A method of Bender applied to groups of J_3-type.

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December 18, 2007

Abstract

We apply a method of Bender [6] to determine the order of a group G of J_3-type. Moreover, we determine the local p-structure of G for every prime p dividing the order of G. The results of this paper are obtained by exploiting the action of G on its geometry [4] and by sophisticated use of elementary group theory.

1 Introduction.

A finite simple group G is said to be of J_3-type provided that all involutions of G are conjugate and the centralizer of an involution is a split extension of an extraspecial group of order 32 by $\mathrm{Alt}(5)$. Z. Janko calculated the order of a group of J_3-type using character theory [11]. There is the Thompson Order Formula which determines the order of a simple group with more than one conjugacy class of involutions by counting involutions, see [1, 45.6]. H. Bender introduced a method of counting involutions which can sometimes be applied to determine the order of a group with just one conjugacy class of involutions, see [6].

In this paper we use this method to prove

Theorem 1 *Let G be a group of J_3-type. Then* $|G| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$.

The method of Bender, which will be introduced in the next section, uses only local information of the group.

Beside the order of a group of J_3-type, we show

Theorem 2 *Let G be a group of J_3-type and p a prime dividing the order of G. Then the local p-structure is as described in Sections 3 and 5.*

In a forthcoming paper we aim to use Theorem 1 and the results of [4] to show the uniqueness of a group of J_3-type, as announced in [3]. For other existence or uniqueness proofs see [5, 9, 12, 8, 2]. There is also a new computer based existence proof by Bradley and Curtis [7].

Let G be a group of J_3-type. In [4] the author showed that G is the completion of an amalgam of J_3-type, i.e. that there are subgroups G_1, G_2 and G_3 in G such that

(i) $G_1 \simeq L_2(16):2,\ G_2 \simeq 2^4 : GL_2(4),\ G_3 \simeq 3 : PGL_2(9)$.

(ii) $G_1 \cap G_2 \simeq 2^4 : (3 \times D_{10}),\ G_2 \cap G_3 \simeq GL_2(4) \simeq 3 \times \mathrm{Alt}(5),\ G_1 \cap G_3 \simeq \mathrm{Sym}(3) \times D_{10}$.

(iii) $G_1 \cap G_2 \cap G_3 \simeq 3 \times D_{10}$.

(iv) $G = \langle G_1, G_2, G_3 \rangle$.

In Section 3 we heavily use the fact that G is a completion of such an amalgam. In Subsections 3.1, 3.5 and 3.2 we determine the local 5, 2-structure and to some extent the local 3-structure of G, respectively.

The local 3-structure of a group of J_3-type is fairly complicated, see for instance [11]. Notice, that Aschbacher used the local 3-structure of a group of J_3-type to embed and recognize groups of J_3-type in E_6 [2]. In order to bound the size of a Sylow 3-subgroup of G, we bound the order of G, see Subsection 3.3.

In the penultimate section we prove Theorem 1. Once we know $|G|$, we are able to complete the determination of the local 3-structure of G, which is done in the last section. There we also describe the local 17 and 19-structures of G. The results of this paper are obtained by exploiting the action of G on its geometry and by sophisticated use of elementary group theory.

Acknowledgement I would like to thank Chris Parker as well as the referee for pointing out a gap in an earlier version of the paper. Moreover, I am grateful to the referee and to Chris Parker for their careful reading of the manuscript and for their comments.

2 The Method of Bender

Bender considers a group G, the set of involutions I of G and a subgroup H of G such that $|I| > |G : H|$. He introduces the following notation.

I_n = set of u in $I \setminus H$ such that $|Hu \cap I| = n$.

b_n = number of cosets $Hg \neq H$ with $|Hg \cap I| = n$.

c = number of u in I_1 such that $C_H(u) \neq 1$.

f = $|I|/|G : H| - 1$.

Notice that $|I_n| = n \cdot b_n$ and, as $|I| > |G : H|$, that $f > 0$. Bender made the following observation.

Lemma 2.1 *[6]*

(i) $|I| = |I \cap H| + b_1 + 2b_2 + 3b_3 + \ldots$.

(ii) $b_1 = c + m|H|$ *for some integer* $m \geq 0$.

(iii) $b_1 < f^{-1}(|I \cap H| + b_2 + 2b_3 + 3b_4 + \ldots) - 1 - b_2 - b_3 - b_4 - \ldots$.

The idea for determining the order of G is to calculate b_i for $i > 1$ and then to use the lemma to determine b_1 and the number of involutions $|I|$ of G. In order to calculate b_i for $i > 1$ it is helpful to use the following fact. Let u be an involution in I_n, that is Hu contains precisely n involutions. Notice, if $a = hu$, with $h \in H$, is an involution, then u inverts h. Thus the number of involutions in Hu equals the number of elements in H which are inverted by u.

3 More about the structure of a group of J_3-type.

In this section we provide the information needed to apply the method of Bender.

Notation Throughout the paper we are using the notation which has been established in the introduction. So G is a group of J_3-type and $G_1 \cong L_2(16) : 2$, $G_2 \cong 2^4 : GL_2(4)$ and $G_3 \cong 3 : PGL_2(9) \cong (3 \times \mathrm{Alt}(6)) : 2$ are subgroups of G such that $G = \langle G_1, G_2, G_3 \rangle$.

For n a natural number and r a prime, we denote by n_r the r-part of n.

For $g \in G$ let $C_g := C_G(g)$ and $N_g := N_G(\langle g \rangle)$. For $i \in G$ an involution, set $Q_i := O_2(C_i)$ and let T_i be a complement to Q_i in C_i. In most of our notation we follow [1]. For instance, for G a group we denote by $G^\#$ the set $G \setminus \{1\}$ and by G^∞ the intersection of the commutator subgroups $G^{(i)}$, see [1, p. 27].

If we have a group G which is isomorphic to an extension $A^{\cdot}B$ and we want to specify the action of B on A only to some extent, then we write $G \simeq A^{\cdot}B$.

3.1 The 5-structure of G.

Lemma 3.1 *Let S be a Sylow 5-subgroup of G. Then*

(i) S is of order 5 and

(ii) $N_G(S) \cong \mathrm{Sym}(3) \times D_{10}$.

In particular, if w is an element of order 5 in N_s, then N_w is a subgroup of N_s.

Proof. Assume that $w \in N_s$. Then it follows $N_{N_s}(\langle w \rangle) \cong D_{10} \times \mathrm{Sym}(3)$. Let $\langle i, j \rangle \cong 2^2$ be a subgroup of $N_{N_s}(\langle w \rangle)$ such that i centralizes w. In C_i we see that $\langle i \rangle$ is a Sylow 2-subgroup of C_w. Therefore, C_w has a 2-complement R. As i inverts $R/\langle w \rangle$, the group $R/\langle w \rangle$ is abelian. Moreover, $R = \langle C_R(x) \mid x \in \{i, j, ij\} \rangle$. Without loss of generality we may assume that $C_R(j) = \langle s \rangle$. It remains to determine $C_R(ij)$. Assume that $C_R(ij) \neq 1$. Then either $|R| = 5 \cdot 3^2$ or $5^2 \cdot 3$. In both cases we would obtain the contradiction that $C_R(s) > \langle w \rangle \times \langle s \rangle = C_s \cap C_w$. Hence $R \leq N_{N_s}(\langle w \rangle)$. In C_i we see that N_w induces only a subgroup of order 2 on $\langle w \rangle$, which implies the assertion in that case.

As, for $w \in N_s$, the order of the normalizer of w in G is only divisible by 5, every Sylow 5-subgroup of G is of size 5. Therefore, the previous paragraph also proves (i) and (ii). $\square$

3.2 The 3-structure of G, part I.

In [4] we also showed the following:

Lemma 3.2 *[4, 2.3, 2.4] $G_3 = N_G(O_3(G_3))$ and $G_2 = N_G(O_2(G_2))$.*

Hence, if $\langle s \rangle$ is a Sylow 3-subgroup of the centralizer of an involution, then

$$N_s \cong 3 : PGL_2(9), \; N_s' \cong 3 \times \text{ Alt}(6) \text{ and } N_s^{\infty} \cong \text{ Alt}(6).$$

In what follows we assume that $G_3 = N_s$.

Let W be a Sylow 3-subgroup of G_3. Then W is elementary abelian of order 27. Let $T = W \cap N_s^{\infty}$. Then T is elementary abelian of order 9 and $N_{N_s}(W) = W : K$, where K is cyclic of order 8 which acts regularly on $T^{\#}$. In the next two lemmas we study the embedding of W in G.

Lemma 3.3 *(i) All the elements of order* 3 *in* $N_s \setminus N_s^{\infty}$ *are conjugate to* s *in* G.

(ii) All the elements of order 3 *in* N_s^{∞} *are in the same conjugacy class* $\mathcal{C}$ *of* G.

(iii) $\mathcal{C}$ *does not contain* s.

Proof. The fact that K acts regularly on $T^{\#}$ implies that all the elements of order 3 in N_s^{∞} are in the same conjugacy class, which is statement (ii). In the proof of Lemma 2.4 in [4] it was shown that s is not conjugate to any element in N_s^{∞} proving (iii).

We have $G_2 \cap G_3 \cong 3 \times \text{ Alt}(5) \cong GL_2(4)$ and, as $G_2 \cong 2^4 : GL_2(4)$, there is a subgroup of order 3 in $(G_2 \cap G_3) \setminus Z(G_2 \cap G_3)$ which centralizes an involution in $O_2(G_2)$. Hence $\langle s \rangle$ is conjugate to a subgroup in $W \setminus \langle s \rangle$. As there is an element in N_s which inverts s, the latter element is conjugate to its inverse. This together with the fact that K is transitive on $T^{\#}$ yields (i). $\square$

Lemma 3.4 *The following hold.*

(i) $N_G(W) = Q : K$ *with* Q *a group of order* 3^5 *and* K *cyclic of order* 8.

(ii) $Z(Q)$ *is the subgroup* T *of* W *and is elementary abelian of order* 9.

(iii) K *acts regularly on* $Z(Q)^{\#}$.

(iv) $N_G(W)/W$ *is isomorphic to* $Frob(3^2 : 8)$ *and acts faithfully on* W.

Proof. By Lemma 3.3 every element v in $W \setminus T$ is conjugate to s. Since W is a Sylow 3-subgroup of C_s as well as of C_v, the elements s and v are conjugate in $N_G(W)$. Hence $N_G(W)$ induces two orbits on the set of subgroups of order 3 of W, which are of length 9 and 4, respectively. So,

$$9 = |\langle s\rangle^{N_G(W)}| = |N_G(W) : N_G(W) \cap N_s| = |N_G(W) : (WK)|,$$

which implies that $|N_G(W)| = 3^5 \cdot 8$. Hence K is a Sylow 2-subgroup of $N_G(W)$. In the centralizer of the unique involution of K in G, we see that K is self-normalizing in $N_G(W)$. Therefore, Burnside's Normal p-complement Theorem [1, 39.1] yields that $N_G(W)$ has a 2-complement Q of order 3^5. This proves (i).

Let $t \in T^\#$. As $4 = |\langle t\rangle^{N_G(W)}|$, it follows that $t \in Z(Q)$ and therefore $T \leq Z(Q)$. Now $C_G(W) = W$ implies $Z(Q) \leq T$, so $T = Z(Q)$. This shows (ii) and (iii).

We have shown that $\overline{N} := N_G(W)/C_G(W) = N_G(W)/W$ is a subgroup of $Aut(W)$ which fixes the subgroup $T = Z(Q)$. Moreover, $\overline{K}$ fixes the subgroups T and $\langle s\rangle$ in W and $O_3(\overline{N})$ fixes T pointwise. This shows that $\overline{N} \cong Frob(3^2 : 8)$. As $|\langle s\rangle^{\overline{N}}| = 9$, the subgroup $O_3(\overline{N})$ acts faithfully on W, which implies (iv). $\square$

3.3 A bound on the order of groups of J_3-type.

Let G be a group of J_3-type. We provide an upper bound for $|G|$ which turns out to be rather good.

As mentionned in the introduction the author showed in [4] that G is the completion of an amalgam $\mathcal{A}$ of J_3-type. Therefore G acts flag-transitively on a rank three geometry Γ, called DEQ (dual extended quadrangle), consisting of points, lines and quads, which are the cosets of G_1, G_2 and G_3 in G, respectively, see [4, 1.1]. In [5] the author constructed a DEQ Γ_K and a completion K of an amalgam of J_3-type which is a non-split extension $3^{\cdot}J_3$ of a group of J_3-type and which acts flag-transitively on Γ_K.

We are able to deduce from [5] an upper and a lower bound for $|K|$. Notice that the upper bound is very good!

Lemma 3.5 *(i)* $|K| \leq 2^{12} \cdot 3^6 \cdot 5 \cdot 19.$

(ii) $|K| \geq 2^8 \cdot 3^6 \cdot 5^2 \cdot 17.$

Proof. We follow the notation of [5] and let V be a 9-dimensional $GF(4)$-space equipped with a unitary form and $X \cong GU_9(2)$ the group of isometries of V. It has been shown that $Z(K) = Z(X) \cong \mathbb{Z}_3$ [5, 7.11]. Let $\hat{X} = X/Z(X)$. Then $\hat{K}_3$ is the stabilizer $\hat{K}_e$ in $\hat{K}$ of a 1-space $\langle e \rangle$ in V where e is a non-isotropic vector [5, 8.1].

As $|\hat{X}| \geq |\hat{X}_e \hat{K}| = |\hat{X}_e||\hat{K}|/|K_e|$ it follows that $|\hat{X} : \hat{X}_e| \geq |\hat{K} : \hat{K}_e|$. We have $\hat{X}_e \cong U_8(2)$ and $|\hat{X} : \hat{X}_e| = 2^8 \cdot 3^2 \cdot 19$. As $\hat{K}_e = \hat{K}_3 \cong 3 : PGL_2(9)$, it follows that

$$|\hat{K}| \leq |\hat{K}_e||\hat{X} : \hat{X}_e| = 2^4 \cdot 3^3 \cdot 5 \cdot 2^8 \cdot 3^2 \cdot 19 = 2^{12} \cdot 3^5 \cdot 5 \cdot 19,$$

which implies $|K| \leq 2^{12} \cdot 3^6 \cdot 5 \cdot 19$.

In order to prove (ii) we need to study the geometry Γ_K more closely. Let $\hat{\Gamma}_K$ be the quotient of Γ_K where we identify the points, lines and planes which are in a common $Z(K)$-orbit, respectively. Then $\hat{\Gamma}_K$ is a DEQ with $\hat{K}$ acting flag-transitively.

We claim that $\hat{\Gamma}_K$ has at least $1+85+680+2 \cdot 1360$ points. We consider the collinearity graph $\mathcal{G}$ of $\hat{\Gamma}_K$ (its vertices and edges are the points and pairs of collinear points, respectively). In our DEQ $\hat{\Gamma}_K$ every two points at distance 2 are on precisely one quad, every quad is a 6×6-grid and every point is contained in 17 lines, see [5]. Therefore, a point p has 85 neighbours in $\mathcal{G}$. Let Q be a quad containing p. Then the stabilizer S of p and Q in $\hat{K}$ is isomorphic to $3 : 2 \times 5 : 2$ and $O_3(S)$ fixes the grid pointwise. Let f be an involution in S with $[O_3(S), f] = 1$. Then f fixes precisley one point q on Q at distance two from p and the stabilizer in $\hat{K}$ of p and this point is isomorphic to $3 : 2 \times 2$. Hence,

$$|q^{\hat{K}_p}| = |\hat{K}_p : \hat{K}_p \cap \hat{K}_q| = |L_2(16) : 2|/|3 : 2 \times 2| = 680.$$

Notice that five of the points of Q are contained in this orbit. The remaining 20 points at distance two from p in Q, fall into two orbits under the action of S, each of size 10. This shows that there are two more $\hat{K}_p$-orbits in $\mathcal{G}$ and that they are of size $|L_2(16) : 2|/6 = 1360$.

Thus we have shown that $\hat{\Gamma}_K$ and Γ_K have at least $1+85+680+2 \cdot 1360 = 3486$ and $3 \cdot 3486$ points, respectively. Hence,

$$|K| \geq 3 \cdot 3486 \cdot |K_p| \geq 2^8 \cdot 3^6 \cdot 5^2 \cdot 17,$$

as claimed. □

With [4, 1.1] we conclude

Proposition 3.6 *The universal completion $\tilde{G}$ of $\mathcal{A}$ acts flag-transitively on the universal cover $\tilde{\Gamma}$ of Γ. The groups G as well as K are quotients of $\tilde{G}$ and the stabilizer $\tilde{G}_1$ of a point in $\tilde{\Gamma}$ is isomorphic to the stabilizer G_1 of a point in Γ.*

Further we know

Lemma 3.7 *[5, 2.2] $\tilde{\Gamma}$ has at most 6^{17} points.*

Proposition 3.8 *Let G be a group of J_3-type. Then $|G| \leq (2^{14} \cdot 3^{12})/5$.*

Proof. By 3.6(i) K and G are composition factors of $\tilde{G}$. If $K/O_3(K) \cong G$, then by 3.5

$$|G| = |K/O_3(K)| \leq 2^{12} \cdot 3^5 \cdot 5 \cdot 19$$

and the assertion holds.

Assume that $K/O_3(K) \not\cong G$. Then $\tilde{G}$ is divisible by $|G||K|$. Moreover, 3.6 and 3.7 yield that $|\tilde{G} : \tilde{G}_1| \leq 6^{17}$. As $\tilde{G}_1 \cong G_1 \cong L_2(16) : 2$ and $|K| \geq 2^8 \cdot 3^6 \cdot 5^2 \cdot 17$ (see 3.6(ii)), we get the assertion. $\square$

Corollary 3.9 *Let G be a group of J_3-type. Then $|G|_3 \leq 3^{10}$.*

Proof. We know that G is divisible by $2^7 \cdot 5 \cdot 17$. Now Proposition 3.8 implies the assertion. $\square$

3.4 The 3-structure of G, part II.

In what follows we are using the notation introduced in Lemma 3.4.

Lemma 3.10 *Let S be a Sylow 3-subgroup of $N_G(T)$. Then S is of order $3^5, 3^7$ or 3^9 and $|N_G(T)| = 8 \cdot |S|$.*

Proof. We may assume that Q is a subgroup of S. In the following we use the fact that $N_G(Q)/Q$ acts faithfully on Q/T. We know that $\overline{Q} = Q/T$ is of order 3^3 and that it is either extraspecial with $Z(\overline{Q}) = \langle s \rangle T/T$ or abelian. If $Q = S$, then the assertion follows from 3.4 and the subgroup structure of $GL_3(3)$.

Assume that Q is a proper subgroup of S. Then $\overline{Q}$ is a proper subgroup of $\overline{S}$ and $\overline{M} := N_{\overline{S}}(\overline{Q}) > \overline{Q}$. Hence, W is not normal in $N_S(Q)$ and therefore $\overline{Q}$

is abelian. Notice, as $T = Z(Q)$, it follows that $[N_G(Q), T] \leq T$. Therefore $M := N_S(Q)$ is a Sylow 3-subgroup of $N_G(Q)$ and $N_G(Q)$ contains by 3.4(i) a cyclic group K of order 8.

Let i be an involution in K. Then every element in T as well as every element in Q/W is inverted by i and $[s, i] = 1$. This shows that sT is not a third power in Q/T. Therefore, $\overline{Q}$ is elementary abelian. By 3.4(iv) the orbits of K on the set of 3-subgroups in $\overline{Q}$ are of length $1, 4$ and 8.

Hence, the orbits of M are of length 5 and 8 or 9 and 4 or M is transitive. In the latter case we get $\overline{M} \cong 3^3 : GL_3(3)$ and the order of $\overline{M}_s$ is divisible by 3^6, which contradicts $N_s \cong 3 : PGL_2(9)$. Assume that the orbits are of length 5 and 8. As $|N_s \cap N_G(Q)|/|W| = 8$, it follows that $|N_{\overline{G}}(\overline{Q})| = 5 \cdot 8$. This is impossible as 5 does not divide $|GL_3(3)|$.

Therefore, the orbits are of length 9 and 4 and therefore,

$$|N_{\overline{G}}(\overline{Q}) : N_{\overline{G}}(\overline{W})| = 3^2.$$

As $N_G(Q)/Q$ acts faithfully on Q/T, we see in $GL_3(3)$ that $N_G(Q)/Q \cong 3^2 : \mathbb{Z}_8$. Thus, it follows that $N_G(Q) = M : K$.

Moreover, the center $\overline{U} := Z(N_{\overline{S}}(\overline{Q}))$ is of order 3^2. Let U be the preimage of $\overline{U}$ in S. Then M/U is of order 3^3 and either extraspecial with $Z(M/U) = Q/U$ or abelian. If M/U is abelian, then we get by the same argument as for $\overline{Q}$ that it is elementary abelian.

Set $P := N_S(M)$. Clearly, $Z(M) = T$. Thus, $[P, T] \leq T$ and P also acts on $Z(M/T) = \overline{U}$, on U and on $Z(M/U)$. Clearly, if $P = M$, then $M = S$ and $N_G(T) = M : K$.

If M/U is extraspecial, then $Z(M/U) = Q/U$. Thus, in that case P acts on Q, which implies $P \leq N_S(Q) = M$ and therefore $M = S$ and $N_G(T) = M : K$.

Assume that $P \neq M$. Then M/U is elementary abelian and $P : K$ acts as a subgroup of $GL_3(3)$ on M/U. By the same argument as above it follows that $N_G(M)/M \cong 3^2 : \mathbb{Z}_8$ and that $N_G(M) = P : K$. Thus P is of order 3^9.

Then $Z(P/U)$ is of order 9. Let V be the preimage of $Z(P/U)$. Then P/V is of order 3^3 and, as before, either elementary abelian or extraspecial.

If $N_S(P) > P$, then as before $N_G(P)/P \cong 3^2 : \mathbb{Z}_8$ and $|G|_3 \geq 3^{11}$ in contradiction with Corollary 3.9. Hence $N_S(P) = P$ and $S = P$, which completes the proof of the lemma. $\square$

Corollary 3.11 *Let R be a Sylow 3-subgroup of G. Then $|R| = 3^i$ where $i \in \{5, 7, 9\}$.*

Proof. Let S be a Sylow 3-subgroup of $N_G(T)$ and assume $S \leq R$. If S is a proper subgroup of R, then $N_R(S) > S$ and, as $T = Z(S)$ char S, we get the contradiction $N_R(S) \leq N_G(T)$. $\square$

Lemma 3.12 *(i) Every element of order 8 in G is self centralizing.*

(ii) Let L be a Sylow 2-subgroup of $N_G(T)$. Then L is cyclic of order 8.

(iii) Let $t \in T^{\#}$. Then every Sylow 2-subgroup of $N_G(\langle t \rangle)$ is of order 2.

Proof. The structure of the centralizer of an involution implies (i). L is of order 8 by 3.10 and cyclic by 3.4. Statement (iii) follows immediately from (ii) and Lemma 3.4(iii). $\square$

3.5 The 2-structure of G.

Next we consider the 2-strucure of G and recall some facts which follow directly from Lemma 2.1 in [4]. We continue to use the notation introduced in the beginning of Section 3.

Lemma 3.13 *Let i be an involution in G. Then the following holds*

(i) $Q_i \cong D_8 * Q_8$.

(ii) $Q_i/\langle i \rangle$ is the even part of the permutation module for $T_i \cong$ Alt(5).

(iii) Let $a \in Q_i \backslash \{i\}$ be an involution. Then $N_{T_i}(\langle i, a \rangle) = C_{T_i}(\langle i, a \rangle) \cong$ Alt(4).

(iv) Let $b \in T_i$ be an involution. Then $C_{Q_i}(\langle b \rangle)$ is elementary abelian of order 4.

(v) All the involutions in $Q_i \setminus \{i\}$ as well as all the involutions in $C_i \setminus Q_i$ are conjugate in C_i.

Remark 3.14 *In the following and specially in Section 4 we calculate in the centralizer C_i of an involution i of G again and again. Therefore, it is helpful to visulize C_i.*

According to 3.13(ii) $Q_i/\langle i\rangle$ is the even part of the permutation module for $T_i \cong$ Alt(5). In $Q_i \setminus \{i\}$ there are 20 and 10 elements of order 4 and 2, respectively. Their images in $Q_i/\langle i\rangle$ are $e_i + e_j$, $1 \le i < j \le 5$ and $\sum_{i\neq j} e_i$ with $1 \le j \le 5$, respectively, where $\{e_1, e_2, \ldots, e_5\}$ is a basis of the permutation module on which T_i acts by permuting the indices.

Lemma 3.15 *Let i be an involution in G and $a \in Q_i$ of order 4. Then*

(i) $N_{C_i}(\langle a\rangle) = Q_i : X$, *where* $X \cong$ Sym(3).

(ii) $\langle a\rangle$ *together with all the involutions in Q_i inverting the element a form a dihedral group Y of order 8.*

(iii) $[Y, X] \le Y$

Proof. 3.13(ii) implies (i) and (iii). (ii) follows from 3.13(i). □

We introduce further notation. Let $a, b \ge 2$ be natural numbers and A a group. Then $2^{a+b} : A$ denotes a 2-group of order 2^{a+b} extended by the group A such that the 2-group has two A-composition factors, which are elementary abelian of order 2^a and 2^b, respectively.

Lemma 3.16 *There are precisely two classes of elementary abelian subgroups of order 4 in G. Let i, a and b be involutions in G such that $a \in Q_i$ and $b \in C_i \setminus Q_i$. Then we have:*

(i) $N_G(\langle i, a\rangle) \simeq 2^{2+4} : (3\times$ Sym(3)).

(ii) $N_G(\langle i, b\rangle) \simeq 2^{2+2} :$ Sym(3).

Proof. The first statement is a consequence of 3.13(v).

By (ii) of 3.13 we have $C_i \cap N_G(\langle i, a\rangle) = Q_i : A$ with $A \cong$ Alt(4) and $C_G(\langle i, a\rangle)$ is a subgroup of index 2 in $Q_i : A$. Let $V := \langle i, a\rangle O_2(A)$. Then $N_G(V) \cong 2^4 : GL_2(4)$ [4] and the normaliser of $\langle i, a\rangle$ in $N_G(V)$ is isomorphic to $2^{2+2} : (3\times$ Alt(4)). Hence $N_G(\langle i, a\rangle)$ is transitive on $\langle i, a\rangle^{\#}$ and $N_G(\langle i, a\rangle)$ induces the full symmetric group Sym(3) on $\langle i, a\rangle$.

We may assume that s is in $C_G(\langle i, a\rangle)$. In $N_s \cong 3 : \mathrm{PGL}_2(9)$ we see that $N_G(\langle i, a\rangle)/O_2(N_G(\langle i, a\rangle)) \cong 3\times$ Sym(3). As $N_G(\langle i, a\rangle)$ does not normalize V, we get (i).

In C_i we see using 3.13(iv) that $C_G(\langle i, b\rangle)$ is elementary abelian of order 16. As $C_G(\langle i, b\rangle)$ is of index 2 in $C_i \cap N_G(\langle i, b\rangle)$, there is an involution in

$N_G(\langle i,b\rangle) \cap C_i$ interchanging ib and b. As $i \in C_b \setminus Q_b$, we can interchange the roles of i and b and it follows that $N_G(\langle i,b\rangle)$ induces the full symmetric group on $\langle i,b\rangle$, which proves assertion (ii). □

Corollary 3.17 *Let $U = \langle i,a\rangle$ be a subgroup as in (i) of 3.16. Then the 3-elements in $N_G(U)'$ are not conjugate to s.*

Proof. The center of $N_G(U)/O_2(U)$ acts trivially on U and is therefore conjugate to $\langle s\rangle$. Hence we may assume that it is $\langle s\rangle$. The normalizer of U in N_s is isomorphic to $3\times$ Sym(4). Therefore, all the 3-elements in $N_G(U)'$ are in N_s^∞ and Lemma 3.3 (ii) and (iii) yield the assertion. □

4 The order of a group of J_3-type.

We continue to use the notation introduced so far. So, let G be a group of J_3-type and let s be an element of 3 in the centralizer of an involution. Set

$$H := N_s.$$

So $H \cong 3 : PGL_2(9)$ and $|H| = 2^4 \cdot 3^3 \cdot 5$. Thus we have

$$f = (|G : C_i|/|G : H|) - 1 = (|H|/|C_i|) - 1 = (2^4 \cdot 3^3 \cdot 5)/(2^7 \cdot 3 \cdot 5) - 1 = 1/8 > 0.$$

We need one further piece of notation. For u an involution in I_n let $S(u)$ be the set of elements of H inverted by u and set

$$H(u) := \langle S(u)\rangle.$$

Then $H(u)$ is a subgroup of H which is normalized by $L := H(u) : \langle u\rangle$. As explained in Section 2 the number of involutions in Hu equals the size of $S(u)$. Notice also, that $H(u) \leq H \cap H^u$, as $x^u = x^{-1} \in H^u \cap H$ for all $x \in S(u)$.

4.1 The subgroups $H(u)$.

Next we determine the subgroups $H(u)$. We show:

(A) If $I_n \neq \emptyset$, then $n \in \{1,2,3,4,6,9\}$.

(B) If $|Hu \cap I| = n$, then $H(u)$ is as listed in Table 1 at the beginning of the next subsection. Distinct rows of Table 1 correspond to distinct conjugacy classes $H(u)^H$. The normalizers of $H(u)$ in H and in G as well as the length of the conjugacy class $|H(u)^H|$ which is the index of $N_H(H(u))$ in H are as listed in Table 1.

To prove this it is helpful to know the properties $(\star)$ and $(\star\star)$ which are listed below. Notice, that $(\star)$ follows from the fact that $H = N_G(O_3(H))$, see Lemma 3.2. Assume that there is a subgroup S in $H(u)$ which is of order 5. Then the Frattini Argument yields $L = H(u)N_L(S)$, which is not possible as $N_G(S)$ is contained in H by Lemma 3.1. This shows $(\star\star)$.

$(\star)$ $O_3(H)$ is not a characteristic subgroup of $H(u)$.

$(\star\star)$ The order of $H(u)$ is not divisible by 5.

Now we go through the list of subgroups of H. Recall that L is the extension of $H(u)$ by u, so $H(u)$ is a subgroup of L of index 2.

Lemma 4.1 *$H(u)$ is not isomorphic to one of the following groups:*

$$3^2:2,\ 3^2:4,\ 3^2:8,\ \mathrm{Sym}(3),\ \mathbb{Z}_4,\ \mathbb{Z}_8,\ \mathrm{Alt}(4),\ \mathrm{Sym}(4),\ D_{16}$$

and $|H(u)|$ is not divisible by 5.

Proof. Assume that $H(u) \cong 3^2 : 2$, $3^2 : 4$ or $3^2 : 8$. Then L contains an elementary abelian subgroup $\langle a, u\rangle$ of order 4 and $O_3(H(u))$ is generated by the centralizers of a, u and au in $O_3(H(u))$ in contradiction to 3.3(iii).

Assume $H(u) \cong \mathrm{Sym}(3)$. Then L again contains an elementary abelian subgroup of order 4 which yields that $O_3(H(u))$ is conjugate to $\langle s\rangle$. As $H(u) \cong \mathrm{Sym}(3)$, in fact $O_3(H(u)) = \langle s\rangle$ in contradiction to $(\star)$.

Further $H(u) \not\cong \mathrm{Sym}(4)$, as there is no subgroup in G isomorphic to $2\times \mathrm{Sym}(4)$.

As elements of order 8 are self-centralizing, $H(u)$ is neither isomorphic to D_{16} nor to $\mathbb{Z}_8$.

Assume $H(u) \cong \mathrm{Alt}(4)$. Then $N_H(O_2(H(u))) \cong 3\times \mathrm{Sym}(4)$. By Corollary 3.17 every element t of order 3 of $H(u)$ which is inverted by u is an element of H^∞. Therefore, $N_t \cap N_G(O_2(H(u))) \cong 3\times \mathrm{Sym}(3)$ is contained in H. Thus u is in H, which is a contradiction to our assumption.

Notice, all the elements of order 4 in H are contained in H' and in particular in a subgroup isomorphic to D_8 which is not lying in H'. Assume that $H(u)$ is cyclic of order 4. Then u inverts every element in $H(u)$. Moreover $N_H(H(u)) \cong 3{\cdot}D_{16}$ and $N_G(H(u)) \cong (D_8{*}Q_8) : \mathrm{Sym}(3)$ by 3.15(i). Let i be the involution in $H(u)$. Then, as $N_H(H(u)) \cap Q_i \cong D_8$, Lemma 3.15(ii) implies that u is an involution in $C_i \setminus Q_i$. By 3.15 u normalizes $N_H(H(u)) \cap Q_i$. It then follows that u commutes with an involution in $(C_i \cap N_H(H(u))) \setminus \{i\}$. Hence $|H(u) \cap I| \geq 6$ and $H(u)$ is not cyclic of order 4. $\square$

Lemma 4.2 *If $H(u)$ is a subgroup of $H^\infty \cong$ Alt(6), then $H(u)$ is neither cyclic nor a dihedral group of order* 8.

Proof. If $H(u)$ is cyclic, then $H(u)$ is of order 2 or 3 by Lemma 4.1 and $(\star\star)$.

Assume $|H(u)| = 2$ and let $H(u) = \langle i \rangle$. Then $C_H(i) \cong (3 \times D_8) : 2 \cong D_8 : \mathrm{Sym}(3)$.

If $u \in Q_i$, then, as $Q_i \cong D_8{*}Q_8$, the involution u centralizes an involution in $C_H(i) \setminus Q_i$ in contradiction to our assumption.

Thus $u \notin Q_i$. Let T_i be a complement to Q_i in C_i and let $u = cy$ with c in Q_i and y in T_i. There are five Sylow 2-subgroups in T_i. Two of them centralize an elementary abelian subgroup of order 4 in $H \cap Q_i$. If y is contained in one of these two, then c is inside $H \cap Q_i$ or an element of order 4 which centralizes $C_{Q_i}(u)$. Thus, in this case $H(u)$ is at least of order 4. Therefore, y is in one of the remaining three Sylow 2-subgroups in T_i. In each of those there is an involution j of $H \cap T_i$, which yields that u is not in T_i. But then there is an element $x \in H \cap Q_i$ such that x and u commute. Again we get that $|H(u)| \geq 4$ in contradiction to our assumption. Thus $H(u)$ is not of order 2.

If $H(u)$ is of order 3, then every involution which inverts $H(u)$, also inverts

$$C_{H'}(H(u)) \cong 3^2.$$

So in fact $H(u)$ is elementary abelian of order 9, which is a contradiction.

Finally assume $H(u) \cong D_8$. The fact that $H(u) \leq H^\infty$ yields that $H(u) = H \cap Q_i$, where i is the central involution in $H(u)$. Therefore, $C_G(H(u)) \cong Q_8 : \mathbb{Z}_3$ and $N_G(H(u)) \cong (D_8{*}Q_8) : \mathrm{Sym}(3)$. If $[H(u), u] = 1$, then u is in Q_i and therefore $u = i \in H$, a contradiction. Thus $[H(u), u] \neq 1$. Therefore, as $H(u)$ is generated by the elements in H which are inverted by u, the involution u inverts every element of order 4 in $H(u)$ and fixes a non-central involution in $H(u)$. But there does not exist such an involution in $C_i \setminus Q_i$, see 3.13(ii), nor in Q_i, see 3.15(ii). This shows the statement of the lemma. $\square$

Lemma 4.3 *If $H(u)$ is a subgroup of $H^\infty \cong$ Alt(6), then $H(u) \cong 2^2, 3^2$ and $N_K(H(u))$ and n are as listed in Table 1 for $K \in \{G, H\}$.*

Proof. By Lemmas 4.1, 4.2 and by the list of subgroups of Alt(6), see [10, II (8.27)], $H(u)$ is elementary abelian of order 4 or 9.

Let $U = \langle i, a\rangle \cong 2^2$ be a subgroup of H^∞. Then $N_H(U) \cong 3 \times$ Sym(4) and by 3.16 $N_G(U) \simeq 2^{2+4} : (3 \times \text{Sym}(3))$. Moreover, $C := C_{Q_i}(U) \cong \mathbb{Z}_2 \times Q_8$. Let v be an element of order 4 in C. Then there is an involution b in $C_{T_i}(U)$ which inverts v. Set $u = vb$. Then u is an involution and, as by the last paragraph $H(u)$ is of order 4 or 9, it follows that $H(u) = U$. As u inverts every element in U, in this case we have $n = 4$.

Now let $U \cong 3^2$ be a subgroup of H^∞. Then $N_H(U) \cong (3 \times 3^2) : \mathbb{Z}_8$ and $N_G(U)$ is of order $3^i \cdot 8$ with $i \in \{5, 7, 9\}$ by Lemma 3.10. Hence we find an involution u in $N_G(U)$ which maps s onto some element in $(\langle s\rangle \times U) \setminus \langle s\rangle$. This involution inverts every element in U and therefore $H(u) = U$ and $n = 9$. $\square$

Lemma 4.4 *If $H(u)$ is not a subgroup of $H^\infty \cong$ Alt(6), then $H(u)$ is isomorphic to one of the following groups:*

$$\mathbb{Z}_2, \mathbb{Z}_3, 2^2, D_8.$$

Proof. Let M be a maximal subgroup of H which contains $H(u)$. Then M is isomorphic to one of the following groups:

$$3 : D_{20}, (3 \times 3^2) : 8, 3 : D_{16}, 3 \times \text{ Alt}(6) \text{ or } PGL_2(9).$$

By $(\star\star)$ $H(u) \not\cong PGL_2(9)$.

Moreover, notice the following: In $N_s \cong 3 : PGL_2(9)$ there is no involution which at the same time inverts s and a 3-element in N_s^∞. Therefore, $|H(u)| \neq 9, 27$.

Now we consider the maximal subgroups M case by case.

If $M \cong PGL_2(9)$, then $H(u)$ is also contained in one of the other maximal subgroups of H.

Assume $M \cong 3 \times \text{Alt}(6)$. Then because of $(\star)$ and $(\star\star)$ $H(u) \cong \mathbb{Z}_3$ or 3^2. As $N_s \cong 3 : PGL_2(q)$, the latter case is not possible.

Next assume $M \cong 3 : D_{16}$. Then $(\star)$ implies that $O_3(H)$ is not the Sylow 3-subgroup of $H(u)$ and that therefore $H(u)$ is isomorphic to $\mathbb{Z}_2, 2^2$ or D_8, see Lemma 4.1.

If $M \cong 3 : D_{20}$, then $(\star)$ and $(\star\star)$ imply $H(u) \cong \mathbb{Z}_2$ or 2^2.

Finally assume $M \cong (3 \times 3^2) : 8$. by the second paragraph $|H(u)| \neq 27$. Then $(\star)$ yields $|H(u)| = 3$ or 9. As $|H(u)| = 9$ is not possible, see above, we get $|H(u)| = 3$. This shows the assertion. $\square$

Lemma 4.5 *If $H(u)$ is not a subgroup of H^∞, then $H(u)$, $N_K(H(u))$ and n are as listed in Table 1 for $K \in \{G, H\}$.*

Proof. By Lemma 4.4 we need only to study the subgroups U of $H \setminus H^\infty$ which are isomorphic to $\mathbb{Z}_2, \mathbb{Z}_3, 2^2$ or D_8. Notice, that in all these cases there is only one conjugacy class of subgroups of the respective type. One by one we consider these four classes.

$|H(u)| = 2$. Let $U = \langle b \rangle$ with b an outer involution of H. Then $N_H(U) \cong D_{20} \cong (2 \times 5) : 2$ and $N_G(U) = C_b$. Notice, that there are five involutions in $N_H(U)$ which are contained in H', but not in $Q_b = O_2(C_b)$. Denote one of those by i.

We next search for an involution $u \in C_b$ which does not centralize any involution in $N_H(U) \setminus \{b\}$. There are

$$|C_b : C_i \cap C_b| = 2^7 \cdot 3 \cdot 5 / 2^4 = 2^3 \cdot 3 \cdot 5$$

involutions in $C_b \setminus Q_b$, see 3.13(iii). On the other hand the number of involutions in $C_b \setminus Q_b$ which centralize an involution in $N_H(U) \setminus \{b\}$ equals

$$2^2 \cdot 3 \cdot 5,$$

as there is precisely one involution in $N_H(U)$ from every Sylow 2-subgroup of

$$C_b/Q_b \cong \text{Alt}(5)$$

and as each of these involutions centralizes precisely 4 elements in Q_b. Thus we are able to choose an involution u in C_b which does not centralize any involution in $N_H(U) \setminus \{b\}$.

As b is an outer involution of H, it is not a square in H. Therefore, by our choice of u, U is a Sylow 2-subgroup of $H(u)$. Thus, it follows with 4.4

that $H(u) = U$. Here, u inverts two elements in $H(u)$, so $n = 2$.

$|H(u)| = 3$. Next let U be a subgroup of $H \setminus H^\infty$ of order 3. Then $U = \langle st \rangle$ with $t \in H^\infty$ an element of order 3 and $N_H(U)$ is a Sylow 3-subgroup of H. By 3.3 U is conjugate to $O_3(H) = \langle s \rangle$. Therefore $N_G(U) = N_{st} \cong 3 : PGL_2(9)$. Let u be an involution in N_{st} which inverts st. Then by Lemma 4.4 $U = H(u)$. Clearly, u inverts the 3-elements in $H(u)$, so $n = 3$.

$|H(u)| = 4$. Now let U be a subgroup of $H \setminus H'$ which is elementary abelian of order 4. Then $U = \langle i, b \rangle$ with i in H^∞ and $b \notin H^\infty$, which implies that b is not in Q_i. By 3.13(v) we may assume that b is contained in T_i. Further $N_H(U) \cong D_8$ and $N := N_G(U) \simeq 2^4 :$ Sym(3), see 3.16.

Let u be an involution in $O_2(N) \setminus U$ which is contained in T_i. Then clearly, U is a subgroup of $H(u)$. Now Lemma 4.4 implies that either $H(u) = U$ or $H(u) \cong D_8$. As elements of order 4 of H are contained in H', the latter case would yield a contradiction to 3.15(i). This shows that $U = H(u)$. In that case u centralizes and inverts every element in $H(u)$, so $n = 4$.

$H(u) \cong D_8$. Finally let U be a subgroup of $H \setminus H'$ isomorphic to D_8. Then $N_H(U) \cong D_{16}$, $N_G(U) \simeq (Q_8 \times 2) : 2$, $C_H(U) = Z(U)$ and $C_G(U)$ is elementary abelian of order 4. Further notice, if $H(u) \cong D_8$, then either $[H(u), u] = 1$ or u inverts every element of order 4 and centralizes an elementary abelian subgroup of order 4 in $H(u)$.

Let u be an involution in $C_G(U) \setminus Z(U)$. Then $U \leq H(u)$ and, by 4.4 we have $U = H(u)$. In that case, u inverts six elements in $H(u)$, so $n = 6$.

Now let u be an involution in $N_G(U) \setminus U$ which inverts the elements of order 4 and centralizes an elementary abelian subgroup of order 4 in U. Then $u \notin H$ and $U \leq H(u)$. So by 4.4 $U = H(u)$. Here, u inverts six elements again, so $n = 6$. □

4.2 The order of a group of J_3-type

Next we calculate I_n by counting all the involutions outside H which invert precisely n elements in H. According to Table 1, n is in $\{9, 6, 4, 3, 2, 1\}$.
We start with $n = 9$ and end at $n = 1$.

$n = 9$. Then $H(u) \cong 3^2$ and u inverts every element in $H(u)$. If v is an involution in $N_G(H(u)) \setminus N_H(H(u))$, then Table 1 implies that $H(v) = H(u)$. Hence

Figure 1: Table 1.

n	$H(u)$	$N_H(H(u))$	$N_G(H(u))$	$\lvert H(u)^H\rvert$
9	$3^2 \leq H'$	$3^3 : 8$	$[3^i] : 8, i \in \{5, 7, 9\}$	$2 \cdot 5$
6	$D_8 \not\leq H'$	D_{16}	$(Q_8 \times 2) : 2$	$3^3 \cdot 5$
4	$2^2, H(u) \leq H'$	$3 \times$ Sym(4)	$2^{2+4} : (3 \times$ Sym(3))	$2 \cdot 3 \cdot 5$
4	$2^2, H(u) \not\leq H'$	D_8	$2^4 :$ Sym(3)	$2 \cdot 3^3 \cdot 5$
3	$3, H(u) \not\leq H^\infty$	3^3	$3 : PGL_2(9)$	$2^4 \cdot 5$
2	$2, H(u) \not\leq H'$	D_{20}	$2^{1+4} :$ Alt(5)	$2^2 \cdot 3^3$

$|I_9| = (|\{\text{involutions in } N_G(H(u))\}| - |\{\text{involutions in } N_H(H(u))\}|) \cdot |\{\text{subgroups } H(u) \text{ in } H \text{ with } |Hu \cap I| = 9\}| = (3^{i-1} - 3^2) \cdot 2 \cdot 5$ with $i \in \{5, 7, 9\}$, see 3.10. Hence we distinguish the three cases.

$i = 5$. $|I_9| = 3^2 \cdot 2^4 \cdot 5 = 720$.

$i = 7$. $|I_9| = 3^2 \cdot 2^5 \cdot 5^2 = 7200$.

$i = 9$. $|I_9| = 3^2 \cdot 2^4 \cdot 5 \cdot 7 \cdot 13 = 65520$.

$n = 6$. Calculating the order of I_6 we have to consider the subgroups $H(u)$ such that $H(u) \cong D_8$. Here again $v \in N_G(H(u)) \setminus N_H(H(u))$ yields $H(v) = H(u)$ (see Table 1). We have to distinguish the two cases $[H(u), u] = 1$ and $[H(u), u] \neq 1$.

$I_6(1)$. Let $I_6(1)$ consist of the elements $u \in I_6$ with $[H(u), u] = 1$. Then $C_G(H(u)) \cong 2^2$ and $|C_H(H(u))| = 2$. Hence

$$|I_6(1)| = 2 \cdot 3^3 \cdot 5.$$

$I_6(2)$. Let $I_6(2)$ consist of the elements $u \in I_6$ with $[H(u), u] \neq 1$. As $u \in I_6$, the involution u induces an inner automorphism on $H(u)$. Since $C_G(H(u)) \cong D_8 : 2$, there are four such involutions in $H \setminus H(u)$ so $|I_6(2)| = 2^2 \cdot 3^3 \cdot 5$. Thus

$$|I_6| = |I_6(1)| + |I_6(2)| = 2 \cdot 3^4 \cdot 5 = 810.$$

$n = 4$. To obtain I_4 we have to consider $H(u)$ to be an elementary abelian group of order 4. Clearly, u centralizes $H(u)$. We have to consider the two cases $H(u) \leq H'$ and $H(u) \not\leq H'$.

$I_4(i)$. Let $I_4(i)$ consist of the elements $u \in I_4$ with $H(u) \le H'$.
We have $C_H(H(u)) \cong 3 \times 2^2$ and $C_G(H(u)) \cong 2^{2+4} : 3 \cong (2 \times Q_8) :$ Alt(4) by Lemma 3.16. Hence there are 27 involutions in $C_G(H(u))$. Since the subgroups of H isomorphic to D_8 considered in the case $n = 6$ intersect H' in a cyclic group of order 4, the subgroups considered here are not contained in such a subgroup isomorphic to D_8. Hence Table 1 yields, if v is an involution in $C_G(H(u)) \setminus H(u)$, then $H(v) = H(u)$. Therefore,

$$|I_4(i)| = (27 - 3) \cdot 2 \cdot 3 \cdot 5 = 2^4 \cdot 3^2 \cdot 5.$$

$I_4(o)$. Let $I_4(o)$ consist of the elements $u \in I_4$ with $H(u) \not\le H'$.
Then $C_H(H(u))$ and $C_G(H(u))$ are elementary abelian of order 2^2 and 2^4, respectively. Every $H(u)$ lies in exactly one group $K \cong D_8$ as considered in case $n = 6$. There are two conjugates of $H(u)$ under of the action of H in K. Notice, that every involution in $I_6(1)$ centralizes these two conjugates of $H(u)$ and every involution in $I_6(2)$ centralizes exactly one subgroup $H(u)$. Hence

$I_4(o) = (|\{\text{involutions in } C_G(H(u))\}| - |\{\text{involutions in } C_H(H(u))\}|) \cdot |\{\text{subgroups } H(u) \text{ in } H \setminus H' \text{ with } |Hu \cap I| = 4\}| - 2I_6(1) - I_6(2) = (15 - 3) \cdot 2 \cdot 3^3 \cdot 5 - 2^3 \cdot 3^3 \cdot 5 = 2^4 \cdot 3^3 \cdot 5$.

Thus

$$|I_4| = |I_4(i)| + |I_4(o)| = 2^6 \cdot 3^2 \cdot 5 = 2880.$$

$n = 3$. Next, we calculate I_3, so $H(u)$ is cyclic of order 3 and not contained in H'. By Table 1 $N_G(H(u)) \cong 3 : PGL_2(9)$ and $N_H(H(u))$ is elementary abelian of order 3^3 which implies
$|I_3| = 108 \cdot 2^4 \cdot 5 = 2^6 \cdot 3^3 \cdot 5 = 8640$.

$n = 2$. Here $H(u)$ is a cyclic group of order 2 which is not contained in H'. Every involution in $I_6(1)$ (resp. $I_6(2)$ or $I_4(o)$) centralizes 4 (resp. 2 or 2) conjugates of $H(u)$ in H. Hence, as there are $1 + 10 + 120 = 131$ involutions in $C_G(H(u))$

$$|I_2| = (131 - 11) \cdot 2^2 \cdot 3^3 - 4|I_6(1)| - 2|I_6(2)| - 2|I_4(o)| = 2^4 \cdot 3^4 \cdot 5 = 6480.$$

It remains to determine b_1. We first calculate the number c. Recall that c is the number of u in I_1 such that $C_H(u) \ne 1$. If $C_H(u) \ne 1$, but u is in I_1, then by 3.1 $C_H(u)$ is cyclic of order 3; moreover $C := C_G(C_H(u)) \cong 3 \times$ Alt(6) and $C_H(C_H(u)) \cong 3^3$. There are 9 involutions in C which invert $C_H(C_H(u)) \cap H^\infty$. Clearly, these 9 involutions are contained in I_9. Moreover, $C_H(u)$ is contained in three subgroups isomorphic to Alt(4) in H. Every such $\mathrm{Alt}(4)$ is centralized by an elementary abelian group of order 4. Hence there are 9 further involutions in C which are contained in I_4. This yields $c = (45-18)\cdot 80 = 2^4\cdot 3^3\cdot 5 = 2160$. Moreover,

$$|I\cap H| = 3\cdot 15 + (3\cdot 2^4\cdot 3^2\cdot 5)/20 = 3(15+36) = 3^2\cdot 17.$$

To determine b_1 we use the Lemma of Bender. Up to now we calculated:

$$f = 1/8,\ b_2 = 2^3\cdot 3^4\cdot 5 = 3240,\ b_3 = 2^6\cdot 3^2\cdot 5 = 2880,\ b_4 = 2^4\cdot 3^2\cdot 5 = 720,$$

$$b_6 = 3^3\cdot 5 = 135 \text{ and } b_9 = 2^4\cdot 5 = 80.$$

Then (ii) and (iii) of Lemma 2.1 imply that

$$c+m|H| = b_1 < f^{-1}(|I\cap H|+b_2+2b_3+3b_4+5b_6+8b_9)-1-b_2-b_3-b_4-b_6-b_9$$

$$= 8|I\cap H| + 7\cdot b_2 + 15\cdot b_3 + 23\cdot b_4 + 39\cdot b_6 + 63\cdot b_9 - 1.$$

Thus

$$2^4\cdot 3^3\cdot 5(1+m) < 8\cdot(3^2\cdot 17)+7\cdot(2^3\cdot 3^4\cdot 5)+15\cdot(2^6\cdot 3^2\cdot 5)+23\cdot(2^4\cdot 3^2\cdot 5)+$$

$$39\cdot(3^3\cdot 5) + 63\cdot(x) - 1,$$

where

$$x = 2\cdot 5 \ \text{ or } \ 2^5\cdot 5^2 \ \text{ or } \ 2\cdot 5\cdot 7\cdot 13.$$

Hence

$$1+m < \frac{17}{2\cdot 3\cdot 5} + \frac{7\cdot 3}{2} + 2^2\cdot 5 + \frac{23}{3} + \frac{39}{2^4} + \frac{y}{3} - \frac{1}{c},$$

where

$$y = 7 \ \text{ or } 2\cdot 5\cdot 7 \ \text{ or } 7^2\cdot 13$$

and m is at most $43, 67$ or 255, respectively.

As 17 divides the order of G, but not the order of the centralizer of an involution, it has to divide $|I|$. Moreover, as 3^5 divides the order of G and every Sylow 3-subgroup of the centralizer of an involution is of order 3, it follows that 3^4 divides $|I|$. Therefore the following equation is helpful to determine b_1, respectively m:

$$|I| - b_1 = |H\cap I| + |I_2| + |I_3| + |I_4| + |I_6| + |I_9|$$

In all cases, $|I| - b_1 \equiv 0\ (3^4)$, which implies that $b_1 = 2^4\cdot 3^3\cdot 5\cdot(1+m)$ is divisibe by 3^4 and $m+1$ by 3. Let us consider the three different cases:

$|G|_3 = 3^7$.

Here $x = 2^5 \cdot 5^2$, $m \leq 67$ and $|I|$ is divisible by 3^6. Further, $|I| - b_1 \equiv 0$ (17) which yields $m \equiv 16$ (17). As $m \equiv 2$ (3) and because of $m \leq 67$, in fact $m = 50$. Thus $b_1 = 2^4 \cdot 3^3 \cdot 5 \cdot 51 \equiv 81$ (3^6) which yields that $|I| - b_1 \equiv 0 - 81 \equiv -81$ (3^6) in contradiction to $|I| - b_1 \equiv 648$ (3^6).

$|G|_3 = 3^9$.

Here $x = 2 \cdot 5 \cdot 7 \cdot 13$, $m \leq 255$ and $|I|$ is divisible by 3^8. Further, $|I| - b_1 \equiv 10$ (17) and therefore $m \equiv 6$ (17). Hence $m \equiv 23$ (51) and, as $m \leq 255$, we get $m = 23 + k \cdot 51$ with k in $\{1, \cdots, 5\}$. Using the numbers we determined we calculate that $|I| - b_1 \equiv 5751$ (3^8), but this does not hold for any k in $\{1, \cdots, 5\}$.

Thus, we get that

$|G|_3 = 3^5$.

Then $x = 2 \cdot 5$ and $m \leq 43$. As $|H| = c \equiv 1$ (17) and $|I| - b_1 \equiv 14$ (17) it follows that $m \equiv 2$ (17). As $1 + m$ is divisible by 3 we get $m \equiv 2$ (51). Now $m \leq 43$ yields $m = 2$. Hence $|I| = 3^4 \cdot 17 \cdot 19$ and $|G| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$, which proves the theorem.

5 The $3, 17$ and 19-Structure of G

Corollary 5.1 *Let Q and K be as introduced in 3.4.*

(i) Q is a Sylow 3-subgroup of G.

(ii) $N_G(Z(Q)) = N_G(Q) = Q : K$.

(iii) $N_G(t) = Q : \langle x^4 \rangle$ with $\langle x \rangle = K$ for all t in $Z(Q)^\#$.

Proof. (i) is a consequence of the main Theorem, (ii) follows from 3.4 and 3.10.

It remains to show (iii). Because of the order of the centralizer of an involution and the order of the normalizer of a Sylow 5-subgroup, $C_G(t)$ is a $\{3, 17, 19\}$-group for every $t \in Z(Q)^\#$. Further, Q is a Sylow 3-subgroup of $C_G(t)$ and (ii) implies that Q is self-normalizing in $C_G(t)$. Sylow‘s Theorem yields that $|C_G(t)| = 3^5$ or $3^5 \cdot 19$. Assume the latter. Then the Theorem of Sylow forces $X := O_{19}(C_G(t))$ to be a Sylow 19-subgroup of G.

As $N_G(X)/C_G(X)$ is a cyclic group of order $3^i \cdot 2^j$ with $i \in \{0,1,2\}$ and $j \in \{0,1\}$, it follows that $|G : N_G(X)| = 2^a \cdot 5 \cdot 17^b$ with a in $\{6,7\}$ and b in $\{0,1\}$, which contradicts Sylow‘s Theorem. Thus $|C_G(t)| = 3^5$ and (iii) holds. □

According to Theorem 1 every Sylow 19-subgroup of G is of order 19. Next we determine the structure of the centralizer and the normalizer of a Sylow 19-subgroup of G.

Corollary 5.2 *Let X be a Sylow 19-subgroup of G. Then $C_G(X) = X$ and $N_G(X)/X$ is cyclic of order 9.*

Proof. As X is of order 19, $N_G(X)/C_G(X)$ is a cyclic group of order $3^i \cdot 2^j$ with $i \in \{0,1,2\}$ and $j \in \{0,1\}$. Moreover, the order of an involution centralizer and 3.1 yield that $|C_G(X)/X|$ is not divisible by 2 or 5. Assume that it is divisible by 3.

Then, applying 1 and 3.2 we see that there is an element u of order 3 in $C_G(X)$ whose centralizer is a $\{3,17,19\}$-group. Assume that 17 divides the centralizer. Then by Burnside Normal p-complement Theorem $C_G(u)$ has a 17-complement [1, 39.1].

So, in any case $C_G(u)$ has a subgroup C which contains a subgroup of order 19 and the center T of a Sylow 3-subgroup of G. More precisely, $|C| = 3^a \cdot 19$ with a in $\{3,4\}$. Then X is normal in C which contradicts 5.1(iii). Thus $|C_G(X)/X| \in \{1,17\}$ and $|G : N_G(X)| = 2^i \cdot 3^j \cdot 5 \cdot 17^k$ with $i \in \{6,7\}$, $j \in \{3,4,5\}$ and $k \in \{0,1\}$. Sylow‘s Theorem yields the assertion. □

Corollary 5.3 *Let X be a Sylow 17-subgroup of G. Then $C_G(X) = X$ and $N_G(X)/X$ is cyclic of order 8.*

Proof. Because of the local p-structure of G for $p \neq 17$ a prime dividing $|G|$, we have $C_G(X) = X$. The fact that $N_G(X)/C_G(X)$ is a cyclic group of order 2^i for some $i \in \{2,3,4\}$ and the Theorem of Sylow yield the assertion. □

Corollary 5.4 *$Q/Z(Q)$ is extraspecial of order 3^3.*

Proof. Assume that $Q/Z(Q)$ is not extraspecial. Then it is abelian. Let s be an element of order 3 in Q whose centralizer has even order. By 3.4(iv) and as, $sZ(Q)$ is not a third power, see the proof of 3.10, it follows that $Q/Z(Q)$ is elementary abelian. Let i be an involution in $C_G(s) \cap N_G(Q)$. Then

$$Q/Z(Q) = [Q/Z(Q), i] \oplus C_{Q/Z(Q)}(i).$$

Let U be the preimage of $[Q/Z(Q), i]$ in Q. Then $Q = U : \langle s \rangle$. As there are $[N_G(Q) : (C_i \cap N_G(Q))] = 3^4$ involutions in $N_G(Q)$, there are at least as many elements of order 3 in $Q \setminus U$. This implies that U is elementary abelian. Therefore Q is of exponent 3 in contradiction to 5.2. $\square$

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