Commuting graphs of odd prime order elements in simple groups*

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December 30, 2009

Abstract

We study the commuting graph on elements of odd prime order in finite simple groups. The results are used in a forthcoming paper describing the structure of finite Bruck loops and of Bol loops of exponent 2.

1 Introduction

Let G be a group and X a *normal subset of* G, that is for all $x \in X, g \in G$ we have $x^g \in X$. The *commuting graph* on X is the undirected graph $\Gamma_{X,G} = \Gamma_X$ with vertex set X such that two different vertices x and y are adjacent if and only if $[x, y] = 1$. The commuting graph of a group is an object which has been studied quite often to obtain strong results on the group G. We give a short overview of some major work on or related to commuting graphs. For more details see the references given below.

Bender noted in his paper on strongly 2-embedded subgroups, [B], the equivalence between the existence of a strongly 2-embedded subgroup and the disconnectedness of the commuting graph of involutions.

At about the same time Fischer determined the groups generated by a class X of 3-transpositions by studying the commuting graph on X [Fi]. Later Stellmacher classified those groups which are generated by a special class of elements of order 3 again by examining the related commuting graph [St].

To prove the uniqueness of the sporadic simple group Ly, Aschbacher and Segev showed that its commuting graph on 3-central elements is simply connected [AS]. In addition, a major breakthrough towards the famous Margulis-Platonov conjecture has been made by Segev by using the commuting graph on the whole set G for G a non-trivial finite group [Se].

Finally Bates et al. [BBPR] determined the diameter of the connected commuting graphs of a conjugacy class of involutions of G where G is a Coxeter group and Perkins [Pe] did the same for the affine groups $\tilde{A}_n$, see also the related work [IJ2]. In [AAM] Abdollahi, Akbari and Maimanithe considered the dual of the commuting graph on $G \setminus Z(G)$. They conjectured that if these graphs are isomorphic for two non-abelian finite groups then the groups have the same order. This conjecture has been checked for some simple groups in [IJ1].

*This research is part of the project "Transversals in Groups with an application to loops" GZ: BA 2200/2-2 funded by the DFG

In this paper we study the connected components of the commuting graph $\Gamma_{\mathcal{O}}$ on the set $\mathcal{O}$ of odd prime order elements of a finite simple group G and of some of its subgraphs. We use our main results in [S] and [BS] to characterize the finite Bruck loops of 2-power exponent. We also consider our theorems to be of independent interest.

Suppose that X is a normal subset of a group G. Then G acts on Γ_X by conjugation. We say that a connected component of Γ_X is *big* if it is invariant under this action. If a connected component of Γ_X is not big then we say that it is *small.* In our first theorem we determine all the finite simple groups whose commuting graph $\Gamma_{\mathcal{O}}$ has a big connected component.

Theorem 1 *Let G be a finite simple group, then either $\Gamma_{\mathcal{O}}$ has a big connected component or G is one of the following groups:*

(a) $A_1(q), {}^2B_2(q), {}^2G_2(q)$ (for any q),

(b) ${}^2A_2(q)$ for q odd with $\frac{q+1}{(q+1,3)}$ a 2-power or

(c) $M_{11}, J_1, A_2(4)$.

Conversely, the groups in (a), (b) and (c) do not have big connected components in $\Gamma_{\mathcal{O}}$.

For ρ a subset of $\pi(G)$, where $\pi(G)$ is the set of primes dividing the order of G, we denote by $\mathcal{E}_\rho(G) := \{x \in G : o(x) \in \rho\}$ the set of elements in G whose order is in ρ. For simplicity we abbreviate $\Gamma_{\mathcal{E}_\rho(G)}$ by Γ_ρ for ρ a subset of $\pi(G)$. Thus for p a prime, Γ_p is the commuting graph on the set of elements of order p of G.

For $x \in \mathcal{O}$ let $\mathcal{C}_x$ be the connected component of $\Gamma_{\mathcal{O}}$ containing x. If $\mathcal{C}_x$ is big, then it follows from 3.3 below that Γ_p is a subgraph of $\mathcal{C}_x$ where p is the order of x. It is reasonable to ask whether or not Γ_p is already connected. For S a subgraph of Γ_X, let $\pi(S)$ be the set of orders of the elements in S. So $\pi(S)$ is a subset of $\pi(G)$.

Theorem 2 *Let G be a finite simple group. If $\Gamma_{\mathcal{O}}$ has a big connected component C, then there is a prime p in $\pi(C)$ such that Γ_p is connected.*

As a consequence we obtain:

Corollary 1.1 *Let G be a finite group and suppose that there is a big connected component C of $\Gamma_{\mathcal{O}}$. Suppose that the Sylow-p-subgroups are cyclic for all the primes p in $\pi(C)$. Then G is not simple.*

A connected component $\mathcal{C}_x$ is big if and only if it contains the full conjugacy class x^G, see 3.1(c). We say that a *conjugacy class x^G is connected* if the commuting graph on x^G is connected. In the next theorem we provide examples of groups having a connected conjugacy class. More precisely, we present all the alternating groups and groups of Lie type in even characteristic which possess such an element x of odd prime order. We will need these results to prove Theorem 5.

Theorem 3 *In Table 1 and below that table we list the simple groups G which are alternating or groups of Lie type in even characteristic and some subsets ω of $\pi(G)$ such that the conjugacy class x^G is connected for some element x in G of order r, $r \in \omega$, with $E(C_G(x))/Z(E(C_G(x)))$ as given in the third column. In the first column of the table we list G, in the second ω is given and in the last further conditions which have to be satisfied.*

Table 1

G	ω	$E(C_G(x))/Z(E(C_G(x)))$	conditions
$\mathrm{Alt}(n), n \geq 8$	$\{3\}$	Alt_{n-3}	
$A_2(q)$	$\pi(\frac{q-1}{(q-1,3)})$	$A_1(q)$	$q > 4$
$A_3(q)$	$\pi(q-1)$	$A_2(q)$	$q > 2$
$A_n(q), n \geq 4$	$\pi(q^2-1)$	$A_{n-2}(q)$	
${}^2A_2(q)$	$\pi(\frac{q+1}{(q+1,3)})$	$A_1(q)$	$q > 2$
${}^2A_3(q)$	$\pi(q+1)$	${}^2A_2(q)$	
${}^2A_n(q), n \geq 4$	$\pi(q^2-1)$	${}^2A_{n-2}(q)$	
$C_n(q), n \geq 3$	$\pi(q^2-1)$	$C_{n-1}(q)$	
$D_n(q), n \geq 4$	$\pi(q-1)$	$D_{n-1}(q)$	$q > 2$
$D_n(q), n \geq 4$	$\pi(q+1)$	${}^2D_{n-1}(q)$	
${}^2D_n(q), n \geq 4$	$\pi(q-1)$	$D_{n-1}(q)$	$q > 2$
${}^2D_n(q), n \geq 4$	$\pi(q+1)$	${}^2D_{n-1}(q)$	
${}^3D_4(q)$	$\pi(q^2-1)$	$A_1(q^3)$	$q > 2$
${}^2F_4(q)$	$\pi(q^2+1)$	${}^2B_2(q)$	$q > 2$
$F_4(q)$	$\pi(q^2-1)$	$C_3(q)$	
$E_6(q)$	$\pi(q^2-1)$	$A_5(q)$	
${}^2E_6(q)$	$\pi(q^2-1)$	${}^2A_5(q)$	
$E_7(q)$	$\pi(q^2-1)$	$D_6(q)$	
$E_8(q)$	$\pi(q^2-1)$	$E_7(q)$	
$G_2(q)$	$\{3\}$	$A_1(4)$	$q = 4$
	$\pi(q^2-1)$	$A_1(q)$	$q > 4$

In the groups ${}^3D_4(2)$ and ${}^2F_4(2)'$ the set of elements of type 3B or 3A (in the notation of [Atlas]) with centralisers isomorphic to $3_+^{1+2} \cdot 2S_4$ or $3_+^{1+2} : 4$, respectively, are connected conjugacy classes.

There is always at most one big connected component in $\Gamma_{\mathcal{O}}$ beside in the case of $O'N$. This yields a new characterisation of the sporadic group $O'N$:

Theorem 4 *Let G be a finite simple group, such that $\Gamma_{\mathcal{O}}$ has a big connected component. Then either G has a unique big connected component or $G = O'N$ and $\Gamma_{\{3,5\}}$ and Γ_7 are the two big connected components.*

Last not least we determine for all the finite simple groups all the elements which are contained in a small connected component of $\Gamma_{\mathcal{O}}$. This information will also be needed to prove Theorem 4.

In Table 3 and also later we use the following notation: Let q be a power of a prime p and $r \neq p$ another prime. Set

$$d_q(r) := \min\{i \in \mathbb{N} : r \mid q^i - 1\}.$$

So $d_q(r)$ is the order of q modulo r.

Theorem 5 *Let G be a finite simple group. Suppose that there is a big connected component in $\Gamma_{\mathcal{O}}$ and let x be an element of G of odd prime order r. If x is not contained in a big connected component of $\Gamma_{\mathcal{O}}$, then G and r are as in Tables 2-4. Conversely if these conditions are satisfied, then x is in a small connected component.*

Table 2

G	r
M_{12}	$r \in \{5, 11\}$
M_{22}	$r \in \{5, 7, 11\}$
J_2	$r = 7$
M_{23}	$r \in \{7, 11, 23\}$
HS	$r \in \{7, 11\}$
J_3	$r \in \{17, 19\}$
M_{24}	$r \in \{11, 23\}$
McL	$r \in \{7, 11\}$
He	$r = 17$
Ru	$r \in \{7, 13, 29\}$
Suz	$r \in \{11, 13\}$
$O'N$	$r \in \{11, 19, 31\}$
Co_3	$r \in \{11, 23\}$
Co_2	$r \in \{7, 11, 23\}$
Fi_{22}	$r \in \{11, 13\}$
HN	$r \in \{11, 19\}$
Ly	$r \in \{31, 37, 67\}$
Th	$r \in \{19, 31\}$
Fi_{23}	$r \in \{11, 17, 23\}$
Co_1	$r = 23$
J_4	$r \in \{23, 29, 31, 37, 43\}$
Fi'_{24}	$r \in \{17, 23, 29\}$
B	$r \in \{17, 19, 23, 31, 47\}$
M	$r \in \{41, 47, 59, 71\}$

Table 3

G	condition on G	r
$\mathrm{Alt}(n)$	$n - t$ a prime , $t \in \{0, 1, 2\}$	$n - t$

Table 4

G	condition on G	$d_q(r)$
$A_2(q)$	$\pi(\frac{q-1}{(q-1,3)}) \subseteq \{2\}, q$ odd	$1,2,3$
		3
$A_3(q)$	$\pi(q-1) \subseteq \{2\}$	3
	$\pi(q+1) \subseteq \{2\}$	4
$A_n(q), n \geq 4$	$q=3, n=4$	4
	n a prime, $\pi(\frac{q-1}{(q-1,n+1)}) \subseteq \{2\}$	n
	$n+1$ a prime	$n+1$
${}^2A_2(q)$	$\pi(\frac{q+1}{(q+1,3)}) \not\subseteq \{2\}$	6
${}^2A_3(q)$	$\pi(q-1) \subseteq \{2\}$	4
	$\pi(q+1) \subseteq \{2\}$	6
${}^2A_n(q), n \geq 4$	$q \in \{3,9\}, n=4$	4
	n a prime $, \pi(\frac{q+1}{(q+1,n+1)}) \subseteq \{2\}$	$2n$
	$n+1$ a prime	$2n+2$
$B_n(q), n \geq 3,\ q$ odd	n a prime, $\pi(q-1) \subseteq \{2\}$	n
	$\pi(n) \subseteq \{2\}$	$2n$
	n a prime, $\pi(q+1) \subseteq \{2\}$	$2n$
$C_2(q)$	$q \neq 2$	4
$C_n(q), n \geq 3$	n a prime, $\pi(q-1) \subseteq \{2\}$	n
	$\pi(n) \subseteq \{2\}$	$2n$
	n a prime, $\pi(q+1) \subseteq \{2\}$	$2n$
$D_n(q), n \geq 4$	$n-1$ a prime, $\pi(q-1) \subseteq \{2\}$	$n-1$
	n a prime, $\pi(q-1) \subseteq \{2\}$	n
	$n-1$ a prime, $\pi(q+1) \subseteq \{2\}$	$2n-2$
	$\pi(n-1) \subseteq \{2\}, \pi(q+1) \subseteq \{2\}$	$2n-2$
${}^2D_n(q), n \geq 4$	$n-1$ a prime, $q=3$	$n-1, 2n-2$
	$\pi(n-1) \subseteq \{2\}, \pi(q-1) \subseteq \{2\}$	$2n-2$
	n a prime, $\pi(q+1) \subseteq \{2\}$	$2n$
	$\pi(n) \subseteq \{2\}$	$2n$
${}^3D_4(q)$		12
$F_4(q)$		$8,12$
${}^2F_4(q)'$	$q=2$	4
		12
$E_6(q)$	$q \in \{3,7\}$	8
		9
${}^2E_6(q)$	$q \in \{2,3,5\}$	8
	$q=2$	12
		18
$E_7(q)$	$\pi(q-1) \subseteq \{2\}$	$7,9$
	$\pi(q+1) \subseteq \{2\}$	$14,18$
$E_8(q)$		$15,24,30$
	$5 \nmid q^2+1$	20
$G_2(q), q \neq 2$	$3 \nmid q-1$	3
	$3 \nmid q+1$	6

These tables imply the following interesting fact:

Corollary 1.2 *Let G be a finite simple group and suppose $\Gamma_{\mathcal{O}}$ has a big connected component. If x is an element of G of prime order r which is in a small connected component, then $O^2(C_G(x))$ is abelian and the Sylow-r-subgroups of G are either cyclic or $G \cong {}^2F_4(2)'$ and $r = 5$.*

The proof of our results requires a close study of the simple groups of Lie type. We use the following sources for information about their maximal subgroups: [Ca1], [Bou] for the general theory, [KL] for classical groups, [LSS] and [CLSS] for exceptional groups of Lie type. Furthermore, the papers [Coo], [K3D4] and [M] were useful.

The structure of the paper is as follows: In Section 2 we provide some facts from number theory and Section 3 contains general results about commuting graphs and big connected components. In Section 4 we prove Theorems 1 and 2 at the same time. Theorem 3 is shown in Section 5 and Theorems 5 and 4 in Section 6. We study separately the sporadics, alternating and groups of Lie type. If G is a group of Lie type, then we show that the elements in the small components are those elements of G which are not contained in certain maximal subgroups of G. These elements are then determined.

We use the classification of the finite simple groups, but we wonder whether there is a proof, which does not use the full classification. This may be complicated as for instance, if $G = \mathrm{PSL}_2(8) \times Sz(8)$, then $\Gamma_{\mathcal{O}}$ is connected and all the Sylow subgroups of G of odd order are cyclic.

2 Facts from number theory

Let q be a power of the prime p and let $r \neq p$ be another prime. By Lagrange's Theorem $d_q(r)|r-1$. Recall Zsigmondy's famous theorem:

Theorem 6 *Let n be a positive integer. There is either an odd prime s with $d_p(s) = n$ or one of the following cases holds.*

(a) p is a Mersenne prime, i.e. $p = 2^m - 1$ for some prime m and $n = 2$.

(b) p is a Fermat prime, i.e. $p = 2^{2^m} + 1$ for some integer m and $n = 1$.

(c) $p = 2$ and $n = 1$ or $n = 6$

Let $\Phi_n(x) \in \mathbb{Z}[x]$ be the n-th cyclotomic polynomial. Then the following lemmata are consequences of Theorem 6.

Lemma 2.1 *Let p be a prime and n an integer. The following holds.*

(a) If $\Phi_n(p)$ is a power of 2, then $n = 1$ and p is 2 or a Fermat prime or $n = 2$ and p is a Mersenne prime.

(b) If $\Phi_n(p)$ is a power of 3, then $p = 2$ and $n \in \{1, 2, 6\}$.

(c) If $\Phi_n(p)$ is a power of 3 times a power of 5, then $p = 2$ and $n \in \{1, 2, 4, 6\}$.

Proof: If $n > 2$ and $(p, n) \neq (2, 6)$ by Theorem 6 there exists a prime r dividing $\Phi_n(p)$, which does not divide $\Phi_m(p)$ for $m < n$. Since 3 divides $(p-1)p(p+1) = \Phi_1(p)p\Phi_2(p)$ we have $r > 3$. So in the first two cases the question reduces to those primes p, for which $p-1$ (in case $n=1$) or $p+1$ (in case $n=2$) is a 2-power or a 3-power. For the third case observe, that $n \mid r-1$, so $n \in \{1,2,4\}$ in this case and we have to determine those primes p, for which one of $p-1, p+1$ or p^2+1 is a 3-power times a 5-power. Since in particular $\Phi_n(p)$ is odd, $p=2$. The statement is immediate. $\square$

Lemma 2.2 *Let q be a prime power. The following holds.*

(a) If $q-1$ is a 2-power, then $q=2$, $q=9$ or q is a Fermat prime.

(b) If $q+1$ is a 2-power, then q is a Mersenne prime.

(c) If q^2-1 is a 2-power, then $q=3$.

(d) If q^2-1 is a 2-power times a 3-power, then $q \in \{2,3,5,7,17\}$.

(e) If q^2-1 is a 3-power times a 5-power, then $q \in \{2,4\}$.

Proof: Let $q=p^e$. Remember the formulas

$$(p^e)^n - 1 = \prod_{d \mid en} \Phi_d(p) \text{ and } (p^e)^n + 1 = \prod_{\substack{d \mid 2en \\ d \nmid en}} \Phi_d(p).$$

If $n=1$, then we get $e \leq 2$ in (i) and (ii) by 2.1. If $n=2$, then we get (iii) again by 2.1.

Let $q^2-1 = 2^a 3^b$. Since 3 divides exactly one of $q-1$, q, $q+1$, we get $q=2$ or q is a Mersenne or Fermat prime by (i) and (ii).

If $p = 2^r - 1$ is a Mersenne prime, then $p-1 = 2(2^{r-1}-1)$ is a 2-power times a 3-power iff $r \leq 2$ by the formula above and by 2.1. If $p = 2^m+1$ is a Fermat primes $p = 2^m+1$, then $p+1 = 2(2^{m-1}+1)$ and we get $m \leq 4$. Also (v) is a consequence of the formula above and 2.1. $\square$

3 Commuting graphs and big connected components

In the following let G be a group and X a non-empty normal subset of G. We begin with some basic but powerful observations.

Lemma 3.1 *Let X be a normal subset of the group G and Γ_X the commuting graph on X.*

(a) G acts by conjugation as a group of automorphisms on Γ_X.

(b) Let $g \in G$. Then the vertices x^g and x in Γ_X are connected or equal if and only if $g \in H_x$.

(c) A connected component of Γ_X is big if and only if it contains a conjugacy class x^G.

The following lemma is helpful as it allows to go from G to a central extension of G.

Lemma 3.2 *Let $\overline{G} := G/Z(G)$. If x and y are elements in X which are connected in Γ_X, then $\overline{x}, \overline{y}$ are connected in $\Gamma_{\overline{X}}$.*

3.1 A nice property of big connected components

Lemma 3.3 *Suppose C is a big connected component in Γ_Y where $Y = \mathcal{E}_\rho(G)$ for some $\rho \subseteq \pi(G)$. If r is in $\pi(C)$, then C contains all the elements of order r.*

Proof: Let $y \in Y$ be any element of order r different from x. We show, that x and y are connected in Γ_Y. Let $R \in \mathrm{Syl}_r(G)$ with $x \in R$. As $\mathcal{E}_r(G) \subseteq Y$, there is some z in $Z(R) \neq 1$. Clearly, z is connected to x, likewise y is connected to a conjugate of z. By 3.1(c) z and this conjugate are connected, which shows the assertion. □

Corollary 3.4 *Let X be a normal subset of $\mathcal{O}$ such that Γ_X is connected. Then there is a subset $\rho \subseteq \pi(G) - \{2\}$ such that $\mathcal{E}_\rho(G) = \Gamma_X$.*

Thus the big connected components of $\Gamma_\mathcal{O}$ are among the subsets $\mathcal{E}_\rho(G)$.

Notice, that the subset ρ for a big connected component C of Γ_π can be determined from the sizes of centralisers only, once the order r of a single element $x \in C$ is known. For this we simply define a graph on the set π by connecting all primes p_1 and p_2, such that p_2 divides the size of a centraliser of an element of order p_1. The connected component of the prime r in this graph is the subset ρ in question.

3.2 The graph Γ_p.

In order to use this method, we have to establish the existence of big connected components. A special case is the connectedness of Γ_p. Following Bender [B], we show, that connectedness of Γ_p is equivalent to the fact that G has no strongly p-embedded subgroup. First we give criteria for the connectedness of Γ_p.

Lemma 3.5 *If one of the following conditions holds then Γ_p is connected.*

(a) $O_p(G) \neq 1$.

(b) $G = \langle N_G(Y) : Y \leq P, Y \neq 1\rangle$ *with* $P \in \mathrm{Syl}_p(G)$.

(c) There exists a prime p and subgroups $A, B \leq G$, such that $G = \langle A, B\rangle$, $A\cap B$ contains elements of order p and both $\Gamma_p(A)$ and $\Gamma_p(B)$ are connected.

Proof: (a) Choose $x \in \Omega_1(Z(O_p(G)))$. Then also $x^g \in \Omega_1(Z(O_p(G)))$ for $g \in G$. So $[x, x^g] = 1$ and $g \in H_x$ by 3.1.

(b) Let $x \in \Omega_1(Z(P)), o(x) = p$. Then $P \leq H_x$. For $1 \neq Y \leq P$ we may choose $1 \neq y \in Y$ with $o(y) = p$. Then $N_G(Y) \leq H_y$ by 3.5. As $H_x = H_y$, $H_x = \langle N_G(Y) : Y \leq P, Y \neq 1\rangle = G$. Therefore all conjugates of x in G are connected, so Γ_p is connected.

(c) Choose $x \in A \cap B, o(x) = p$. Consider H_x in Γ_p. As $\Gamma_p(A)$ is connected, $A \leq H_x$. As $\Gamma_p(B)$ is connected, $B \leq H_x$. Therefore $G = \langle A, B\rangle \leq H_x$, so Γ_x is connected. □

Next we show that (b) characterises the connected Γ_p. Recall that a subgroup $U \leq G$ is *strongly p-embedded*, if $U \neq G$, $p \in \pi(U)$ and $p \notin \pi(U \cap U^g)$ for all $g \in G - U$, cf. [B]. The equivalence of (a) and (b) is already shown by Bender for $p = 2$ as well as essentially the equivalence of (b) and (c), see [B].

Lemma 3.6 *Let $p \in \pi(G)$. The following statements are equivalent:*

(a) The graph Γ_p is connected.

(b) G has no strongly p-embedded subgroup.

(c) $G = \langle N_G(Y) : 1 \neq Y \leq P\rangle$ for some $P \in Syl_p(G)$.

Proof: Suppose Γ_p is connected, but there exists a strongly p-embedded subgroup U. Let $x \in U, o(x) = p$. As U is strongly p-embedded, U is the stabiliser of a unique point in the action of G on the U-cosets and this is the unique fixed point of x. Therefore $C_G(x)$ fixes this unique point, so $C_G(y) \leq U$ for every $y \in U$ of order p. This gives a contradiction to Γ_p connected, as $G - U$ contains elements of order p.

Suppose $U := \langle N_G(Y) : 1 \neq Y \leq P\rangle \neq G$, but G has no strongly p-embedded subgroup. Let $g \in G - U$ with $|U \cap U^g|_p$ maximal and $X \in \mathrm{Syl}_p(U \cap U^g)$. If $X = 1$, then U is strongly p-embedded, contrary to the assumption.

If $X \in \mathrm{Syl}_p(G)$, we find some $u \in U$ with $X^u = P$, so $U = \langle N_G(Y) : 1 \neq Y \leq X\rangle$. Likewise we find some $v \in U^g$ with $X^v = P^g$. Then also $U^g = \langle N_G(Y) : 1 \neq Y \leq X\rangle$, so $U = U^g$. Then $g \in N_G(U)$. As $N_G(P) \leq U$, $N_G(U) = U$ by Frattini, so $g \in U$, a contradiction.

So $1 < |X| < |G|_p$. Let $A, B \in \mathrm{Syl}_p(N_G(X))$ with $A \leq U$ and $B \leq U^g$. As $|A| > |X|$, $B \not\leq U$. We can choose a $Q \in \mathrm{Syl}_p(U)$ with $X \leq Q$. There exists a $w \in U$ with $P^w = Q$, so $U = \langle N_G(Y) : 1 \neq Y \leq Q\rangle$. Then $N_G(X) \leq U$ contradicts $B \not\leq U$.

If $G = \langle N_G(Y) : 1 \neq Y \leq P\rangle$, then Γ_p is connected by 3.5. □

Moreover, we get the following

Corollary 3.7 *Let $p \in \pi(G)$. If Γ_p is connected, then Sylow-p-subgroups of G are noncyclic or $O_p(G) \neq 1$.*

Proof: As Γ_p is connected, $G = \langle N_G(Y) : 1 \neq Y \leq P\rangle$. If Sylow-$p$-subgroups are cyclic, all those subgroups $N_G(Y)$ are contained in the subgroup $N_G(Y_1)$ for $Y_1 = \Omega_1(P)$, so $O_p(G)$ contains $\Omega_1(P)$. □

3.3 The graph Γ_{x^G}.

In this paper we also study the stronger condition that x^g is connected for some $x \in G$. In this section we present some helpful criterions. Let $X := x^G$.

Lemma 3.8 *Let $X = x^G$ with $x \in \mathcal{O}$. Suppose U is a subgroup of G such that $U = AB$ for two commuting subgroups A and B of U with $x \in A$ and such that there is a $g \in G$ with $A^g \leq B$. Then $H_x \geq \langle U, g\rangle > U$.*

Proof: It follows that $B \leq C_U(x)$ is contained in H_x. As there is $g \in G$ with $A^g \leq B$ and as $[x, x^g] = 1$, we also get that $A \leq C_U(x^g)$ is contained in H_x. By 3.1(b) g is in H_x as well. As $U \leq N_G(A)$, but $g \notin N_G(A)$, we have $\langle U, g\rangle > U$. □

We can strengthen our criterion on Γ_p:

Lemma 3.9 *Let $x \in G$ be an element of order p. If $G = \langle N_G(A) : A \leq G, x \in A, A' = 1 \rangle$, then x^G is connected.*

Proof: Let $\Gamma = \Gamma_X$ for $X = x^G$. If $x \in A$ with $A' = 1$, then $N_G(A) \leq H_x$. So $G \leq H_x$ and Γ_X is connected. □

We end this section with a criterion for the nonexistence of big connected components.

Lemma 3.10 *Let C be a big connected component of Γ_X. Then either some $x \in C$ exists such that $C_G(x)$ is not abelian or $\langle C \rangle \leq F(G)$.*

Proof: Suppose $C_G(x)$ is abelian for every $x \in C$. Let $x, y, z \in C$ with $[x, y] = 1 = [y, z]$. As $C_G(y)$ is abelian and $x, z \in C_G(y)$, $[x, z] = 1$. As C is a connected component, any two elements of C commute, so $A := \langle C \rangle$ is abelian. As C is a big connected component, A is G-invariant and $A \leq F(G)$. □

Notice, that groups with abelian centralisers were considered already by Weisner [W] and Suzuki [Sz1]. We wonder, whether it is possible to classify those finite simple groups without big connected component in $\Gamma_{\mathcal{O}}$ without using the classification.

4 Proofs of Theorems 1 and 2

In this section we show that if G is a simple group not listed in Theorem 1, then $\Gamma_{\mathcal{O}}$ has at least one big connected component. At the same time we show that there is a prime p such that Γ_p is connected.

Let p divide $|G|$. By 3.6 Γ_p is connected if and only if $G = \langle N_G(Y) \ : \ 1 \neq Y \leq P \rangle$ for each Sylow p subgroup P of G. Moreover, by 3.7 if P is cyclic, then Γ_p is not connected. Now assume that the p-rank $m_p(P)$ of P is at least 2. Then Theorem (7.6.1) of [GLS, Vol. III] gives an answer:

Theorem 7 *[GLS, Vol. III,(7.6.1)] Let G be a finite simple group ($\mathcal{K}$-group in the theorem) and p a prime such that for $P \in Syl_p(G)$ it holds $m_p(P) \geq 2$ and $\langle N_G(Y) \ : \ 1 \neq Y \leq P \rangle < P$. Then p and G are one of the following:*

(1) p is arbitrary and $G \in Lie(p)$ with $G \cong A_1(p^a), {}^2A_2(p^a), {}^2B_2(p^a), {}^2G_2(p^a)$

(2) $p > 2$ and $G \cong A_{2p}$

(3) $p = 3$ and $G \cong A_2(4)$ or M_{11}

(4) $p = 5$ and $G \cong {}^2F_4(2), Mc$ or Fi_{22}.

(5) $p = 11$ and $G \cong J_4$.

This theorem classifies the finite simple groups of p-rank at least 2 such that Γ_p is not connected. If G has a Sylow p-subgroup of p-rank at least 2 and the pair (p, G) is not listed in the theorem, then Γ_p is connected. This shows that if $G \cong A_{2p}$, $p > 3$, ${}^2F_4(2), Mc, Fi_{22}, J_4$ (i.e. if (2), (4) or (5) holds), then Γ_p is connected for some prime p. We study (1) and (3) case by case.

$G \cong A_1(p^a), {}^2B_2(p^a)$. We use Dixon's Theorem for $A_1(p^a)$ and [Sz] in case of ${}^2B_2(p^a)$ for the list of maximal subgroups. Then 3.10 shows, that G has no big connected component.

$G \cong {}^2A_2(p^a)$. There is a subgroup in G isomorphic to $((\mathbb{Z}_{q+1} \times \mathbb{Z}_{q+1})/(q+1,3))$: Sym(3), see [K]. If $(q+1)/(q+1,3)$ is not a 2-power, then there is an element x in G of prime order r such that r divides $(q+1)/(q+1,3)$. By Theorem 7 Γ_r is connected.

If $(q+1)/(q+1,3)$ is a 2-power, then all semisimple elements have an abelian centraliser, see [K]. By 3.10 $\Gamma_{\mathcal{O}}$ has no big connected component.

$G \cong {}^2G_2(q), q = 3^a$. The case $q = 3$ has been treated as $A_1(8)$ above. We use the list of maximal subgroups in [K2G2]. In particular as centralisers of semisimple elements are reductive, centralisers of elements of odd prime order in G are abelian $3'$-groups. Thus by 3.10 the commuting graph on the set of prime elements of order $\pi(G) \setminus \{2,3\}$ is not connected. Moreover, it follows that centralisers of elements of order 3 are $\{2,3\}$-groups. This shows that $\Gamma_{\mathcal{O}}$ has a big connected component if and only if Γ_3 is connected. As by Theorem 7 Γ_3 is not connected, the assertion holds for $G \cong {}^2G_2(q)$.

$G \cong A_2(4)$. According to [Atlas, p. 24] all the centralisers of elements of odd prime order of G are abelian. Therefore, 3.10 implies the assertion.

$G \cong M_{11}$. By [Atlas, p. 18] the centralisers of elements of odd prime order p, $p \neq 3$, of G are of order p. Therefore, the argumentation is here as in the case $G \cong^2 G_2(q)$.

This shows Theorem 1 for all the finite simple groups which possess a non-cyclic Sylow subgroup for some odd prime. Thus we do need to collect all the finite simple groups whose odd order Sylow subgroups are cyclic.

Lemma 4.1 *Let G be a finite simple group such that all the Sylow p-subgroups of G are cyclic for every odd prime divisor p of $|G|$. Then G is one of the following:*

$$A_1(p), A_1(2^n), {}^2B_2(2^n), A_2(2) \text{ or } J_1.$$

Proof: If G is a sporadic group, then the assertion follows from Table 5 and the fact that M_{11} contains a subgroup isomorphic to Alt(6). If G is alternating, then clearly $G =$ Alt(5).

So let G be a group of Lie type in characteristic p. If the rank d of G is bigger than 2, then a group isomorphic to $A_1(q) \times A_1(q)$ is involved in G. Hence d is at most 2.

Let $d = 1$. If G is not a twisted group, then $G \cong A_1(p)$ or $A_1(2^n)$. If G is a twisted group, then every Sylow p-subgroup of G is not abelian, see [Ca1, 13.6.4]. Therefore, $p = 2$. Considering the structure of the maximal tori of G, we get $G \cong {}^2B_2(2^n)$.

If $d = 2$. Then, again by the maximal tori of G, it is not a twisted group and $G \cong A_2(2)$. $\square$

We already showed above that all these groups, notice $A_2(2) \cong A_1(7)$, do not contain a big connected component. Moreover, for these groups Γ_p is never connected,by 3.7. This proves Theorems 1 and 2.

5 The Proof of Theorem 3

The strategy of the proof is to present an element x of prime order r such that r is in ω and such that there are conjugates x_i in $C_G(x)$ so that

(a) $E(C_G(x))/Z(E(C_G(x)))$ is as given Table 1,

(b) the subgroups $C_G(x)$ and $C_G(x_i)$ generate G.

We require property (b), as then $G = \langle C_G(x), C_G(x_i)\rangle \leq H_x$.

If $G \cong \mathrm{Alt}(n)$, $n \geq 8$, then given two 3-cycles, there is a 3-cycle which commutes with both 3-cycles. Hence Theorem 3 holds in that case.

So let G be a group of Lie type in even characteristic q. In the following we use the notation introduced in [Ca1]. In particular we use the notation for the root systems Φ for the Lie type groups as given in 3.6 of [Ca1]. We also denote the generators of the group by $x_s(t)$, with $s \in \Phi$ and $t \in \mathrm{GF}(q)$, and set $X_s := \langle x_s(t) : t \in \mathrm{GF}(q)\rangle$. Moreover, we use the elements $h_s(t)$ of the diagonal subgroup H of G, see [Ca1].

Case I G not a twisted group. Let u be the highest root in Φ, see for instance [Ca1, 2.2.6] or [Bou]. In [Bou] the extended Dynkin diagram, which is the Dynkin diagram extended by the node $-u$ such that two nodes are connected if and only if the corresponding roots are not perpendicular, of the different types are given.

In the following assume $G \ncong D_n(q)$ and if $G \cong A_n(q)$, then $n \geq 4$. Let $H := \langle X_u, X_{-u}\rangle \cong \mathrm{SL}_2(q)$. Then it follows that $D_H := \langle X_s, X_{-s}$: s is not connected with $-u$ in the extended Dynkin diagram $\rangle$ is contained in $C_G(H)$.

Now let x be an element in H of order r where r divides q^2-1 (if $G \cong G_2(q)$, then assume that r divides $(q-1)$), then it follows from the respective root system that $E(C_G(x))/Z(E(C_G(x)))$ is in the respective case as given in Table 1. Therefore notice if G is not of type G_2, then the subgroup D_H of $C_G(H)$ is immediatly seen to be normal in $C_G(H)$. Set $C := C_G(x)$. The case $G \cong G_2(q)$ and $r \mid (q+1)$ will be treated a bit differently, see below.

It remains to find conjugates x_i of x such that the subgroups $C_G(x), C_G(x_i)$ generate G. As the Weyl group is transitive on all the roots in Φ which are of the same length, X_u is conjugate to all the X_s with s a root of the same length as u. Now we consider case by case.

$G \cong A_n(q)$. As $n \geq 4$, there are subgroups $\langle X_s, X_{-s}\rangle$ in C and elements x_1, x_2 in C which are conjugate to x such that every X_s either commutes with x, x_1 or x_2. This shows that G is generated by the centralisers $C_G(x), C_G(x_1)$ and $C_G(x_2)$ and therefore x^G is connected.

$G \cong C_n(q), F_4(q), E_m(q), m = 6, 7, 8$. The argumentation is precisely as in the case $G \cong A_n(q)$.

$G \cong G_2(q)$. As $G_2(2)' \cong {}^2A_2(3)$ by Theorem 1, G has no big connected component and we assume $q > 2$.

Let $q > 4$. Let the Dynkin diagram of type G_2 be labelled with the short root a and the long root b as in [Ca1] . Then $u = 3a+2b$ and $C_G(H) = \langle X_a, X_{-a}\rangle \leq$

$X := \langle X_u, X_{-u}\rangle \times \langle X_a, X_{-a}\rangle \cong \mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$, see [Coo]. Moreover, notice that $L := \langle X_s, X_{-s} \mid s \text{ a long root } \rangle \leq G$ is isomorphic to $\mathrm{SL}_3(q)$.

If $r \mid q-1$ or $r \mid q+1$, then x is contained in a maximal torus $T \cong (q-1)^2$ of L or in $T \cong (q+1)^2$ of a subgroup $M \cong \mathrm{SU}_3(q).2$ of G, see [Coo]. As T is abelian $C_L(T) \cong (q-1)^2 :\ \mathrm{Sym}(3)$ and $N_M(T) \cong (q+1)^2 : D_{12}$ are contained in H_x, respectively. As $C_G(H)$ is neither contained in L nor in M (there is a long root element in M, see [Coo, 3.1], so M contains without loss of generality X_u. As M is a rank one group, X_a is not contained in M but in H_x. This shows that $H_x = G$ in both cases, see [Coo, (2.3)].

Finally let $q = 4$. Due to the J_2-maximal subgroup, we had to perform computer calculations. We calculated in MAGMA, using the 6-dimensional representation of G over $\mathrm{GF}(4)$, that G has a connected conjugacy class of elements of order 3 with the given centraliser structure. There is no connected conjugacy class of elements of order 5, though Γ_5 is connected.

Now we still need to consider the linear groups of small rank and the orthogonal groups. Here the argumentation is different. For the orthogonal groups we need a bigger centraliser than we would get using the methods above.

$G \cong A_2(q), A_3(q)$. Let $G \cong A_2(q)$, $q \geq 8$ and set

$$x := h_{r_1}(t)h_{r_2}(t^2) \ \text{ and } x_2 := h_{r_1}(t^2)h_{r_2}(t)$$

with t an element of order r in $\mathrm{GF}(q)$. Then x and x_2 are conjugate in G and $[x, x_2] = 1$. Moreover, X_{r_1} and X_{r_2} is contained in $C_G(x)$ and $C_G(x_2)$, respectively. Hence, $G = \langle X_{r_1}, X_{r_2}\rangle \leq H_x$ which implies the assertion.

Let $G \cong A_3(q)$. Set

$$x := h_{r_1}(t)h_{r_2}(t^2)h_{r_3}(t^3) \ \text{ and } x_2 := h_{r_1}(t^3)h_{r_2}(t^2)h_{r_3}(t)$$

with t an element of order r in $\mathrm{GF}(q)$. Again x and x_2 are conjugate in G and $[x, x_2] = 1$. As $[X_{r_i}, x] = 1$ with $i \leq 2$ and $[X_{r_j}, x_2] = 1$ with $j > 1$, the assertion follows.

$G \cong D_n^\varepsilon(q), \varepsilon \in \{+,-\}$ with $D_n^+ = D_n(q)$ and $D_n^- = {}^2D_n(q)$. By [KL] there exist maximal subgroups U_+^ε of type $O_2^+(q) \perp O_{2n-2}^\varepsilon(q)$ and U_-^ε of type $O_2^-(q) \perp O_{2n-2}^{-\varepsilon}(q)$ in $\Omega^\varepsilon(q)$, provided $q > 2$ in case U_+^ε. For $q = 2$ we exclude the cases U_+^ε, as then $q - 1 = 1$.

Let $A_+^\varepsilon \cong \mathbb{Z}_{q-1}, B_+^\varepsilon \cong \Omega_{2n-2}^\varepsilon(q)$ be normal subgroups of U_+^ε and $A_-^\varepsilon \cong \mathbb{Z}_{q+1}, B_-^\varepsilon \cong \Omega_{2n-2}^{-\varepsilon}(q)$ be normal subgroups of U_-^ε. From the action of G on its natural module we conclude, that in any case some $g_{\varepsilon_2}^{\varepsilon_1}$ with exist, such that $(U_{\varepsilon_2}^{\varepsilon_1}, A_{\varepsilon_2}^{\varepsilon_1}, B_{\varepsilon_2}^{\varepsilon_1}, \mathrm{End} g_{\varepsilon_2}^{\varepsilon_1})$ satisfy the conditions of 3.8 for x any element of order r, $r \in \pi(q - (\varepsilon_2 1))$. As $U_{\varepsilon_2}^{\varepsilon_1}$ is a maximal subgroup of G, Γ_X is connected for $X = x^G$.

Case II G is a twisted group. Let $T = \mathcal{L}(q^i)$ be a Chevalley group over $\mathrm{GF}(q)$, q a power of the prime p, and $G = {}^i\mathcal{L}(q^i)$ the twisted group. Assume first that all the roots in Φ of the untwisted group T are of the same length. Let g be the graph automorphism of T which is related to the symmetry ρ of the diagram defining G and let $\sigma = gf$, f a field automorphism of T of order i. Then $G = \langle C_U(\sigma), C_V(\sigma)\rangle$ where U and V are the opposite Sylow p-subgroups $U = \langle X_s \mid s \in \Phi^+\rangle$ and $V = \langle X_s \mid s \in \Phi^-\rangle$.

Let τ be the isometry of the vector space spanned by Φ which extends ρ. Then τ permutes the fundamental system Π and acts therefore on Φ^+. Hence, the highest root u of Φ is fixed by τ. Moreover, by [Ca1, 12.2.3] $g(x_s(t)) = x_{\tau(s)}(\gamma_s t)$ with $\gamma_s = \pm 1$. As q is a power of 2, we have $\gamma_s = 1$ for all $s \in \Phi$.

In particular, $\sigma(X_s) = X_{\tau(s)}$ for all s in Φ and $\sigma(x_u(t)) = x_u(t)$ for all $t \in \mathrm{GF}(q)$. Hence $H := \langle X_u(t_1), X_{-u}(t_2) \mid t_1, t_2 \in \mathrm{GF}(q)\rangle \cong \mathrm{SL}_2(q)$ is contained in G.

Assume that $n \geq 3$ in case $T = A_n(q^2)$ and that $T \ncong D_n(q)$. The latter has already been done above. Let x be an element in H of prime order r where r divides $q^2 - 1$. Then we see immediately in the extended Dynkin diagram of T that $E(C_G(x))/Z(E(C_G(x)))$ is as indicated in Table 1. We still need to show that x^G is connected.

$G \cong {}^2A_n(q)$. Let $\Pi = \{\alpha_i \mid 1 \leq i \leq n\}$ so that ρ interchanges α_i and α_{n+1-i}. Let $w = (1i)(n+1-i, n)$. Then $[\rho, w] = 1$ and there is an n_w in G such that n_w permutes X_1 and X_i and X_n and X_{n+1-i}. Hence $x_i := x^{n_w}$ is an element in $\langle x_{\beta_i}, x_{-\beta_i}\rangle$ which commutes with x if $2 \leq i \leq n-1$ where $x_{\beta_i}(t) = x_{\alpha_i}(t)x_{\alpha_{\tau(i)}}(t^q)$ if $i \neq n/2$ and $X_{\beta_i}(t) = X_{\alpha_i}(t)$ if $i = n/2$, see [Ca, 13.6.3].

If $n \geq 5$, then $\langle C_G(x), C_G(x_i) \mid 2 \leq i \leq n-1\rangle$ contains the root subgroups $X_{\beta_1}, \ldots, X_{\beta_j}$ where $j = n/2$ if n is even and else $j = (n-1)/2 + 1$, which generate G by [Ca1, 13.6.5].

If $n = 4$, then $\langle C_G(x), C_G(x_2)\rangle = G$ by [K].

$G \cong {}^3D_4(q)$. Number the roots of Π such that α_2 is the middle node of the diagram. Then $M := \langle X_u(t_1), X_{-u}(t_2), X_{\alpha_2}(t_3), X_{-\alpha_2}(t_4) \mid t_1, \ldots, t_4 \in \mathrm{GF}(q)\rangle \leq G$ is isomorphic to $\mathrm{SL}_3(q)$. If r divides $q-1$, then there is an element g in M such that $x_1 := x^g$ is not contained in H and such that $[x, x_1] = 1$. Then according to [K3D4] $\langle C_G(x), C_G(x_1)\rangle = G$.

Now assume that r divides $q+1$. For $q = 2$ we use the list of maximal subgroups in [ATLAS]. By 3.5, Γ_3 and Γ_7 are connected. As G has three 3-local maximal subgroups, but only two classes of elements of order 3, G has a connected conjugacy class of elements of order 3 by 3.9. However it is class 3B, which is not the class we use in case of $q > 2$.

Let $q > 2$ and assume that r divides $q+1$. Then $C_G(x) \geq C_H(x) \times C_G(H) \cong \mathbb{Z}_{q+1} \times A_1(q^3)$. The list of maximal subgroups [K3D4] yields that $\mathbb{Z}_{q+1} \times A_1(q^3) \cong C_G(x) \leq H \times C_G(H)$. By [LSS] there is a torus normaliser $N_\varepsilon \cong \mathbb{Z}_{q^3-\varepsilon} \times \mathbb{Z}_{q-\varepsilon}.D_{12}$ in G. As $|N|_r = |G|_r$, we may assume that x is in N. Then by 3.8 N is a subgroup of H_x. The assertion follows, as $\langle C_G(x), N_\varepsilon\rangle = G$.

$G \cong {}^2E_6(q)$. Here H is contained in a subgroup M of G isomorphic to ${}^2D_5(q)$. If r divides $q-1$ or $q+1$, then there is an element g in M such that $x_1 := x^g$ is not contained in H and such that $[x, x_1] = 1$. Then by 3.8 H_x contains beside $\langle C_G(x^h) \mid h \in G\rangle$ the element g. As $H \times C_G(H)$ is a maximal subgroup of G by [LSS], it follows that $H_x = G$, the assertion.

Assume that $G \cong {}^2A_n(q)$ with $n = 2$ or 3. If $q = 2$ and $n = 2$, then G is soluble and if $n = 3$, then $G \cong C_2(3)$. Therefore, let $q > 2$.

Let $n = 2$ and let r be a prime divisor of $(q+1)/(q+1, 3)$. Then $M \cong \mathbb{Z}_{(q+1)/(q+1,3)} \circ A_1(q)$ is a maximal subgroup of G, see [K]. Let x be a central element of order r in M. Then, as the normaliser of the subgroup $T \cong (q+1)^2/(q+1,3)$ is not contained in M, it follows that $x^G \cap M$ contains an element which is not in $\langle x \rangle$. Therefore x satisfies the conditions given in Table 1.

Now let $n = 3$ and let r be a prime divisor of $q+1$. Here there is a maximal subgroup $M \cong \mathbb{Z}_{q+1} \circ \mathrm{SU}_3(q)$ in G, see [K], and we can use the same argumentation as in the case $n = 2$.

Finally assume that the roots of Φ are not all of the same length. Then according to Theorem 1 Φ is of type F_4.

$G \cong {}^2F_4(q)$. If $q = 2$ we use this list of maximal subgroups in [ATLAS]. By 3.5 the graph Γ_3 is connected. Notice, that Γ_5 is not connected, as a Sylow-5-subgroup is normal in the centraliser of a 5-element.

Let $q > 2$. It follows that $r \neq 3$. We can factorise $q^2+1 = (q-\sqrt{2q}+1)(q+\sqrt{2q}+1)$. Let $\varepsilon \in \{+,-\}$, such that r is a divisor of $q+\varepsilon\sqrt{2q}+1$ and let $x \in G$ be an element of order r with $C_G(x) \cong \mathbb{Z}_{q+\varepsilon\sqrt{2q}+1} \times {}^2B_2(q)$, see [M, 1.3]. Notice that $|{}^2F_4(2)|_r = (q+1)_r^2 = (q+\varepsilon\sqrt{2q}+1)_r^2$. Then x is also contained in a maximal subgroup $M_1 \cong ({}^2B_2(q) \times {}^2B_2(q)).2$. Notice, that the outer involution of M_1 interchanges the two components, as ${}^2B_2(q)$ has no outer automorphism of order 2. Therefore, $M_1 \leq H_x$.

There is a subgroup $N \cong (\mathbb{Z}_{q+\varepsilon\sqrt{2q}+1} \times \mathbb{Z}_{q+\varepsilon\sqrt{2q}+1}).[96]$ in G, which contains x but is not contained in M_1. Therefore, $H_x \geq G$ and Γ_X is connected for $X = x^G$.

6 The proof of Theorems 5 and 4

Clearly, we consider only those groups, which have a big connected component in Γ_{calO}. We classify the small connected components and we thereby show the uniqueness of the big connected components (if possible).

The nice fact 3.4 is a basic tool to find the small components. Let C be a big connected component of $\Gamma_{\mathcal{O}}$. Then according to that corollary there is a subset ρ of $\pi(G)$ such that C consists of all the elements of order s with s in ρ.

6.1 The alternating groups

Let G be an alternating group. If x is an element in G of order r which is in a small connected component, then there is no 3-cycle which commutes with x. This yields that $r = n - t$ is a prime and $t \leq 2$. Clearly, whenever this holds, then x is in a small component, which proves the assertion for the alternating groups.

6.2 The sporadic groups

By Theorem 1, we can exclude M_{11} and J_1. By the centraliser sizes in [ATLAS], all primes listed in Table 2 are in some $\pi(S)$ where S runs through the small connected components of Γ_{calO}.

It remains to show, that the big connected component(s) contains all other primes. In Table 5 we give the set of primes $\pi(\mathcal{C})$ of the orders of elements in the big connected component together with elements x whose centraliser size shows that the elements of G of order r, r in $\pi(\mathcal{C})$, form indeed a connected component of $\Gamma_{\mathcal{O}}$.

This also shows, that the big connected component is unique, apart from the case $G = O'N$.

Table 5

Group	$\pi(\mathcal{C})$	x
M_{12}	$\{3\}$	$3A$
M_{22}	$\{3\}$	$3A$
J_2	$\{3,5\}$	$3A$
M_{23}	$\{3,5\}$	$3A$
HS	$\{3,5\}$	$3A$
J_3	$\{3,5\}$	$3A$
M_{24}	$\{3,5,7\}$	$3A, 3B$
McL	$\{3,5\}$	$3A$
He	$\{3,5,7\}$	$3A$
Ru	$\{3,5\}$	$3A$
Suz	$\{3,5,7\}$	$3A$
$O'N$	$\{3,5\}$	$3A$
	$\{7\}$	$7A$
Co_3	$\{3,5,7\}$	$3A, 3C$
Co_2	$\{3,5\}$	$3A$
Fi_{22}	$\{3,5,7\}$	$3A$
HN	$\{3,5,7\}$	$5A$
Ly	$\{3,5,7,11\}$	$3A$
Th	$\{3,5,7,13\}$	$3A, 3C$
Fi_{23}	$\{3,5,7,13\}$	$3A$
Co_1	$\{3,5,7,11,13\}$	$3A$
J_4	$\{3,5,7,11\}$	$3A$
Fi'_{24}	$\{3,5,7,11,13\}$	$3A, 3B$
B	$\{3,5,7,11,13\}$	$3A$
M	$\{3,5,7,11,13,17,19,23,29,31\}$	$3A, 3C$

6.3 Groups of Lie type

In this section $G = \mathcal{L}(q)$ is a group of Lie type of rank n defined over the field $\mathrm{GF}(q)$, where $q = p^a$ for a prime p. We continue to use the notation introduced in the previous sections.

Let x be an element in a small component of $\Gamma_{\mathcal{O}}$ of order r. Before we consider as above first the untwisted and then the twisted groups we study the groups of small dimension, which escape our general methods. They are the groups of type A_2, ${}^2A_2$, C_2.

If $G \cong {}^2A_2(q)$, we see as in the proof of Theorem 1 that elements of prime order whose centraliser is isomorphic with $(q+1)/(q+1,3) \circ A_1(q)$ are in a common connected component C. It follows that also those elements whose order divides $q(q-1)$ are contained in C. As the torus of type $(q^2-q+1)/(q+1,3)$ is self centralising, it follows that all the primes of the small components are precisley the divisors of $(q^2-q+1)/(q+1,3)$.

If $G \cong A_2(q)$, then the torus of size $(q^2+q+1)/(q-1,3)$ is always self centralising. Hence if $d_q(r) = 3$, then we have no further condition on q.

Let $d_q(r) \leq 2$. Then x centralises a cyclic subgroup of size $(q-1)/(q-1,3)$. By 3.9 the conjugacy classes of the elements of odd prime order s, where s divides $(q-1)/(q-1,3)$, are connected. This and as by Theorem 1 $q \neq 2, 4$ we get that q is odd and $(q-1)/(q-1,3)$ a 2-power.

Let $G \cong C_2(q)$. By Theorem 1 $q > 2$. According to [K] there is a self centralising torus of size $(q^2+1)/(q-1,2)$, which gives small connected components.

Let y be an element of prime order dividing $(q-1)(q+1)$. Then according to [K] there exist subgroups M and N of type $(\mathrm{Sp}_2(q) \times \mathrm{Sp}_2(q)).2$ and $O_4^+(q)$ in G which contain y. By 3.8 the isomorphic subgroups M and N are contained in H_y and therefore all prime order elements dividing $(q-1)q(q+1)$ are in a unique big connected component.

Now we study the remaining groups.

Case I. Let G be a non-twisted group of Lie type with root system Φ. Let Π be a fundamental system of Φ and J a subsystem of Φ such that Φ contains a subsystem $J \times K$ with $|K| \geq 1$. For instance if Φ is of type A_n, then the set J of the first $n-2$-nodes of Φ and K the last node satisfy this condition.

Moreover, let

$$P_J = \langle X_j, X_{-j}, X_i \mid j \in J \text{ and } i \in \{1, \dots, n\} \setminus J \rangle.$$

Then the following holds:

Lemma 6.1 *The elements of prime order dividing $|P_J|$ for some J as introduced above are all contained in the same big connected component of $\Gamma_{\mathcal{O}}$.*

Proof: Assume that r divides $|P_J|$. By assumption P_J commutes with

$$\langle X_k, X_{-k} \mid k \in K \rangle.$$

As $n \geq 2$, we have $\langle X_k, X_{-k} \rangle \cong \mathrm{SL}_2(q)$ for every k in K.

If p is odd, then p is in ρ by 3.6. As p divides $|\mathrm{SL}_2(q)|$, it follows that r is in ρ as well in contradiction to the fact that x is not in the big component.

Hence $p = 2$. Then according to Theorem 3 there is an element of order s in G where s is an odd prime dividing $q^2 - 1$ such that its conjugacy class is connected. This means $s \in \rho$. As s divides $|\mathrm{SL}_2(q)|$, we get the same contradiction as in the last paragraph. Thus r does not divide $|P_J|$. $\square$

By considering the order of G and of its parabolic subgroups we are now able to determine r in a case by case analysis.

$G \cong A_n(q)$, $n \geq 3$. Here 6.1 implies $d_q(r) = n$ or $n+1$.

Assume $d_q(r) = n+1$. If $n+1$ is a prime, a torus of size $(q^{n+1}-1)/[(q-1)(q^{n+1}-1, n+1)]$ is self centralising and gives a small connected component.

If $n+1 = a_1 \cdot a_2$ with $a_i \neq 1$, then $q^{a_i} - 1$ divides $q^{n+1} - 1$. It follows, as r does not divide $|A_{n-2}(q)|$ by 6.1 and as $a_i \leq n-1$, that

$$m := (q^{a_i} - 1)/[(q-1)(q^{n+1} - 1, n+1)]$$

is a 2-power. By Theorem 6 there is either a Zsigmondy prime s with $s = d_q(a_i)$ or $a_i = 2$. If there is such a prime, then s divides $a_i - 1$. As m is a power of 2, the latter case implies that s divides $n+1$ as well, which is not possible. So $a_1 = a_2 = 2$, $n+1 = 4$ and $q+1$ is a 2-power.

Now assume $d_q(r) = n$. Then the natural G-module V splits under the action of x into the direct sum $V = U \oplus W$ of a 1-dimensional and an n-dimensional subspace and $C_G(x) \cong \mathbb{Z}_{(q^n-1)/(q-1,n+1)}$. This implies that $(q-1)/(q-1, n+1)$ is a power of 2. If n is not a prime, then by the same argumentation as above, $n = 4$ and $q^2 - 11$ is a 2-power, which yields $q = 3$. Then $(q^n-1)/(q-1, n+1) = 3^4 - 1 = 2^4 \cdot 5$. If n is a prime and $(q-1)/(q-1, n+1)$ a 2-power, then again the odd order elements in $C_G(x)$ form a small connected component in $\Gamma_{\mathcal{O}}$.

$G \cong B_n(q), n \geq 3, q$ odd. By Lemma 6.1 and as there are subgroups in G isomorphic to $O_3(q) \perp O^+_{2n-2}(q)$ or $O_3(q) \perp O^-_{2n-2}(q)$ it follows that $d_q(r) = n$ or $2n$.

Assume $d_q(r) = n$. Then, by order reasons, x fixes a totally isotropic n-space of the natural G-module V. It follows that $C_G(x) \cong \mathbb{Z}_{(q^n-1)/2}$. Hence n is a prime and $q-1$ a 2-power.

Assume $d_q(r) = 2n$. Then x fixes an $2n$-dimensional subspace of $-$-type of the natural G-module V and $C_G(x)$ is cyclic of order $q^n + 1/2$. According to 6.1 $q^n + 1$ is not divisible by some odd number which divides $q^{2i} - 1$ with $i \leq n-2$. By this fact and as, if a is odd $q^{ab} + 1 = (q^b + 1)(q^{(a-1)b} - q^{(a-2)b} \pm +1)$, we get that either n is a 2-power or that n is a prime and $q+1$ a 2-power, see the proof of 2.2.

$G \cong C_n(q)$, $n \geq 3$. By 6.1 and as there is a subgroup isomorphic to $\mathrm{Sp}_2(q) \perp \mathrm{Sp}_{2n-2}(q)$, we have $d_q(r) = n$ or $2n$.

Assume first $d_q(r) = n$. Then n is odd, since else $r \mid |\mathrm{Sp}_n(q)|$ in contradiction to 6.1. It follows that x fixes a totally isotropic n-subspace of the natural G-module and that $C_G(x)$ is cyclic of order $(q^n-1)/(2, q-1)$. Hence it follows that n is a prime and $q-1$ a 2-power.

Now let $d_q(r) = 2n$. Then $C_G(x) = (q^n+1)/(q-1, 2)$ as can be seen in the subgroup $C_2(q^n)$. Hence n is a 2-power or an odd prime and $q+1$ is a 2-power.

$G \cong D_n(q)$ By 6.1 and as there is a subgroup M isomorphic to $O^-_2(q) \perp O^-_{2n-2}(q)$ in G as well as if q is odd, one isomorphic to $O_3(q) \perp O_{2n-3}(q)$ – recall $O^-_2(q) \cong D_{2(q+1)}$ and $O_3(q) \cong L_2(q)$ – the order r divides $(q^n-1)(q^{2(n-1)}-1)$. If n is even, then (q^n-1) divides the order of $D_{n-2}(q)$. So by 6.1 $r \mid (q^{2(n-1)}-1)$.

Suppose $d_q(r) = n-1$. There is a torus of type $q^{n-1} - 1$ in a subgroup M_2 of type $\mathrm{GL}_n(q).2$, see 4.2.7 of [KL]. If $q-1$ is not a 2-power, then $Z(F^*(M_2))$ contains elements of odd order and x is in the big connected component. If $(q-1)$ is a 2-power, then x is contained in a small connected component if and only if $n-1$ is a prime.

Suppose $d_q(r) = 2(n-1)$. There is a torus of type $q^{n-1}+1$ contained in a subgroup of type $\mathrm{GU}_n(q)$ in class $\mathcal{C}_3$ in G, see Proposition 4.3.18 of [KL]. If $q+1$ is not a 2-power, $Z(F^*(M_3))$ contains elements of odd order and x is in the big connected component. We get that $q+1$ is a 2-power and $n-1$ a prime.

Let n be odd.

Suppose $d_q(r) = n$. There is torus of type $q^n - 1$ in a subgroup of type $\mathrm{GL}_n(q).2$ in class $\mathcal{C}_2$, see Proposition 4.2.7 of [KL]. It follows that $q-1$ is a 2-power and n a prime.

Suppose $d_q(r) = n-1$. If q is odd, then $q^{n-1}-1$ divides $|\Omega_n(q)|$ and if q is even, then $q^{n-1}-1$ divides $|\Omega^-_{n+1}(q)|$. This contradicts that x is an element of order r which is contained in a small connected component.

Suppose $d_q(r) = 2(n-1)$. Then because of the subgroup M, $(q+1)$ is a power of 2. We claim that $n-1$ is 2-power as well. Let $n-1 = a \cdot b$ with $a > 1$ a 2-power and b odd. Then there is a torus of type $q^{n-1}+1$ in a subgroup M_2 of type $\mathrm{GU}_b(q^a)$, which is a subgroup of $O^-_2(q) \perp O^-_{2n-2}(q) \leq G$. If $b = 1$, then the torus is centralised by a 2-group of size $q+1$. If $b \neq 1$, then $Z(F^*(M_2))$ contains elements of odd order, see 4.3.18 of [KL], which yields a contradiction.

$G \cong E_6(q)$. Here we get $d_q(r) \in \{8, 9, 12\}$. As there is a subgroup of type ${}^3D_4(q) \circ \frac{q^2+q+1}{(q-1,3)}$ in G, see [LSS], in fact $d_q(r) \neq 12$.

Assume $d_q(r) = 8$. Then, again by the list of subgroups of G given in [LSS], $(q+1)(q-1)/(3, q-1)$ is a 2-power, which yields $q = 3$ or 7 by Lemma 2.2.

Assume $d_q(r) = 9$. Then there is a self-centralising torus of size $q^6 + q^3 + 1$. So x is in a small component.

$G \cong E_7(q)$. By [LSS] and 6.1 $d_q(r) \in \{7, 9, 14, 18\}$. A subgroup of type $A_1(q^7)$, see [LSS] gives small connected components for $d_q(r) \in \{7, 14\}$, if $q-1$ resp. $q+1$ is a 2-power. If $d_q(r) = 9$ or 18, then, as there are subgroups isomorphic to $E_6(q) \circ (q-1)$ and ${}^2E_6(q) \circ (q+1)$ in G, it follows that $q-1$ resp. $q+1$ is a 2-power.

$G \cong E_8(q)$. Here, as there is a subgroup isomorphic to $\mathrm{SL}_2(q) \circ E_7(q)$, see [LSS], $d_q(r) \in \{15, 20, 24, 30\}$.

Let $d_q(r) = 20$. By [LSS] there exists a subgroup M in G of type $\mathrm{SU}_5(q^2)$, which contains a torus isomorphic to $\frac{q^{10}+1}{q^2+1}$. As M has a nontrivial center of odd order, if $5 \mid q^2+1$, it follows that x is in a small component iff 5 does not divide q^2+1.

In G there are self centralising tori of size $\Phi_d(q)$ for $d = 15, 24, 30$, see [LSS], which implies that Table 3 holds for G.

$G \cong F_4(q)$. By [LSS] $d_q(r) \in \{8, 12\}$. According to the list of maximal subgroups which contain all centralisers of elements in G given in [CLSS] we get that these conditions indeed give small connected components.

$G \cong G_2(q)$. By Theorem 1 $q > 2$ and by [LSS] $d_q(r) \in \{3, 6\}$. By [LSS] there exist subgroups of type $\mathrm{SL}_3(q)$ and $\mathrm{SU}_3(q)$ in G, which have a nontrivial center, if $3 \mid q-1$ or $3 \mid q+1$, respectively. This together with the list given in [CLSS] shows the assertion for $d_q(r) = 3$ or 6, respectively.

Case II. Let G be a twisted group of Lie type. Here again the analogous version of 6.1 holds. Precisely in the same way as for the untwisted groups we get the conditions on q, n and r as given in Table 4 (we quote [Ca1], [K] und [LSS]).

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