \subset \{1, \dots, d\}$ is independent in G, then we write $\mathscr{H}_T$ for the hyperplane,

$$\mathscr{H}_T = \left\{(x_1, \dots, x_d) \in \mathbb{R}^d;\ \sum_{i \in T} x_i = \sum_{j \in N(G;T)} x_j\right\},$$

in $\mathbb{R}^d$ and $\mathscr{H}_T^{(-)}$ for the closed half-space,

$$\mathscr{H}_T^{(-)} = \left\{(x_1, \dots, x_d) \in \mathbb{R}^d;\ \sum_{i \in T} x_i \leq \sum_{j \in N(G;T)} x_j\right\},$$

in $\mathbb{R}^d$. It then follows that $\mathscr{P}_G \subset \mathscr{H}_T^{(-)}$ because T is independent in G.

THEOREM 1.7. (a) *Let G be a finite connected graph on the vertex set $\{1, \dots, d\}$ allowing loops and having no multiple edge and suppose that G has at least one odd cycle, i.e.,* $\dim \mathscr{P}_G = d - 1$. *Let Ψ denote the set of those hyperplanes $\mathscr{H}_i$ such that i is regular in G and of those hyperplanes $\mathscr{H}_T$ such that T is fundamental in G. Then, the set of facets of the edge polytope $\mathscr{P}_G$ coincides with $\{\mathscr{H} \cap \mathscr{P}_G;\ \mathscr{H} \in \Psi\}$.*

(b) *Let G be a finite connected bipartite graph on the vertex set $V(G) = \{1, \ldots, d\}$, i.e.,* $\dim \mathscr{P}_G = d - 2$, *and let $V(G) = V_1 \cup V_2$ be the partition of $V(G)$. Let Ψ denote the set of those hyperplanes $\mathscr{H}_i$ such that i is ordinary in G and of those hyperplanes $\mathscr{H}_T$ such that T is acceptable in G with $T \subset V_1$. Then, the set of facets of the edge polytope $\mathscr{P}_G$ coincides with $\{\mathscr{H} \cap \mathscr{P}_G;\ \mathscr{H} \in \Psi\}$.*

Proof. (a) First, we show that each hyperplane $\mathscr{H} \in \Psi(G)$ is a supporting hyperplane of $\mathscr{P}_G$ such that $\mathscr{H} \cap \mathscr{P}_G$ is a facet of $\mathscr{P}_G$.

Let $i \in \{1, \ldots, d\}$ be regular in G. Because every connected component of $G_{\{1,\ldots,d\}\setminus\{i\}}$ has at least one odd cycle, we can find a reduced subgraph G' of G satisfying (i)–(iv) in Lemma 1.5 such that $\mathscr{P}_{G'} \subset \mathscr{H}_i \cap \mathscr{P}_G$. Moreover, $\mathscr{H}_i \cap \mathscr{P}_G \neq \mathscr{P}_G$ and $\mathscr{P}_G \subset \mathscr{H}_i^{(+)}$. Hence, $\mathscr{H}_i \cap \mathscr{P}_G$ is a facet of $\mathscr{P}_G$.

Let $\emptyset \neq T \subset \{1, \ldots, d\}$ be fundamental in G. Because the bipartite graph induced by T in G is connected and because every connected component of $G_{\{1,\ldots,d\}\setminus(T \cup N(G;T))}$ has at least one odd cycle if $T \cup N(G;T) \neq \{1, \ldots, d\}$, we can find a reduced subgraph G' of G satisfying (i′)–(v′) in Lemma 1.5 such that $\mathscr{P}_{G'} \subset \mathscr{H}_T \cap \mathscr{P}_G$. Moreover, $\mathscr{H}_T \cap \mathscr{P}_G \neq \mathscr{P}_G$ because $\dim \mathscr{P}_G = d - 1$, and $\mathscr{P}_G \subset \mathscr{H}_T^{(-)}$ because T is independent in G. Hence, $\mathscr{H}_T \cap \mathscr{P}_G$ is a facet of $\mathscr{P}_G$.

Second, to see why each facet F of $\mathscr{P}_G$ is of the form $F = \mathscr{H} \cap \mathscr{P}_G$ for some $\mathscr{H} \in \Psi$, given a facet F of $\mathscr{P}_G$, we choose a reduced subgraph G' of G with $\mathscr{P}_{G'} \subset F$ for which $\mathscr{P}_{G'}$ is a $(d-2)$-simplex.

If G' satisfies (i)–(iv) in Lemma 1.5 and if $i \notin V(G')$ with $1 \leq i \leq d$, then i is regular in G and $\mathscr{P}_{G'} \subset \mathscr{H}_i$. We already proved that $\mathscr{H}_i \cap \mathscr{P}_G$ is a facet of $\mathscr{P}_G$. Hence, both F and $\mathscr{H}_i \cap \mathscr{P}_G$ are facets of $\mathscr{P}_G$ with $\mathscr{P}_{G'} \subset F$ and $\mathscr{P}_{G'} \subset \mathscr{H}_i \cap \mathscr{P}_G$. Thus, $F = \mathscr{H}_i \cap \mathscr{P}_G$ as desired.

If G' satisfies (i′)–(v′) in Lemma 1.5 and if a connected component H of G' is a tree having the vertex set $V(H) = T \cup S$ with $T \neq \emptyset$, $S \neq \emptyset$, and $T \cap S = \emptyset$ such that each edge of H is of the form $\{i, j\}$ with $i \in T$ and $j \in S$. We then define the hyperplane $\mathscr{H} \subset \mathbb{R}^d$ by

$$\mathscr{H} = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d;\ \sum_{i \in T} x_i = \sum_{j \in S} x_j \right\},$$

and the closed half-spaces $\mathscr{H}^{(+)}$ and $\mathscr{H}^{(-)}$ in $\mathbb{R}^d$ by

$$\mathscr{H}^{(+)} = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d;\ \sum_{i \in T} x_i \geq \sum_{j \in S} x_j \right\};$$

$$\mathscr{H}^{(-)} = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d;\ \sum_{i \in T} x_i \leq \sum_{j \in S} x_j \right\}.$$

Then, $\mathscr{P}_{G'} \subset \mathscr{H}$. If neither $N(G;T) = S$ nor $N(G;S) = T$ arises, then $\mathscr{P}_G \cap (\mathscr{H}^{(+)} \setminus \mathscr{H}) \neq \emptyset$ and $\mathscr{P}_G \cap (\mathscr{H}^{(-)} \setminus \mathscr{H}) \neq \emptyset$. Thus, none of the facets of $\mathscr{P}_G$ can contain $\mathscr{P}_{G'}$ because $\dim \mathscr{P}_{G'} = \dim \mathscr{P}_G - 1$. Thus, either

$N(G;T) = S$ or $N(G;S) = T$ arises. Let us assume that $N(G;T) = S$. In particular, T is fundamental in G and $\mathscr{H} = \mathscr{H}_T$. Thus, $\mathscr{P}_{G'} \subset \mathscr{H}_T$. We already proved that $\mathscr{H}_T \cap \mathscr{P}_G$ is a facet of $\mathscr{P}_G$. Hence, both F and $\mathscr{H}_T \cap \mathscr{P}_G$ are facets of $\mathscr{P}_G$ with $\mathscr{P}_{G'} \subset F$ and $\mathscr{P}_{G'} \subset \mathscr{H}_T \cap \mathscr{P}_G$. Thus, $F = \mathscr{H}_T \cap \mathscr{P}_G$ as required.

(b) If we replace "regular" with "ordinary" and if we replace "fundamental" with "acceptable" in the preceding proof of (a) and if we employ Lemma 1.6 instead of Lemma 1.5, then we immediately obtain the proof of (b). Note that if $S \subset V_2$ is acceptable, then $V_1 \setminus N(G;S)$ is also acceptable and $\mathscr{H}_S \cap \mathscr{P}_G = \mathscr{H}_{V_1 \setminus N(G;S)} \cap \mathscr{P}_G$. Q.E.D.

COROLLARY 1.8. *(a) Let G be a finite connected graph on the vertex set* $\{1, \ldots, d\}$ *allowing loops and having no multiple edge and suppose that G has at least one odd cycle. Let* $\mathscr{H}$ *denote the hyperplane,*

$$\mathscr{H} = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d;\ \sum_{i=1}^{d} x_i = 2 \right\},$$

in $\mathbb{R}^d$. *Then, we have*

$$\mathscr{P}_G = \mathscr{H} \cap \left(\bigcap_{i=1}^{d} \mathscr{H}_i^{(+)} \right) \cap \left(\bigcap_{T} \mathscr{H}_T^{(-)} \right),$$

where T ranges over all independent subsets $T \subset \{1, \ldots, d\}$.

(b) Let G be a finite connected bipartite graph on the vertex set $V(G) = \{1, \ldots, d\}$ *and let* $V(G) = V_1 \cup V_2$ *be the partition of* $V(G)$. *Let* $\mathscr{H}_k$ $(k = 1, 2)$ *denote the hyperplanes,*

$$\mathscr{H}_k = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d;\ \sum_{i \in V_k} x_i = 1 \right\},$$

in $\mathbb{R}^d$. *Then, we have*

$$\mathscr{P}_G = (\mathscr{H}_1 \cap \mathscr{H}_2) \cap \left(\bigcap_{i=1}^{d} \mathscr{H}_i^{(+)} \right) \cap \left(\bigcap_{T} \mathscr{H}_T^{(-)} \right),$$

where T ranges over all independent subsets $T \subset V_1$.

2. NORMALIZATIONS OF EDGE RINGS

Let $\mathscr{P} \subset \mathbb{R}^N$ denote a convex polytope of dimension d and suppose that $\mathscr{P}$ is integral, i.e., each vertex of $\mathscr{P}$ has integer coordinates. Let $t_1, \ldots, t_N$

and s be indeterminates over a field K. Given an integer $n \geq 1$, we write $A(\mathscr{P})_n$ for the vector space over K which is spanned by those monomials $t^\alpha s^n = t_1^{\alpha_1} \cdots t_N^{\alpha_N} s^n$ such that $\alpha = (\alpha_1, \dots, \alpha_N) \cap n\mathscr{P} \cap \mathbb{Z}^N$. Here $n\mathscr{P} := \{n\alpha;\ \alpha \in \mathscr{P}\}$. Because $\mathscr{P}$ is convex, $A(\mathscr{P})_n A(\mathscr{P})_m \subset A(\mathscr{P})_{n+m}$ for all n and m. It then follows that the graded algebra $A(\mathscr{P}) := \bigoplus_{n=0}^{\infty} A(\mathscr{P})_n$ is finitely generated over $K = A(\mathscr{P})_0$ with Krull-dim $A(\mathscr{P}) = d + 1$. Moreover, $A(\mathscr{P})$ is a normal domain. We say that $A(\mathscr{P})$ is the *Ehrhart ring* associated with an integral convex polytope $\mathscr{P} \subset \mathbb{R}^N$. Consult [1] and [3] for the detailed information about algebra and combinatorics on Ehrhart rings. Let $K[\mathscr{P}]$ denote the subalgebra of $A(\mathscr{P})$ which is generated by $A(\mathscr{P})_1$, i.e., generated by all monomials $t_1^{\alpha_1} \cdots t_N^{\alpha_N} s$ with $(\alpha_1, \dots, \alpha_N) \in \mathscr{P} \cap \mathbb{Z}^N$. An integral convex polytope $\mathscr{P}$ is called *normal* if $K[\mathscr{P}]$ is a normal domain. Because the Ehrhart ring $A(\mathscr{P})$ of $\mathscr{P}$ is normal and is integral over $K[\mathscr{P}]$, it follows that, when $A(\mathscr{P})$ is contained in the quotient field of $K[\mathscr{P}]$, $K[\mathscr{P}]$ is normal if and only if $K[\mathscr{P}] = A(\mathscr{P})$.

Let G be a finite connected graph on the vertex set $\{1, \dots, d\}$ allowing loops and having no multiple edge. Let $K[t_1, \dots, t_d]$ denote the polynomial ring in d indeterminates over a field K and let $K[G]$ denote the subalgebra of $K[t_1, \dots, t_d]$ generated by all quadratic monomials $t_i t_j$ such that $\{i, j\}$ is an edge of G and by all quadratic monomials t_i^2 such that G has a loop at i. The affine semigroup ring $K[G]$ is called the *edge ring* of G.

A cycle C in a graph is called *minimal* if C possesses no chord. Thus, in particular, a loop is minimal and is an odd cycle (of length 1). If C and C' are cycles in a graph G having no common vertex, then an edge $\{i, j\}$ of G is called a *bridge* of C and C' if i is a vertex of C and if j is a vertex of C'. We say that a graph G satisfies the *odd cycle condition* [2] if, for arbitrary two minimal odd cycles C and C' in G, either C and C' have a common vertex or there exists a bridge of C and C'.

Proposition 2.1 is discussed in [5, Proposition 6.8].

PROPOSITION 2.1. *If the edge ring $K[G]$ of a graph G is normal, then G satisfies the odd cycle condition.*

Proof. Suppose that G does not satisfy the odd cycle condition and choose minimal odd cycles C_1 and C_2 in G having no common vertex such that there exists no bridge of C_1 and C_2. Let us assume that $V(C_1) = \{1, 2, \dots, 2m-1\}$, $V(C_2) = \{2m, 2m+1, \dots, 2n\}$, $E(C_1) = \{\{1, 2\}, \{2, 3\}, \dots, \{2m-2, 2m-1\}, \{2m-1, 1\}\}$ and $E(C_2) = \{\{2m, 2m+1\}, \{2m+1, 2m+2\}, \dots, \{2n-1, 2n\}, \{2n, 2m\}\}$. Let

$$\alpha = \Big\{\underbrace{1, \dots, 1}_{2n}, 0, \dots, 0\Big\} \in n\mathscr{P} \cap \mathbb{Z}^d,$$

and

$$t^{\alpha} = t_1 t_2 \cdots t_{2n} \in K[t_1, \dots, t_d].$$

Because

$$(t^{\alpha})^2 = \left(\prod_{j=1}^{2m-2} t_j t_{j+1} \right) (t_{2m-1} t_1) \left(\prod_{j=2m}^{2n-1} t_j t_{j+1} \right) (t_{2n} t_{2m})$$

belongs to $K[G]$, we know that t^{α} is integral over $K[G]$. Because G is connected, we can find a *walk* (or *path*) of odd length in G, say

$$\left(\{1, i_1\}, \{i_1, i_2\}, \dots, \{i_{2p-1}, i_{2p}\}, \{i_{2p}, 2n\}\right),$$

from $1 \in V(C_1)$ to $2n \in V(C_2)$. Then,

$$t^{\alpha} = t_1 t_{2n} \prod_{i=1}^{n-1} t_{2j} t_{2j+1},$$

and

$$t_1 t_{2n} = \frac{t_1 t_{i_1}}{t_{i_1} t_{i_2}} \frac{t_{i_2} t_{i_3}}{t_{i_3} t_{i_4}} \cdots \frac{t_{i_{2p-2}} t_{i_{2p-1}}}{t_{i_{2p-1}} t_{i_{2p}}} t_{i_{2p}} t_{2n}.$$

Thus, t^{α} belongs to the quotient field of $K[G]$. Because $t^{\alpha} \notin K[G]$, the edge ring $K[G]$ is not normal. Hence, if the edge ring $K[G]$ of a graph G is normal, then G must satisfy the odd cycle condition. Q.E.D.

We are now in the position to describe the normalization of the edge ring $K[G]$ of a graph G explicitly in terms of combinatorics on G and to give a combinatorial criterion for the edge polytope $\mathscr{P}_G$ to be normal.

Let, as before, G be a finite connected graph on the vertex set $\{1, \dots, d\}$ allowing loops and having no multiple edge. A pair $\Pi = \{C, C'\}$, where C and C' are minimal odd cycles in G with $V(C) \cap V(C') = \emptyset$, is called *exceptional* if there exists no bridge of C and C' in G. Given an exceptional pair $\Pi = \{C, C'\}$ of minimal odd cycles C and C' in G, we write $\mathscr{M}_{\Pi}$ for the monomial $(\prod_{i \in V(C)} t_i)(\prod_{j \in V(C')} t_j)$ in $K[t_1, \dots, t_d]$.

THEOREM 2.2. *Let G be a finite connected graph on the vertex set $\{1, \dots, d\}$ allowing loops and having no multiple edge and let $K[G]$ be the edge ring of G. Let $\Pi_1 = \{C_1, C_1'\}$, $\Pi_2 = \{C_2, C_2'\}, \dots, \Pi_q = \{C_q, C_q'\}$ denote the exceptional pairs of minimal odd cycles in G. Then, the normalization of $K[G]$ is generated by the monomials $\mathscr{M}_{\Pi_1}, \mathscr{M}_{\Pi_2}, \dots, \mathscr{M}_{\Pi_q}$ as an algebra over $K[G]$. More precisely, as a module over $K[G]$, the normalization of $K[G]$ is generated by those (square-free) monomials of the form $\mathscr{M}_{\Pi_{i_1}} \mathscr{M}_{\Pi_{i_2}} \cdots \mathscr{M}_{\Pi_{i_l}}$*

with $1 \leq i_1 < i_2 < \cdots < i_l \leq q$ *such that* $(V(C_{i_f}) \cup V(C'_{i_f})) \cap (V(C_{i_g}) \cup V(C'_{i_g})) = \emptyset$ *for all* $1 \leq f < g \leq l$.

Our proofs of Theorem 2.2 and Corollary 2.3 are given after preparing Lemmas 2.4 and 2.5 together with Corollary 2.6.

Let, in general, $\mathscr{P} \subset \mathbb{R}^N$ be given an integral convex polytope of dimension d. We say that a finite set Δ consisting of integral d-simplices $\mathscr{Q} \subset \mathbb{R}^N$ with $\mathscr{Q} \subset \mathscr{P}$ is a *unimodular covering* of $\mathscr{P}$ if the normalized volume (e.g., [7, p. 36]) of each simplex $\mathscr{Q} \in \Delta$ is equal to 1 and $\mathscr{P} = \bigcup_{\mathscr{Q} \in \Delta} \mathscr{Q}$. Thus, in particular, a *unimodular triangulation* (e.g., [7, p. 69]) of $\mathscr{P}$ is a unimodular covering of $\mathscr{P}$. It follows that an integral convex polytope which possesses a unimodular covering is a normal polytope.

Recall that $\tilde{G}$ is the graph obtained by adding those edges $\{i, j\}$ to G (if $\{i, j\} \notin E(G)$) such that G has both a loop at i and a loop at j. Thus, $\mathscr{P}_G = \mathscr{P}_{\tilde{G}}$ and $K[\mathscr{P}_G]$ is isomorphic to $K[\tilde{G}]$.

COROLLARY 2.3. *Let G be a finite connected graph allowing loops and having no multiple edge. Then the following conditions are equivalent*:

(i) *the edge ring* $K[G]$ *is normal*;

(ii) *the edge polytope* $\mathscr{P}_G$ *possesses a unimodular covering*;

(iii) *the graph G satisfies the odd cycle condition.*

In particular, the edge polytope $\mathscr{P}_G$ *of G is a normal polytope if and only if the graph* $\tilde{G}$ *satisfies the odd cycle condition.*

Corollary 2.3 is an affirmative answer to the conjecture in Simis, Vasconcelos, and Villarreal [5, p. 412]. In [4] we discover a simple graph G satisfying the odd cycle condition such that $\mathscr{P}_G$ possesses no regular unimodular triangulation. It seems, however, an open question if there exists a normal polytope which possesses no unimodular covering. We refer the reader to Fulkerson, Hoffman, and McAndrew [2] and Stanley [6] for some information about the graphs satisfying the odd cycle condition.

Lemma 2.4 explains the essential role of a bridge of odd cycles in a graph. Some combinatorial techniques appearing in its proof will be indispensable to understand the proof of Theorem 2.2.

LEMMA 2.4. *Let* G_1 (*resp*., G_2) *be a finite connected graph allowing loops and having no multiple edge on the vertex set* $\{1, \ldots, d'\}$ (*resp*., $\{d' + 1, \ldots, d\}$). *Suppose that* G_1 (*resp*., G_2) *has exactly one odd cycle* C_1 (*resp*., C_2) *and* C_1 (*resp*., C_2) *is a unique cycle in* G_1 (*resp*., G_2). *Let G be a graph on the vertex set* $\{1, \ldots, d', d' + 1, \ldots, d\}$ *with* $E(G) = E(G_1) \cup E(G_2) \cup \{e_{(C_1, C_2)}\}$, *where* $e_{(C_1, C_2)}$ *is a bridge of* C_1 *and* C_2. *Then, the edge polytope* $\mathscr{P}_G$ *of G is normal.*

Proof. Because the Ehrhart ring $A(\mathscr{P}_G)$ is normal, in order to see why the edge polytope $\mathscr{P}_G$ of G is normal, we may show that $K[\mathscr{P}_G]$ coincides with $A(\mathscr{P}_G)$. It follows that $K[\mathscr{P}_G] = A(\mathscr{P}_G)$ if and only if every $\alpha \in n\mathscr{P}_G \cap \mathbb{Z}^d$ can be expressed in the form $\alpha = \sum_{e \in E(G)} a_e \rho(e)$ with each $0 \le a_e \in \mathbb{Z}$ and with $\sum_{e \in E(G)} a_e = n$.

First, every $\alpha \in n\mathscr{P}_G \cap \mathbb{Z}^d$ can be expressed in the form $\alpha = \sum_{e \in E(G)} a_e \rho(e)$ with each $0 \le a_e \in \mathbb{R}$ and with $\sum_{e \in E(G)} a_e = n$. Because

$$\alpha = \sum_{e \in E(G)} \lfloor a_e \rfloor \rho(e) + \sum_{e \in E(G)} (a_e - \lfloor a_e \rfloor) \rho(e)$$

belongs to $\mathbb{Z}^d$, we have $\sum_{e \in E(G)} (a_e - \lfloor a_e \rfloor) \rho(e) \in \mathbb{Z}^d$. Thus, if $\deg_G i = 1$ and $e = \{i, j\} \in E(G)$, then $a_e - \lfloor a_e \rfloor = 0$, i.e., $0 \le a_e \in \mathbb{Z}$. Let G' denote the subgraph of G obtained by removing all vertices i of G with $\deg_G i = 1$ and by removing all edges $e = \{i, j\}$ of G with $\deg_G i = 1$. Then, because $\sum_{e \in E(G')} (a_e - \lfloor a_e \rfloor) \rho(e) \in \mathbb{Z}^d$, if $\deg_{G'} i = 1$ and if $e = \{i, j\} \in E(G')$, then $a_e - \lfloor a_e \rfloor = 0$, i.e., $0 \le a_e \in \mathbb{Z}$. Hence, repeated applications of such techniques enable us to see that if e is an edge of G with $e \notin E(C_1) \cup E(C_2) \cup \{e_{(C_1, C_2)}\}$, then $0 \le a_e \in \mathbb{Z}$. Thus,

$$\sum_{e \in E(C_1) \cup E(C_2) \cup \{e_{(C_1, C_2)}\}} (a_e - \lfloor a_e \rfloor) \rho(e) \in \mathbb{Z}^d.$$

Let us assume that $V(C_1) = \{1, 2, \ldots, 2l - 1\}$, $V(C_2) = \{2l, 2l + 1, \ldots, 2m\}$, $e_{(C_1, C_2)} = \{1, 2m\}$ and set $e_1 = \{1, 2\}$, $e_2 = \{2, 3\}, \ldots, e_{2l-2} = \{2l - 2, 2l - 1\}$, $e_{2l-1} = \{2l - 1, 1\}$, $e_{2l} = \{2m, 2l\}$, $e_{2l+1} = \{2l, 2l + 1\}, \ldots, e_{2m-1} = \{2m - 2, 2m - 1\}$, and $e_{2m} = \{2m - 1, 2m\}$. Note that if $l = 1$ (resp., $l = m$), then e_1 (resp., e_{2m}) is a loop at 1 (resp., m). Let

$$\begin{aligned} \beta &= (\beta_1, \ldots, \beta_{2m}, 0, \ldots, 0) \\ &= \sum_{e \in E(C_1) \cup E(C_2) \cup \{e_{(C_1, C_2)}\}} (a_e - \lfloor a_e \rfloor) \rho(e) \in \mathbb{Z}^d. \end{aligned}$$

then, because $\sum_{i=1}^d \beta_i$ is even, it follows easily that $(\beta_1, \ldots, \beta_{2m})$ is $(0, 0, \ldots, 0)$, or $(1, 1, \ldots, 1)$, or $(2, 1, 1, \ldots, 1, 2)$. Hence, β is equal to $(0, 0, \ldots, 0)$, or

$$\sum_{k=1}^{l-1} \rho(e_{2k}) + \sum_{k=l}^{m-1} \rho(e_{2k+1}) + \rho(e_{(C_1, C_2)}),$$

or

$$\sum_{k=1}^{l-1} \rho(e_{2k}) + \sum_{k=l}^{m-1} \rho(e_{2k+1}) + 2\rho(e_{(C_1, C_2)}).$$

Thus, every $\alpha \in n\mathscr{P}_G \cap \mathbb{Z}^d$ can be expressed in the form $\alpha = \sum_{e \in E(G)} a_e \rho(e)$ with each $0 \leq a_e \in \mathbb{Z}$ as required. Q.E.D.

LEMMA 2.5. *Work with the same situation as in Lemma* 2.4. *Given an expression in the form* $\alpha = \sum_{e \in E(G)} a_e \rho(e) \in \mathscr{P}_G$ *with each* $0 \leq a_e \in \mathbb{R}$ *and with* $\sum_{e \in E(G)} a_e = 1$, *we may assume that* $a_e = 0$ *for at least one edge* $e \in E(C_1) \cup E(C_2)$.

Proof. In fact, keeping the notation on the vertices and edges of C_1 and C_2, we define $\delta \geq 0$ by

$$\delta = \min\left(\left\{a_{e_{2i-1}};\ 1 \leq i \leq l\right\} \cup \left\{a_{e_{2i}};\ l \leq i \leq m\right\}\right).$$

Then, replacing a_{e_i} with $a_{e_i} + (-1)^i\delta$ if $1 \leq i \leq 2l-1$ and with $a_{e_i} - (-1)^i\delta$ if $2l \leq i \leq 2m$ and replacing $a_{e_{(C_1, C_2)}}$ with $a_{e_{(C_1, C_2)}} + 2\delta$ in the previous expression for α produces a required expression. Q.E.D.

COROLLARY 2.6. *Let G be a finite connected graph on the vertex set* $\{1, \ldots, d\}$ *allowing loops and having no multiple edge and suppose that G has at least one odd cycle. Let Ω denote the set of all reduced subgraphs G' of G satisfying the conditions* (i)–(iv) *in Lemma* 1.4 *such that every odd cycle in G' is minimal in G and that, for arbitrary two odd cycles C and C' in G' with $C \neq C'$, the pair $\Pi = \{C, C'\}$ of minimal odd cycles in G is exceptional. Then, we have*

$$\mathscr{P}_G = \bigcup_{G' \in \Omega} \mathscr{P}_{G'}.$$

Proof. Let Ω' denote the set of all reduced subgraphs G' of G satisfying the conditions (i)–(iv) in Lemma 1.4. It then follows that $\mathscr{P}_G = \bigcup_{G' \in \Omega'} \mathscr{P}_{G'}$. Moreover, we write Ω'' for the set of subgraphs $G' \in \Omega'$ any of whose odd cycles is minimal in G. Thus, $\Omega \subset \Omega'' \subset \Omega'$.

First, we show that $\mathscr{P}_G = \bigcup_{G' \in \Omega''} \mathscr{P}_{G'}$. Because $\mathscr{P}_G = \bigcup_{G' \in \Omega'} \mathscr{P}_{G'}$, given $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathscr{P}_G$, we can choose $G_0 \in \Omega'$ with $\alpha \in \mathscr{P}_{G_0}$ and then we can express α in the form $\alpha = \sum_{e \in E(G_0)} a_e \rho(e)$ with each $0 \leq a_e \in \mathbb{R}$ and with $\sum_{e \in E(G_0)} a_e = 1$. Let us, for a while, assume that G_0 contains at least one nonminimal odd cycle C in G. We work with the notation $V(C) = \{1, 2, \ldots, 2l-1\}$ and $E(C) = \{e_1, e_2, \ldots, e_{2l-1}\}$ with $e_1 = \{1, 2\}$, $e_2 = \{2, 3\}, \ldots, e_{2l-2} = \{2l-2, 2l-1\}$, $e_{2l-1} = \{2l-1, 1\}$. Let $e' = \{1, 2k\}$ be a chord of C, where $2 \leq k \leq l-1$. We define $\delta \geq 0$ by

$$\delta = \min\left\{a_{e_{2i-1}};\ 1 \leq i \leq k\right\}.$$

Then, replacing a_{e_i} with $a_{e_i} + (-1)^i\delta$ for each $1 \leq i \leq 2k-1$ in the foregoing expression for α and setting $a_{e'} = \delta$, we obtain an expression,

$$\alpha = \sum_{e \in E(G_0) \cup \{e'\}} a_e \rho(e). \tag{1}$$

In expression (1), we have $a_{e_i} = 0$ for at least one edge $e_i \in E(C)$ with $1 \leq i \leq 2k-1$. Fix such an edge e_i and construct the subgraph G_1 by deleting e_i from G_0 and adding e' to G_0. Then, $G_1 \in \Omega'$ with $\alpha \in \mathscr{P}_{G_1}$. Hence, repeated applications of such techniques enable us to find a desired subgraph $G' \in \Omega''$ with $\alpha \in \mathscr{P}_{G'}$.

Now, we prove that $\mathscr{P}_G = \bigcup_{G' \in \Omega} \mathscr{P}_{G'}$. Because we know $\mathscr{P}_G = \bigcup_{G' \in \Omega''} \mathscr{P}_{G'}$, given $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathscr{P}_G$, we can choose $G_0 \in \Omega''$ with $\alpha \in \mathscr{P}_{G_0}$. If G_0 contains no nonexceptional pair of minimal odd cycles in G, then G_0 belongs to Ω. Let us assume that G_0 contains at least one nonexceptional pair of minimal odd cycles in G. Then, by virtue of Lemma 2.5, we can find $G_1 \in \Omega''$ with $\alpha \in \mathscr{P}_{G_1}$ such that the number of nonexceptional pairs of minimal odd cycles which G_1 contains is less than the number of nonexceptional pairs of minimal odd cycles which G_0 contains. Hence, repeated applications of Lemma 2.5 guarantee the existence of a required subgraph $G' \in \Omega$ with $\alpha \in \mathscr{P}_{G'}$. Thus, $\mathscr{P}_G = \bigcup_{G' \in \Omega} \mathscr{P}_{G'}$ as desired. Q.E.D.

We now come to the proof of Theorem 2.2.

Proof of Theorem 2.2. The proof of Proposition 2.1 guarantees that each $\mathscr{M}_{\Pi_i}$ belongs to the quotient field of the edge ring $K[G]$ and $(\mathscr{M}_{\Pi i})^2 \in K[G]$. Because the Ehrhart ring $A(\mathscr{P}_G)$ of the edge polytope $\mathscr{P}_G$ of G is normal, our goal is to prove that $A(\mathscr{P}_G)$ is generated by $\mathscr{M}_{\Pi_1} s^{n_1}, \mathscr{M}_{\Pi_1} s^{n_2}, \dots, \mathscr{M}_{\Pi_1} s^{n_q}$ as an algebra over $K[\mathscr{P}_G]$, where $n_i = |V(C_i) \cup V(C_i')|$.

(*First Step*). First, we assume that there exists at least one odd cycle in the graph G, i.e., $\dim \mathscr{P}_G = d-1$. By virtue of Corollary 2.6, given $\alpha = (\alpha_1, \dots, \alpha_d) \in n\mathscr{P}_G \cap \mathbb{Z}^d$, we can find $G' \in \Omega$ with $\alpha \in n\mathscr{P}_{G'}$. Let $B_1, \dots, B_m$ denote the odd cycles in G'. Then, the technique appearing in the proof of Lemma 2.4 enables us to see that α can be expressed in the form,

$$\alpha = \sum_{e \in E(G')} a_e \rho(e) + \sum_{k=1}^{m} \varepsilon_k \left(\sum_{e \in E(B_k)} \rho(e) \right),$$

with each $0 \leq a_e \in \mathbb{Z}$ and with each $\varepsilon_k \in \{0, \frac{1}{2}\}$. Let V_1 (resp., V_2) denote the subset of $\{1, \dots, m\}$ consisting of all k for which $\varepsilon_k = \frac{1}{2}$ (resp., $\varepsilon_k = 0$). Because $\sum_{i=1}^d \alpha_i = 2n$, the cardinality of V_1 must be even. Let us assume that $V_1 = \{1, 2, \dots, 2l\}$ and $V_2 = \{2l+1, 2l+2, \dots, m\}$. Thus, α can be expressed in the form,

$$\begin{aligned}\alpha &= \sum_{e \in E(G')} a_e \rho(e) + \sum_{k=1}^{2l} \frac{1}{2}\left(\sum_{e \in E(B_k)} \rho(e) \right) \\ &= \sum_{e \in E(G')} a_e \rho(e) + \sum_{k=1}^{l} \frac{1}{2}\left(\sum_{e \in E(B_{2k-1}) \cup E(B_{2k})} \rho(e) \right),\end{aligned}$$

with each $0 \le a_e \in \mathbb{Z}$. It then follows that, as a module over $K[\mathscr{P}_G]$, the Ehrhart ring $A(\mathscr{P}_G)$ is generated by all square-free monomials in the form $\mathscr{M}_{\Pi_{i_1}} s^{n_{i_1}} \mathscr{M}_{\Pi_{i_2}} s^{n_{i_2}} \cdots \mathscr{M}_{\Pi_{i_l}} s^{n_{i_l}}$ with $1 \le i_1 < i_2 < \cdots < i_l \le q$ such that $(V(C_{i_f}) \cup V(C'_{i_f})) \cap (V(C_{i_g}) \cup V(C'_{i_g})) = \emptyset$ for all $1 \le f < g \le l$, where $n_{i_j} = |V(C_{i_j}) \cup V(C'_{i_j})|$.

(*Second Step*). If a connected graph G has no odd cycle, then $\mathscr{P}_G$ is a normal polytope, e.g., [5, Theorem 5.9]. In fact, $\dim \mathscr{P}_G = d - 2$ if G has no odd cycle. By Lemma 1.5, for a subgraph $G' \subset G$, the edge polytope $\mathscr{P}_{G'}$ of G' is a $(d-2)$-simplex if and only if G' is a spanning subtree of G. It then follows easily that $K[\mathscr{P}_G]$ coincides with the Ehrhart ring $A(\mathscr{P}_G)$. Hence, $\mathscr{P}_G$ is a normal polytope. Q.E.D.

We conclude the present article with the proof of Corollary 2.3.

Proof of Corollary 2.3. If the edge ring $K[G]$ of a graph G is normal, then G must satisfy the odd cycle condition (Proposition 2.1). If G satisfies the odd cycle condition, then there exists no exceptional pair of minimal odd cycles in G. Hence, Theorem 2.2 guarantees that $K[G]$ is normal. We now prove the existence of a unimodular covering of the edge polytope $\mathscr{P}_G$ of a graph G which satisfies the odd cycle condition.

(*First Step*). Suppose that G satisfies the odd cycle condition and that G has at least one odd cycle. Let Λ denote the set of all reduced *connected* subgraphs G' of G satisfying the conditions (i)–(iv) in Lemma 1.4 such that every (in fact, a unique) odd cycle in G' is minimal in G. Then, Corollary 2.6 guarantees that $\mathscr{P}_G = \bigcup_{G' \in \Lambda} \mathscr{P}_{G'}$ because $\Lambda = \Omega$. Moreover, it follows from [7, Lemma 9.5] that the normalized volume of the edge polytope $\mathscr{P}_{G'}$ of each $G' \in \Lambda$ is equal to 1. Hence, $\{\mathscr{P}_{G'};\, G' \in \Lambda\}$ is a unimodular covering of $\mathscr{P}_G$ as desired.

(*Second Step*). Because the incidence matrix of a bipartite graph is totally unimodular, the edge polytope of a bipartite graph possesses a unimodular triangulation. See [7, pp. 69–70]. Q.E.D.

APPENDIX

We give here some supplements related with Fulkerson, Hoffman, and McAndrew [2]. First, we present a quick proof of the result in the following text generalizes the "marriage theorem" as well as [2, Theorem 2.5]. Recall that a *perfect matching* of a simple graph G on $[d] = \{1, \dots, d\}$ is a subset $\mathscr{M}$ of $E(G)$ such that each $i \in [d]$ is a vertex of a unique edge belonging to $\mathscr{M}$. In particular, if G possesses a perfect matching, then d must be even.

PROPOSITION A.1. *Let G be a finite simple connected graph on $[d]$, where d is even, and suppose that G satisfies the odd cycle condition. Then, G possesses a perfect matching if (and only if) $\#(T) \leq \#(N(G; T))$ for all independent subsets $T \subset [d]$ in G. Here, $\#(T)$ is the cardinality of a finite set T.*

Proof. In general, G possesses a perfect matching if and only if $t_1 t_2 \cdots t_d$ belongs to $K[G]$. Because $\#(T) \leq \#(N(G; T))$ for all independent subsets $T \subset [d]$ in G, Corollary 1.8 enables us to see that $(2/d, \dots, 2/d) \in \mathbb{R}^d$ belongs to $\mathscr{P}_G$. Hence, $t_1 t_2 \cdots t_d$ is integral over $K[G]$. Moreover, because G is connected, it follows that $t_1 t_2 \cdots t_d$ belongs to the quotient field of $K[G]$. Because $K[G]$ is normal, $t_1 t_2 \cdots t_d$ belongs to $K[G]$ as required. Q.E.D.

Second, we present a direct proof, which follows from [2, Theorem 2.1], of (iii) $\Rightarrow$ (i) of Corollary 2.3. We are grateful to the referee for pointing out the discussion in the following text.

LEMMA A.2. *Let $\alpha_1, \alpha_2, \dots, \alpha_n$ belong to $\mathbb{Z}^d$. Let L denote the lattice generated by the α_is and let Z denote the zonotope generated by the α_is, i.e., $L = \{\sum_{i=1}^n a_i \alpha_i;\ a_i \in \mathbb{Z}\}$ and $Z = \{\sum_{i=1}^n x_i \alpha_i;\ x_i \in \mathbb{R},\ 0 \leq x_i \leq 1\}$. Suppose that every $\beta \in Z \cap L$ is a nonnegative integer linear combination of the α_is. Then, the affine semigroup ring $K[t^{\alpha_1}, t^{\alpha_2}, \dots, t^{\alpha_n}]$ is normal.*

Proof. Let $\gamma = \sum_{i=1}^n a_i \alpha_i$, with each $a_i \in \mathbb{Z}$ and suppose that $m\gamma = \sum_{i=1}^n b_i \alpha_i$, where m is a positive integer and where each b_i is a nonnegative integer. Thus, t^γ belongs to the quotient field of $K[t^{\alpha_1}, t^{\alpha_2}, \dots, t^{\alpha_n}]$ and is integral over $K[t^{\alpha_1}, t^{\alpha_2}, \dots, t^{\alpha_n}]$. What we must prove is that $\gamma = \sum_{i=1}^n c_i \alpha_i$ for some nonnegative integers c_is. Let $b_i = mq_i + r_i$, where $q_i \geq 0$ and $0 \leq r_i < m$ are integers. Thus, we have

$$\gamma - \sum_{i=1}^n q_i \alpha_i = \sum_{i=1}^n \frac{r_i}{m} \alpha_i \in Z \cap L.$$

It then follows from the hypothesis that $\gamma - \sum_{i=1}^n q_i \alpha_i = \sum_{i=1}^n d_i \alpha_i$ for some nonnegative integers d_is. Hence $\gamma = \sum_{i=1}^n (q_i + d_i) \alpha_i$ as desired. Q.E.D.

Now, if G is a finite connected graph on $[d]$, then the lattice L_G generated by $\{\rho(e);\ e \in E(G)\}$ is

$$L_G = \{(a_1, a_2, \ldots, a_d) \in \mathbb{Z};\ a_1 + a_2 + \cdots + a_d \text{ is even}\}.$$

(In fact, if G' is a spanning tree of G, then $\{\rho(e);\ e \in E(G')\}$ is a $\mathbb{Z}$-basis for L_G.) By virtue of [2, Theorem 2.1], the hypothesis of Lemma A.2 is guaranteed if G satisfies the odd cycle condition. This completes the direct proof of (iii) $\Rightarrow$ (i) of Corollary 2.3.

Note added in proof. After this paper was submitted, the authors learned that the normalization of $K[G]$ is also obtained in [8].

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