A remark about Müntz spaces

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Wojtaszczyk [3] recently proved the following theorem.

**Theorem 1** If \( X \) is a closed subspace of \( C[0,1] \) such that each \( f \in X \) is continuously differentiable on \([0,1)\), then \( X \) is isomorphic to a subspace of \( c_0 \).

The proof to follow is clearly modelled after Wojtaszczyk’s proof, but somewhat more pedestrian. Let us show:

**Theorem 2** Under the assumptions of Theorem 1, \( X \) is almost isometric to a subspace of \( c \). That is, for every \( \varepsilon > 0 \) there is an operator \( J_\varepsilon: X \to c \) such that

\[
(1 - \varepsilon) \|f\|_\infty \leq \|J_\varepsilon f\| \leq \|f\|_\infty \quad \forall f \in X.
\]

**Proof.** The idea is to define \( J_\varepsilon f = (f(s_n)) \) for a suitably chosen sequence in \([0,1)\).

Let \( \varepsilon > 0 \). For \( 0 < a < 1 \), the operator \( D: X \to C[0,a], \ f \mapsto f'|_{[0,a]} \) is well-defined and has a closed graph; therefore it is continuous. Thus, there is some \( K(a) > 0 \) such that

\[
\sup_{0 \leq t \leq a} |f'(t)| \leq K(a)\|f\|_\infty \quad \forall f \in X.
\]

We now fix a sequence \( 0 = a_0 < a_1 < a_2 < \ldots < 1 \) converging to 1. Pick points \( 0 = s_0 < s_1 < \ldots < s_{n_1} = a_1 < s_{n_1+1} < \ldots < s_{n_2} = a_2 < \ldots \) in such a way that \( (n_0 = 0) \)

\[
s_{j+1} - s_j \leq \frac{\varepsilon}{K(a_{k+1})} \quad \text{for } n_k \leq j < n_{k+1},
\]

i.e., two consecutive points of \( (s_n) \) in \([a_k, a_{k+1}]\) are at most \( \varepsilon/K(a_{k+1}) \) apart.

Put \( J_\varepsilon f = (f(s_n))_n \); then \( J_\varepsilon \) maps \( X \) into \( c \) with \( \|J_\varepsilon\| \leq 1 \). On the other hand, let \( f \in X \) and \( s \in [0,1) \), say \( a_k \leq s \leq a_{k+1} \). If \( s_m \in [a_k, a_{k+1}] \) is the member of \( (s_n) \) closest to \( s \), then

\[
|f(s)| \leq |f(s) - f(s_m)| + |f(s_m)|
\]

\[
\leq \sup_{a_k \leq t \leq a_{k+1}} |f'(t)||s - s_m| + |f(s_m)|
\]

\[
\leq K(a_{k+1})\|f\|_\infty \cdot \frac{\varepsilon}{K(a_{k+1})} + \|J_\varepsilon f\|
\]
and therefore
\[(1 - \varepsilon)\|f\|_\infty \leq \|J_\varepsilon f\| \quad \forall f \in X. \quad \Box\]

Remarks. (1) If \([0,1)\) is replaced by \([0,1]\) in Theorem 1, then \(X\) is finite-dimensional.

(2) Actually Wojtaszczyk’s proof is more general in that he only assumes existence of \(f'\) on \([0,1)\), but not its continuity; we are using the local boundedness of \(f'\).

(3) Natural examples of spaces satisfying the assumptions of Theorem 1 are the Müntz spaces: If \(0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots\) is an increasing sequence of integers with \(\sum_{n=1}^{\infty} 1/\lambda_n < \infty\), then every function \(f\) in the closed linear span of the monomials \(t^{\lambda_n}\) \((n \in \mathbb{N}_0)\) in \(C[0,1]\) has a power series representation
\[f(t) = \sum_{n=0}^{\infty} c_n t^{\lambda_n}\]
which converges on \([0,1)\). This is a result due to Clarkson and Erdős, see [1, p. 350].

(4) There is also an \(L^p\)-version of Theorem 2: If \(X\) is a closed subspace of \(L^p[0,1]\) such that each \(f \in X\) is continuously differentiable on \([0,1)\), then \(X\) is almost isometric to a subspace of \(\ell^p\). The proof is similar to the one above. Again, Müntz spaces serve as natural examples. (This also follows from [2].)

References

