The Advanced Australian Shuffle

Ehrhard Behrends

Abstract. The "classical" Australian under-down shuffle starts with a deck of n cards. Then one proceeds as follows: one card under the deck, one on the table, one under the deck, one on the table, etc. One continues until only one card remains. There is an explicit formula to calculate the number of the card in the original deck that survives, and this is the basis of several mathematical magic tricks.

Here we study the following variant. We have a deck of n cards, on each card is written a number: $a_1, a_2, \ldots, a_n \in \mathbb{N}_0$. (Number a_i is written on card number i; the top card has number 1.) The procedure is as follows: Have a look at the top card, suppose that it bears the number a. Deal a cards one by one from the top to the bottom of the deck and put the card that is now on top on the table. Repeat this procedure until only one card remains.

There is no general result by which the remaining card can be predicted. But, surprisingly, there are sequences $a_1, a_2, \ldots, a_n$ such that the same card survives regardless whether one starts with $a_1, a_2, \ldots, a_n$ or with a cyclic translation, i.e., with the sequence $a_k, a_{k+1}, \ldots, a_n, a_1, \ldots, a_{k-1}$ for any $k \in \{2, \ldots, n\}$. To state it otherwise: the original deck can be cut at an arbitrary position, the last card will always be the same. We investigate some properties of such sequences and indicate how they can be used for mathematical magic tricks.

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1. The phenomenon

The starting point of my investigations was the following magic trick that was communicated by the magician Henning Köhlert (who attributes the trick to Werner Miller) at a seminar for magicians in 2015. We note that it will be necessary to know that the German translations of the English words *diamonds, hearts, clubs and spades* for the card suits are *Karo, Herz, Kreuz* and *Pik*.

- Four cards are given to a spectator, one of each suit. From now on the magician gives only instructions, he has no further information concerning the cards.
- The spectator is invited to arrange these cards arbitrarily to form a little deck. The only condition: red and black cards must alternate.

- Then three times the following happens: Look at the top card. Put – one by one – as many cards from the top to the bottom of the deck as the number of letters in the suit of this card indicates. (I.e., three cards for „Pik“, five cards for „Kreuz“ and four cards for „Herz“ and „Karo“.) The card that is now on top is removed.

- One card will remain, and the magician knows for sure in advance that it will be "Kreuz", the club card.

We will investigate here a generalization of this phenomenon. First we will introduce some notation. Suppose that $n \in \mathbb{N}$ with $n \geq 2$ and $(a_1, \ldots, a_n) \in \mathbb{N}_0^n$ are given. Now a_1 numbers from the beginning of the sequences move one-by-one to the end and the number that is the now the first one is removed. (The cases $a_1 = 0$ and $a_1 > n$ are expressly admissible.)
Call the result $(b_1, \ldots, b_{n-1})$ and write $(a_1, \ldots, a_n) \to (b_1, \ldots, b_{n-1})$.

We can iterate this procedure $n-1$ times, finally a single integer will remain. Here are some examples:

$(4, 3, 4, 5) \to (3, 4, 5) \mapsto (4, 5) \to (5)$; $(5, 2, 4) \to (5, 2) \to (5)$.

$(0, 2, 4) \to (2, 4) \to (4)$; $(1, 2, 3, 4, 1) \to (3, 4, 1, 1) \to (3, 4, 1) \to (4, 1) \to (1)$.

The last example reveals a little difficulty: we see that the remaining number is "1", but it is not clear whether it is the first or the last 1 in the sequence we started with. Since this can be relevant later we repeat the calculation where we distinguish the two 1's:

$$(1_1, 2, 3, 4, 1_2) \to (3, 4, 1_2, 1_1) \to (3, 4, 1_2) \to (4, 1_2) \to (1_2).$$

Thus the second 1 persists.

In order to omit this ambiguity and to spare whenever possible the intermediate calculations in this article it will be convenient to use the following notation: the number that survives the procedure will be underlined. In this way we can write the preceding examples as follows:

$$(4, 3, 4, \underline{5}), (\underline{5}, 2, 4), (0, 2, \underline{4}), (1, 2, 3, 4, \underline{1}).$$

We will be interested in the following type of sequences:

Definition 1.1. *A sequence $(a_1, \ldots, a_n) \in \mathbb{N}_0^n$ will be called a* good sequence, *if there is a $k \in \{1, \ldots, n\}$ such that a_k is underlined in all cyclic translations, i.e. the k'th element in $(a_1, \ldots, a_n)$, the $(k-1)$-th in $(a_2, \ldots, a_n, a_1)$, etc. If this is the case, we will write*

$$(a_1, \ldots, a_{k-1}, a_k^*, a_{k+1}, \ldots, a_n),$$

i.e., the relevant a_k is marked with a $*$.

With this notation the phenomenon that we described at the beginning implies that $(3, 4, 5^*, 4)$. Of course this is the same assertion as $(4, 5^*, 4, 3)$, $(5^*, 4, 3, 4)$, or $(4, 3, 4, 5^*)$.

In the case $(a_1, \ldots, a_n) = (1, \ldots, 1)$ we arrive at the *Australian under-down-shuffle*. (This fact motivates why we have chosen the above title for the present article.) It can easily be proved by induction that the number at position $2r+1$ has to be underlined, where r is chosen such that $n = 2^s + r$ with a maximal s (see [2], section 2.2, or [3], chapter 6; see also [1]).

Even in the case $(2, \ldots, 2) \in \mathbb{N}_0^n$ there seems to be no simple way to predict which 2 has to be underlined. Thus computer simulations will play an important role in the sequel, and theoretical results will concern only rather special examples.

We start our investigations of good sequences in *section 2*. The main result will be that for each n "nontrivial" good sequences $(a_1, \ldots, a_n)$ exist. *Section 3* deals with good sequences where also the reflected sequence is good (with the "*" at the same number): *very good sequences*. We describe a procedure by which (rather special) "long" very good sequences can be constructed. Finally, in *section 4*, we provide some proposals how to use good and very good sequences for mathematical magic tricks.

2. Good sequences

In the following lemma we collect some facts that can easily be proved:

Lemma 2.1. *(i) Let λ_n be the least common multiple of the numbers $1, 2, \ldots, n$. Then, for $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{N}_0^n$ with $a_i = b_i \bmod \lambda_n$ for all i, the sequence $(a_1, \ldots, a_n)$ is good if and anly if also $(b_1, \ldots, b_n)$ is. If this is the case, the element with its "*" is at the same position in both sequences.*
Therefore it suffices to consider only those $(a_1, \ldots, a_n)$ where $0 \leq a_i < \lambda_n$ for all i.

(ii) Arbitrary long good sequences exist: Let $\alpha \in \mathbb{N}$ be such that $\alpha \neq 0 \bmod j$ for $j = 2, \ldots, n$. Then $(0, \ldots, 0, \alpha)$ is a good sequence: $(0, \ldots, 0, \alpha^*)$. Conversely, if $\alpha = 0 \bmod j$ for some $j \in \{2, \ldots, n\}$, then $(0, \ldots, \alpha)$ is not good.

(iii) Suppose that there is a $k_0 \in \{2, \ldots, n\}$ such that

$$(a_1, \ldots, a_n) = (a_{k_0}, \ldots, a_n, a_1, \ldots, a_{k_0-1}),$$

i.e., the sequence admits a nontrivial shift symmetry. Then $(a_1, \ldots, a_n)$ is not a good sequence.

In particular there is no good sequence of the form $(a, a, \ldots, a)$, and the number of good sequences in $\{0, \ldots, \lambda_n - 1\}^n$ is always divisible by n.

(iv) Suppose that $(a_1, \ldots, a_n)$ is good with the "" at a_k. Then $a_k \bmod n \neq 0$ and it is not possible that both $a_k \bmod (n-1) = 0$ and $a_k \bmod n = n-1$ hold.*

Proof. (i) and the first part of (ii) are obvious. For the second suppose that $\alpha \bmod j = 0$ for some $j \geq 2$. Then α would disappear when considering the cyclic translation $(0, \ldots, \alpha, 0, \ldots, 0)$ (with $j-1$ zeros after α.)

For the proof of (iii) suppose that the k'th element of $(a_1, \ldots, a_n)$ has the "*" and that $k_0 \leq k$. This implies that the element number k of $(a_1, \ldots, a_n)$ and element number $(k - k_0 + 1)$ of $(a_{k_0}, \dots, a_n, a_1, \dots, a_{k_0-1})$ are underlined. But these are different elements (since $k_0 \neq 1$)in contradiction to the fact that only one element can be underlined in the case of good sequences. (In the case $k < k_0$ one argues similarly.)

It remains to prove (iv). We may assume that $k = 1$. In the case $a_1 \bmod n = 0$ (resp. $a_1 \bmod (n-1) = 0$ and $a_1 \bmod n$=n-1) the number a_1 would disappear in the first (resp. the second) iteration so that a_1 cannot have a "*". □

Remark: We note that λ_n can not be replaced by n in part (i) of the lemma: we have $(0, 0, 1^*, 1)$, but $(0, 4, 1, 1)$ is not a good sequence. (Since $(0, 4, 1, \underline{1})$, but $(4, \underline{1}, 1, 0)$.)

Let us consider some *examples*:

The case $n = 2$. We have $\lambda_2 = 2$ so that, by lemma 2.1(i), we only have to check $2^2 = 4$ sequences. Two of them are good sequences, namely $(0, 1^*)$ and $(1^*, 0)$. (In fact, one of these sequences is the cyclic translate of the other so that there is essentially only one good (a_1, a_2).)

The case $n = 3$. Since $\lambda_3 = 6$ there are $6^3 = 216$ candidates. The following 20 sequences are good:

$(0, 0, 1^*), (0, 0, 5^*), (0, 1^*, 1), (0, 1^*, 3), (0, 1^*, 4), (0, 2, 1^*), (0, 2, 4^*),$
$(0, 2, 5^*), (0, 5^*, 1), (0, 5^*, 3), (0, 5^*, 4), (1^*, 1, 2), (1, 2, 5^*), (1^*, 3, 2),$
$(1, 3, 5^*), (1^*, 4, 2), (2, 4^*, 3), (2, 5^*, 3), (2, 5^*, 4), (3, 5^*, 4).$

Note that each of these examples gives rise to three different good sequences: We have listed here among the three cyclic translates only the sequence that is minimal in the lexikographic order. So there are 60 good sequences in $\{0, 1, 2, 3, 4, 5\}^3$.

The case $n = 4$. From $\lambda_4 = 12$ we conclude that we have to examine $12^4 = 20.736$ sequences (a_1, a_2, a_3, a_4). With the help of a computer one checks easily that 3924 (or 18.92 percent) of them are good. Here are some examples. (One recognizes the second one as the sequence that motivated the present investigations.)

$$(3, 1^*, 1, 4), (4, 5^*, 4, 3), (2^*, 1, 3, 2), (3, 5^*, 4, 7), (4, 6, 5^*, 5), (8, 8, 10^*, 10).$$

For larger n the number of candidates grows rapidly: $60^5 = 777.600.000$ for $n = 5$, $60^6 = 46.656.000.000$ for $n = 6$, $420^7 \approx 2.30 \cdot 10^{18}$ for $n = 7$, etc. We therefore have determined the percentage of good sequences stochastically: we have generated "very often" n random numbers $a_1, \dots, a_n$ in $\{0, \dots, \lambda_n - 1\}$ and checked whether $(a_1, \dots, a_n)$ is good or not. Here is our list:

n	Percentage of good $(a_1, \ldots, a_n)$ $(0 \le a_i < \lambda_n)$
2	50 %
3	$60/216 \approx 27.7$ %
4	$3924/20763 \approx 18.92$ %
5	≈ 9.9 %
6	≈ 5.5 %
7	≈ 2.8 %
8	≈ 1.4 %
9	≈ 0.7 %
10	≈ 0.3 %

We complement this table with some concrete examples. For later use we have chosen them such that the a_i are not too large and not zero.

n=5: $(5,2,8,3,7^*), (5,3,8,7^*,6), (6,5,8,3,1^*), (1,4,3,4,1^*), (3,3^*,2,1,8)$

n=6: $(9,5,2,6,2,1^*), (3,9^*,8,1,6,4), (8,1,8,3,2,9^*), (4,9,9^*,8,5,5)$

n=7: $(4,6,9^*,3,1,7,8), (1^*,2,2,6,7,5,4), (10,2,1^*,8,2,1,2), (: 2,3^*,2,6,6,4,7)$

n=8: $(7,3,9,7,3,2,3^*,2), (1^*,9,4,1,6,6,4,6), (2,1^*,3,9,9,7,6,3)$

n=9: $(5,4,7,6,1^*3,9,1,7), (6,1,4,1^*,2,1,3,7,1), (9,1,7,7,4,2,3^*,9,9)$

n=10: $(3^*,6,9,2,1,7,6,5,3,2), (2,2,1,6,6,4,3,8,4^*,5), (3,7,6,8,6,6,2,1^*,6,1)$

Several natural questions come to mind:

1. Is there a formula for the proportion of good sequences?
2. Is there a procedure to derive longer good sequences from known ones?
3. Can one construct systematically all good sequences of length n ?
4. Is there an easy-to-apply criterion to decide whether a sequence is good or not?
5. Are there nontrivial good sequences for every n ? (I.e., sequences that are different from the sequences in lemma 2.1(ii).)

For *question 1* we have no answer. In view of part (i) and (iv) of lemma 2.1 it is unlikly that an answer to *question 2* exists. (If a_k has a "*" it will lose it for sufficiently large n). *Question 3* has in fact a positive answer, the "construction", however, is – taking into account the huge numbers under consideration – of no practical use:

– Fix an n and a "target number" $a \in \{1, \ldots, \lambda_n - 1\}$.
– Find all (a_1, a_2) with $a_i \in \{0, \ldots, \lambda_n - 1\}$ such that one of the a_i equals a and such that this a_i is underlined in (a_1, a_2). Call this collection Δ_2.
– Next construct all (a_1, a_2, a_3) that are mapped by "$\to$" to an element of Δ_2. Call this collection Δ_3.
– Continue in this way until Δ_n is constructed. Now find all $(a_1, \ldots, a_n) \in \Delta_n$ such that all cyclic translates $(a_k, \ldots, a_n, a_1, \ldots, a_{k-1})$ also belong to this set.

For an answer to *question 4* one should note that according to the definition of "good sequence" one needs n^2 steps to check this property. For a sequence chosen at random the test will in many cases stop much earlier since there is no need to continue the calculations as soon as there are two cyclic translates with different underlined numbers. But it is unlikely that a criterion exists that needs significantly less than n^2 steps for arbitray $(a_1, \ldots, a_n)$.

Question 5 is answered affirmatively in the next proposition. We start with a sequence $(d_r, d_{r-1}, \ldots, d_1)$ of length r and we will find conditions that imply $(d_r, d_{r-1}, \ldots, d_1^*)$. (It will be convenient here to let the indices decrease.)

First we introduce a *further notation*. Recall that $(a_1, \ldots, a_n) \to (b_1, \ldots, b_{n-1})$ means that $(b_1, \ldots, b_{n-1})$ is derived in one step from $(a_1, \ldots, a_n)$. If one or several "$\to$" are necessary to come from $(a_1, \ldots, a_n)$ to a sequence $(c_1, \ldots, c_m)$ we will write $(a_1, \ldots, a_n) \Rightarrow (c_1, \ldots, c_m)$. (So that, e.g., $(5, 2, 4, 1) \Rightarrow (5, 4)$.) We have to guarantee that $(d_s, d_{s-1}, \ldots, d_1, d_r, \ldots, d_{s-1}) \Rightarrow (d_1)$ for every s.

We start with $(d_r, d_{r-1}, \ldots, d_1)$ in its original order. Obviously the condition $d_s \bmod s = 0$ for $s \geq 2$ (*condition 1*) implies $(d_r, d_{r-1}, \ldots, d_1) \Rightarrow (d_1)$. Now we translate this sequence cyclically. Let's first deal with $(d_s, \ldots, d_1, d_r, \ldots, d_{s+1})$, where $s \in \{2, \ldots, r-1\}$. If we assume that $d_s \bmod r = s$ for these s (*condition 2*) we conclude that $(d_s, \ldots, d_1, d_r, \ldots, d_{s+1}) \to (d_{r-1}, \ldots, d_1)$. And since $(d_{r-1}, \ldots, d_1) \Rightarrow (d_1)$ (by condition 1) it follows that
$(d_s, \ldots, d_1, d_r, \ldots, d_{s+1}) \Rightarrow (d_1)$.

It remains to treat $(d_1, d_r, d_{r-1}, \ldots, d_2)$, this is the most difficult case. We will assume that $\beta := d_1 \bmod r \in \{1, \ldots, r-2\}$ (*condition 3*). When $\beta = 1$ we arrive at $(d_1, d_r, d_{r-1}, \ldots, d_2) \to (d_{r-1}, \ldots, d_2, d_1)$, and $(d_{r-1}, \ldots, d_2, d_1) \Rightarrow (d_1)$ (by condition 1) so that we are done. More interesting is the case $\beta \in \{2, \ldots, r-2\}$. Then $(d_1, d_r, d_{r-1}, \ldots, d_2) \to (d_s, \ldots, d_2, d_1, d_r, \ldots, d_{s+2})$, where $s \in \{2, \ldots, r-2\}$. If we knew that $d_s \bmod (r-1) = s$ for $s \in \{2, \ldots, r-2\}$ (*condition 4*) it would follow that $(d_s, \ldots, d_2, d_1, d_r, \ldots, d_{s+2}) \to (d_{r-1}, \ldots, d_{s+2}, d_s, d_{s-1}, \ldots, d_2, d_1)$. (This is the original sequence, but d_r and d_{s+1} disappeared.) In order to guarantee that d_1 survives at this stage we now assume that $d_{r-1} \bmod (r-2) = 1$ (*condition 5*). Then

- $(d_{r-1}, \ldots, d_{s+2}, d_s, d_{s-1}, \ldots, d_2, d_1) \to (d_{s-1}, \ldots, d_1, d_{r-1})$
(if $r-1 = s+2$, case A) or

- $(d_{r-1}, \ldots, d_{s+2}, d_s, d_{s-1}, \ldots, d_2, d_1) \to (d_{r-3}, \ldots, d_{s+2}, d_s, \ldots, d_1, d_{r-1})$
(if $r-1 > s+2$, case B)

will hold. Let us treat case A first. If we assume that $r \geq 6$ (*condition 6*) it follows from $r-1 = s+2$ that $s \geq 3$ so that $s-1 \geq 2$. Consequently, if $d_t \bmod (t+1) = t$ for $t \in \{2, \ldots, r-2\}$ would hold (*condition 7*, we will use it first for $t = s-1$) we could conclude that $(d_{s-1}, \ldots, d_1, d_{r-1}) \to (d_{s-1}, \ldots, d_1)$. Condition 1 now guarantees that $(d_{s-1}, \ldots, d_1) \Rightarrow (d_1)$, i.e, we have shown that $(d_1, d_r, d_{r-1}, \ldots, d_2) \Rightarrow (d_1)$ in case A.

In order to treat case B we observe that $(d_{r-3}, \ldots, d_{s+2}, d_s, \ldots, d_1, d_{r-1})$ has $r-3$ elements, and thus, by condition 1, $d_{r-3}, \ldots, d_{s+2}$ disappear:

$$(d_{r-3}, \ldots, d_{s+2}, d_s, \ldots, d_1, d_{r-1}) \Rightarrow (d_s, \ldots, d_1, d_{r-1}).$$

Once more condition 7 comes into play (this time applied for $t = s$). We conclude that $(d_s, \ldots, d_1, d_{r-1}) \to (d_s, \ldots, d_1)$. But $(d_s, \ldots, d_1) \Rightarrow (d_1)$ by condition 1 so that we have in fact shown that $(d_1, d_r, d_{r-1}, \ldots, d_2) \Rightarrow (d_1)$ also in case B.

We summarize our investigations in

Proposition 2.2. *Fix $r \in \mathbb{N}$. One has $(d_r, d_{r-1}, \ldots, d_1^*)$ provided that the following conditions are satisfied:*
$d_s \bmod s = 0$ for $s \geq 2$; $d_s \bmod r = s$ for $2 \leq s < r$; $d_1 \bmod r \in \{1, \ldots, r-2\}$; $d_s \bmod (r-1) = s$ for $s \geq 2$; $d_{r-1} \bmod (r-2) = 1$; $r \geq 6$; $d_t \bmod (t+1) = t$ for $t \in \{2, \ldots, r-2\}$.

Examples: Suppose that $r \geq 6$. The conditions of the proposition are obviously met if $d_s = s$ for $s \geq 2$ and $d_1 \bmod r \in \{1, \ldots, r-2\}$. Thus, e.g., $(20, 19, \ldots, 3, 2, \alpha^*)$ will hold for every α such that $\alpha \bmod 20 \in \{1, \ldots, 18\}$. Examples with $d_i < \lambda_r$ and $d_i \neq i$ can also easily be found. For example, if $r = 8$, we have $\lambda_8 = 840$, and d_4 will have to satisfy the conditions $d_4 \bmod 4 = 0$, $d_4 \bmod 8 = 4$, $d_4 \bmod 7 = 4$, $d_4 \bmod 5 = 4$. Not only $d_4 = 4$ is admissible, but also $d_4 = 284$.

With a similar analysis one can prove:

Proposition 2.3. *The sequence $(0, \ldots, 0, d_r, \ldots, d_1)$ starts with m zeros, and $n := m + r$. Then $(0, \ldots, 0, d_r, \ldots, d_1^*)$ provided that:*
$d_s \bmod s = 0$ for $s \geq 2$; $d_r \bmod (r+m_2) = r$ for $m_2 = 1, \ldots, m$; $d_s \bmod n = s$ for $s \in \{2, \ldots, r-2\}$; $d_1 \bmod n \in \{1, \ldots, n-2\}$; $d_s \bmod (n-1) = s$ for $s = 2, \ldots, r-1$; $d_r \bmod r-1 = 1$; $r \geq 5$; $d_{r-3} \bmod (r-2) = r-3$; $d_t \bmod (t+1) = t$ for $t \in \{2, \ldots, r-2\}$.

Examples: As in the preceding proposition the choice $d_s = s$ for $s = 2, \ldots, r$ is admissible. For large m, however, it might be difficult to find further examples.

3. Very good sequences

Sometimes good sequences have a special property:

Definition 3.1. *If both $(a_1, \ldots, a_{k-1}, a_k^*, a_{k+1}, \ldots, a_n)$ and $(a_n, \ldots, a_{k+1}, a_k^*, a_{k-1}, \ldots, a_1)$ hold, then $(a_1, \ldots, a_{k-1}, a_k, a_{k+1}, \ldots, a_n)$ is called a* very good *sequence. To state it otherwise: $(b_1, \ldots, b_n) \Rightarrow (a_k)$, where $(b_1, \ldots, b_n)$ runs through all translates of the sequence and its reflection. In this case we will write $(a_1, \ldots, a_{k-1}, a_k^{**}, a_{k+1}, \ldots, a_n)$.*

It should be stressed that the condition is stronger than the requirement that both $(a_1, \ldots, a_n)$ and $(a_n, \ldots, a_1)$ are good sequences: the number with the "*" must be the same. (E.g. $(1^*, 7, 4, 3, 4)$ and $(4, 3, 4, 7^*, 1)$ hold, but $(1, 7, 4, 3, 4)$ is not very good.)

With the help of a computer it is not difficult to find examples:

n=3: $(0, 0, 1^{**}), (0, 0, 5^{**})$. (In fact these are essentially the only very good sequences in $\{0, 1, 2, 3, 4, 5\}^3$.)

n=4: $(10, 1, 4, 11^{**})$, $(2, 5^{**}, 8, 9)$, $(6, 5^{**}, 2, 7)$, $(5^{**}, 8, 3, 4)$.
n=5: $(4, 1, 2, 11^{**}, 2)$, $(7^{**}, 2, 4, 4, 12)$, $(4, 2, 9^{**}, 2, 4)$, $(2, 4, 6, 2, 1^{**})$.
n=6: $(2, 12, 4, 6, 12, 1^{**})$, $(3, 1, 6, 12, 10, 11^{**})$, $(9^{**}, 4, 1, 8, 1, 4)$.
n=7: $(2, 11^{**}, 3, 6, 6, 9, 7)$, $(10, 1, 7, 6, 5, 11^{**}, 2)$, $(5, 6, 12, 9, 7, 14, 11^{**})$, .
n=8: $(1, 6, 1, 8, 12, 11^{**}, 10, 8)$, $(4, 6, 11^{**}, 4, 3, 6, 6, 4)$, $(1, 10, 3, 3, 17^{**}, 14, 12, 20)$.
n=9: $(17, 16, 1, 9, 10, 4, 2, 11^{**}, 14)$, $(10, 9, 7, 1, 4, 18, 17^{**}, 6, 15)$.
n=10: $(5, 4, 7, 19, 2, 18, 3, 2, 17^{**}, 8)$, $(8, 23^{**}, 2, 25, 6, 2, 1, 18, 15, 25)$.

Certain very good sequences have a special structure: one has a good sequence where the length n is odd, the central element has the "*" and the sequence is symmetric with respect to the center. An example is the very good sequence $(4, 2, 9^{**}2, 4)$ above, more – even rather long – such sequences can easily be found by checking "many" randomly generated symmetric n-tuples with a computer. (In this way we found, $(1, 2, 0, 8, 8, 7, 7^{**}, 7, 8, 8, 0, 2, 1)$, $(1, 1, 1, 7, 7, 6, 9^{**}, 6, 7, 7, 1, 1, 1)$, $(0, 0, 0, 10, 8, 8, 18, 10, 7^{**}, 10, 18, 8, 8, 10, 0, 0, 0)$ and many others.) This motivates the strategy to find very good sequences by identifying good sequences with odd n that are symmetric with respect to the center and where the central element has the "*".

Lemma 3.2. *(i) Let $(a_1, \ldots, a_r) \in \mathbb{N}_0^n$ be a sequence where the k'th element is underlined such that $a_1 = r$ and $a_i \geq r$ for all i. Expand $(a_1, \ldots, a_r)$ by writing s_1 zeros at the beginning and s_2 zeros at the end:*

$$(0, \ldots, 0, a_1, \ldots, a_r, 0, \ldots, 0)$$

Then $(0, \ldots, 0, a_1, \ldots, a_{k-1}, a_k^*, a_{k+1}, \ldots, a_r, 0, \ldots, 0)$ provided that $s_1 + s_2 \geq M + 1$, where $M = \max_{i=1\ldots,r} a_i$.
(ii) Suppose that in addition $r = 2l + 1$ is odd, that a_k is the central element a_{l+1} and that the sequence is symmetric: $a_{l+1+i} = a_{l+1-i}$ for $i = 1, \ldots, l$.
Then $(0, \ldots, 0, a_1, \ldots, a_l, a_{l+1}^{**}, a_{l+2}, \ldots, a_r, 0, \ldots, 0)$ holds.

Proof. (i) We have to show that in all cyclic translates the number a_k survives. This is clear by our assumption for the translate $(0, \ldots, 0, a_1, \ldots, a_r)$. We consider next a translate $(0, \ldots, 0, a_1, \ldots, a_r, 0, \ldots, 0)$. Since $a_1 = r$ we conclude that

$$\begin{aligned} (0, \ldots, 0, a_1, \ldots, a_r, 0, \ldots, 0) &\Rightarrow (a_1, \ldots, a_r, 0, \ldots, 0) \\ &\rightarrow (0, \ldots, 0, a_1, \ldots, a_r) \\ &\Rightarrow (a_k). \end{aligned}$$

It remains to deal with translates of the form $(a_l, \ldots, a_r, 0, \ldots, 0, a_1, \ldots, a_{l-1})$ for some l with at least $M + 1$ zeros in the middle. Since $r \leq a_l \leq M$ we conclude that

$$\begin{aligned} (a_l, \ldots, a_r, 0, \ldots, 0, a_r, \ldots, a_{l+1}) &\rightarrow (0, \ldots, 0, a_1, \ldots, a_r, 0, \ldots, 0) \\ &\Rightarrow (a_1, \ldots, a_r, 0, \ldots, 0) \\ &\rightarrow (a_1, \ldots, a_r) \\ &\Rightarrow (a_k). \end{aligned}$$

(ii) follows at once from (i). $\square$

Proposition 3.3. *For every n_0 there exist very good sequences with at least n_0 nonzero elements.*

Proof. We show that one can find sequences $(a_1, \ldots, a_r)$ for every $r = 2l+1$ that satisfy the conditions of lemma 3.2 (ii). Thus it suffices to choose $r \geq n_0$.

We start by setting $a_1 = a_r = r$. Then $(a_1, \ldots, a_r) \to (a_2, \ldots, a_r)$. Now we choose any $a_2 \geq r$ such that $a_2 \bmod (r-1) = (r-2)$ and $a_2 \bmod (r-2) = 0$ (e.g., $a_2 = (r-1)(r-2) + (r-2) = r(r-2)$), and we put $a_{r-1} := a_2$. This guarantees that

$$(a_2, \ldots, a_r) \to (a_2, \ldots, a_{r-1}) \to (a_3, \ldots, a_{r-1}).$$

We continue with this strategy: a_3 satisfies $a_3 \geq r$, $a_3 \bmod (r-2) = (r-3)$, $a_3 \bmod (r-3) = 0$, $a_{r-2} := a_3$ etc. In this way we arrive at $(a_1, \ldots, a_r) \Rightarrow (a_{l+1}, a_{l+2})$, and thus it suffices to choose an odd a_{l+1} such that $a_{l+1} \geq r$. This shows that $(a_1, \ldots, a_l, \underline{a_{l+1}}, a_{l+2}, \ldots, a_r)$. Lemma 3.2 (ii) now provides the very good sequence

$$(0, \ldots, 0, a_1, \ldots, a_l, a_{l+1}^{**}, a_{l+2}, \ldots, a_r, 0, \ldots, 0),$$

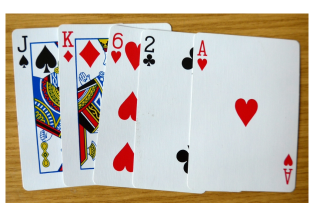
where the sequence starts and ends with s zeros (s arbitrary such that $2s > \max a_i$). $\square$

Remark: A "minimal" candidate for which lemma 3.2 (ii) together with the preceding proposition could possibly be applied would be the sequence $(r, \ldots, r) \in \mathbb{N}_0^r$ for odd $r > 2$. For which r is it true that the central element is underlined? Among the odd r in $\{3, \ldots, 20.000\}$ the numbers $r = 3, 7, 171, 513, 517, 519, 529, 531$ are the only examples. It is open whether infinitely many such r exist and why there is this mysterious accumulation near 520.

4. Mathematical magic

Our starting point was a magic trick. By our investigations variations are possible. Here are some proposals.

1. Choose any good or very good sequence from our supply, e.g. $(2, 4, 6, 2, 1^{**})$, and realize it by using playing cards or blank cards with numbers written on them. (We will assume that Ace, Jack, Queen and King count $1, 2, 3, 4$, respectively):



The very good sequence $(2, 4, 6, 2, 1^{**})$.

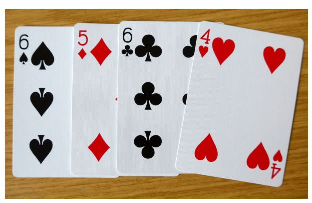
This little deck might be cut by a spectator arbitrarily often, also – in the case of very good sequences – the order might be reversed by counting the cards one by one on the table. The magician knows the card that will survive the advanced Australian shuffle (in our example it will be the Ace).

2. The three numbers $7, 1, 2$ have the property that they give rise to good sequences in their original and in their reflected order: $(7^*, 1, 2)$ and $(2, 1^*, 7)$. Put three cards that represent these values faceup side by side on the table. A random number k is generated by throwing a dice. Now the magician turns his back to the table and asks a spectator to interchange arbitrarily k times two of the three cards. Then he knows: In the case of even (resp. odd) k the cards now form a cyclic translate of $7, 2, 1$ (resp. of $2, 1, 7$). It is therefore guaranteed that the number 7 (resp. the number 1) will survive our advanced Australian shuffle when the cards on the table are put together properly (from top to bottom the left, middle and right card).

Here are some other triples that could also be used: $(6, 7^*, 1)$ and $(1^*, 7, 6)$; $(7, 8, 1^*)$ and $(1, 8, 7^*)$; $(6, 1^*, 7)$ and $(7^*, 1, 6)$.

Very good sequences with three elements exist also. Then the spectator can choose the order of the three cards completely arbitrarily. But there are essentially only two candidates, namely $(0, 0, 1^{**})$ and $(0, 0, 5^{**})$. By lemma 2.1(i) $(6, 6, 1^{**})$, $(6, 12, 1^{**})$, $(6, 6, 5^{**})$ and $(6, 12, 1^{**})$ work similarly well. But 12 is too large for our purpose and a repeated 6 decreases the number of different permutations. (This fact, however, will be overlooked by most spectators.)

3. Choose a very good sequence of four elements where the numbers that are (cyclically) neighbours of the element with the "**" are identical. As in illustration we consider the sequence $(6, 5^{**}, 6, 4)$.



The very good sequence $(6, 5^{**}, 6, 4)$.

Ask a spectator to arrange them in any order subject to the condition that red and black cards alternate. The advanced Australian shuffle will end with the five.

If there are German speaking people in the audience one can repeat the trick with the same cards, this time using the numbers of the letters in the German words of the suits: then one deals with $(3, 4, 5^{**}, 4)$ in disguise. Provided that red and black cards alternate one can be sure that the Kreuz-card (the 6 of clubs) will survive.

Similarly well one could deal with $(4, 5^{**}, 4, 7)$, $(6, 1^{**}, 6, 4)$, $(2, 3, 2, 5^{**})$ or $(6, 4, 6, 1^{**})$,

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Ehrhard Behrends
Mathematisches Institut, Freie Universität Berlin
Arnimallee 6
D-14 195 Berlin
Germany
e-mail: behrends@math.fu-berlin.de