Preface

This monograph investigates operators on a Banach space satisfying the norm equation

$$\|\mathrm{Id} + T\| = 1 + \|T\| \tag{DE}$$

and the Banach spaces where they are defined.

The first paper to study this phenomenon is Igor K. Daugavet's note [3] from 1963, where he writes (see Figure 0.1):

> The purpose of this note is to observe an almost obvious, but at the same time unexpected property of completely continuous operators on the space C of continuous functions on an interval $[a, b]$.

This almost obvious, but unexpected property of such operators is that they satisfy the above equation (DE) which has come to be known as the *Daugavet equation*. (We shall present his argument in Section 1.1.)

ОБ ОДНОМ СВОЙСТВЕ ВПОЛНЕ НЕПРЕРЫВНЫХ ОПЕРАТОРОВ В ПРОСТРАНСТВЕ C

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Цель настоящей заметки — отметить одно почти очевидное и в то же время несколько неожиданное свойство вполне непрерывных операторов в пространстве C функций, непрерывных на промежутке $[a, b]$. Именно, имеет место следующая

Теорема. *Пусть I — тождественный, а A — вполне непрерывный операторы в пространстве C. Тогда*

$$\|I + A\| = 1 + \|A\|. \tag{1}$$

Figure 0.1. Daugavet's 1963 paper

Daugavet's theorem spawns a number of natural questions that were addressed by various authors over the years: On which spaces do compact operators satisfy (DE)? Which other operators satisfy the Daugavet equation on $C[a, b]$ or other spaces? What are the consequences when one knows that "many" or "few" operators satisfy (DE)? The paper [1] suggests to say "X has the Daugavet property" as a shorthand notation for the statement that all (weakly) compact operators $T\colon X \to X$ satisfy (DE).

Our own approach to defining the Daugavet property is different, following the 2000 paper [4] of the same title as the monograph at hand. Namely, we say that a Banach space X has the *Daugavet property* if every rank-one operator $T = x^* \otimes x$ satisfies (DE). The benefit of this definition is that one can reformulate it geometrically in terms of slices of the unit ball, a reformulation that can be checked in concrete examples and can be used to derive nontrivial conclusions. One of them is a transfer theorem saying that on a Banach space with the Daugavet property, automatically every weakly compact operator (indeed, every strong Radon-Nikodým operator) satisfies (DE), thus reconciling the Daugavet property from [1] with ours.

Let us now describe the contents of this volume in more detail. We begin with a chapter that surveys results on the Daugavet equation obtained by various techniques before [4] appeared; it is, as it were, rooted in the 20th century. In particular, it is shown that $C[a,b]$, $L_1[a,b]$, and $L_\infty[a,b]$ enjoy the Daugavet property (in its incarnation from [1]), and more generally do $C(K)$, $L_1(\mu)$, and $L_\infty(\mu)$ if the compact Hausdorff space K does not contain isolated points ("K is perfect") and μ is atomless.

Chapter 2 collects prerequisites that will be used throughout our text, from ultrafilters to tensor products, passing through some classical topics of the geometry of Banach spaces and some lesser known isometric concepts. Given the survey character of this chapter, we have omitted many proofs; at times we offer arguments when they are slightly different from the traditional ones, or because the results we need are modified versions of the classical theorems.

The core text starts in Chapter 3 where we define "our" 21st century Daugavet property, revisit the above examples and present many others, prove the indicated transfer theorem and construct ℓ_1-subspaces in a space with the Daugavet property. Further, more specialised ramifications are discussed in Chapter 4 where we deal for instance with tensor products, rearrangement invariant function spaces, L-orthogonal elements, and polynomials on Banach spaces.

One of the remarkable consequences the Daugavet property entails for a Banach space is its lack of unconditional structure. More precisely, a Banach space with the Daugavet property cannot be embedded isomorphically into a space with an unconditional basis. This is discussed in Chapter 5, and now one can see the common background for the classical theorem that neither $C[0,1]$ nor $L_1[0,1]$ can be embedded into a space with an unconditional basis. Chapter 5 also looks at operators that can take the role of the identity operator in (DE), called *Daugavet centres*.

One can view the philosophy of the Daugavet property as dealing with Banach spaces which are sort of "large", like $L_1[0,1]$ vs. ℓ_1, on which operators with a sort of "small" range, e.g., (weakly) compact operators, satisfy the Daugavet equation.[1] Taking this viewpoint to a rather abstract level, we introduce *narrow operators* in an algebraic way in Chapter 6. This idea extends the notion of a narrow operator as advocated by Plichko and Popov [5], a class which already appears in Chapter 5 under the name PP-narrow operators to distinguish them from the ones in the present class. It turns out that every narrow operator satisfies the Daugavet equation, and that weakly compact, strong Radon-Nikodým and ℓ_1-singular operators, i.e., those that do not fix copies of ℓ_1, on a space with the Daugavet property are narrow. (One caveat: If X fails the Daugavet property, then no operator on X is narrow, not even $T = 0$.) The dual notion is that of a *rich subspace*, a subspace where the corresponding quotient map is narrow; for example, a uniform algebra is a rich subspace of the space of continuous functions on its Shilov boundary (provided the latter is perfect). Rich subspaces inherit the Daugavet property.

[1] A. Pełczyński, in a colloquium talk, once put forward the maxim that all of mathematics is about deciding whether items are large or small.

In Chapter 7 we look at the stability of the Daugavet property by taking absolute sums, M-ideals, and ultrapowers. The case of ultrapowers is particularly rewarding since it inspires us to study a quantitative version of the Daugavet property, viz., the *uniform Daugavet property*. Adapting an example due to Bourgain and Rosenthal [2] we show that the uniform Daugavet property is strictly stronger. This also leads to an example of a space with both the Daugavet and the Schur property (weakly convergent sequences are norm convergent), two properties which seem to be incompatible at first sight.

We continue the study of narrow operators in the setting of vector-valued $C(K)$- or L_1-spaces in Chapter 8.

A variant of the Daugavet property is the theme of Chapter 9, namely, the *almost Daugavet property* that requires the validity of the Daugavet equation only for rank-one operators $x^* \otimes x$ with x^* in some given one-norming subspace of X^* rather than for all $x^* \in X^*$. Such spaces can neatly be characterised, at least in the separable case, and ℓ_1 (for real scalars) is one of them. We also define the notion of a *poor subspace* in analogy with and opposite to that of a rich one. We can use this machinery to eventually answer a question posed by A. Pełczyński: There is a subspace E of $L_1[0,1]$ isomorphic to ℓ_1 such that $L_1[0,1]/E$ fails the Daugavet property. (It is a theorem due to Shvydkoy [6] that X/E inherits the Daugavet property from X if E is reflexive, which hence does not extend to RNP subspaces.)

A new idea is the subject matter of Chapter 10, i.e., that of slicely countably determined sets, spaces, and operators. A bounded subset U of a Banach space is *slicely countably determined* (SCD for short) if there is a countable family $\{S_n\colon n \in \mathbb{N}\}$ of slices with the property that every slice of U contains one of the S_n, and a Banach space is an SCD-space if every convex bounded subset is SCD. This property is opposite to the Daugavet property; for example, (separable) RNP spaces and those without ℓ_1-subspaces are SCD whereas the only known non-SCD spaces are those with the Daugavet property in some equivalent norm. Eventually, this provides a new approach to the narrowness of the operators mentioned in Chapter 6 and their linear combinations.

Chapter 11 deals with the Daugavet property for Lipschitz spaces and their preduals and the Daugavet equation for Lipschitz maps. The setting is that of a complete pointed metric space M (i.e., a metric space with a distinguished point $p \in M$); the Lipschitz space $\mathrm{Lip}_0(M)$ consists of all real-valued Lipschitz functions on M mapping p to 0, equipped with the optimal Lipschitz constant as the norm. While it is clear (modulo results in classical real analysis) that $\mathrm{Lip}_0([0,1])$ has the Daugavet property, already the case of the unit square, $M = [0,1] \times [0,1]$, is hardly obvious. Still, this chapter provides a geometric characterisation of those metric spaces M for which $\mathrm{Lip}_0(M)$ has the Daugavet property (they are precisely the length spaces), and indeed the unit square is one of them. We also extend this result to the vector-valued case. Another topic in this chapter is the Daugavet equation for Lipschitz maps between Banach spaces when the operator norms in (DE) are replaced by the Lipschitz norms.

The final Chapter 12 surveys a potpourri of properties related to the Daugavet property: norm equations different from (DE), e.g., $\|\mathrm{Id} + T\| = \|\mathrm{Id} - T\|$; various diameter two properties (a necessary, but not sufficient condition for the Daugavet property is that every slice S of the unit ball, indeed every convex combination of slices, has diameter $\mathrm{diam}(S) = 2$); the alternative Daugavet property and the numerical index of a Banach space; the anti-Daugavet property when only the smallest collection possible of operators satisfies the Daugavet equation, and the acs-type properties.

Each chapter concludes with a “Notes and remarks” section detailing the sources for the displayed material and additional results, and a section listing open problems suggested by the main text.

The work on this project started when one of us (V.K.) prepared lecture notes for an intended graduate course on all things Daugavetian to be taught at V. N. Karazin Kharkiv National University (Ukraine). Over time, he attracted the other coauthors to the project with the objective to produce a monograph on the modern geometric theory of the Daugavet equation and the Daugavet property. When the terror regime in Moscow started the full-scale war against Ukraine, which has led to uncountably many victims and the destruction of Ukrainian infrastructure, like the university of Kharkiv [7] (see Figure 0.2), progress got disrupted. Eventually, we were able to finish the manuscript, remembering the slogan from many years back, ¡No pasarán!

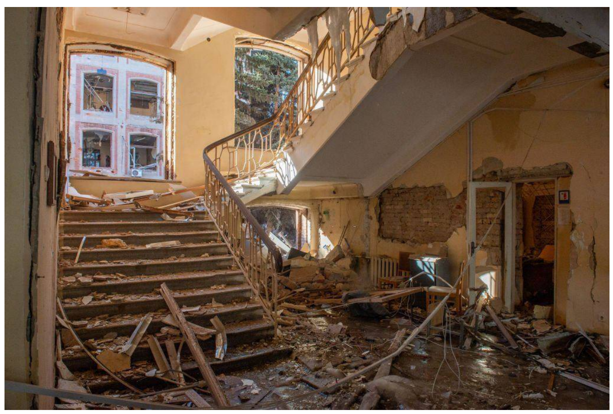

FIGURE 0.2. Kharkiv University, 2022

Thanks to ♣♣

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