A remark about Müntz spaces

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Wojtaszczyk [3] recently proved the following theorem.

Theorem 1 *If X is a closed subspace of $C[0,1]$ such that each $f \in X$ is continuously differentiable on $[0,1)$, then X is isomorphic to a subspace of c_0.*

The proof to follow is clearly modelled after Wojtaszczyk's proof, but somewhat more pedestrian. Let us show:

Theorem 2 *Under the assumptions of Theorem 1, X is almost isometric to a subspace of c. That is, for every $\varepsilon > 0$ there is an operator $J_\varepsilon\colon X \to c$ such that*

$$(1-\varepsilon)\|f\|_\infty \le \|J_\varepsilon f\| \le \|f\|_\infty \qquad \forall f \in X.$$

Proof. The idea is to define $J_\varepsilon f = \bigl(f(s_n)\bigr)$ for a suitably chosen sequence in $[0,1)$.

Let $\varepsilon > 0$. For $0 < a < 1$, the operator

$$D\colon X \to C[0,a], \quad f \mapsto f'|_{[0,a]}$$

is well-defined and has a closed graph; therefore it is continuous. Thus, there is some $K(a) > 0$ such that

$$\sup_{0\le t\le a} |f'(t)| \le K(a)\|f\|_\infty \qquad \forall f \in X.$$

We now fix a sequence $0 = a_0 < a_1 < a_2 < \ldots < 1$ converging to 1. Pick points $0 = s_0 < s_1 < \ldots < s_{n_1} = a_1 < s_{n_1+1} < \ldots < s_{n_2} = a_2 < \ldots$ in such a way that $(n_0 = 0)$

$$s_{j+1} - s_j \le \frac{\varepsilon}{K(a_{k+1})} \quad \text{for } n_k \le j < n_{k+1},$$

i.e., two consecutive points of (s_n) in $[a_k, a_{k+1}]$ are at most $\varepsilon/K(a_{k+1})$ apart.

Put $J_\varepsilon f = \bigl(f(s_n)\bigr)_n$; then J_ε maps X into c with $\|J_\varepsilon\| \le 1$. On the other hand, let $f \in X$ and $s \in [0,1)$, say $a_k \le s \le a_{k+1}$. If $s_m \in [a_k, a_{k+1}]$ is the member of (s_n) closest to s, then

$$\begin{aligned} |f(s)| &\le |f(s) - f(s_m)| + |f(s_m)| \\ &\le \sup_{a_k \le t \le a_{k+1}} |f'(t)|\,|s - s_m| + |f(s_m)| \\ &\le K(a_{k+1})\|f\|_\infty \frac{\varepsilon}{K(a_{k+1})} + \|J_\varepsilon f\| \end{aligned}$$

and therefore

$$(1-\varepsilon)\|f\|_\infty \le \|J_\varepsilon f\| \qquad \forall f \in X. \qquad \Box$$

Remarks. (1) If $[0,1)$ is replaced by $[0,1]$ in Theorem 1, then X is finite-dimensional.

(2) Actually Wojtaszczyk's proof is more general in that he only assumes existence of f' on $[0,1)$, but not its continuity; we are using the local boundedness of f'.

(3) Natural examples of spaces satisfying the assumptions of Theorem 1 are the Müntz spaces: If $0=\lambda_0<\lambda_1<\lambda_2<\dots$ is an increasing sequence of integers with $\sum_{n=1}^\infty 1/\lambda_n<\infty$, then every function f in the closed linear span of the monomials t^{λ_n} $(n\in\mathbf{N}_0)$ in $C[0,1]$ has a power series representation

$$f(t)=\sum_{n=0}^\infty c_n t^{\lambda_n}$$

which converges on $[0,1)$. This is a result due to Clarkson and Erdős, see [1, p. 350].

(4) There is also an L_p-version of Theorem 2: *If X is a closed subspace of $L_p[0,1]$ such that each $f\in X$ is continuously differentiable on $[0,1)$, then X is almost isometric to a subspace of ℓ_p.* The proof is similar to the one above. Again, Müntz spaces serve as natural examples. (This also follows from [2].)

References

[1] R. A. DeVore, G. G. Lorentz. *Constructive Approximation.* Springer 1993.

[2] N. Kalton, D. Werner. *Property (M), M-ideals and almost isometric structure of Banach spaces.* J. reine und angew. Mathematik **461** (1995), 137–178.

[3] P. Wojtaszczyk. *Letter to V. Gurarii.*