NORMAL POLYTOPES, TRIANGULATIONS, AND KOSZUL ALGEBRAS

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This paper is devoted to the algebraic and combinatorial properties of polytopal semigroup rings defined as follows. Let P be a lattice polytope in $\mathbb{R}^n$, i. e. a polytope whose vertices have integral coordinates, and K a field. Then one considers the embedding $\iota\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$, $\iota(x) = (x, 1)$, and defines S_P to be the semigroup generated by the lattice points in $\iota(P)$; the K-algebra $K[S_P]$ is called a *polytopal semigroup ring*. Such a ring can be characterized as an affine semigroup ring that is generated by its degree 1 elements and coincides with its normalization in degree 1.

The first question to be asked about $K[S_P]$ is whether it is normal, and a geometric or combinatorial characterization of normality is certainly the most important problem in the theory of polytopal semigroup rings. (By a theorem of Hochster [18], the normality of $K[S_P]$ implies the Cohen–Macaulay property.) However, it is by no means clear whether such a characterization exists. The best known upper approximation to normality is the existence of a unimodular lattice covering (that is, a covering by lattice simplices of normalized volume 1). In Section 1 we show that the homothetic images cP of an arbitrary lattice polytope have such a covering for $c \gg 0$. The existence of a unimodular covering is derived from a unimodular triangulation of the unit n-cube.

The second ring-theoretic question we are interested in is the Koszul property: a graded K-algebra R is called *Koszul* if K has a linear free resolution as an R-module. (The resolution is linear if all the entries of its matrices are forms of degree 1; see Backelin and Fröberg [3] for a discussion of the basic properties of Koszul algebras.) It is immediate that a Koszul algebra is generated by its degree 1 component and is defined by degree 2 relations. (Though these properties do in general not imply that R is Koszul, no counterexample seems to be known among the semigroup rings.) A sufficient condition for the Koszul property is the existence of a Gröbner basis of degree 2 elements for the defining ideal of R (for example, see [9]).

An algebraic approach to the multiples cP yields that $K[S_{cP}]$ is normal for $c \geq \dim P - 1$, a Koszul algebra for $c \geq \dim P$, and a level ring of a-invariant -1 for $c \geq \dim P + 1$ (this means that the canonical module is generated by elements of degree 1). The Koszul property is proved by the Gröbner basis argument just mentioned; actually we generalize the theorem on the Koszul property of high Veronese subrings of an algebras generated in degree 1 (Eisenbud, Reeves, and Totaro [12]) to algebras that are just finite modules over a subalgebra generated in degree 1. This algebraic result is of general interest.

A basic tool for the study of polytopal semigroup rings is the connection between regular triangulations of P and Gröbner bases of the defining ideal I_P of $K[S_P]$

established by Sturmfels [23]. (All the triangulations of lattice polytopes to be considered in this paper are triangulations into lattice simplices.) After a discussion of some auxiliary results for the manipulation of regular triangulations, we show in Section 2 that polytopes whose facets are parallel to the hyperplanes given by the equations $X_i = 0$ and $X_i - X_j = 0$ have regular unimodular triangulations such that the minimal non-faces of the associated simplicial complexes are edges. It follows that these polytopes are normal and Koszul.

Let us call the maximal number of vertices of a minimal non-face of a triangulation Δ its degree. Then it is clear that a triangulation of an n-polytope P is of degree at most $n+1$, and there obviously exist lattice n-polytopes P for which every full triangulation, i. e. a triangulation for which every lattice point is a vertex, has degree $n+1$: if ∂P contains exactly $n+1$ lattice points and P has at least one interior lattice point, then the boundary lattice points form a minimal non-face. (A unimodular triangulation is evidently full). However, it will be shown in Section 3 that these obvious exceptions are the only ones: if P has at least $n+2$ lattice points in its boundary or no interior lattice point, then P has a regular full triangulation of degree at most n. If P is a polygon (i. e. of dimension 2), then every full triangulation of P is unimodular, and it follows that I_P has a Gröbner basis of degree 2, provided that P has at least 4 lattice points in its boundary; in particular such polygons yield Koszul algebras. Furthermore, in this case one can always find a unimodular triangulation of degree 2 that is induced by a lexicographic term order.

Bruns and Gubeladze [6] discuss the semigroup rings defined by rectangular simplices. Despite of their 'simplicity', these rings illustrate many of the phenomena discussed in the following.

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1. Polytopal semigroup rings

1.1. **Preliminaries.** We use the following notation: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are the additive groups of integral, rational, and real numbers, respectively; $\mathbb{Z}_+$, $\mathbb{Q}_+$ and $\mathbb{R}_+$ denote the corresponding additive subsemigroups of non-negative numbers, and $\mathbb{N} = \{1, 2, \dots\}$. An *affine semigroup* is a semigroup (always containing a neutral element) which is finitely generated and can be embedded in $\mathbb{Z}^n$ for some $n \in \mathbb{N}$.

We write $\mathrm{gp}(S)$ for the group of differences of S, i. e. $\mathrm{gp}(S)$ is the smallest group (up to isomorphism) which contains S. Thus every element $x \in \mathrm{gp}(S)$ can be presented as $s - t$ for some $s, t \in S$.

An affine semigroup S is called *normal* if every element $x \in \mathrm{gp}(S)$ such that $cx \in S$ for some $c \in \mathbb{N}$ belongs to S. It is well known that for any field K and any affine semigroup S the normality of the semigroup ring $K[S]$ is equivalent to the normality of S (see Hochster [18] or Bruns and Herzog [7], 6.1.4). The normalization

$\overline{S}$ of a semigroup S is the set of all $x \in \mathrm{gp}(S)$ for which there exists $c \in \mathbb{N}$ with $cx \in S$; it follows that $\overline{S}$ is a normal semigroup.

Let M be a subset of $\mathbb{R}^n$. We set

$$L_M = M \cap \mathbb{Z}^n,$$
$$E_M = \{(x, 1) \colon x \in L_M\} \subset \mathbb{Z}^{n+1};$$

so L_M is the set of lattice points in M, and E_M is the image of L_M under the embedding $\mathbb{R}^n \mapsto \mathbb{R}^{n+1}$, $x \mapsto (x, 1)$. Very frequently we will consider $\mathbb{R}^n$ as a hyperplane of $\mathbb{R}^{n+1}$ under this embedding; then we may identify L_M and E_M. By S_M we denote the subsemigroup of $\mathbb{Z}^{n+1}$ generated by E_M.

Now suppose that P is a (finite convex) lattice polytope in $\mathbb{R}^n$, where 'lattice' means all the vertices of P belong to the integral lattice $\mathbb{Z}^n$. The affine semigroups of the type S_P will be called *polytopal semigroups*. A lattice polytope P is *normal* if S_P is a normal semigroup.

It follows immediately from the dimension theory of commutative semigroup rings that

$$\dim K[S_P] = \dim(P) + 1.$$

for an arbitrary field K. Note that S_P (or, more generally, S_M) is a graded semigroup, i. e. $S_P = \bigcup_{i=0}^{\infty} (S_P)_i$ such that $(S_P)_i + (S_P)_j \subset (S_P)_{i+j}$; its i-th graded component $(S_P)_i$ consists of all the elements $(x, i) \in S_P$. Therefore $R = K[S_P]$ is a graded K-algebra in a natural way. Its i-th graded component R_i is the K-vector space generated by $(S_P)_i$. The elements of $E_P = (S_P)_1$ have degree 1, and therefore R is a homogeneous K-algebra in the terminology of [7].

Remark 1.1.1. If P and P' are two lattice polytopes in $\mathbb{R}^n$ that are integral-affinely equivalent, then $S_P \cong S_{P'}$.

Integral-affine equivalence means the equivalence under some affine transformation $\psi \in \mathrm{Aff}(\mathbb{R}^n)$ carrying $\mathbb{Z}^n$ onto $\mathbb{Z}^n$. The remark follows from the fact that such an integral-affine transformation of $\mathbb{R}^n$ can be lifted to (a uniquely determined) linear automorphism of $\mathbb{R}^{n+1}$ given by a matrix $\alpha \in \mathrm{GL}_{n+1}(\mathbb{Z})$. (Of course, we understand that $\mathbb{R}^n$ is embedded in $\mathbb{R}^{n+1}$ by the assignment $x \mapsto (x, 1)$).

Next we describe the normalization of a semigroup ring that is 'almost' a polytopal semigroup ring.

Proposition 1.1.2. *Let M be a finite subset of $\mathbb{Z}^n$. Let $C_M \subset \mathbb{R}^n$ be the (convex) cone generated by E_M. Then the normalization of $R = K[S_M]$ is the semigroup ring $\overline{R} = K[\mathrm{gp}(S_M) \cap C_M]$. Furthermore, with respect to the natural gradings of R and $\overline{R}$, one has $R_1 = \overline{R}_1$ if and only if $M = P \cap \mathbb{Z}^n$ for some lattice polytope P.*

Proof. It is an elementary observation that $G \cap C$ is a normal semigroup for every subgroup G of $\mathbb{R}^{n+1}$ and that every element $x \in \mathrm{gp}(S_P) \cap C$ satisfies the condition $cx \in S_P$ for some $c \in \mathbb{N}$.

Consider $\mathbb{R}^n$ as a hyperplane in $\mathbb{R}^{n+1}$ as above. Then the degree 1 elements of $\mathrm{gp}(S_P) \cap C$ are exactly those in the lattice polytope generated by $\mathrm{gp}(S_P) \cap C \cap \mathbb{R}^n$. This implies the second assertion. □

The class of polytopal semigroup rings can now be characterized in purely ring-theoretic terms.

Proposition 1.1.3. *Let R be a domain. Then R is (isomorphic to) a polytopal semigroup ring if and only if it has a grading $R = \bigoplus_{i=0}^{\infty} R_i$ such that*

(i) *$K = R_0$ is a field, and R is a K-algebra generated by finitely many elements $x_1, \dots, x_m \in R_1$;*
(ii) *the kernel of the natural epimorphism $\varphi\colon K[X_1, \dots, X_m] \to R$, $\varphi(X_i) = x_i$, is generated by binomials $X^{\mathbf{a}} - X^{\mathbf{b}}$ where $X^{\mathbf{a}} = X_1^{a_1} \dots X_m^{a_m}$ for $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{Z}_+^m$;*
(iii) *$R_1 = \overline{R}_1$ where $\overline{R}$ is the normalization of R (with the grading induced by that of R).*

Proof. We have seen above that a polytopal semigroup ring has the properties (i) and (iii). Let $E_M = \{x_1, \dots, x_m\}$. Then the kernel I_P of the natural projection $K[X_1, \dots, X_m] \mapsto K[x_1, \dots, x_m]$, $X_i \mapsto x_i$, is generated by binomials (see Gilmer [16], §7).

Conversely, a ring with property (ii) is a semigroup ring over K with semigroup H equal to the quotient of $\mathbb{Z}_+^m$ modulo the congruence relation defined by the pairs $(\mathbf{a}, \mathbf{b})$ associated with the binomial generators of $\operatorname{Ker}\varphi$ ([16], §7); in particular, H is finitely generated. Since R is a domain, H is cancellative and torsion-free, and 0 is its only invertible element. Thus it can be embedded in $\mathbb{Z}_+^n$ for a suitable n (for example see [7], 6.1.5), and we may consider $x_1, \dots, x_m$ as points of $\mathbb{Z}_+^n$. Set $x_i' = (x_i, 1) \in \mathbb{Z}_+^{n+1}$ and S equal to the semigroup generated by the x_i'. We claim that R is isomorphic to $K[S]$. In fact, let $\psi\colon K[X_1, \dots, X_m] \to K[S]$ be the epimorphism given by $\psi(X_i) = x_i'$. We obviously have $\operatorname{Ker}\psi \subset \operatorname{Ker}\varphi$, but the converse inclusion is also true: if $X^{\mathbf{a}} - X^{\mathbf{b}}$ is one of the generators of $\operatorname{Ker}\varphi$, then $X^{\mathbf{a}}$ and $X^{\mathbf{b}}$ have the same total degree, and therefore they are in $\operatorname{Ker}\psi$, too.

Finally it remains to be shown that $x_1', \dots, x_n'$ are exactly the lattice points in the polytope spanned by them. This, however, follows directly from (iii) and 1.1.2 above. □

1.2. **Normality and unimodular coverings.** We begin with a sufficient condition for the normality of a polytopal semigroup ring. (Not all polytopal semigroups are normal as will be demonstrated by some examples in Section 1.2; see also Hoa [17].)

Proposition 1.2.1. *If an affine semigroup S is a union (set-theoretically) of normal subsemigroups S_α which have the same groups of differences $\operatorname{gp}(S_\alpha)$ (in $\operatorname{gp}(S)$), then S itself is normal.*

In fact, $\operatorname{gp}(S_\alpha) = \operatorname{gp}(S)$ for all indices α, and the proof is straightforward.

Recall that an n-dimensional lattice simplex Δ in $\mathbb{R}^n$ is called a *unimodular* simplex if its volume has the smallest possible value $1/n!$ (or normalized volume 1; we fix on $\mathbb{R}^n$ the standard translation invariant volume function). The verification of the equivalence of the following three conditions is left to the reader.

(i) Δ is a unimodular lattice simplex in $\mathbb{R}^n$;
(ii) Δ is a lattice simplex in $\mathbb{R}^n$ and $\operatorname{gp}(S_\Delta) = \mathbb{Z}^{n+1}$;

(iii) Δ is a lattice simplex in $\mathbb{R}^n$ and for some (equivalently, any) vertex v_0 of Δ the elements

$$v_1 - v_0, \dots, v_n - v_0 \in \mathbb{Z}^n$$

form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$, where $v_1, \dots, v_n$ are the other vertices of Δ.

A collection of (unimodular) lattice simplices covering P is called a (*unimodular*) *covering* of P.

Proposition 1.2.2. *Let P be an n-polytope in $\mathbb{R}^n$. If P has a unimodular covering, then it is normal.*

Proof. Assume $P = \bigcup_\alpha \Delta_\alpha$ where the Δ_α are unimodular lattice simplices. Then $S_{\Delta_\alpha} \cong \mathbb{Z}_+^{n+1}$ and $\operatorname{gp}(S(\Delta_\alpha)) = \mathbb{Z}^{n+1}$ for all α. Since free semigroups are normal, the proof is complete in view of the previous proposition. □

Let P be a n-dimensional polytope in $\mathbb{R}^n$. Clearly, the equality $\operatorname{gp}(S_P) = \mathbb{Z}^{n+1}$ holds for the polytopes P which are covered by lattice unimodular simplices (as we have mentioned in the proof of Proposition 1.2.2). However, it is not true in general that $\operatorname{gp}(S_P) = \mathbb{Z}^{n+1}$. For instance, for any $n \geq 3$ and $c \in \mathbb{N}$ there exists a lattice simplex $\delta \subset \mathbb{R}^n$ such that $\delta \cap \mathbb{Z}^n$ is just the vertex set of δ and $\operatorname{vol}(\delta) = c/n!$; in this situation $\operatorname{gp}(S_\delta)$ is a subgroup of $\mathbb{Z}^{n+1}$ of index c.

However, after changing the lattice of reference, we can always assume that $\operatorname{gp}(S_P) = \mathbb{Z}^{n+1}$. Let $M \subset \mathbb{R}^n$ be the lattice generated by the differences of the vertices of P. Then we replace $\mathbb{Z}^{n+1}$ by $M \oplus \mathbb{Z}$.

Question 1.2.3. Let the n-dimensional lattice polytope $P \subset \mathbb{R}^n$ satisfy the conditions

$$\text{(i) } \operatorname{gp}(S_P) = \mathbb{Z}^{n+1}, \qquad \text{(ii) } P \text{ is normal.}$$

Does P then have a unimodular covering?

In other words, is the existence of a unimodular covering a necessary (and sufficient) condition for the normality of P?

The answer to this question seems to be open. A more special case of a covering by unimodular lattice simplices is a triangulation (Δ_α) into such simplices, called a *unimodular triangulation*. It follows immediately that every integral point of P is a vertex of at least one Δ_α. In general, if we speak of a *triangulation* (Δ_α), then we always require that the simplices of (Δ_α) are lattice simplices; it is called *full* if every lattice point in P is the vertex of some simplex Δ_α.

Proposition 1.2.4. (a) *A lattice polytope that has a unimodular triangulation is normal.*
(b) *Every full triangulation of a lattice polygon (2-dimensional polytope) is unimodular.*
(c) *There exists a normal 4-dimensional lattice polytope P that has no unimodular triangulation.*

Proof. (a) This is just a special case of Proposition 1.2.2.

Part (b) follows from the observation that a lattice triangle Δ has area $1/2$ if and only if its vertices are the only integral points of Δ. In particular, a triangulation (Δ_α) of a lattice polygon is unimodular if and only if it is full.

(c) will be discussed after the proof of Theorem 1.3.1. $\square$

In the last part of this subsection we describe the connection between unimodular coverings and the canonical module of a polytopal semigroup ring. Let P be a lattice polytope in $\mathbb{R}^n$ and $\mathcal{C}$ a covering of P by (not necessarily unimodular) lattice simplices. We say that a face $\sigma \in \mathcal{C}$ is *interior* if $\sigma \not\subset \partial P$, and we call the number

$$\operatorname{int}\deg \mathcal{C} := \min\{\dim \sigma \mid \sigma \text{ is an interior face of } \mathcal{C}\} + 1$$

the interior degree of $\mathcal{C}$. For every affinely independent set $\{x_1, \dots, x_t\}$ of points we denote by $\langle x_1, \dots, x_t\rangle$ the simplex spanned by them. Let $M_{\mathcal{C}}$ denote the ideal generated by the monomials corresponding to the sums $x_1 + \dots + x_t$ with $x_i \in E_P$ such that the simplex $\langle x_1, \dots, x_t\rangle$ is a minimal interior face of $\mathcal{C}$ with respect to inclusion. By a theorem of Danilov [10] and Stanley [22] the canonical module ω_R of a normal affine semigroup ring $R = K[S]$ is spanned over K by the monomials corresponding to the points of $\operatorname{gp}(S)$, equivalently: of S, inside the relative interior of the cone C generated by S; see also [7], 6.3.5 or Trung and Hoa [25]. This applies in particular to $K[S_P]$ where P is a normal polytope.

For a Cohen-Macaulay graded ring R the number

$$a(R) = -\min\{i\colon (\omega_R)_i \neq 0\}$$

is called the *a-invariant* of R (see [7], Chapters 3 and 4). If $R = K[S_P]$ is normal, then $a(R) < 0$, since the monomials spanning ω_R have positive degrees, as pointed out above.

Proposition 1.2.5. *Let P be a lattice polytope with a unimodular covering $\mathcal{C}$. Then the ideal $M_{\mathcal{C}}$ is the canonical module ω_R of $R = K[S_P]$, and*

$$a(R) = -\operatorname{int}\deg \mathcal{C}.$$

Proof. Let C_P denote the convex cone spanned by E_P in $\mathbb{R}^{n+1}$. The conclusion will follow if every lattice point x in the interior of C_P can be written as a sum $x_1 + \dots + x_t + y$ for some minimal interior face $\langle x_1, \dots, x_t\rangle$ of $\mathcal{C}$ and $y \in S_P$. Let $\sigma \in \mathcal{C}$ be a unimodular lattice simplex that covers the intersection point of P with the line passing through x and the origin. Then we may write x as a sum $x_1 + \dots + x_r$ of vertices of σ ($x_1, \dots, x_r$ need not be different). Let ρ be the convex hull of these vertices. Since x is in the interior of C_P, $\rho \not\subseteq \partial P$. Hence ρ is an interior face of $\mathcal{C}$. Let ε be a minimal interior face of $\mathcal{C}$ in ρ, say $\varepsilon = \langle x_1, \dots, x_t\rangle$. Put $y = x_{t+1} + \dots + x_r$. Then we get $x = x_1 + \dots + x_t + y$, as required. $\square$

Recall that a graded algebra R is called *level* if the canonical module ω_R of R is generated by elements of the same degree. This notion leads us to call a unimodular covering $\mathcal{C}$ of P *s-level* if the dimension of every minimal interior face of $\mathcal{C}$ is $s - 1$.

Corollary 1.2.6. *If P has an s-level unimodular covering, then $R = K[S_P]$ is level with $a(R) = -s$.*

Now we will use the above result to prove the level property of polygon semigroup rings. ($\sharp M$ denotes the cardinality of the set M.)

Theorem 1.2.7. *Let P be a lattice polygon with $\sharp L_P \geq 4$. Then $R = k[S_P]$ is level with $a(R) = -2$ if P has no interior lattice points, and $a(R) = -1$ else.*

Proof. By Proposition 1.2.4 (b) and Lemma 1.2.6 we only need to show that P has an s-level triangulation $\mathcal{C}$ for the appropriate integer $s = 1$ or $s = 2$. If P has no interior lattice points, we choose any triangulation $\mathcal{C}$ of P. Since every lattice point of $\mathcal{C}$ lies on ∂P, every edge not contained in ∂P is a minimal interior face of $\mathcal{C}$. Moreover, since $\sharp L_P \geq 4$, every triangle of $\mathcal{C}$ is not a minimal interior face of $\mathcal{C}$. Therefore, $\mathcal{C}$ is 2-level. If P has an interior lattice point, say x, then we connect x with the vertices of P. As a consequence we obtain a triangulation of P. Let $\mathcal{C}$ be any full triangulation of P which is finer than this triangulation. Then every edge of $\mathcal{C}$ which does not lie on ∂P must have a vertex not contained in ∂P. Therefore, no edge of $\mathcal{C}$ is a minimal interior face of $\mathcal{C}$. Thus $\mathcal{C}$ is 1-level. □

1.3. **High multiples of polytopes.** Let $P \subset \mathbb{R}^n$ be a polytope. Then cP denotes the image of P under the homothetic transformation of $\mathbb{R}^n$ with factor c and centre at the origin $0 \in \mathbb{R}^n$.

Theorem 1.3.1. *For any lattice polytope P there exists $c_0 > 0$ such that cP has a unimodular covering (and, hence, is normal by Proposition 1.2.2) for all $c \in \mathbb{N}$, $c > c_0$.*

Proof. We will use the well-known (and easy) observation that any finite convex rational polyhedral cone in $\mathbb{R}^n$ admits a finite subdivision into simplicial cones C_α such that the edges of each C_α correspond to a basis of $\mathbb{Z}^n$; more precisely, we obtain a basis of $\mathbb{Z}^n$ if we choose on each edge of C_α the first integral point different from 0. (Equivalently, toric varieties admit equivariant resolutions of singularities; Kempf et al. [19] or Fulton [14].) 'Subdivision' here means that the intersection $C_\alpha \cap C_{\alpha'}$ is a face (of arbitrary dimension) of both C_α and $C_{\alpha'}$.

Now let P be our polytope (of arbitrary dimension n) and v be an arbitrary vertex of it. Since the properties of P we are dealing with are invariant under integral-affine transformations (see above), we can assume $v = 0 \in \mathbb{Z}^n$. Let C be the cone in $\mathbb{R}^n$ spanned by 0 as its vertex and P itself, i. e. C corresponds to the corner of P at v. Let $C = \bigcup_\alpha C_\alpha$ be a subdivision into simplicial cones C_α as above. So the edges of C_α for each α are determined by the radial directions of some basis $\{e_{\alpha 1}, \dots, e_{\alpha n}\}$ of $\mathbb{Z}^n$. Denote by Λ_α the parallelepiped in $\mathbb{R}^n$ spanned by the edges $[0, e_{\alpha 1}], \dots, [0, e_{\alpha n}] \subset \mathbb{R}^n$. Thus $\operatorname{vol}(\Lambda_\alpha) = 1$ for all α. Equivalently, $\Lambda_\alpha \cap \mathbb{Z}^n$ coincides with the vertex set of Λ_α. Clearly, each of the C_α is covered by parallel translations of Λ_α (precisely as $\mathbb{R}^n_+$ is covered by parallel translations of the standard unit n-cube).

For each α and each $c \in \mathbb{N}$ let $Q_{\alpha c}$ be the union of the parallel translations of Λ_α inside $C_\alpha \cap cP$. Clearly, $Q_{\alpha c}$ is not convex in general. By $c^{-1}Q_{\alpha c}$ we denote the homothetic image of $Q_{\alpha c}$ centered at $v = 0$ with factor c^{-1}. The detailed verification of the following claim is left to the reader.

Claim. *Let F_v^{op} denote the union of all the facets of P not containing v (i. e. 0 in our case). Then for any real $\varepsilon > 0$ there exists $c_\varepsilon \in \mathbb{N}$ such that*

$$P \setminus U_\varepsilon(F_v^{\text{op}}) \subset \bigcup_\alpha c^{-1} Q_{\alpha c}$$

whenever $c > c_\varepsilon$ ($U_\varepsilon(F_v^{\text{op}})$ denotes the ε-neighbourhood of F_v^{op} in $\mathbb{R}^n$).

Let us just remark that the crucial point in showing this inclusion is that the covering of each C_α by parallel translations of the $c^{-1}\,\Lambda_\alpha$ becomes finer in the appropriate sense when c tends to ∞. (The finiteness of the collection $\{C_\alpha\}$ is of course essential).

For an arbitrary vertex w of P we define F_w^{op} analogously.

Claim. *There exists $\varepsilon > 0$ such that*

$$\bigcap_w U_\varepsilon(F_w^{\text{op}}) = \emptyset,$$

where w runs over all vertices of P.

Indeed, first one easily observes that

$$\bigcap_w U_\varepsilon(F_w^{\text{op}}) = \bigcap_F U_\varepsilon(F),$$

where on the right hand side F ranges over the set of facets of P, while $U_\varepsilon(F)$ is the ε-neighbourhood of F, and then one completes the proof as follows. Consider the function

$$d\colon P \to \mathbb{R}_+, \qquad d(x) = \max(\operatorname{dist}(x, F)),$$

where F ranges over the facets of P and $\operatorname{dist}(x, F)$ stands for the (Euclidean) distance from x to F. The function d is continuous and strictly positive. So, by the compactness of P, it attains its minimal value at some $x_0 \in P$. Now it is enough to choose $\varepsilon < d(x_0)$.

Summing up the two claims, one is directly lead to the conclusion that, for $c \in \mathbb{N}$ sufficiently large, cP is covered by lattice n-parallelepipeds which are integral-affinely equivalent to the standard unit cube, i. e. they have volume 1. Now the proof of our theorem is finished by the well-known fact that the standard unit cube has a unimodular triangulation (see Subsection 2.3 for more details.) □

We have still to provide a justification for part (c) of 1.2.4 in which we have stated that a normal polytope P does not always have a unimodular triangulation. Bouvier and Gonzalez–Sprinberg [5] have found that the cone D in $\mathbb{R}^4$ spanned by $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,0,1)$, and $(1,3,4,7)$ does not have a subdivision into simplicial cones C_α satisfying the following conditions: (i) the edges of C_α correspond to a basis of $\mathbb{Z}^4$ (as described in the proof of 1.3.1); (ii) each edge of C_α is a ray from 0 through an element of the (uniquely determined) minimal set E of generators of the semigroup $D \cap \mathbb{Z}^4$.

Let the polytope $P \in \mathbb{R}^4$ be spanned by $E \cup \{0\}$. It can be checked numerically that P is a normal polytope and $E \cup \{0\} = P \cap \mathbb{Z}^4$. If P had a unimodular triangulation (Δ_α), then the cones (with vertices in $0 \in \mathbb{R}^5$) over those Δ_α that contain $0 \in \mathbb{R}^4$ would constitute a subdivison of D satisfying the conditions (i) and (ii) above. (See also Sturmfels [24], 13.17.)

As will be discussed in Subsection 2.3, the subsets $Q_{\alpha c} \subset cP$, mentioned in the proof of Proposition 1.3.1 have a unimodular triangulation; moreover, these triangulations are regular and the minimal non-faces of the corresponding simplicial complexes are necessarily edges (i. e. have dimension 1). (For all of these notions see Subsection 2.1). This observation suggests the following

Question 1.3.2. Let P be a lattice polytope (of arbitrary dimension). Does the polytope cP then have a unimodular triangulation for $c \in \mathbb{N}$ sufficiently large? Can such a triangulation be chosen to be regular? Can it be chosen such that the minimal non-faces of the corresponding simplicial complex are edges, and furthermore level of interior degree 1?

We will see below that all the algebraic properties one can derive from the existence of such a triangulation are indeed satisfied. Furthermore, the existence of a regular unimodular triangulation of cP for *some* $c \gg 0$ is a major result of [19], p. 161, Theorem 4.1. It is however by no means clear that the existence of such a triangulation for cP has anything to do with its existence for $(c+1)P$.

One can give an algebraic proof of the normality of cP for c sufficiently large, avoiding a reference to the triangulations of cubes. The algebraic approach not only yields an explicit range for c, but also several other properties of $K[S_{cP}]$. Altogether, these properties give a rather complete structural description of the rings $K[S_{cP}]$.

Recall from the introduction that a graded K-algebra R is called a *Koszul algebra* if K (considered as the R-module $R/\mathfrak{m}$ where $\mathfrak{m}$ is the maximal homogeneous ideal) has a linear free resolution over R. Clearly, a Koszul algebra is generated over K by its degree 1 elements, and the defining ideal of every representation $K[X_1, \dots, X_m] \to R$ that maps $X_1, \dots, X_m$ to a basis of the vector space R_1 is generated by homogeneous polynomials of degree 2. We call a polytope P *Koszul* if $K[S_P]$ is Koszul for every field K. (See also Remark 1.3.5 below.)

The c-th Veronese subring $\bigoplus_i R_{ic}$ of a graded ring R is denoted by $R^{(c)}$. If $x \in R^{(c)}$ is homogeneous of degree kc as an element of R, then its *normalized* degree as an element of $R^{(c)}$ is k.

Theorem 1.3.3. *Let P be a lattice n-polytope with* $\operatorname{gp}(S_P) = \mathbb{Z}^{n+1}$.
(a) *Then cP is normal for $c \geq n-1$, Koszul for $c \geq n$, and level of a-invariant -1 for $c \geq n+1$.*
(b) *If P is normal, then cP is Koszul for $c \geq (n+1)/2$.*

Proof. We may assume that K is infinite. If K should be finite, then we pass to some infinite extension field L of K; for each c we have $L[S_{cP}] = K[S_{cP}] \otimes_K L$, and all the properties considered in the theorem are invariant under an extension of K.

A key point of the proof is the relationship between $K[S_{cP}]$ and the c-th Veronese subrings of $R = K[S_P]$ and its normalization S (see 1.1.2 for the description of S): one has the inclusions

$$R^{(c)} \subset K[S_{cP}] \subset S^{(c)}$$

of graded K-algebras. (In general both of these inclusions are strict, and one can give examples where the first inclusion is strict for all c.) It is easy to see that

$K[S_{cP}]$ is normal if and only if $K[S_{cP}] = S^{(c)}$, equivalently, if $S^{(c)}$ is generated by its elements of normalized degree 1. If P is normal then this equality holds for all c.

Let us first show that cP is normal for $c \geq n$. Afterwards we will improve the bound. We choose a graded Noether normalization $R_0 \subset R$. Then S is a finite R_0-module generated by $1 \in S$ and homogeneous elements $y_1, \dots, y_w$ of positive degree. Since S is Cohen-Macaulay by Hochster's theorem, S is a free module over the polynomial ring R_0, and thus these elements can even be chosen such that $1, y_1, \dots, y_w$ form a basis of S.

In order to bound the degree of the y_t we look at the Hilbert series

$$H_S(t) = \frac{1 + h_1 t^1 + \cdots + h_s t^s}{(1-t)^{n+1}}, \qquad h_s \neq 0.$$

Then $h_i = \sharp\{j\colon \deg y_j = i\}$. Furthermore, $a(S)$ is the degree of $H_S(t)$ as a rational function, so that $s = n + 1 + a(S) \leq n$, since $a(S) < 0$ (see the discussion preceding 1.2.5). Thus $\deg y_j \leq n$ for all j.

It follows easily that $S^{(c)}$, $c \geq n$, is generated by its elements of normalized degree 1: every element of $S^{(c)}$ is a K-linear combination of the monomials $x_{i_1} \dots x_{i_v} y_j$ where $v + \deg y_j$ is a multiple of c. Therefore, if $c \geq \deg y_j$, then, as a K-algebra, $S^{(c)}$ is generated by the monomials $x_{i_1} \dots x_{i_v} y_j$ with $v + \deg y_j = c$. This proves the normality of $K[S_{cP}]$ for $c \geq n$.

In order to derive the level property we use a similar argument. Let ω be the canonical module of S. Its Hilbert series is

$$H_\omega(t) = \frac{t^{n+1} + h_1 t^n + \cdots + h_s t^{n+1-s}}{(1-t)^{n+1}}$$

(see [22] or [7], 4.3.8). It is also a free R_0-module, and as such a module it has a basis of elements of degree at most $n+1$ (and $n+1$ is indeed attained as such a degree). Since $K[S_{cP}] = S^{(c)}$ for $c \geq n$, its canonical module is $\omega^{(c)}$. (This follows either by general algebraic arguments or by the description of the canonical module given above 1.2.5.) Similarly as above we conclude that the canonical module of $K[S_{cP}]$ is generated by its elements of normalized degree 1 if $c \geq n+1$ (even as a $R_0^{(c)}$-module).

Let us now show that cP is also normal for $c = n-1$. This is clear from the previous arguments if P has no interior lattice point: in that case one has $a(R) \leq -2$. In the general case we start with a full lattice triangulation (Δ_α) of P. (Such a triangulations always exists; see the discussion preceding Lemma 3.1.1.) Then each simplex Δ_α has no interior lattice point. The normalization $\overline{S}_P$ of S_P is the union of the integral closures

$$\hat{S}_\alpha = \{x \in \mathbb{Z}^{n+1}\colon kx \in S_\alpha \text{ for some } k \in \mathbb{N}\}$$

of the semigroups $S_\alpha = S_{\Delta_\alpha}$ in $\operatorname{gp}(S_P) = \mathbb{Z}^{n+1}$. (Note that $\operatorname{gp}(S_\alpha) = \mathbb{Z}^{n+1}$ if and only if Δ_α is unimodular.) The assertions on the Hilbert series and the canonical module of S hold analogously for $K[\hat{S}_\alpha]$ (for example, see [7], p. 265). Since $a(K[\hat{S}_\alpha]) \leq -2$, it follows that $K[\hat{S}_\alpha]$ has a basis of degree at most $n-1$ as a $K[S_\alpha]$-module (the latter ring is a polynomial ring whose indeterminates correspond to the vertices of

Δ_α). Since such a basis can always be chosen to consist of monomials, we conclude that each of the semigroups $\hat{S}_\alpha$ has a description as follows: there exist elements y_k of degree at most $n-1$ such that each element z of $\hat{S}_\alpha$ is a product $x_1 \cdots x_l y_k$ with elements x_i corresponding to the vertices of Δ_α. Consequently this holds for the union $\overline{S}_P$ of the $\hat{S}_\alpha$ with respect to the set of lattice points of P, whence S is generated as an R-module by elements of degree at most $n-1$, as was to be shown. (However, note that one can replace R by R_0 in the last statement only if P has no interior lattice point.)

Part (a) is complete once we have proved the Koszul property of $K[S_{cP}]$ for $c \geq n$. However, it is useful to treat (b) first. If P is normal, then one has $R = S$ so that S is generated by its degree 1 elements. The Castelnuovo-Mumford regularity $\operatorname{reg}(R)$ (see [11]) is given by

$$\operatorname{reg}(R) = \max\{i + j : H^i_{\mathfrak{m}}(R)_j \neq 0\}$$

($\mathfrak{m}$ is the irrelevant maximal ideal of R). Since (for example by local duality) $a(R) = \max_j\{H^{n+1}_{\mathfrak{m}}(R) \neq 0\}$ and $H^i_{\mathfrak{m}}(R) = 0$ for $i < n+1$ because of the Cohen-Macaulay property of R, we see that $\operatorname{reg}(R) = n + 1 + a(R) = s \leq n$ (with the notation introduced above).

Now we use the theorem of Eisenbud, Reeves, and Totaro [12] by which $R^{(c)}$ is Koszul for $c \geq (\operatorname{reg}(R) + 1)/2)$. This completes the proof of (b). (Note that the results in [12] are formulated in terms of $\operatorname{reg}(I_P) = \operatorname{reg}(R) + 1$).

If P is not normal, then S is not generated by its degree 1 elements, but it is Cohen-Macaulay and a finitely generated R-module, and this is sufficient to make its Veronese subalgebras Koszul for $c \geq \operatorname{reg}(S)$; see Theorem 1.4.1(b) below. □

We single out a result derived in the previous proof:

Corollary 1.3.4. *Let P be a lattice n-polytope. Then the normalization of $K[S_P]$ is generated as a $K[S_P]$-module by elements of degree at most $n-1$.*

One should note that 1.3.3 and 1.3.4 include the normality of lattice polygons stated in 1.2.4.

Remark 1.3.5. (a) The theorem of Eisenbud, Reeves, and Totaro and Theorem 1.4.1 even says that the defining ideal of $K[S_{cP}]$ has a Gröbner basis (see Eisenbud [11] for an introduction to Gröbner bases) of degree 2 for $c \geq (n+1)/2$ if P is normal and for $c \geq n$ in general; if we could find such a Gröbner basis with squarefree initial monomials, then we could draw strong combinatorial consequences for cP. (See Subsection 2.1 for the connection between Gröbner bases and regular triangulations.) However, we do not see how to modify the proof of 1.4.1 in order to achieve such an improvement.

(b) We do not know an example of a polytope P for which the Koszul property of $K[S_P]$ depends on K. However, in general the graded Betti numbers of K as a $K[S_P]$-module depend on K. Such an example is given by the affine semigroup ring R associated with the minimal triangulation of the real projective plane as described in Bruns and Herzog [8], Theorem 2.1. This semigroup ring is polytopal (with a grading different from that in [8]), and the third Betti number of K in characteristic 2 is greater by 1 than that in any other characteristic.

(c) The assertion on the normality of cP in 1.3.3 is essentially equivalent to the results of Ewald and Wessels [13] and Liu, Trotter, and Ziegler [21] which, however, have been derived by different methods. Modifying an example of [13], one sees easily that the bound $c \geq \dim P - 1$ for the normality is sharp. In fact, let $P \subset \mathbb{R}^n$ be the polytope whose vertices are $e_0 = 0$, $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$, and $a_n = (1, \dots, 1, n)$; then $\operatorname{gp}(S_P) = \mathbb{Z}^{n+1}$, the vertices are the only lattice points in P, and $(1, \dots, 1, n-1) \in \mathbb{Z}^{n+1}$ belongs to a minimal system of generators of $\overline{S}_P$. (We are grateful to G. Ziegler for informing us about the results of [13] and [21].)

1.4. **High Veronese subrings are Koszul.** The following theorem and its proof generalize the main result of Eisenbud, Reeves, and Totaro [12] who showed it for the case $R = S$ (and $c \geq (\operatorname{reg}(R) + 1)/2$). Unfortunately the proof given in [12] requires several modifications, forcing us to include all the details. For the application to polytopal semigroup rings part (b) of the theorem is sufficient. Since the theorem is of independent interest, we treat the general case.

Theorem 1.4.1. *Let K be an infinite field and S graded K-algebra that is a finitely generated module over a graded subalgebra R generated by its degree 1 elements. Let $y_1, \dots, y_n \in S$ be homogeneous elements such that $y_1, \dots, y_n, y_{n+1} = 1$ is a minimal system of generators of the R-module S. Furthermore, let $\mathfrak{a}_j \subset R$, $j = i+1, \dots, n+1$, denote the annihilator of y_j modulo the R-submodule of S generated by $y_{j+1}, \dots, y_{n+1}$ (thus $\mathfrak{a}_{n+1}$ is the kernel of the structure morphism $R \to S$). We set*

$$e = \max_j \deg y_j \quad \textit{and} \quad d = \max_j \Bigl(\deg y_j + \operatorname{reg}(R/\mathfrak{a}_j) \Bigr).$$

(a) *Then the following hold:*

(i) *for $c \geq e$ the Veronese subring $S^{(c)}$ is generated as a K-algebra by its elements of normalized degree 1;*

(ii) *for $c \geq d+1$ the defining ideal of $S^{(c)}$ with respect to a suitable representation as a quotient of a polynomial ring has a Gröbner basis of elements of degree at most 2;*

(iii) *$S^{(c)}$ is a Koszul algebra for $c \geq d + 1$.*

(b) *Suppose that S is a Cohen-Macaulay ring. Then the bounds e and $d + 1$ in* (a) *can be replaced by* $\operatorname{reg}(S)$.

Proof. Part (a)(i) appears already in Bourbaki [4], Chap. III, §1, Lemme 1. It is easily seen as follows. Let $x_1, \dots, x_m$ be a vector space basis of R_1. Every homogeneous element $x \in S$ with $u = \deg x \geq e$ is a K-linear combination of the products $x_{k_1} \dots x_{k_u}$ and $y_j x_{k_1} \dots x_{k_r}$ with $u = \deg y_j + r$, and therefore $R^{(c)}$ is generated as a K-algebra by such products of total degree c and normalized degree 1. Below we will always use the notation just introduced.

For (a)(ii) and (a)(iii) we consider the epimorphism

$$\rho\colon Q \to S, \qquad Q = K[X_1, \dots, X_m, Y_1, \dots, Y_n], \quad X_i \mapsto x_i, Y_i \mapsto y_i;$$

we set $\deg X_i = 1$, $\deg Y_j = \deg y_j$. That (a)(iii) follows from (a)(ii) (separately for each c) has been shown in Bruns, Herzog, and Vetter [9]. Part (a)(ii) requires some auxiliary results, and we will prove it later.

It is essential for (b) that we can replace R by a Noether normalization R_0 of R that is generated by degree 1 elements. Then S is generated as an R_0-module by elements of degree at most $\operatorname{reg}(S)$, as shown in the proof of 1.3.3. Therefore we can replace e by the (possibly worse) bound $\operatorname{reg}(S)$ in (a)(i). Furthermore we have $\mathfrak{a}_j = 0$ for all j, and therefore $d = e = \operatorname{reg}(S)$ after the replacement of R by R_0. The rest of the proof of (b) is also postponed. $\square$

In the following we will freely use that a term order on Q induces a term order on each of its subrings generated by monomials. On Q we set up a term order evaluating the following rules in the sequence given; in (ii) we denote by $\sharp_Y$ the number of factors Y_j: $\mu \prec \nu$ if (i) $\deg\mu < \deg\nu$, (ii) $\sharp_Y\mu < \sharp_Y\nu$, (iii) the Y-factor of μ is reverse-lexicographically smaller than that of ν, (iv) the X-factor of μ is reverse-lexicographically smaller than that of ν. The variables are ordered by $Y_1 \prec \cdots \prec Y_n$ and $X_1 \prec \cdots \prec X_m$; we use the term 'reverse lexicographic order' as in [11].

We introduce some further notation: $P = K[X_1, \dots, X_m]$, and $Q^{\langle c\rangle}$ is the subalgebra of Q generated by the monomials $X_{k_1} \dots X_{k_c}$ and $Y_j X_{k_1} \dots X_{k_r}$ with $\deg Y_j + r = c$. The epimorphism $\rho\colon Q \to S$ introduced above induces an epimorphism $\rho^{\langle c\rangle}\colon Q^{\langle c\rangle} \to S^{(c)}$ for $c \ge e$ (as seen above). We set $J = \operatorname{Ker}\rho$ and $J^{\langle c\rangle} = \operatorname{Ker}\rho^{\langle c\rangle} = J \cap Q^{\langle c\rangle}$.

Let I_j be the preimage of $\mathfrak{a}_j$ with respect to the restriction of ρ to P. Then $R/\mathfrak{a}_j \cong P/I_j$, and in particular $\operatorname{reg}(R/\mathfrak{a}_j) = \operatorname{reg}(P/I_j)$. By a theorem of Bayer and Stillman [1], after a generic change of variables in P we may assume that $\operatorname{in}(I_j)$ is generated by elements of degree $\le \operatorname{reg}(P/I_j) + 1$.

Lemma 1.4.2. (a) *Let $c \ge d+1$. Then the ideal* $\operatorname{in}(J^{\langle c\rangle}) = \operatorname{in}(J) \cap Q^{\langle c\rangle}$ *of* $Q^{\langle c\rangle}$ *is generated by monomials of the following type:*

(i) $(\mu_1 Y_j)(\mu_2 Y_k)$ *with* $\mu_1, \mu_2 \in P$ *and* $\deg\mu_1 Y_j = \deg\mu_2 Y_k = c$*,*
(ii) μY_j *with* $\mu \in \operatorname{in}(I_j)$, $\deg\mu Y_j = c$*;*
(iii) $\nu \in \operatorname{in}(I_{n+1})$, $\deg\nu = c$.

Moreover, all monomials of type (i) *are contained in* $\operatorname{in}(J^{\langle c\rangle})$.

(b) *If* $R = K[X_1, \dots, X_m]$ *and S is a free R-module, then* $\operatorname{in}(J^{\langle c\rangle})$ *is generated by the monomials of type* (i) *for all* $c \ge \operatorname{reg}(S)$.

Proof. A monomial $\lambda \in Q$ belongs to $Q^{\langle c\rangle}$ exactly when $\deg\lambda = kc$ for some k and $\sharp_Y\lambda \le k$. The way we have ordered the monomials of Q guarantees that the initial monomial of a homogeneous element $f \in Q$ has the highest number of factors Y_j among all its monomials. Thus, $f \in Q^{\langle c\rangle}$ if and only if $\operatorname{in}(f) \in Q^{\langle c\rangle}$. This implies the equation $\operatorname{in}(J^{\langle c\rangle}) = \operatorname{in}(J) \cap Q^{\langle c\rangle}$ since J is generated by homogeneous elements.

Since $y_1, \dots, y_n, 1$ generate S as an R-module, and thus as a P-module, it follows that J has a system of generators consisting of polynomials

(1) $f_0 + f_1\ell(Y) + Y_k Y_l$ with $f_i \in P$ and a linear form ℓ (we include the case $k = l$).
(2) $f_0 + f_1\ell(Y)$ with $f_0, f_1 \in P$, and

(3) $f \in P$.

If we replace Y_i by e_i, then (2) and (3) yield a system of generators of W above. It is clear that the elements of (1) belong to a Gröbner basis $\mathcal{G}$ of J with respect to our term order, and that the leading monomial of every other element of a (reduced) Gröbner basis $\mathcal{G}$ has at most 1 factor Y_j. Thus the elements of $\mathcal{G}$ are again of the types (1), (2), and (3). The leading monomial of (1) is Y_kY_l, that of (2) has the form μY_j with $\mu \in \operatorname{in}(I_j)$, and that of (3) belongs to $\operatorname{in}(I_{n+1})$.

Since $Y_kY_l \in \operatorname{in}(J)$ for all k, l, it follows that every monomial of type (i) belongs to $\operatorname{in}(J^{\langle c\rangle})$. Thus it remains to show that every monomial of $\operatorname{in}(J^{\langle c\rangle})$ with at most 1 factor Y_j is a multiple of one of the monomials of type (ii) or (iii). This is evident if one uses the inequalities for c and the fact that I_j is generated by monomials of degree at most $c - \deg y_j$.

For (b) we note that $\mathcal{G}$ consists of elements of type (1) so that only the bound $c \geq e = \operatorname{reg}(S)$ is needed. □

Let V be the polynomial ring over K whose indeterminates Z_μ are indexed by the monomials $\mu = X_{k_1} \cdots X_{k_c}$, $k_1 \leq \cdots \leq k_c$, and $\mu = Y_jX_{k_1} \cdots X_{k_r}$, $k_1 \leq \cdots \leq k_r$, $c = \deg Y_j + r$. For $c \geq e$ we define the epimorphism $\psi\colon V \to Q^{\langle c\rangle}$ by the substitution $\psi(T_\mu) = \mu$. Then $S^{(c)}$ is a homomorphic image of V via the composition $\rho^{\langle c\rangle} \circ \psi$. If we let U be the polynomial subring of V generated by the indeterminates T_μ with $\sharp_Y \mu = 0$, then we obtain the following commutative diagram in which the vertical arrows denote the natural inclusions:

$$\begin{array}{ccccc} U & \longrightarrow & P^{(c)} & \longrightarrow & R^{(c)} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ V & \xrightarrow{\psi} & Q^{\langle c\rangle} & \xrightarrow{\rho^{\langle c\rangle}} & S^{(c)}. \end{array}$$

We introduce a term order on V as follows. Let M and N be monomials of V. In the case in which $\psi(M) \neq \psi(N)$ we set $M \prec N$ if $\psi(M) \prec \psi(N)$. This defines an order on the indeterminates T_μ, so that the case $\psi(M) = \psi(N)$ can be covered by letting $M \prec N$ if M precedes N in the reverse lexicographic order.

The next task is the analysis of $\mathfrak{a} = \operatorname{Ker} \psi$. For this purpose one introduce the K-linear map $\tau\colon V \to V$ by setting $\tau(M)$ for a monomial M to be the smallest monomial N with respect to $\prec$ such that $\psi(N) = \psi(M)$. A monomial M is called *standard* if $M = \tau(M)$. By the definition of τ it is obvious that $\psi(\tau(f)) = \psi(f)$ for every element $f \in V$. Furthermore each monomial dividing a standard monomial is standard. Therefore the vector subspace spanned by the non-standard monomials is an ideal H of V.

It is useful to describe $\tau(M)$ explicitly. We list all the factors X_i and Y_j of $\psi(M)$ as follows:

$$X_{i_1}, \dots, X_{i_s}, \quad i_1 \leq \cdots \leq i_s, \qquad Y_{j_1}, \dots, Y_{j_t}, \quad j_1 \leq \cdots \leq j_t.$$

Then we arrange them in the following sequence. The first factor is Y_{j_1}. It is followed by $X_{i_1} \cdots X_{i_u}$ with $u = c - \deg Y_{j_1}$. Then we proceed with Y_{j_2} followed by $X_{i_{u+1}} \cdots X_{i_v}$ with $v = u + c - \deg Y_{j_2}$ etc. Then we cut the total product into monomials $\mu_1, \dots, \mu_k$ of degree c. It is not hard to see that $\tau(M) = Z_{\mu_1} \cdots Z_{\mu_k}$.

In fact, suppose $\tau(M) = Z_{\nu_1} \cdots Z_{\nu_k}$. We may assume that $\nu_1, \dots, \nu_k$ are in descending order with respect to $\prec$, and, furthermore, that each μ_i has its factors ordered as just described: the potential factor Y_j first, and then the factors X_l in descending order with respect to $\prec$. If their product written out in this order is not the sequence described above, then we can pass to a smaller product $Z_{\pi_1} \cdots Z_{\pi_k}$ by exchanging factors between ν_i and ν_{i+1} for some i. This contradicts the choice of $\tau(M)$. (We leave it to the reader to check all the combinatorial details.)

The previous argument also shows that a non-standard monomial contains a non-standard monomial of degree 2. In other words, H is generated by monomials of degree 2.

Lemma 1.4.3. *$H = \operatorname{in}(\mathfrak{a})$, and H is generated by monomials of degree 2.*

Proof. By definition the standard monomials correspond bijectively to the monomials in $Q^{\langle c\rangle}$, and they also correspond bijectively to the monomial basis of V/H. It follows that Q/H and $Q^{\langle c\rangle}$ have the same Hilbert function. Thus the equation $H = \operatorname{in}(\mathfrak{a})$ is proved, once we know that $H \subset \operatorname{in}(\mathfrak{a})$. But this is also clear: if M is non-standard, then it is the leading monomial of $M - \tau(M) \in \mathfrak{a}$. That H is generated by degree 2 monomials, has been seen above. $\square$

It is useful to introduce the K-linear map $\sigma\colon Q^{\langle c\rangle} \to V$ by assigning each monomial $\mu \in Q^{\langle c\rangle}$ the unique standard monomial M with $\psi(M) = \mu$.

Now we look at the initial ideal of $\mathfrak{b} = \operatorname{Ker} \rho^{\langle c\rangle} \circ \psi$ of which we claim that it has a Gröbner basis of degree 2.

Lemma 1.4.4. (a) *For $c \geq d+1$ the initial ideal* $\operatorname{in}(\mathfrak{b})$ *is generated by* (i) $\operatorname{in}(\mathfrak{a})$, (ii) *the monomials $Z_\kappa Z_\lambda$ with $\sharp_Y \kappa = \sharp_Y \lambda = 1$,* (iii) *the monomials $\sigma(\mu)$ with $\mu \in Q^{\langle c\rangle}$, $\mu = \nu Y_j$, $\nu \in \operatorname{in}(I_j)$, and* (iv) *the monomials $\sigma(\pi)$ with $\pi \in \operatorname{in}(I_{n+1}) \cap Q^{\langle c\rangle}$.*
(b) *Under the hypothesis of 1.4.2*(b) $\operatorname{in}(\mathfrak{b})$ *is generated by the monomials of type* (i) *and type* (ii) *for all $c \geq \operatorname{reg}(S)$.*

Proof. (a) Pick $f \in \mathfrak{b}$. If the initial monomial A of f is non-standard, then it belongs to $\operatorname{in}(\mathfrak{a})$. If $\sharp_Y(\psi(A)) \geq 2$, then it is of type (ii). In the remaining case we note that $\psi(A)$ is the leading monomial of $\psi(f)$. In fact, if A is standard, then $A = \operatorname{in}(\tau(f))$ as well, and since $\psi(\tau(f)) = \psi(f)$, we may assume that $f = \tau(f)$. In this case the monomials of f are mapped to pairwise different monomials of $Q^{\langle c\rangle}$, and the leading monomial cannot be cancelled by the application of ψ. That A goes to the leading monomial of $\psi(f)$ follows from the definition of the term order on V. Since $\sharp_Y \psi(A) \leq 1$, the rest follows from Lemma 1.4.2 and the fact that $A = \sigma(\tau(A))$.

(b) In this case there are no monomials of type (iii) or (iv). Furthermore, note that 1.4.2(b) and 1.4.3 hold for $c \geq \operatorname{reg}(S)$. $\square$

Part (b) of 1.4.4 completes the proof of 1.4.1(b): $\operatorname{in}(\mathfrak{b})$ has indeed a Gröbner basis of degree 2 elements for $c \geq \operatorname{reg}(S)$, since S is a free R_0-module with a basis in degrees $\leq \operatorname{reg}(S)$.

The reader may ask why 1.4.4 does not yet prove our claim in the general case: in 1.4.2 we have shown that the ideal $\mathrm{in}(I_j)Y_j \cap Q^{\langle c\rangle}$ is generated by its elements of normalized degree 1. Therefore 1.4.4 should yield that $\mathrm{in}(\mathfrak{b})$ has a Gröbner basis of degree 1 and 2 elements. However, if $\mu \mid \nu$ for monomials $\mu, \nu \in Q^{\langle c\rangle}$, then it does by no means follow that $\sigma(\mu) \mid \sigma(\nu)$. Fortunately this obstruction can be overcome. As usual we call a monomial ideal $I \subset K[X_1, \dots, X_m]$ *combinatorially stable* if it contains with each monomial $X_{i_1} \cdots X_{i_k}$, $i_1 \le \cdots \le i_k$, all the monomials $X_j X_{i_1} \cdots X_{i_{k-1}}$ with $j \le i_k$.

According to a theorem of Bayer and Stillman [2] we may assume: $\mathrm{in}(I_j)$ is invariant under the action of the group of upper triangular matrices. Proposition 10 of [12] then implies that $\mathrm{in}(I_j)_{\ge r}$ is combinatorially stable for all $r \ge \mathrm{reg}(P/I_j) + 1$. Here $I_{\ge r}$ denotes the ideal generated by all elements of I that have degree $\ge r$.

We can now conclude the proof of Theorem 1.4.1. According to 1.4.4 we look at a monomial $\mu = \nu Y_j \in Q^{\langle c\rangle}$ with $\nu \in \mathrm{in}(I_j)$ (the case $j = n+1$ is covered if we let $Y_{n+1} = 1$). By 1.4.2 there exists a monomial $\nu' \in \mathrm{in}(I_j)$ that divides ν and for which $\nu' Y_j$ has normalized degree 1. We write μ as a product in the order from which $\sigma(\mu)$ is computed:

$$\mu = Y_j X_{i_1} \dots X_{i_r}, \qquad r = kc - \deg Y_j.$$

Let $s = c - \deg Y_j$. Then ν' is a product of s indeterminates among the X_{i_t}. Because of the combinatorial stability of $\mathrm{in}(I_j)$ in degrees $\ge c - \deg y_j \ge \mathrm{reg}(P/I_j) + 1$ it follows that $X_{i_1} \dots X_{i_s} \in \mathrm{in}(I_j)$, and $\sigma(Y_j X_{i_1} \dots X_{i_s})$ divides $\sigma(\mu)$. Altogether this shows that $\mathrm{in}(\mathfrak{b})$ is generated in degrees 1 and 2.

2. Regular triangulations

2.1. **Regular triangulations and Gröbner bases.** In this subsection we recall the notion of a regular polyhedral subdivision (called ‘projective’ in [19] and ‘coherent’ in Gelfand, Kapranov, and Zelevinsky [15]) and review the connection between the regular triangulations of a polytope P and the Gröbner bases of the defining ideal I_P of $K[S_P]$.

Let n be a natural number and $P \subset \mathbb{R}^n$ a polytope (of dimension n). A *polyhedral subdivision* of P is a finite system (Q_α) of subpolytopes of P such that $\dim Q_\alpha = \dim Q_\beta$ for all α and β and, furthermore, $Q_\alpha \cap Q_\beta$ is a face both of Q_α and Q_β (of arbitrary dimension, maybe empty). A polyhedral subdivision that consists of simplices only is called a *triangulation*.

Assume $Q \subset \mathbb{R}^n$ is a polytope of dimension n. A function $G\colon Q \to \mathbb{R}$ is said to be *linear* if there exists a function $G'\colon \mathbb{R}^n \to \mathbb{R}$ with

$$G'(x_1, \dots, x_n) = C_1 x_1 + \cdots + C_n x_n + C$$

for some $C_1, \dots, C_n, C \in \mathbb{R}$, such that $G'|Q = G$. Clearly, if G' exists, it is uniquely determined by G. The functions $\mathbb{R}^n \to \mathbb{R}$ of the type of G' are called *affine.*

Now assume $X \subset \mathbb{R}^n$ is a convex set. A function $F\colon X \to \mathbb{R}$ is *convex* if

$$F\Bigl(\sum_{i=1}^k \lambda_i x_i\Bigr) \ge \sum_{i=1}^k \lambda_i F(x_i) \quad \text{for all } k \in \mathbb{N}, \lambda_i \in [0,1],\ \sum_{i=1}^k \lambda_i = 1,\ x_i \in X.$$

(Sometimes such functions are called concave, and the convex ones are defined by the opposite inequality.)

Let (Q_α) be a polyhedral subdivision of P. A function $F\colon P \to \mathbb{R}$ is called *piecewise linear with domains of linearity* (Q_α) if the restrictions $F|Q_\alpha$ are linear for all α, and F is not linear on an arbitrary subpolytope of P strictly containing one of the Q_α.

Definition 2.1.1. A polyhedral subdivision $\{Q_\alpha\}$ of P is *regular* if there exists a piecewise linear convex function $F\colon P \to \mathbb{R}_+$ with domains of linearity (Q_α). Such a function F is called a *realizing function* of the subdivision (Q_α).

We have the following obvious observation: if F is a realizing function of the subdivision (Q_α), then $C_1F + C_2$ is so as well for arbitrary real numbers $C_1 > 0$ and $C_2 \geq 0$.

Not all polyhedral subdivisions of P are regular. Below we shall give an example in connection with the patching lemma 2.2.2.

Assume (Q_α) is a regular polyhedral subdivision of P with realizing function F. The subset

$$\{(x, F(x))\colon x \in P\} \subset \mathbb{R}^{n+1}$$

is a polyhedral ball (of dimension n) mapping isomorphically into P via projection. Let H_α denote the affine hull of $\{(x, F(x))\colon x \in Q_\alpha\}$. It is an n-dimensional affine hyperplane of $\mathbb{R}^{n+1}$. One easily sees that for any point $x \in P \setminus Q_\alpha$ the inequality

$$F(x) < h_\alpha(x)$$

holds, where $h_\alpha\colon \mathbb{R}^n \to \mathbb{R}$ is the unique affine function whose graph is H_α (this means $h_\alpha(x) \in H_\alpha$ for all $x \in \mathbb{R}^n$). Conversely, it is also true (and easily seen) that, for a polyhedral subdivision $\{Q_\alpha\}$ of P and a piece-wise linear function $F\colon P \to \mathbb{R}_+$ with domains of linearity (Q_α), the validity of the inequalities

$$F(x) < h_\alpha(x)$$

for all α and all $x \in P \setminus Q_\alpha$ implies the regularity of the subdivision (Q_α). In this situation F is a realizing function of this subdivision (h_α again denotes an affine continuation of $F|Q_\alpha$). Moreover, we could require the validity of these inequalities only for the points $x \in P \setminus Q_\alpha$ which appear as a vertex of some Q_β – the subdivision would again be regular and F would again realize it. This holds true because a polytope is the convex hull of its vertices, and an affine function preserves barycentric coordinates. We will freely use these observations in the sequel.

Now assume $A \subset \mathbb{R}^n$ is a finite subset whose convex hull ($\operatorname{conv}(A)$ for short) has dimension n. Let $\varphi\colon A \to \mathbb{R}_+$ be an arbitrary function; later on it will be called a *height function* and $\varphi(a)$ will be called the *height* at a. Consider the convex hull W (in $\mathbb{R}^{n+1}$) of the set

$$\{(a, 0)\colon a \in A\} \cup \{(a, \varphi(a))\colon a \in A\}.$$

Below $\mathbb{R}^n$ is identified with $\mathbb{R}^n \oplus 0 \subset \mathbb{R}^{n+1}$. We see that W is an $(n+1)$-dimensional polytope, and $\operatorname{conv}(A)$ is one of its facets. Let $\pi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ denote the projection with respect to the last coordinate. Then any facet of W different from $\operatorname{conv}(A)$ will project under π either onto a facet of $\operatorname{conv}(A)$ or onto an n-dimensional subpolytope of $\operatorname{conv}(A)$. The latter facets form the *roof* A_φ of W. (If $A = L_P$, then we write P_φ for A_φ.) Clearly, the vertices of the subpolytopes thus obtained will belong to A, but in general not all elements of A will appear as such vertices.

Using the general observations above we conclude that the subpolytopes of $\operatorname{conv}(A)$ which are n-dimensional π-images of the facets of W constitute a *regular* polyhedral subdivision of $\operatorname{conv}(A)$ (see [15]). Thus any height function $\varphi\colon A \to \mathbb{R}_+$ defines a regular subdivision of $\operatorname{conv}(A)$. The piecewise linear function $F\colon \operatorname{conv}(A) \to \mathbb{R}_+$ naturally determined by the facets of A_φ is convex and, moreover, is a realizing function of the regular subdivision of $\operatorname{conv}(A)$ defermined by φ (in the way described above). We say that *F is spanned by φ.* We see that for arbitrary real numbers $C_1 > 0$ and $C_2 \geq 0$ the two height functions φ and $C_1\varphi + C_2$ determine the same regular subdivision of $\operatorname{conv}(A)$.

If A and φ are as above and d is a facet or, more generally, an arbitrary face of $\operatorname{conv}(A)$ then $\varphi | A \cap d$ is a height function for $A \cap d$ and, thus, defines a regular polyhedral subdivision of d with vertices in $A \cap d$. Strictly speaking, we should first identify $\operatorname{Aff}(d)$ (the affine hull of d) with $\mathbb{R}^{\dim(d)}$, but in this situation the identification will be tacitly understood (if no confusion arises). It is obvious that this regular subdivision of d is nothing but the subdivision induced in a natural way by the regular subdivision of $\operatorname{conv}(A)$ determined by φ. Therefore we arrive at the following conclusion: if $\varphi, \varphi'\colon A \to \mathbb{R}_+$ are two height functions which agree on $A\cap d$ for some face d of $\operatorname{conv}(A)$, then both of the subdivisions of $\operatorname{conv}(A)$, determined by φ and φ' respectively, induce the same regular polyhedral subdivision of d.

The importance of regular triangulations for polytopal semigroup rings stems from their connection with Gröbner bases. For the convenience of the reader we briefly review this connection; see Sturmfels [24] for a detailed treatment. Let $m = \sharp L_P$. As discussed in the proof of Proposition 1.1.3, the semigroup ring $k[S_P]$ has a presentation

$$k[S_P] = k[X_1, \dots, X_m]/I_P,$$

where $X_1, \dots, X_m$ correspond bijectively to the lattice points $x_1, \dots, x_m$ of P, and I_P is an ideal generated by binomials. For any term order $\prec$ on $K[X_1, \dots, X_m]$ there exists a weight function φ on $\{X_1, \dots, X_m\}$ such that the initial ideal of I_P with respect to $\prec$ equals the initial ideal $\operatorname{in}_\varphi(I_P)$ of I_P with respect to the partial term order determined by φ. In this case we say that φ *determines a Gröbner basis* of I_P. Considered as a height function on $\{x_1, \dots, x_m\}$, φ determines a regular subdivision Δ_φ of P, which in this case is actually a triangulation. This triangulation represents a simplicial complex on the set $\{x_1, \dots, x_m\}$ which we also denote by Δ_φ (in general not every x_i is a vertex of Δ_φ). The squarefree monomials $X_{i_1} \dots X_{i_r}$ for which $\{x_{i_1}, \dots, x_{i_r}\}$ is a non-face of Δ_φ generate the Stanley-Reisner ideal I_{Δ_φ}; as its generators are squarefree, it is a radical ideal. The quotient $K[X_1, \dots, X_m]/I_{\Delta_\varphi}$ is the Stanley-Reisner ring of Δ_φ. The connection between $\operatorname{in}_\varphi(I_P)$ and I_{Δ_φ} is given by the following theorem of Sturmfels (see [23] and [24], 8.3 and 8.8).

Theorem 2.1.2. (a) *Let φ be a weight function on $\{X_1, \dots, X_m\}$ that determines a Gröbner basis. Then* $\operatorname{Rad} \operatorname{in}_\varphi(I_P) = I_{\Delta_\varphi}$.

(b) *Conversely, given a regular triangulation* Δ *of* P*, there exists a weight function* φ *on* $\{X_1, \dots, X_m\}$ *with* $\Delta_\varphi = \Delta$ *that determines a Gröbner basis.*
(c) Δ_φ *is unimodular if and only if* $\operatorname{in}_\varphi(I_P) = I_{\Delta_\varphi}$.

Part (c) explains the special interest in unimodular triangulations. (In general the Gröbner basis associated with a regular triangulation is not uniquely determined.)

Corollary 2.1.3. *If* P *has a regular unimodular triangulation* Δ *whose minimal non-faces are edges (i. e. of dimension* 1*), then* P *is Koszul.*

Proof. If the minimal non-faces of Δ are edges, then I_Δ is generated by monomials of degree 2, and if, in addition, Δ is unimodular, then 2.1.2(c) implies that I_P has a Gröbner basis of degree 2. Algebras defined by a Gröbner basis of degree 2 are Koszul according to [9]. □

2.2. **Perturbation and patching of regular triangulations.** In this subsection we prove three lemmas (on perturbation, patching and direct products) which will be useful in the construction of regular triangulations.

In the following a representation of $x \in \mathbb{R}^n$ as a linear combination $x = \sum_{i=1}^m \lambda_i x_i$ will be called *barycentric* if $\sum_{i=1}^m \lambda_i = 1$, and *convex* if additionally $\lambda_i \geq 0$ for all i.

Lemma 2.2.1. (Perturbation Lemma) *Let* $A \subset \mathbb{R}^n$ *be a finite subset,* $\varphi\colon A \to \mathbb{R}_+$ *a height function and* $a \in A$*. Suppose further that* (Q_α) *is the regular polyhedral subdivision of* $\operatorname{conv}(A)$ *determined by* φ*. Then there exists* $\delta \in \mathbb{R}$*,* $\delta > 0$*, such that for any height function* $\varphi'\colon A \to \mathbb{R}_+$ *with*

$$\text{(i) } \varphi'(b) = \varphi(b) \text{ for } b \in A \setminus \{a\}, \quad \text{and} \quad \text{(ii) } |\varphi'(a) - \varphi(a)| < \delta$$

a polytope Q *from* (Q_α) *survives in (i. e. is an element of) the regular polyhedral subdivision of* $\operatorname{conv}(A)$ *determined by* φ' *whenever*
(a) $a \notin Q$*, or*
(b) Q *is a simplex and* $Q \cap A$ *is the vertex set of* Q*.*

Proof. Let π be as above. If $a \notin Q$, then $\varphi(a) < h_Q(a)$, where $h_Q\colon \mathbb{R}^{n+1} \to \mathbb{R}$ is the affine linear function corresponding to the facet of A_φ which is mapped isomorphically to Q by π. So $\varphi'(a) < h_Q(a)$ for δ sufficiently small. Since

$$\varphi'|A \setminus \{a\} = \varphi|A \setminus \{a\},$$

we automatically have $\varphi'(b) < h_Q(b)$ for any element $b \in A \setminus (Q \cup \{a\})$. Every $x \in \operatorname{conv}(A) \setminus Q$ has a convex presentation $x = \sum_{i=1}^k \lambda_i a_i$ with $\lambda_1, \dots, \lambda_k > 0$, $F'(x) = \sum_{i=1}^k \lambda_i \varphi'(a_i)$, and $a_1, \dots, a_k \in A$ not all belonging to Q. Let F' be the piece-wise linear convex function spanned by φ'. Then every element $x \in \operatorname{conv}(A) \setminus Q$ has a convex presentation $x = \sum_{i=1}^k \lambda_i a_i$ with $\lambda_1, \dots, \lambda_k > 0$ and $a_1, \dots .a_k \in A$ not all belonging to Q, but belonging to some domain linearity of F'. It follows that

$$F'(x) = \sum_{i=1}^k \lambda_i \varphi'(a_i) < \sum_{i=1}^k \lambda_i h_Q(a_i) = h_Q(x)$$

(note that h_Q preserves barycentric representations). As we have observed, this inequality presisely means Q is involved in the regular subdivision of $\operatorname{conv}(A)$ determined by φ'.

Now assume Q is a simplex and $Q \cap A$ is the vertex set of Q. Let $a_1, \dots, a_{n+1} \in A$ be the vertices of Q. Then there exists a unique affine hyperplane H_Q in $\mathbb{R}^{n+1}$ passing through all the points $(a_i, \varphi(a_i)) \in \mathbb{R}^{n+1}$. Similarly, there exists a unique affine hyperplane H'_Q passing through the points $(a_i, \varphi'(a_i))$. Let h_Q and h'_Q denote the corresponding affine linear functions. Assume

$$h_Q(x_1, \dots, x_n) = C_1x_1 + \dots + C_nx_n + C,$$
$$h'_Q(x_1, \dots, x_n) = C'_1x_1 + \dots + C'_nx_n + C'$$

Clearly, $C'_1 \to C_1, \dots, C'_n \to C_n$ and $C' \to C$ when $\delta \to 0$. On the other hand $\varphi(a) < h_Q(a)$ for any $a \in A \setminus Q$. Therefore $\varphi'(a) < h'_Q(a)$ for $a \in A \setminus Q$ whenever δ is sufficiently small. As above, this means Q is involved in the regular subdivision of $\operatorname{conv}(A)$ corresponding to φ'. □

It follows immediately from Lemma 2.2.1(b) that if (Q_α) is a regular triangulation of $\operatorname{conv}(A)$ (notation as in the lemma), which is determined by the height function

$$\varphi\colon A \to \mathbb{R}_+$$

and satisfies the condition that any element of A is a vertex of some Q_α, then for any sufficiently small perturbation ψ of the height function φ (at all points of A) the corresponding regular polyhedral subdivision of $\operatorname{conv}(A)$ will be the same triangulation (Q_α). This observation is frequently used in the literature dealing with regular polyhedral subdivisions ([15], [23], [24]).

The next lemma is equivalent to [19], p. 115, Corollary 1.12. In view of the different terminology and for the convenience of the reader, we include a proof.

Lemma 2.2.2. (Patching Lemma) *Let $P \subset \mathbb{R}^n$ be a finite convex n-dimensional polyhedron, (Q_α) a regular polyhedral subdivision, and $(Q_{\alpha\beta})$ be a regular polyhedral subdivision of Q_α for each α. If there exist realizing functions*

$$F_\alpha\colon Q_\alpha \to \mathbb{R}_+$$

of the regular subdivisions $(Q_{\alpha\beta})$ (of Q_α) such that

$$F_\alpha | Q_\alpha \cap Q_{\alpha'} = F_{\alpha'} | Q_\alpha \cap Q_{\alpha'}$$

for all indices α and α', then $(Q_{\alpha\beta})_{\alpha,\beta}$ is a regular polyhedral subdivision of P.

Proof. Consider the function $F\colon P \to \mathbb{R}_+$ defined by $F(x) = F_\alpha(x)$ for $x \in Q_\alpha$. By our hypothesis F is well defined. Let $G\colon P \to \mathbb{R}_+$ be any realizing function of the regular subdivision (Q_α). For any $t > 0$ we consider the function

$$\Phi_t = G + (1/t)F.$$

We claim that $(Q_{\alpha\beta})_{\alpha,\beta}$ is a regular polyhedral subdivision of P and that Φ_t is its realizing function for all sufficiently large t.

We know that Φ_t is convex on each of Q_α and that for all α, β the restriction $\Phi_t | Q_{\alpha\beta}$ is affine for all natural $t \in \mathbb{N}$. To prove our claim it suffices to show that for t sufficiently large the following inequality holds:

$$\Phi_t(x) < H_{t\alpha\beta}(x) \quad \text{for all } \alpha, \beta \text{ and } x \in P \setminus Q_{\alpha\beta}$$

where $H_{t\alpha\beta}\colon \mathbb{R}^n \to \mathbb{R}$ denotes the unique affine continuation of $\Phi_t | Q_{\alpha\beta}$.

Let us introduce some more notation. For any α we denote by g_α the affine continuation of $G|Q_\alpha$ to $\mathbb{R}^n$, and that of $F_\alpha|Q_{\alpha\beta}(= F|Q_{\alpha\beta})$ by $f_{\alpha\beta}\colon \mathbb{R}^n \to \mathbb{R}$. Thus $H_{t\alpha\beta} = g_\alpha + (1/t)f_{\alpha\beta}$.

Let V denote the union of the vertex sets of all the $Q_{\alpha\beta}$. Then $g_\alpha(v) > G(v)$ whenever $v \in V$ is not a vertex of Q_α. Clearly for all $x \in P$ and all α, β we have

$$\Phi_t(x) \to G(x) \quad \text{and} \quad H_{t\alpha\beta}(x) \to g_\alpha(x)$$

when $t \to \infty$. Since V is a finite set, there exists t_0 for which

$$\Phi_t(v) < H_{t\alpha\beta}(v)$$

whenever $t > t_0$, $v \in V \setminus Q_\alpha$. Now suppose $v \in (V \cap Q_\alpha) \setminus Q_{\alpha\beta}$. Then

$$\Phi_t(v) = G(v) + (1/t)F(v) = g_\alpha(v) + (1/t)F_\alpha(v) < g_\alpha(v) + (1/t)f_{\alpha\beta}(v) = H_{t\alpha\beta}(v).$$

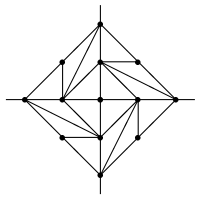
From now on we assume $t > t_0$.

Choose an arbitrary point $x \in P \setminus Q_{\alpha\beta}$. Then there exists a pair $(\alpha', \beta') \neq (\alpha, \beta)$ such that $x \in Q_{\alpha'\beta'}$. Therefore $x = \sum_{i=1}^k \lambda_i v_i$ has a convex representation with $v_i \in V \cap Q_{\alpha'\beta'}$ and $v_j \notin Q_{\alpha\beta}$ for at least one j. Note that Φ_t and $H_{t\alpha\beta}$ preserve barycentric coordinates on $Q_{\alpha'\beta'}$. In view of the fact that $v_j \notin Q_{\alpha\beta}$ for at least one j, the inequalities above yield

$$\Phi_t(x) = \sum_{i=1}^k \lambda_i \Phi_t(v_i) < \sum_{i=1}^k \lambda_i H_{t\alpha\beta}(v_i) = H_{t\alpha\beta}(\sum_{i=1}^k \lambda_i v_i) = H_{t\alpha\beta}(x).$$

□

Remark 2.2.3. In order to patch regular polyhedral subdivisions in the way described in Lemma 2.2.2 it is necessary that $(Q_{\alpha\beta})_\beta$ and $(Q_{\alpha'\beta'})_{\beta'}$ induce the same polyhedral subdivisions on $Q_\alpha \cap Q_{\alpha'}$ (notation as in the lemma). However this condition is not sufficient, not even in the planar case ($n = 2$). We consider the following triangulation Δ:



One can obtain Δ from the regular triangulation of the triangle in the first quadrant by two successive patchings. (The triangulation of the triangle in the first quadrant corresponds to a lexicographic term order; see 3.2.4 below.) However, it has the same characteristic function as its mirror image Δ' with respect to the x-axis. (The characteristic function assigns to each vertex v the sum of the volumes of the facets adjacent to v.) Since $\Delta \neq \Delta'$, it is not regular (see [15], Chapter 7, Theorem 1.7).

Lemma 2.2.4. (Direct Product Lemma) *Let $P_1 \subset \mathbb{R}^{n_1}, \dots, P_k \subset \mathbb{R}^n_k$ be polytopes of dimensions $n_1, \dots, n_k$ respectively. Suppose further $(Q_\alpha^{(1)}), \dots, (Q_\alpha^{(k)})$ are regular polyhedral subdivisions of $P_1, \dots, P_k$ respectively. Then*

$$\{Q_1 \times \dots \times Q_k \colon Q_i \in (Q_\alpha^{(i)}), i = 1 \dots, k\}$$

is a regular polyhedral subdivision of $P_1 \times \dots \times P_k$.

Proof. By induction on k (which is used only for simplicity of the notation) we can asume $k = 2$. Let $F_1\colon P_1 \to \mathbb{R}_+$ and $F_2\colon P_2 \to \mathbb{R}_+$ be realizing functions of the subdivisions $(Q_\alpha^{(1)})$ and $(Q_\alpha^{(2)})$ respectively. Consider the function

$$F\colon P_1 \times P_2 \to \mathbb{R}_+, \quad F(x,y) = F_1(x) + F_2(y).$$

It is now easy to show that F is convex and that its domains of linearity are precisely the products $Q_1 \times Q_2$, $Q_1 \in (Q_\alpha^{(1)})$, $Q_2 \in (Q_\alpha^{(2)})$ (use barycentric coordinates). $\square$

2.3. **Polytopes related to rectangular parallelepipeds.** A *standard n-dimensional rectangular parallelepiped* Λ is a polytope given by

$$\{x \in \mathbb{R}^n\colon 0 \le x_i \le a_i\}$$

for real numbers $a_1, \dots, a_n > 0$. Thus the points $\alpha_i = (0, \dots, 0, a_i, , 0, \dots, 0)$ are vertices of Q, and the vertex set $\operatorname{vert}(\Lambda)$ of Λ is given by

$$\Big\{\sum_{i \in S} \alpha_i\colon S \subset \{1, \dots, n\}\Big\},$$

where $\sum_{i \in \emptyset} \alpha_i = 0$. If all the a_i are equal to 1, then Λ will be called the *standard unit cube (of dimension n)*. A *unit cube* in $\mathbb{R}^n$ is defined as a subset of the type $x + \Lambda$ for some $x \in \mathbb{R}^n$ where Λ is the standard unit cube. Later on we shall use the notation Λ_n for the n-dimensional standard unit cube.

We fix the partial order on $\mathbb{R}^n$ under which $(a_1, \dots, a_n) \le (b_1, \dots, b_n)$ if $a_i \le b_i$ for all $i = 1, \dots n$.

We have the following obvious

Lemma 2.3.1. *Suppose $n \ge 2$. Then for any $i = 1, \dots, n$ the i-th coordinate embedding*

$$C_i\colon \mathbb{R}^{n-1} \to \mathbb{R}^n, \qquad C_i(a_1, \dots, a_{n-1}) = (a_1, \dots, a_{i-1}, 0, a_i, \dots, a_{n-1})$$

respects the order structures of $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$.

The results of this subsection are based on a unimodular triangulation of the unit cube that we are going to construct now. Presumably this triangulation is well-known, but we have no reference covering the details needed below.

The system T_n of n-dimensional simplices with vertices from $\operatorname{vert}(\Lambda_n)$ is inductively defined as follows. For $n = 1$ put $T_1 = \{\Lambda_1\}$. Assume $n > 1$. Then T_n is defined as the system of simplices each of which is the convex hull of some $\delta \in C_i(T_{n-1})$ and the vertex $(1, \dots, 1) \in \Lambda_n$ where i runs over $1, \dots, n$.

By induction on n one sees easily that $\dim(\Delta) = n$ and $\operatorname{vol}(\Delta) = 1/n!$ for all $\Delta \in T_n$. So T_n consists of unimodular lattice simplices (see subsection 1.1).

Here is an alternative description of T_n.

Lemma 2.3.2. *T_n consists precisely of those simplices whose vertex set is a maximal chain (i. e. linearly ordered subset) of $\operatorname{vert}(\Lambda_n)$.*

We leave the easy proof of the lemma to the reader; in conjunction with induction, the essential point is that $(1, \dots, 1)$ is the unique maximal element of $\operatorname{vert}(\Lambda_n)$.

Now we define the system $\mathcal{D}_n$ of $(n-1)$-dimensional hyperplanes in $\mathbb{R}^n$, $n \ge 2$, inductively as follows. For $n = 2$ set $\mathcal{D}_2 = \{\mathbb{R}(1,1)\}$, i. e. $\mathcal{D}_2$ consists of the single line passing through the diagonal $[(0,0),(1,1)]$ of Λ_2. Let $n \geq 3$. Then $\mathcal{D}_n$ is defined as the system of $(n-1)$-dimensional $\mathbb{R}$-subspaces of $\mathbb{R}^n$ spanned by $C_i(D)$ and $e_i = (0,0,\dots,0,1,0,\dots,0)$ for each $i = 1\dots,n$ and $D \in \mathcal{D}_{n-1}$. (For $n = 1$ we set $\mathcal{D}_1 = \emptyset$.) Observe that in the definition of $\mathcal{D}_n$ we could equivalently consider affine hulls; they are actually vector subspaces. Straightforward arguments show

Lemma 2.3.3. *The system $\mathcal{D}_n$ consists of the $(n-1)$-dimensional hyperplanes in $\mathbb{R}^n$ determined by all the linear equations $X_i = X_j$, $i,j = 1,\dots,n$, $i \neq j$. In particular $\sharp\mathcal{D}_n = \binom{n}{2}$.*

A facet F of a simplex $\Delta \in T_n$ is called *non-coordinate* if it is not parallel to $C_i(\mathbb{R}^{n-1})$ for any $i = 1,\dots,n$.

Lemma 2.3.4. (a) *The non-coordinate facets of the simplices belonging to T_n coincide with the simplices (of dimension $n-1$) spanned by the vertex sets $\{v_1, v_2, \dots, v_n\}$ such that $(0,0,\dots,0) = v_1 < v_2 < \dots < v_n = (1,1,\dots,1) \in \Lambda_n$.*
(b) *$\mathcal{D}_n$ coincides with the set of affine hulls of the non-coordinate facets of simplices from T_n.*

Proof. It follows from 2.3.2 that the facets of the simplices in T_n correspond bijectively to the chains $v_1 < \dots < v_n$ of vertices of Λ_n. If $(0,\dots,0) \neq v_1$, then all the v_i lie in a hyperplane given by the equation $X_j = 1$ where j is the unique index such that $v_{1j} = 1$. If $(1,\dots,1) \neq v_n$, then all the v_i lie in a hyperplane with the equation $X_j = 0$ where j is the unique index such that $v_{nj} = 0$.

Conversely suppose that $(0,0,\dots,0) = v_1 < v_2 < \dots < v_n = (1,1,\dots,1)$. Then there exist uniquely determined indices i,j,k such that $v_{i+1} = v_i + e_j + e_k$. Therefore the corresponding facet is contained in the hyperplane given by $X_j = X_k$, and it even spans this hyperplane as a vector space since $v_2,\dots,v_n$ are linearly independent. This shows the first claim and part of the second.

Finally, if we are given a hyperplane with equation $X_j = X_k$, then we can of course find a chain of vertices with exactly the data of the previous paragraph. $\square$

Lemma 2.3.5. (a) *For all $\Delta \in T_n$ and all $D \in \mathcal{D}_n$ the intersection $D \cap \Delta$ is a face (not necessarily a facet) of Δ,*
(b) *For $D \in \mathcal{D}_n$ and $x \in \mathbb{Z}^n$ the intersection $(x + D) \cap \Lambda_n$ contains an interior point of Λ_n if and only if $x + D = D$ (i. e. $x \in D$).*

Proof. (a) We use induction on n. For $n = 2$ the claim is clear. Suppose $n > 2$, and let D be the affine hull of $C_i(\partial)$ and e_i for some $i = 1\dots,n$ and $\partial \in \mathcal{D}_{n-1}$. That $D \cap \Delta$ is not a face of Δ means precisely that $D \cap \Delta$ contains an internal point of Δ. Consider the projection

$$\pi_i \colon \mathbb{R}^n \to \mathbb{R}^{n-1}, \qquad \pi_i(a_1,\dots,a_n) = (a_1,\dots,\check{a}_i,\dots,a_n).$$

We have $\pi_i \circ C_i = \mathrm{id}_{\mathbb{R}^{n-1}}$. So $\pi_i \circ C_i(D) = \partial$. By Lemma 2.3.2, $\delta = \pi_i(\Delta) \in T_{n-1}$. Since internal points of Δ project into internal points of δ we see that ∂ meets the interior of δ whenever D meets that of Δ. Hence the induction hypothesis applies.

(b) Since D is given by an equation $X_j - X_k = 0$, the translate $x + D$ has the equation $X_j - X_k = x_j - x_k$. If $x_j - x_k \neq 0$, then $|x_j - x_k| \geq 1$, but we have $|y_j - y_k| < 1$ for all interior points y of Λ_n. $\square$

Let $\mathcal{F}_n$ denote the system of coordinate hyperplanes in $\mathbb{R}^n$. These hyperplanes correspond to the equations $X_i = 0$, $i = 1 \dots, n$.

Definition 2.3.6. A finite convex polyhedron P in $\mathbb{R}^n$ is called *FD-bounded* if any facet of P is parallel to some hyperplane from $\mathcal{F}_n \cup \mathcal{D}_n$.

Lemma 2.3.7. *For any $n \geq 2$ the polyhedral subdivision of Λ_n determined by the system of hyperplanes $\mathcal{D}_n$ is the triangulation T_n.*

Moreover, for any pair of opposite facets of Λ_n the induced triangulations are the same modulo the corresponding unit coordinate parallel translation.

Proof. That T_n is the corresponding polyhedral subdivision follows directly from Lemma 2.3.4 and Lemma 2.3.5(a), and the assertion about the induced triangulations of opposite facets follows from the previous claim and Lemma 2.3.2. $\square$

Now let Λ be a lattice standard rectangular n-parallelepiped. For each unit lattice cube $\Lambda' \subset \Lambda$ there exists a unique $x \in \mathbb{Z}_+^n$ such that $\Lambda' = x + \Lambda_n$. For each such unit cube Λ' we fix its triangulation $x + T_n$. (Recall that $T_1 = \{\Lambda_1\}$ for $n = 1$). It follows immediately from Lemma 2.3.7 that the fixed system of triangulations defines a global triangulation of Λ. This triangulation will be denoted by $T(\Lambda)$. In particular $T(\Lambda_n) = T_n$.

Now let P be any FD-bounded lattice n-polyhedron. There exists $x \in \mathbb{Z}^n$ and a standard rectangular lattice parallelepiped Λ such that $x + P \subset \Lambda$. By Lemma 2.3.5(b) and 2.3.7 the triangulation $T(\Lambda)$ induces a triangulation of $x + P$, say T'. Thus T' is a triangulation of $x + P$ consisting of those simplices from $T(\Lambda)$ which are included in $x + P$. One easily observes that this triangulation is independent of the choices of x and Λ. It will be denoted by $T(P)$. Clearly, all the lattice points of P are involved in $T(P)$, and $T(P)$ consists of unimodular lattice simplices.

Lemma 2.3.8. *For P as above the minimal non-faces of the simplical complex associated with the triangulation $T(P)$ are edges.*

Proof. As above we can assume $P \subset \Lambda$ for some standard rectangular lattice parallelepiped Λ. The minimal non-faces of our simplicial complex will be minimal non-faces of the simplicial complex associated with $T(\Lambda)$. So we can assume $P = \Lambda$. Let $z_1, \dots, z_k \in \Lambda \cap \mathbb{Z}^n$, $k > 2$, determine a minimal non-face. Then for each pair $i, j = 1, \dots, k$ the points z_i and z_j must be connected by an edge involved in $T(P)$. This is only possible if all the z_i belong to the same unit lattice cube in Λ (by the definition of $T(\Lambda)$). Without loss of generality we can assume $z_1, \dots, z_k \in \Lambda_n$. By Lemma 2.3.2 the points $z_1, \dots, z_k$ determine a non-face if and only if they do not constitute a chain. Hence z_i and z_j are incomparable for some $i, j = 1, \dots, k$. This contradicts the minimality of our non-face. $\square$

Lemma 2.3.9. *For all suffficiently large positive real numbers ω the polyhedral subdivision of Λ_n, determined by the height function*

$$\varphi_\omega \colon \Lambda_n \cap \mathbb{Z}^n \to \mathbb{R}_+, \qquad \varphi_\omega(x_1, \dots, x_n) = \omega^{x_1 + \cdots + x_n},$$

is the triangulation T_n.

Proof. We use induction on n. For $n = 1$ there is nothing to show. Assume $n \geq 2$. For each $i = 1, \ldots, n$ we let $\varphi_{\omega,i}$ denote the composite map

$$\Lambda_{n-1} \cap \mathbb{Z}^{n-1} \xrightarrow{C_i} \Lambda_n \cap \mathbb{Z}^n \xrightarrow{\varphi_\omega} \mathbb{R}_+.$$

By the induction hypothesis the polyhedral subdivisions of Λ_{n-1} determined by all the $\varphi_{\omega,i}$ are the same T_{n-1} for ω sufficiently large.

In view of the induction hypothesis, the opposite of the claim is clearly the following statement: there exist an infinite sequence $(\omega_k)_{k\in\mathbb{N}}$ and a polytope Γ spanned by some vertices of Λ_n different from $(1, \ldots, 1)$ such that $\omega_k \to \infty$, $\dim \Gamma = n$, and Γ is a polytope from the polyhedral subdivision of Λ_n determined by $\varphi_k = \varphi_{\omega_k}$. Let $\gamma_1, \ldots, \gamma_{n+1}$ be affinely independent vertices of Γ (i. e. $\operatorname{conv}(\gamma_1, \ldots, \gamma_{n+1}) = \mathbb{R}^n$). Thus we have a barycentric representation

$$(1, \ldots, 1) = \lambda_1\gamma_1 + \cdots + \lambda_{n+1}\gamma_{n+1}.$$

Let Φ_k be the realizing function of the polyhedral subdivision of Λ_n, spanned by φ_k. So Γ is a domain of linearity for Φ_k. Let $L_k\colon \mathbb{R}^n \to \mathbb{R}$, be the affine continuation of $\Phi_k|\Gamma$. Then we know (see subsection 2.1) that for each point γ of Λ_n one has $\Phi_k(\gamma) \leq L_k(\gamma)$. Since affine functions preserve barycentric coordinates, we obtain

$$\varphi_k(1, \ldots, 1) = \Phi_k(1, \ldots, 1) \leq L_k(1, \ldots, 1) = \sum_{i=1}^{n+1} \lambda_i L_k(\gamma_i) = \sum_{i=1}^{n+1} \lambda_i \varphi_k(\gamma_i),$$

that is $\omega_k^n \leq \sum_{i=1}^{n+1} \lambda_i \omega_k^{|\gamma_i|}$, where for $x = (x_1, \ldots, x_n) \in \mathbb{R}$ we put $|x| = x_1 + \cdots + x_n$. But the last inequality is obviously violated for k sufficiently large. $\square$

Theorem 2.3.10. *Any FD-bounded lattice polyhedron P (of arbitrary dimension) is Koszul.*

Proof. Let P be such a polyhedron. We can assume $P \subset \Lambda$ for some standard rectangular lattice parallelepiped Λ. By Theorem 2.1.3 and Lemma 2.3.8 we only have to show the regularity of the triangulation $T(P)$. Since $T(P) \subset T(\Lambda)$ we can also assume $P = \Lambda$.

Let ω be a positive real number and consider the function

$$\varphi_\omega\colon\ \Lambda \cap \mathbb{Z}^n \to \mathbb{R}_+, \qquad \varphi_\omega(x) = \omega^{|x|}.$$

Any unit lattice cube in Λ has the form $z + \Lambda_n$ for some $z \in \mathbb{Z}^n \cap \Lambda$. We shall use the notation Λ_n^z for $z + \Lambda_n$. So $\Lambda_n^0 = \Lambda_n$. For any unit lattice cube Λ_n^z in Λ we set

$$\varphi_{\omega,z} = \varphi_\omega | \Lambda_n^z \cap \mathbb{Z}^n$$

for any $z, z' \in \mathbb{Z}^n \cap C$. We know that $\varphi_{\omega,z}$ and $\omega^{-|z|}\varphi_{\omega,z}$ define the same polyhedral subdivision of Λ_n^z. But the polyhedral subdivision of Λ_n^z determined by $\omega^{-|z|}\varphi_{\omega,z}$ is obtained by the polyhedral subdivision of Λ_n determined by $\varphi_{\omega,0}$ shifted by the vector z. By Lemma 2.3.9 the latter is nothing else but T_n for ω sufficiently large. Thus for ω sufficiently large, the polyhedral subdivision of Λ_n^z determined by $\varphi_{\omega,z}$ is $z + T_n$. Now assume ω is sufficiently large, and $\Phi_{\omega,z}$ are the realizing functions of the triangulizations $z + T_n$, spanned by $\varphi_{\omega,z}$. Clearly

$$\Phi_{\omega,z} | \Lambda_n^z \cap \Lambda_n^{z'} = \Phi_{\omega,z'} | \Lambda_n^z \cap \Lambda_n^{z'}\ .$$

Therefore the patching lemma will complete the proof once we know the regularity of the subdivision of Λ into unit lattice cubes. But the latter follows from the observation that $\{[0,1],[1,2],\dots,[a-1,a]\}$ is a regular subdivision of the segment $[0,a]$ (with the realizing function corresponding to $k \mapsto \sin\frac{\pi k}{a}, k = 0,\dots,a$) and the direct product lemma. □

Question 2.3.11. Let $\mathcal{E}_n$ be the collection of hyperplanes given by the equations

$$\sum_{i=1}^{n} \varepsilon_i X_i = 0, \qquad \varepsilon_i \in \{0, \pm 1\}.$$

Suppose P is an n-dimensional lattice polytope satisfying the following condition: if $P \cap \Lambda_n^z$, $z \in \mathbb{Z}^n$, has dimension n, then the facets of $P \cap \Lambda_n^z$, except at most one, are parallel to the coordinate hyperplanes, and the remaining one is parallel to some other member of $\mathcal{E}_n$. (In other words, up to reversing the directions of the basis vectors, $P \cap \Lambda_n^z$ has the form $\Delta \times \mathcal{I}$ where Δ is a rectangular 'unit' simplex spanned by k basis vectors and the origin and $\mathcal{I}$ is an $(n-k)$-dimensional unit interval representing the remaining directions.) Is such a polytope Koszul? That the answer is positive in the case $n = 2$, follows from Theorem 3.2.5 below. But more significantly, it can also be shown by a slight extension of the triangulation argument above.

In dimension 2 the condition above is equivalent to the weaker property that every facet of P is parallel to one of the hyperplanes in $\mathcal{E}_n$. In dimension $n \geq 3$ this weaker property is not sufficient for the Koszul property, as demonstrated by the polytope with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(0,0,0)$, and $(1,1,1)$ (the authors are grateful to B. Sturmfels for this example).

3. Degrees of triangulations and Koszul semigroup rings

3.1. **The degree of a triangulation.** Let Δ be a simplicial complex with vertex set V. Note that Δ has no non-face only if it is a simplex. If Δ is not a simplex, we set

$$\deg\Delta := \max\{\dim\sigma \,|\, \sigma \text{ is a minimal non-face of } \Delta\} + 1,$$

and call it the *degree* of Δ.

This notion is closely related to the Stanley-Reisner ring $k[\Delta] = k[X_1,\dots,X_n]/I_\Delta$ of Δ (see the discussion preceding Theorem 2.1.2). Since I_Δ is the ideal generated by the monomials $X_{i_1}\cdots X_{i_r}$ for which $\{X_{i_1},\dots,X_{i_r}\}$ is a non-face of Δ, $\deg\Delta$ is the maximal degree of the elements of a minimal system of generators of I_Δ. Thus $\deg\Delta \leq \dim\Delta + 1$.

It follows that the degree of any triangulation of a lattice polytope P in $\mathbb{R}^n$ is at most $n+1$. We shall characterize the lattice polytopes which have regular triangulations of degree $\leq n$.

In the following we will frequently use that every lattice polytope $P \subset \mathbb{R}^n$ has a regular full triangulation Δ (recall that for Δ to be full every lattice point of P must be a vertex of some simplex of Δ). We simply take the lexicographic order on $\mathbb{Z}^n$. It induces an order on the variables $X_1,\dots,X_m$ corresponding to the lattice points of P. The initial ideal of I_P with respect to the induced term order can not contain a monomial X_i^j since in a binomial $X_i^j - X_{k_1}^{e_1} \dots X_{k_r}^{e_r}$ the second term is the leading one. It follows from 2.1.2 that the regular triangulation associated with this term order is a full triangulation.

Lemma 3.1.1. *A lattice n-simplex τ in P is a minimal non-face of a full triangulation Δ of P if and only if all facets of τ belong to Δ and τ has interior lattice points.*

Proof. Let τ be a minimal non-face of Δ. By the minimality all facets of τ must belong to Δ. Suppose that τ has no lattice point in its interior. Fix any point x in the interior of τ. Since Δ is a triangulation of P, x must be covered by a simplex σ of Δ with $\dim \sigma = n$. Since $\sigma \neq \tau$, σ must have a vertex y lying outside of τ. The line segment $[x, y]$ must meet a proper face ε of τ in the interior of $[x, y]$. Since $\varepsilon \in \Delta$ and $\varepsilon \cap \sigma \neq \emptyset$, ε is a face of σ. On the other hand, $x, y \notin \varepsilon$, so that the hyperplane through ε separates the points x and y of σ. This is a contradiction.

The converse implication is obvious. □

We note that the case $\sharp L_P = n + 1$ is trivial because $k[S_P] = k[X_1, \dots, X_{n+1}]$ in this case.

Theorem 3.1.2. *Let P be a lattice polytope in $\mathbb{R}^n$ with $\sharp L_P \geq n + 2$. Then P has a regular full triangulation Δ with $\deg \Delta \leq n$ if and only if $\sharp L_{\partial P} \geq n + 2$.*

Proof. Assume that $\sharp L_{\partial P} = n + 1$. Then P is an n-simplex whose facets have no lattice points in their interiors. For any full triangulation Δ of P, the facets of P must appear in Δ. Since $\sharp L_P \geq n + 2$, P has an interior lattice point. Therefore, P must be a minimal non-face of Δ, hence $\deg \Delta = n + 1$.

Conversely, assume that $\sharp L_{\partial P} \geq n + 2$. If P has no interior lattice points, we choose any regular full triangulation Δ of P. By Lemma 3.1.1, Δ has no minimal non-face of dimension n, hence $\deg \Delta \leq n$. If P has interior lattice points, we apply Theorem 3.3.1 below which is stronger than the 'if' part of 3.1.2. □

In Subsection 3.2 we will prove a refinement of Theorem 3.1.2 in dimension 2. This case is significantly simpler than the general one, and its proof is independent from 3.1.2.

We say that a triangulation Δ is *n-restricted* if every minimal interior face has at most $n-1$ vertices. In the following figure the triangulation on the left is 2-restricted, that on the right is not.

3.2. **Lattice polygons.** The case of lattice polygons is of particular interest because of its relationship to the Koszul property of semigroup rings. Furthermore one can refine Theorem 3.1.2 in the planar case by showing that there exists a degree 2 lexicographic unimodular triangulation for a lattice polygon P with at least 4 lattice points in its boundary.

Let $L_P = \{x_1, \dots, x_m\}$. Given a total order $x_1 > \dots > x_m$ on L_P, the lexicographic term order induced by $>$ yields a regular triangulation $\Delta_{>\mathrm{lex}}(P)$ of P which we call the *lexicographic triangulation* of P. In combinatorics this triangulation is known as the placing triangulation, see [23]. It can be described recursively as follows (see [24], Proposition 8.6).

Lemma 3.2.1.

$$\Delta_{>\mathrm{lex}}(P) = \Delta_{>\mathrm{lex}}(L_P \setminus \{x_1\}) \cup \Big\{\{x_1\} \cup \varepsilon\colon \varepsilon \in \Delta_{>\mathrm{lex}}(L_P \setminus \{x_1\}),\ \varepsilon \textit{ visible from } x_1\Big\}$$

Let P_i be the convex hull of the set $\{x_i, \dots , x_m\}$, $i \geq 1$. We call the total order $>$ on L_P an *exterior order* if x_i is a vertex of the polytope P_i for all $i = 1, \dots , m$.

Lemma 3.2.2. *Let $x_1 > \ldots > x_m$ be an exterior order on L_P. Put $\Delta = \Delta_{>\mathrm{lex}}(P)$ and $\Gamma = \Delta_{>\mathrm{lex}}(P_2)$. Then $\deg \Delta \leq n$ if the following conditions are satisfied:*
(i) $\deg \Gamma \leq n$,
(ii) Γ *is n-restricted on Q_{x_1}.*

Proof. Assume the contrary. Then Δ has a minimal non-face σ with $\dim \sigma = n+1$. By (i), σ is not a minimal non-face of Γ. Since x_1 is a vertex of P, x_1 lies outside of P_2. Hence every minimal non-face of Δ with vertices in P_2 is also a minimal non-face of Γ. It follows that x_1 is a vertex of σ. Let ε be the $(n-1)$-dimensional face of σ which does not contain x_1. Since σ is a minimal non-face of Δ, ε is a face of Δ and therefore of Γ.

All the other $(n-1)$-dimensional faces of σ are also faces of Δ, and therefore the $(n-2)$-dimensional faces of σ that do not contain x_1 are visible from x_1. Hence they belong to Q_{x_1}. Since they constitute $\partial\varepsilon$ and since Γ is n-restricted on Q_{x_1}, it follows that $\varepsilon \subset Q_{x_1}$. Hence ε is visible from x_1, and Lemma 3.2.1 implies $\sigma \in \Delta$, which is a contradiction. □

If P is a polygon in $\mathbb{R}^2$, condition (ii) of the previous lemma just means that no edge of Γ connects two non-neighbouring lattice points of Q_{x_1}. In the case of polygons we call a connected part of ∂P that starts and ends at vertices of P a *path* of ∂P. Notice that Q_{x_1} is a path of ∂P. Two paths are said to be *disjoint* if they have at most one common point.

Theorem 3.2.3. *Let P be a lattice polygon with $\sharp L_{\partial P} \geq 4$. Let $\partial P = C_1 \cup \ldots \cup C_r$ be a decomposition of ∂P into $r \geq 3$ disjoint paths. Then there exists an exterior order $>$ on L_P such that $\Delta = \Delta_{>\mathrm{lex}}(P)$ is unimodular and satifies the following conditions:*
(i) $\deg \Delta = 2$,
(ii) *every edge of Δ with vertices on C_i lies on C_i, $i = 1, \dots , r,$.*
In particular, there exists an exterior order $>$ on L_P such that $\Delta_{>\mathrm{lex}}(P)$ is unimodular and of degree 2.

Proof. The triangulation Δ will be constructed inductively as indicated by Lemma 3.2.1. It is easy to see that one obtains a full lattice triangulation by this construction, and in dimension 2 every full lattice triangulation is unimodular. Therefore it is not necessary to mention unimodularity any further.

Case 1: $\sharp L_{\partial P} = 4$. Then ∂P is either a triangle $\langle x_1, x_2, x_3\rangle$ with a lattice point x_4 on the edge $[x_1, x_3]$ or a quadrangle $\langle x_1, x_2, x_3, x_4\rangle$. Since the number of the paths $C_1, \dots , C_r$ is at least 3, we may assume that x_2, x_4 do not belong to the same path.

If $\sharp L_P = 4$, i. e. P has no interior point, we obtain the lexicographic triangulation of P which corresponds to the exterior order $x_1 > x_2 > x_3 > x_4$ by connecting x_2, x_4.

If $\sharp L_P > 4$, then P has an interior lattice point. Without restriction we may assume that the triangle $\langle x_1, x_2, x_4\rangle$ contains an interior point of P. For $x = x_1$ we have $\sharp L_{Q_x} \geq 3$, hence $\sharp L_{\partial P_x} \geq 4$. Moreover, $Q_x \cup [x_2, x_3] \cup [x_3, x_4]$ is a decomposition of ∂P_x into 3 disjoint paths. By induction on the number $\sharp L_P$ we may assume that there is an exterior order $>$ on L_{P_x} such that the corresponding lexicographic triangulation Γ of P_x satisfies the conditions:

(i) $\deg \Gamma = 2$,

(ii) every edge of Γ with vertices on Q_x lies on Q_x.

By Lemma 3.2.2, the resulting exterior order $>$ on L_P with x_1 as the maximal element induces a lexicographic triangulation Δ of P with $\deg \Delta = 2$. Neither $[x_2, x_4]$ nor $[x_1, x_3]$ are faces of Δ so that condition (ii) of the theorem is trivially satisfied. In fact, x_3 is not visible from x_1, and $[x_2, x_4]$ has both its vertices in Q_x.

Case 2: $\sharp L_{\partial P} > 4$. Choose x_1 to be the common vertex x of C_1 and C_r. We have $\sharp L_{\partial P_x} \geq \sharp L_{\partial P} - 1 \geq 4$. If $\sharp L_{C_i} = 2$, i.e. C_i has no lattice points in its interior, for all $i = 1, \dots , r$, then $r = \sharp L_{\partial P}$ and ∂P_x has a decomposition into $r - 1 \geq 3$ disjoint paths $D_1 \cup \cdots \cup D_{r-1}$ with

$$D_i = \begin{cases} Q_x, & i = 1, \\ C_i, & i = 2, \dots , r-1. \end{cases}$$

If there exists a path C_i with $\sharp L_{C_i} > 2$, we may assume that it is C_1. Set

$$D_i = \begin{cases} C_1 \cap \partial P_x, & i = 1, \\ C_i, & i = 2, \dots , r-1 \\ C_r \cap \partial P_x, & i = r, \\ Q_x, & i = r+1. \end{cases}$$

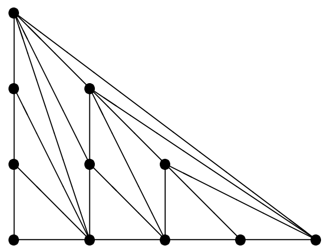
If $\sharp L_{C_r} = 2$, then ∂P_x has the decomposition $D_1 \cup \ldots \cup D_{r-1} \cup D_{r+1}$ and, if $\sharp L_{C_r} > 2$, the decomposition $D_1 \cup \ldots \cup D_{r+1}$ into disjoint paths. In any case, by induction on the number $\sharp L_P$ we may assume that there is an exterior order $>$ on L_{P_x} such that the corresponding lexicographic triangulation Γ of P_x satisfies the following conditions:

(i) $\deg \Gamma = 2$,

(ii) any edge of Γ with vertices on a path D_i lies on D_i.

Note that Q_x is one of the paths D_i. By Lemma 3.2.2, the resulting exterior order $>$ on L_P with x_1 as its maximal element induces a lexicographic triangulation Δ of P with $\deg \Delta = 2$. Due to the definition of D_i and the hypothesis (ii), any edge of Δ with vertices on a path C_i, $i = 2, \dots , r-1$, must lie on C_i. For $i = 1, r$, such an edge must have x_1 as a vertex. The other vertex must be the only lattice point of C_i visible from x_1. Hence this edge lies on C_i. $\square$

Remark 3.2.4. For certain classes of lattice polygons one can explicitly describe term orders that yield unimodular lexicographic triangulations of degree 2. For example let P be a rectangular lattice triangle with the vertices $(0,0)$, $(\lambda_1,0)$, and $(0,\lambda_2)$. By symmetry we may assume $\lambda_1 \geq \lambda_2$. Then we define the order $\prec$ on $\mathbb{R}^2$ by setting $(x_1,x_2) \prec (y_1,y_2)$ if $x_1 > y_1$ or $x_1 = y_1$, $x_2 < y_2$. Then we extend $\prec$ to a lexicographic term order. It can be shown that the associated triangulation is unimodular and of degree 2. For $\lambda_1 = 4$ and $\lambda_2 = 3$ it is given by the following figure:



We now draw the consequences of Theorem 3.1.2.

Corollary 3.2.5. *Let P be a lattice polytope in $\mathbb{R}^2$ with $\sharp L_P \geq 4$. Then the following are equivalent:*
(a) $\sharp L_{\partial P} \geq 4$,
(b) *P has a regular full, equivalently: unimodular, triangulation Δ with* $\deg \Delta = 2$,
(c) *I_P has a Gröbner basis of elements of degree* 2,
(d) *P is Koszul,*
(e) *I_P is generated by elements of degree* 2.

Proof. (e) $\Rightarrow$ (a) Let $m = \sharp L_P$. We use the presentation $K[S_P] = K[X_1,\dots,X_m]/I_P$. Assume that $\sharp L_{\partial P} = 3$, say $L_{\partial P} = \{x_1,x_2,x_3\}$. Then P is a triangle with vertices x_1,x_2,x_3 and P has at least a lattice point in its interior, say x_4. There exist positive integers α, α_1,α_2, α_3 such that $\alpha x_4 = \alpha_1x_1+\alpha_2x_2+\alpha_3x_3$. Hence I_P contains a monomial of the form $X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3} - X_4^{\alpha}$. This monomial cannot be generated by forms of degree 2 of I_P. For, I_P must contain, by a permutation of X_1,X_2,X_3, a non-trivial monomial of the form $X_1^2 - F$ or $X_1X_2 - G$ for some monomials F or G in $K[X_1,\dots,X_m]$. The variables of F or G must correspond to a lattice point on the segment $[x_1,x_2]$ by affine dependence. But this segment has only x_1,x_2 as lattice points because $x_1x_2 \in \partial P$ and $L_{\partial P} = \{x_1,x_2,x_3\}$. Therefore I_P can not be generated by forms of degree 2.

(a) $\Rightarrow$ (b) If $\sharp L_{\partial P} \geq 4$, then L_P has a regular full triangulation Δ with $\deg \Delta = 2$ by Theorem 3.2.3. By Proposition 1.2.4(b) Δ is a unimodular triangulation.

(b) $\Rightarrow$ (c) follows immediately from Theorem 2.1.2.

(c) $\Rightarrow$ (d) is a result of [9], which we have already used several times above.

(d) $\Rightarrow$ (e) is trivial. $\square$

R. Koelman [20] has proved a weaker version of Corollary 3.2.5, namely, if P is a lattice polygon with $\sharp L_{\partial P} \geq 4$, then I_P can be generated by forms of degree 2.

3.3. **The general case.** It remains to prove the following theorem.

Theorem 3.3.1. *Let P be a lattice polytope in $\mathbb{R}^n$ with $\sharp L_{\partial P} \geq n+2$. Suppose that P has interior lattice points. Then P has an n-restricted regular full triangulation Δ with* $\deg \Delta \leq n$.

The proof of Theorem 3.3.1 uses induction on $\sharp L_P$. For this we need some preparation.

For any vertex x of P we denote by P_x the convex hull of the set $L_{P\setminus\{x\}}$ and by Q_x the part of ∂P_x which can be seen from x. We shall see that the n-restrictedness can be passed from P_x to P. Let U be a union of faces of P. A triangulation Δ of P is called *n-restricted on U* if $\varepsilon \subset U$ for every $(n-1)$-simplex ε of Δ with $\partial\varepsilon \subset U$.

Lemma 3.3.2. *Let P be a lattice n-polytope in $\mathbb{R}^n$ with $\sharp L_P \geq n+3$. Let U be a union of facets of P. Assume that U has an interior point x which is a vertex of P such that* $\dim P_x = n$ *and P_x has a regular full triangulation Γ with* $\deg \Gamma \leq n$ *which is n-restricted on $U \cap P_x$ and Q_x. Then P has a regular full triangulation Δ with* $\deg \Delta \leq n$ *which is n-restricted on U.*

Proof. There are two cases.

Case 1: $\sharp L_{Q_x} = n$. Then Q_x is a lattice $(n-1)$-simplex which has no other lattice points than its vertices. Since Q_x is a facet of P_x, Q_x is a facet of an n-simplex of Γ, say σ. Let y be the vertex of σ not contained in Q_x. We denote by T the convex hull of x, y and Q_x. Let z be the intersection point of the segment $[x, y]$ with Q_x. If z is a vertex of Q_x, T is an n-simplex. The facets of T which contain x lie on facets of P. Since x is an interior point of U, these facets of P are also facets of U. Therefore, if ρ is the facet of T which does not contains x, then $\partial\rho \subset U$. Since ρ is also a facet of σ, $\rho \in \Gamma$. Using the n-restrictedness of Γ on $U \cap \partial P_x$ we get $\rho \subset U$. From this it follws that all facets of T are contained in ∂P. Hence $P = T$. So we get $\sharp L_P = n+2$, a contradiction. Thus, z is not a vertex of Q_x. Connecting x with y we obtain a triangulation of T into n-simplices which are spanned by x and the facets of σ containing y. Since z is not a vertex of Q_x, these simplices involve all lattice points T. Together with the simplices of Γ other than σ they compose a full triangulation Δ of P.

If Δ is not n-restricted on U, there exists an $(n-1)$-simplex ε of Δ such that $\varepsilon \not\subset U$ but $\partial\varepsilon \subset U$. If $\varepsilon \in \Gamma$, then $\partial\varepsilon \subset U \cap P_x$. Using the n-restrictedness of Γ on $U \cap \partial P_x$ we get $\varepsilon \subset U$, a contradiction. If $\varepsilon \notin \Gamma$, then x is a vertex of ε. If $y \notin \varepsilon$, ε is spanned by x and a facet of Q_x. Hence ε is contained in a facet of P which contains x. Such a facet of P is also a facet of U because x is an interior point of U. Therefore we have $\varepsilon \subset U$, a contradiction. If $y \in \varepsilon$, then ε is spanned by x, y and $n-2$ vertices of Q_x, say $z_1, \dots, z_{n-2}$. The facets of ε are the simplex $\langle y, z_1, \dots, z_{n-2}\rangle$ spanned by $y, z_1, \dots, z_{n-2}$ and the simplices spanned by x, y and $n-3$ elements of the set $\{z_1, \dots, z_{n-2}\}$. Since $\partial\varepsilon \subset U$, they are all contained in U. As a consequence, every simplex spanned by y, z and $n-3$ elements of the set $\{z_1, \dots, z_{n-2}\}$ is contained in U, too. Let z_{n-1}, z_n be the other vertices of Q_x.

Claim. z is contained in the edge $z_{n-1}z_n$.

Proof. Let $\langle z_1, \dots, z_{n-1}\rangle$ denote the simplex spanned by the points $z_1, \dots, z_{n-1}$. We have $z \notin \langle z_1, \dots, z_{n-1}\rangle$ because otherwise ε is contained in the facet of P

which contains x and $\langle z_1, \dots, z_{n-1}\rangle$, hence $\varepsilon \subset U$, a contradiction. Similarly we also have $z \notin \langle z_1, \dots, z_{n-2}, z_n\rangle$. This means that in the barycentric representation $z = \lambda_1 z_1 + \dots + \lambda_n z_n$ the coefficients λ_{n-1} and λ_n are positive. The claim amounts to the equations $\lambda_i = 0$ for $i = 1, \dots, n-2$.

As stated just above the claim, the simplex $\langle z, z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{n-2}\rangle$ is contained in U. This means that every convex representation $w = \mu_1 z_1 + \dots + \mu_{i-1} z_{i-1} + \mu_{i+1} z_{i+1} + \dots \mu_{n-2} z_{n-2}$ is a point of U. If we choose the μ_j strictly positive, then the segment $[w, z]$ contains an interior point of $Q_x = \langle z_1, \dots, z_n\rangle$. Since $w, z \in U$, it follows that U contains an interior point of Q_x, which is impossible. $\square$

Now we continue the proof of Lemma 3.3.2. By the above claim, all facets of T which contain y, z_{n-1}, z_n also contain x, z_{n-1}, z_n. Hence they lie on the facets of U which contain x, z_{n-1}, z_n. In particular, all $(n-2)$-dimensional faces of σ which contain y and z_{n-1} are contained in U. Since the faces $\langle y, z_1, \dots, z_{n-2}\rangle$ and $\langle z_1, \dots, z_{n-1}\rangle$ are also contained in U, the boundary of the $(n-1)$-simplex $\langle y, z_1, \dots, z_{n-1}\rangle$ is contained in U. Hence $\langle y, z_1, \dots, z_{n-1}\rangle \subset U$ by the n-restrictedness of Γ on $U \cap P_x$. Similarly we can also show that $\langle z_1, \dots, z_{n-3}, z_n\rangle \subset U$. From this it follows that all facets of T are contained in $U \subset \partial P$. Hence $P = T$. This implies $\sharp L_P = n+2$, a contradiction. So we have proved that Δ is n-restricted on U.

If $\deg \Delta = n+1$, there is a lattice n-simplex τ in P such that every facet of τ belongs to Δ and τ has an interior lattice point. Since $\deg \Gamma \le n$, τ is not contained in P_x (in fact, otherwise τ would be contained in $P_x \setminus \sigma$). Hence x is a vertex of τ. From this it follows that τ is the n-simplex spanned by x and Q_x. By the definition of Q_x, this simplex has no interior lattice points. So we obtain a contradiction. Hence $\deg \Delta \le n$.

Let φ be a height function on L_{P_x} for Γ. By choosing $\varphi(x)$ such that $(x, \varphi(x))$ is above the hyperplane of $\mathbb{R}^{n+1}$ containing the facet of $(P_x)_\varphi$ corresponding to σ but below the hyperplanes containing the other facets of $(P_x)_\varphi$ we obtain a height function φ on L_P. Clearly, P_φ coincides with $(P_x)_\varphi$ on all simplices of Γ other than σ. From this we can conclude that $\Delta = \Delta_\varphi$. Hence Δ is a regular triangulation of P.

Case 2: $\sharp L_{Q_x} \ge n+1$. Consider the full triangulation of Q_x into $(n-1)$-simplices induced by Γ. The n-simplices spanned by x and these $(n-1)$-simplices compose a triangulation of the convex hull of $P \setminus P_x$. This triangulation together with Γ forms a lattice triangulation Δ of P.

If Δ is not n-restricted on U, there exists an $(n-1)$-simplex ε of Δ such that $\varepsilon \not\subset U$ but $\partial\varepsilon \subset U$. If $\varepsilon \in \Gamma$, then $\partial\varepsilon \subset U \cap P_x$. Hence we have $\varepsilon \subset U$, a contradiction. If $\varepsilon \notin \Gamma$, then ε is spanned by x and an $(n-2)$-simplex of Γ on Q_x. Since this $(n-2)$-simplex of Γ is contained in U, it lies on the boundary of Q_x. Therefore ε lies on a facet of P which contains x. Since x is an interior point of U, this facet of P is a facet of U. Hence we have $\varepsilon \subset U$, a contradiction.

If $\deg \Delta = n+1$, there is a lattice n-simplex τ in P such that every facet of τ belongs to Δ and τ has an interior lattice point. Since $\deg \Gamma \le n$, τ is not contained in P_x. Hence x is a vertex of τ. Let ε be the facet of τ not containing x. Then

$\partial\varepsilon \subset Q_x$. Hence $\varepsilon \subset Q_x$ by the n-restrictedness of Γ on Q_x. Since there is no lattice point inbetween x and Q_x, τ would have no interior lattice points, a contradiction. Therefore, we have $\deg \Delta \leq n$.

It remains to show that Δ is a regular triangulation of P. If we choose $\varphi(x)$ small enough, the height function on L_P which extends the height function of Γ will keep $(P_x)_\varphi$. Hence Δ_φ coincides with Γ on P_x. From this it follows that $\Delta = \Delta_\varphi$. The proof of Lemma 3.3.2 is now complete. □

A special case of Lemma 3.3.2 is the case $U = \partial P$.

Corollary 3.3.3. *Let P be a lattice n-polytope in $\mathbb{R}^n$ with $\sharp L_P \geq n+3$. Let x be a vertex of P such that $\dim P_x = n$ and P_x has a regular full triangulation Γ with $\deg \Gamma \leq n$ which is n-restricted on $\partial P \cap P_x$ and Q_x. Then P has an n-restricted regular full triangulation Δ with $\deg \Delta \leq n$.*

We shall need the following lemma for the existence of a regular triangulation Γ of P_x with the above properties.

Lemma 3.3.4. *Let P be a lattice polytope in $\mathbb{R}^n$ with $\sharp L_P \geq n+2$ which has no interior lattice point. Let U be a union of facets of P which is homeomorphic to an $(n-1)$-dimensional ball. Assume that $\sharp L_{U^c} \geq n+1$, where U^c is the closure of the complement of U on ∂P. Then P has a regular full triangulation Δ with $\deg \Delta \leq n$ which is n-restricted on U.*

Proof. By Lemma 3.1.1 every regular full triangulation Δ of P has $\deg \Delta \leq n$. Hence we only need to find such a triangulation Δ of P which is n-restricted on U.

We first consider the case in which U contains no interior lattice point which is a vertex of P. Choose any regular triangulation Δ of P. If Δ is not n-restricted on U, there exists an $(n-1)$-simplex $\varepsilon \not\subset U$ of Δ such that $\partial\varepsilon \subset U$. Since U is homeomorphic to an $(n-1)$-dimensional ball, $\partial\varepsilon$ divides ∂P into two parts one of which is contained in U. This part has an interior lattice point of U which is a vertex of P, a contradiction.

Now assume that U contains an interior lattice point x which is a vertex of P. There are two cases.

Case 1: $\dim P_x = n-1$. Then $P_x = Q_x$ and it is a facet of P. Therefore, U must be the union of the facets of P which contain x, and $U^c = Q_x$. Choose any regular full triangulation Δ of P. If Δ is not n-restricted on U, there exists an $(n-1)$-simplex ε of Δ such that $\varepsilon \not\subset U$ but $\partial\varepsilon \subset U$. If ε is contained in Q_x, then $\partial\varepsilon \subset U \cap Q_x = \partial Q_x$. Since $\partial\varepsilon$ and ∂Q_x are both homeomorphic to the $(n-1)$-dimensional sphere, we must have $\varepsilon = Q_x = U^c$, a contradiction to the assumption that $\sharp L_{U^c} \geq n+1$. If ε is not contained in Q_x, ε is spanned by x and an $(n-2)$-simplex of Δ on Q_x. This $(n-2)$-simplex of Δ is contained in $U \cap Q_x = \partial Q_x$. Hence ε is a facet of U, a contradiction.

Case 2: $\dim P_x = n$. We distinguish two subcases.

Subcase 1: $\sharp L_P = n+2$. Then the assumption $\sharp U^c \geq n+1$ implies that U has only one interior lattice point, namely x. We have $\sharp L_{P_x} = n+1$. Together with the assumption $\dim P_x = n$ this implies that P_x is an n-simplex.

If $\sharp Q_x = n$, let y be the vertex of P_x not contained in Q_x. Let z be the intersection point of the segment $[x, y]$ with Q_x. Then z is not a vertex of Q_x because otherwise $y \in U$ and all facets of P containing $[x, y]$ are facets of U, hence z would be an interior lattice point of U. Connecting x and y we obtain a full triangulation Δ of P each of whose simplices is spanned by x, y and a facet of Q_x. If Δ is not n-restricted on U, there is an $(n-1)$-simplex ε of Δ such that $\varepsilon \not\subset U$ but $\partial\varepsilon \subset U$. Since U is homeomorphic to an $(n-1)$-dimensional ball, $\partial\varepsilon$ divides ∂P into two parts one of which is contained in U. If ε is a face of P, we have $U^c = \varepsilon$ and therefore $\sharp L_{U^c} = n$, a contradiction. So ε is not a face of P. The two lattice points of P which are not contained in ε must lie on different sides of ε. One of these two points must be an interior point of U, hence it is x. From this it follows that $Q_x = \varepsilon$. So we get $Q_x \in \Delta$, a contradiction. By choosing a sufficiently general height function φ on L_P with $\varphi(x), \varphi(y)$ greater than the other values of φ, we will obtain $\Delta_\varphi = \Delta$. Hence Δ is a regular triangulation of P which is n-restricted on U.

If $\sharp Q_x = n + 1$, choose any regular triangulation Δ of P which contains the n-simplex P_x. (The existence of such a triangulation is easy to show.) If Δ is not n-restricted on U, there is an $(n-1)$-simplex ε of Δ such that $\varepsilon \not\subset U$ but $\partial\varepsilon \subset U$. As in the case $\sharp Q_x = n$, we can show that $Q_x = \varepsilon$. From this it follows that $\sharp Q_x = n$, a contradiction.

Subcase 2: $\sharp L_P \geq n + 3$. We have $\sharp L_{P_x} \geq n + 2$. Put $U_x = (U \cap P_x) \cup Q_x$. Then U_x is a union of facets of P_x which is homeomorphic to an $(n-1)$-dimensional ball and $(U_x)^c = U^c$. Using induction we may assume that P_x has a regular full triangulation Γ with $\deg \Gamma \leq n$ such that Γ is n-restricted on U_x. By Lemma 3.3.2, P has a regular full triangulation Δ which is n-restricted on U. The proof of Lemma 3.3.4 is now complete. $\square$

Proof of Theorem 3.3.1. There are two cases.

Case 1: $\sharp L_{\partial P} = n+2$. We will first show that there exists a vertex x of P such that $\sharp L_{\partial P_x} \geq n + 2$.

If P is not an n-simplex, then all $n+2$ lattice points of ∂P are vertices of P. Thus P is the union of $n + 2$ simplices σ_j each of which is spanned by $n + 1$ vertices of P. Let y be an interior lattice point of P (which exists by hypothesis). Elementary arguments show that there exists k such that y is not contained in the interior of σ_k. Now we choose x to be the vertex not involved in σ_k.

If P is an n-simplex, let y be the remaining point of $L_{\partial P}$ and ρ the facet of P which contains y. Let x be the vertex of P not contained in ρ. Since P has interior lattice points, P_x has a vertex not contained in ρ. Hence $\sharp L_{\partial P_x} \geq n + 2$.

Let x be a vertex of P such that $\sharp L_{\partial P_x} \geq n + 2$. If P_x has interior lattice points, using induction we may asssume that P_x has an n-restricted regular full triangulation Γ with $\deg \Gamma \leq n$. Then so does P by Corollary 3.3.3. Now assume that P_x has no interior lattice points. We put $U = Q_x$. Then U is a union of facets of P_x which is homeomorphic to an $(n-1)$-dimensional ball. Moreover, $U^c = \partial P \cap P_x$. Hence $\sharp L_{U^c} = \sharp L_{\partial P} - 1 = n + 1$. By Lemma 3.3.4 there is a regular full triangulation Γ of P_x with $\deg \Gamma \leq n$ which is n-restricted on Q_x. By Corollary 3.3.3, P has an n-restricted regular full triangulation Δ with $\deg \Delta \leq n$ if Γ is n-restricted on

$\partial P \cap P_x$. If the latter condition is not satisfied, there exists an $(n-1)$-simplex ε of Γ such that $\varepsilon \not\subset \partial P \cap P_x$ but $\partial\varepsilon \subset \partial P \cap P_x$. Then we can find a vertex y of P on the other side of ε than that of x. Since $\sharp L_{\partial P} = n+2$, x, y and the vertices of ε fill out $L_{\partial P}$. Therefore, there is no lattice point inbetween x and $\partial\varepsilon$. This implies $\partial\varepsilon = \partial Q_x$. Hence $\varepsilon \subset Q_x$ by the n-restrictedness of Γ on Q_x. From this it follows that $\varepsilon = Q_x$, which is impossible because Q_x must contain interior lattice points of P.

Case 2: $\sharp L_{\partial P} \geq n+3$. For any vertex x of P we have $\sharp L_{\partial P_x} \geq n+2$. If P_x has interior lattice points, using induction we may asssume that P_x has an n-restricted full regular triangulation Γ with $\deg \Gamma \leq n$. Then so does P by Corollary 3.3.3. Now assume that there is no vertex x such that P_x has interior lattice points. Fix any vertex x of P. Put $U = Q_x$. Then U is a union of facets of P which is homeomorphic to an $(n-1)$-dimensional ball. Moreover, $U^c = \partial P \cap P_x$. Hence $\sharp L_{U^c} = \sharp L_{\partial P} - 1 \geq n+2$. By Lemma 3.3.4 there is a regular triangulation Γ of P_x with $\deg \Gamma \leq n$ which is n-restricted on Q_x. By Corollary 3.3.3, P has an n-restricted regular triangulation Δ with $\deg \Delta \leq n$ if Γ is n-restricted on $\partial P \cap P_x$. If the latter condition is not satisfied, there exists an $(n-1)$-simplex ε of Γ such that $\varepsilon \not\subset \partial P$ but $\partial\varepsilon \subset \partial P$. Then we can find a vertex y of P on the other side of ε relative to x. P_y contains the convex hull of all lattice points of P on the side of ε that contains x. But this convex hull contains Q_x. Since Q_x contains interior lattice points of P, P_y has interior lattice points, a contradiction. The proof of Theorem 3.3.1 is now complete.

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